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New Study of Prabhakar Operators Associated With Inequalities and Its Significant Applications With Different Convexity

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Abstract. Convexity plays a dominant role in the modification of fractional inequalities. Most fractional inequalities are proved based on different types of convexity and fractional operators, which have immense applications in various areas of mathematics. This article aims to investigate the Hermite-Hadamard type inequalities with a different kind of convexity by the implementation of Prabhakar fractional operators. Moreover, we discuss the behavior of trapezoidal type inequalities for the h-Godunova-Levin pre-invex function through Prabhakar fractional operators. Additionally, we present a comparison of our findings with existing literature, which are summarized through corollaries.

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1. Introduction

Fractional integrals and inequalities define an essential area of research in the field of mathematical analysis and its great applications [1, 16–19, 21–23]. These notions are a generalization of classical integral inequalities and provide important new ideas with wide areas of eventual application. With regard to the integral type of inequalities, it is also observed that an impressive development has been achieved in this area of study as these inequalities are becoming more significant to pure and applied mathematics [6, 14].

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The development of the theory of convexity, which is very closely linked to inequality, has progressed considerably. It is worth noting that convex functions are very important in the study and derivation of many integral inequalities [3, 7, 8]. One such is the well-known Hermite-Hadamard inequality. Formulated by Charles Hermite in 1881 and modernized by Jacques Hadamard in 1893, the inequality has been a part of fundamental concepts of convexity . The historical background of the Hermite-Hadamard inequality would suggest that it thrived long before the mid-1970s revival by Dragoslav Mitrinović, who published several books on the history of mathematics in general and convex inequalities in particular [9, 15].

Recently, various generalizations have turned the studies in inequality areas to another range. The classical structure of differentiable convex functions was broadened by the term invexity, introduced in 1981 by Robert Hanson [13], and so new optimization and analysis areas could be developed . Building on this, Mond [28] and Weir [27] further developed the notion of preinvexity, which has been helpful in perfecting optimization theory .

Further advancements have been inclusive of various generalized concepts of convexity that were defined. Dragomir [7] presented the s-Godunova-Levin type convexity, which has been the focus of many studies in the subsequent period [20]. Moreover, the productive concept of h-convexity by Varošanec [26] and also h-Godunova-Levin convexity and preinvexity by Almutari [4] opened new avenues and methods in this area .

This paper study h-Godunova-Levin types of convex and preinvex functions and the fractional integral operators, including the Mittag-Leffler functions, to achieve the new fractional Hermite-Hadamard and the trapezoid inequalities.

2. Preliminaries

In this section, we discuss the basic definitions which help to understand our main results.

Definition 1. *[2] A function $\S : I \to \mathbb{R}$ is termed convex if it satisfies the following condition:*

$$\S[t_o\mathring{s} + (1-t_o)æ] \le t_o\S(\mathring{s}) + (1-t_o)\S(æ),$$

for all $t_o \in [0,1]$, $\mathring{s}, æ \in I$.
Building upon this concept of convexity, we can establish the Hermite-Hadamard (H-H) type inequality as:

$$\S\left(\frac{\mathring{s}+æ}{2}\right) \le \frac{1}{æ-\mathring{s}}\int_{\mathring{s}}^{æ} \S(t_o)\, dt_o \le \frac{\S(\mathring{s})+\S(æ)}{2}. \tag{1}$$

Numerous related results are presented in [25], assuming $\mathring{s}, æ \in I \subseteq \mathbb{R} and \mathring{s} < æ$.

Definition 2. *[24] Consider an invex set $I \subseteq \mathbb{R}$ defined in relation to a bifunction $\S : I \times I \to \mathbb{R}$. For $æ, \mathring{s} \in I$ and $\lambda \in [0,1]$, we define:*

$$\mathring{s} + \lambda\S(æ, \mathring{s}) \in I$$

Definition 3. *[24] A function* $\S : I \to \mathbb{R}$ *is called preinvex for* $æ, \mathring{s} \in I$ *and* $t_o \in [0,1]$ *if:*

$$\S(\mathring{s} + t_o\zeta(æ, \mathring{s})) \leq t_o\S(æ) + (1-t_o)\S(\mathring{s}),$$

where I *is an invex set relative to the binary function* ζ*.*

Definition 4. *[10] A function* $\S : I \subseteq \mathbb{R} \to \mathbb{R}$*, which takes only positive values, is known as a Godunova-Levin function if, for all* $æ, \mathring{s} \in I$ *and* $t_o \in (0,1)$*, the following inequality holds:*

$$\S(t_oæ + (1-t_o)\mathring{s}) \leq \frac{\S(æ)}{t_o} + \frac{\S(\mathring{s})}{1-t_o},$$

for all $æ, \mathring{s} \in I, t_o \in (0,1)$*.*

Definition 5. *[4] Assume* $h : (0,1) \to \mathbb{R}$ *is a non-negative function. We say a function* $\S : I \to \mathbb{R}$ *is h-Godunova-Levin if, for any* $æ, \mathring{s} \in I$ *and* $t_o \in (0,1)$*, the following inequality holds:*

$$\S(t_oæ + (1-t_o)\mathring{s}) \leq \frac{\S(æ)}{h(t_o)} + \frac{\S(\mathring{s})}{h(1-t_o)}.$$

Definition 6. *[4] A function* $\S : I \to \mathbb{R}$ *is called h-Godunova-Levin preinvex with respect to* ζ *if, for any* $æ, \mathring{s} \in I$ *and* $t_o \in (0,1)$*, the inequality*

$$\S(æ + t_o\zeta(\mathring{s}, æ)) \leq \frac{\S(æ)}{h(1-t_o)} + \frac{\S(\mathring{s})}{h(t_o)},$$

is satisfied.

Definition 7. *[12] Let* $\S \in L^1[æ, \mathring{s}]$*, then the Riemann-Liouville left and right fractional integrals are defined as follows:*

$$I^{\alpha}_{æ+}\S(z) = \frac{1}{\Gamma(\alpha)}\int_{æ}^{z}(z-u)^{\alpha-1}\S(u)\,du, \quad z > æ,$$
$$I^{\alpha}_{\mathring{s}-}\S(z) = \frac{1}{\Gamma(\alpha)}\int_{z}^{\mathring{s}}(u-z)^{\alpha-1}\S(u)\,du, \quad z < \mathring{s}.$$

Definition 8. *[29] The gamma function is defined by the following integral representation:*

$$\Gamma(z) = \int_0^{+\infty} u^{z-1}e^{-u}du,$$

for $\Re(z) > 0$*.*

Definition 9. *[29] The Pochhammer symbol is defined as follows:*

$$(z)_k = \begin{cases} 1, & \text{for } k = 0, z \neq 0, \\ z(z+1)\cdots(z+k-1), & \text{for } k \geq 1. \end{cases}$$

For $k \in \mathbb{N}$ *and* $z \in \mathbb{C}$*.*

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)},$$

where Γ *is the gamma function.*

Definition 10. *[5] The Mittag-Leffler function of three parameters:*

$$E^{\alpha}_{\beta,\gamma}(w;p) = \sum_{n=0}^{+\infty} \frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}, \quad (w,\alpha,\beta,\gamma \in \mathbb{C}, \Re(\beta) > 0).$$

Definition 11. *[11] Let $\alpha,\beta,\gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$. Let $\S \in L_1[\ae,\mathring{s}]$ and $x \in [\ae,\mathring{s}]$. Then the left-sided and the right-sided Prabhakar fractional operators $\mathfrak{J}^{\alpha,\gamma}_{\beta;\ae^+}\S$ and $\mathfrak{J}^{\alpha,\gamma}_{\beta;\mathring{s}-}\S$, are defined by*

$$\left(\mathfrak{J}^{\alpha,\gamma}_{\beta;\ae^+}\S\right)(x;r) = \int_{\ae}^{x} (x-t)^{\beta-1} E^{\alpha}_{\beta,\gamma}\left(\omega(x-t)^{\alpha};r\right)\S(t)dt,$$
$$\left(\mathfrak{J}^{\alpha,\gamma}_{\beta;\mathring{s}-}\S\right)(x;r) = \int_{x}^{\mathring{s}} (t-x)^{\beta-1} E^{\alpha}_{\beta,\gamma}\left(\omega(t-x)^{\alpha};r\right)\S(t)dt.$$

In this work, the following notations will be used:

$$\left(\mathfrak{J}^{\ae^+}_{\mathring{s},\beta}\right)(\omega,\S) = \left(\mathfrak{J}^{\alpha,\gamma}_{\beta;\ae^+}\S\right)(\mathring{s},p)$$
$$\left(\mathfrak{J}^{\mathring{s}-}_{\ae,\beta}\right)(\omega,\S) = \left(\mathfrak{J}^{\alpha,\gamma}_{\beta;\mathring{s}-}\S\right)(\ae;p)$$

3. Fractional Analysis of the Hermite-Hadamard (H-H) type Inequalities via h-Godunova-Levin convexity (h-GL).

This section focuses on deriving Hermite-Hadamard type inequalities for h-Godunova-Levin convex functions by means of the Fractional Function Operator, which is detailed below.

Theorem 1. *Let $\S : [\ae,\mathring{s}] \to \mathbb{R}$ be an h-Godunova-Levin convex function, where $0 < \ae < \mathring{s}$ and $\S \in L_1[\ae,\mathring{s}]$. Assume $h : (0,1) \to \mathbb{R}$ is a positive function with $h(t_o) \neq 0$; then,*

$$\frac{h(1/2)}{2}\S\left(\frac{\ae+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{\ae,\beta}\right)(\omega',1) \leq \frac{1}{2}\left[\left(\Im^{\mathring{s}-}_{\ae,\beta}\right)(\omega',\S) + \left(\mathfrak{J}^{\ae^+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq \frac{\S(\ae)+\S(\mathring{s})}{2}\int_0^1 \left[\frac{1}{h(t_o)} + \frac{1}{h(1-t_o)}\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o,$$
$$\omega' = \frac{\omega}{(\mathring{s}-\ae)^{\alpha}}.$$

Proof. Using the h -Godunova-Levin convexity of $\S$ on $[\ae,\mathring{s}]$, let $m,n \in [\ae,\mathring{s}]$, and we obtain

$$\S((\eta)m + (1-\eta)n) \leq \frac{\S(m)}{h(\eta)} + \frac{\S(n)}{h(1-\eta)} \tag{2}$$

For putting the values $m = t_o\ae + (1-t_o)\mathring{s}$, $n = (1-t_o)\ae + t_o\mathring{s}$ and $\eta = \frac{1}{2}$ in equation (2), we have

$$\S\left(\frac{\ae+\mathring{s}}{2}\right) \leq \frac{1}{h(1/2)}[\S(t_o\ae + (1-t_o)\mathring{s}) + \S((1-t_o)\ae + t_o\mathring{s})] \tag{3}$$

Multiplying each side by $(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)$ and integrate the resultant inequality on $[0,1]$ in terms of t_o in the equation (3), we have

$$\begin{aligned}
&§\left(\frac{æ+\mathring{s}}{2}\right)\int_0^1(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o \leq \frac{1}{h(\frac{1}{2})}\times\\
&\left[\int_0^1(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)§(t_oæ+(1-t_o)\mathring{s})\mathrm{d}t_o\right.\\
&\left.+\int_0^1(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)§((1-t_o)æ+t_o\mathring{s})\mathrm{d}t_o\right]\\
&§\left(\frac{æ+\mathring{s}}{2}\right)\sum_{n=0}^{+\infty}\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)}\frac{w^n}{n!}\int_0^1(1-t_o)^{\alpha n+\beta-1}\mathrm{d}t_o \leq \frac{1}{h(1/2)}\sum_{n=0}^{+\infty}\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)}\frac{w^n}{n!}\\
&\quad\times\left[\int_0^1(1-t_o)^{\alpha n+\beta-1}§(t_oæ+(1-t_o)\mathring{s})\mathrm{d}t_o+\int_0^1(1-t_o)^{\alpha n+\beta-1}§((1-t_o)æ+t_o\mathring{s})\mathrm{d}t_o\right]
\end{aligned} \tag{4}$$

By evaluating the integrals in inequality (4), we obtain

$$\frac{h(1/2)}{2}§\left(\frac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\left(\omega',1\right)\right)\leq\frac{1}{2}\left[\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega';§)+\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega';§)\right] \tag{5}$$

For the second half of the inequality, we similarly employ the h-Godunova-Levin convexity of §, we have

$$\begin{aligned}
§(t_oæ+(1-t_o)\mathring{s}) &\leq \frac{§(æ)}{h(t_o)}+\frac{§(\mathring{s})}{h(1-t_o)}\\
§((1-t_o)æ+t_o\mathring{s}) &\leq \frac{§(æ)}{h(1-t_o)}+\frac{§(\mathring{s})}{h(t_o)}
\end{aligned}$$

After adding the above inequalities, we have

$$§(t_oæ+(1-t_o)\mathring{s})+§((1-t_o)æ+t_o\mathring{s})\leq(§(æ)+§(\mathring{s}))\left[\frac{1}{h(t_o)}+\frac{1}{h(1-t_o)}\right] \tag{6}$$

Multiplying both sides by $(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)$ and integrating the resultant inequality on $[0,1]$ with respect to t_o in equation (6), we obtain

$$\begin{aligned}
&\left[\int_0^1(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)§(t_oæ+(1-t_o)\mathring{s})\mathrm{d}t_o\right]\\
&+\left[\int_0^1(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)§((1-t_o)æ+t_o\mathring{s})\mathrm{d}t_o\right]\\
&\quad\leq(§(æ)+§(\mathring{s}))\int_0^1\left[\frac{1}{h(t_o)}+\frac{1}{h(1-t_o)}\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o
\end{aligned} \tag{7}$$

After solving the equation (7), we have

$$\frac{1}{2}\left[\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)+\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega';\S)\right]\leq\frac{\S(æ)+\S(\mathring{s})}{2}\int_0^1\left[\frac{1}{h(t_o)}+\frac{1}{h(1-t_o)}\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o \tag{8}$$

Combining the equations (5) and (8), we obtain

$$\frac{h(1/2)}{2}\S\left(\frac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\frac{1}{2}\left[\left(\Im^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)+\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq\frac{\S(æ)+\S(\mathring{s})}{2}\int_0^1\left[\frac{1}{h(t_o)}+\frac{1}{h(1-t_o)}\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o$$

Corollary 1. *Taking $h(t_o)=t_o^s$ in Theorem (1), we derive an inequality of the (H-H) type for s-Godunova-Levin (GL) type convex functions:*

$$\frac{(1/2)^s}{2}\S\left(\frac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\frac{1}{2}\left[\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)\right.$$
$$\leq\frac{\S(æ)+\S(\mathring{s})}{2}\int_0^1\left[\frac{1}{t_o^s}+\frac{1}{(1-t_o)^s}\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o$$

Corollary 2. *By selecting $h(t_o)=1$ in Theorem (1), we derive an inequality of the (H-H) type for the p function:*

$$\tfrac{1}{2}\S\left(\tfrac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\tfrac{1}{2}\left[\left(\Im^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)+\left(\Im^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq\left(\S(æ)+\S(\mathring{s})\right)\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',1)$$

Corollary 3. *Selecting $h(t_o)=1/t_o$ in Theorem (1), an inequality of the (H-H) type is derived for functions that are convex:*

$$\S\left(\tfrac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\tfrac{1}{2}\left[\left(\Im^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)+\left(\Im^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq\tfrac{\S(æ)+\S(\mathring{s})}{2}\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',1)$$

Corollary 4. *By choosing $h(t_o)=t_o$ in Theorem (1), we obtain an (H-H) type inequality for (G-L) functions:*

$$\tfrac{1}{4}\S\left(\tfrac{æ+\mathring{s}}{2}\right)\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\tfrac{1}{2}\left[\left(\mathfrak{J}^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)+\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq\tfrac{\S(æ)+\S(\mathring{s})}{2}\int_0^1\left[\tfrac{(1-t_o)^{\beta-2}}{1-t_o}\right]\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o$$

Corollary 5. *Choosing $h(t_o)=1/t_o^s$ in Theorem (1), we obtain an (H-H) type inequality for s-convex functions:*

$$2^{s-1}\S\left(\tfrac{æ+\mathring{s}}{2}\right)\left(\Im^{\mathring{s}-}_{æ,\beta}\right)(\omega',1)\leq\tfrac{1}{2}\left[\left(\Im^{\mathring{s}-}_{æ,\beta}\right)(\omega',\S)+\left(\mathfrak{J}^{æ+}_{\mathring{s},\beta}\right)(\omega',\S)\right]$$
$$\leq\tfrac{\S(æ)+\S(\mathring{s})}{2}\int_0^1\left[t_o^s+(1-t_o)^s\right](1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)\mathrm{d}t_o$$

4. On Trapezoidal-Type Inequalities for Prabhakar Functions with Preinvexity Properties of the h-Godunova-Levin Type

In this section, we prove a lemma concerning Prabhakar fractional operators that possess the h-Godunova-Levin preinvexity property. This lemma is crucial for supporting the derivation of our main results.

Lemma 1. *Let $\S : I = [æ, æ + \zeta(\mathring{s}, æ)] \to \mathbb{R}$ be a differentiable function, and let I be a set that is invex with respect to $\zeta : I \times I \to \mathbb{R}$, where $\zeta(\mathring{s}, æ) > 0$ for all $\mathring{s}, æ \in I$. Then*

$$\begin{aligned}&\frac{\S(æ) + \S(æ + \zeta(\mathring{s}, æ))}{2}\Im^{\alpha}_{\beta,\gamma}(\omega; p) - \frac{1}{2\zeta(\mathring{s}, æ)^{\beta-1}} \\ &\times \left[\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega'; \S) + \left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega'; \S)\right] \\ &\quad = \frac{\zeta(\mathring{s}, æ)}{2} I\end{aligned} \tag{9}$$

where

$$I = \int_0^1 (1 - t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1 - t_o)^{\alpha}; p\right) \S'(æ + t_o\zeta(\mathring{s}, æ))dt_o + \int_0^1 (-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha}; p\right) \S'(æ + t_o\zeta(\mathring{s}, æ))dt_o,$$

and $\omega' = (\omega/\zeta(\mathring{s}, æ)^{\alpha})$.

Proof. Consider the integral

$$\begin{aligned}I = &\int_0^1 (1 - t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1 - t_o)^{\alpha}; p\right) \S'(æ + t_o\zeta(\mathring{s}, æ))\mathrm{dt_o} \\ &+ \int_0^1 (-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha}; p\right) \S'(æ + t_o\zeta(\mathring{s}, æ))\mathrm{dt_o}\end{aligned} \tag{10}$$

Let

$$I = I_1 + I_2$$

First, we take the fractional integral I_1, we have

$$I_1 = \sum_{\mathring{s}=0}^{+\infty} \frac{(\gamma)_n}{\Gamma(\beta n + \alpha)} \frac{w^n}{n!} \int_0^1 (1 - t_o)^{\beta-1+\alpha n}\S'(æ + t_o\zeta(\mathring{s}, æ))\mathrm{d}t_o$$

Taking the integration by parts, we have

$$I_1 = \sum_{n=0}^{+\infty} \frac{(\gamma)_n}{\Gamma(\beta n + \alpha)} \frac{w^n}{n!} \times$$

$$\left[(1 - t_o)^{\beta+\alpha n-1} \frac{\S(æ + t_o\zeta(\mathring{s}, æ))}{\zeta(\mathring{s}, æ)}\Bigg|_0^1 - \frac{\beta + \alpha n - 1}{\zeta(\mathring{s}, æ)} \int_0^1 (1 - t_o)^{\beta+\alpha n-2}\S(æ + t_o\zeta(\mathring{s}, æ))\mathrm{d}t_o\right]$$

$$I_1 = \sum_{n=0}^{+\infty} \frac{(\gamma)_n}{\Gamma(\beta n + \alpha)} \frac{w^n}{n!}$$

$$\times\left[\frac{(æ+\zeta(\mathring{s},æ))}{\zeta(\mathring{s},æ)}-\frac{\beta+\alpha n-1}{\zeta(\mathring{s},æ)}\int_0^1(1-t_o)^{\beta+\alpha n-2}\S(æ+t_o\zeta(\mathring{s},æ))\mathrm{d}t_o\right]$$

$$I_1=\frac{\S(æ+\zeta(\mathring{s},æ))}{\zeta(\mathring{s},æ)}\Im^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{(\zeta(\mathring{s},æ))^{\beta}}\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)$$

Continuing in the same manner, we obtain

$$\begin{aligned}I_2=&\frac{\S(æ)}{\zeta(\mathring{s},æ)}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{(\zeta(\mathring{s},æ))^{\beta}}\left(\mathfrak{J}^{æ+\zeta(\mathring{s},æ)^-}_{æ,\beta-1}\right)(\omega',\S)\\ I=&\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{\zeta(\mathring{s},æ)}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{(\zeta(\mathring{s},æ))^{\beta}}\\ &\times\left[\left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega',\S)+\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)\right]\end{aligned}$$

By multiplying by $\zeta(\mathring{s},æ)/2$*, we obtain*

$$\begin{aligned}&\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{2}\Im^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\\ &\times\left[\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega';\S)+\left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega';\S)\right]\\ &=\frac{\zeta(\mathring{s},æ)}{2}I\end{aligned}$$

By Lemma 1, we present the following theorem.

Theorem 2. *Consider a function* $\S$*:* $I=[æ, æ+\zeta(\mathring{s},æ)]\longrightarrow(0,+\infty) with I\in\mathbb{R}$*, and let it be a differentiable function on* I*. Also, suppose that* $|\S'|$ *is a h-Godunova-Levin preinvex function on* I*; then,*

$$\begin{aligned}&\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\\ &\quad\times\left[\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)+\left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega',\S)\right]\\ &\le\frac{\zeta(\mathring{s},æ)}{2}\left(\left|\S'(æ)\right|+\left|\S'(\mathring{s})\right|\right)\int_0^1\sum_{n=0}^{+\infty}\left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)}\frac{w^n}{n!}\right|\\ &\quad\times\left|\frac{(1-t_o)^{\beta+\alpha n-1}-(t_o)^{\beta+\alpha n-1}}{h(t_o)}\right|\mathrm{d}t_o.\end{aligned}$$

Proof.

$$\begin{aligned}&\Bigg|\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\\ &\times\left[\left(\Im^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)+\left(\Im^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega',\S)\right]\Bigg|\end{aligned}$$

$$= \left|\frac{\zeta(\mathring{s}, æ)}{2} I\right|$$
$$\leq \frac{\zeta(\mathring{s}, æ)}{2} \sum_{n=0}^{+\infty} \left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}\right| \int_0^1 \left|(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}\right| \left|§'(æ + t_o \zeta(\mathring{s}, æ))\right| \mathrm{d}t_o$$
$$\leq \frac{\zeta(\mathring{s}, æ)}{2} \sum_{n=0}^{+\infty} \left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}\right|$$
$$\times \int_0^1 \left|(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}\right| \left|\frac{§'(æ)}{h(t_o)} + \frac{§'(\mathring{s})}{h(1-t_o)}\right| \mathrm{d}t_o$$
$$\leq \frac{\zeta(\mathring{s}, æ)}{2} \sum_{n=0}^{+\infty} \left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}\right|$$
$$\times \left[\left|§'(æ)\right| \int_0^1 \left|(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}\right| \frac{1}{h(t_o)} \mathrm{d}t_o + \left|§'(\mathring{s})\right| \right.$$
$$\left. \int_0^1 \left|(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}\right| \frac{1}{h(1-t_o)} \mathrm{d}t_o \right.$$
$$= \frac{\zeta(\mathring{s}, æ)}{2} \left(\left|§'(æ)\right| + \left|§'(\mathring{s})\right|\right) \int_0^1 \sum_{n=0}^{+\infty} \left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}\right|$$
$$\times \left|\frac{(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}}{h(t_o)}\right| \mathrm{d}t_o.$$

Corollary 6. *Taking $\zeta(\mathring{s}, æ) = \mathring{s} - æ$ in Theorem (2), we derive the following inequality:*

$$\frac{§(æ) + §(\mathring{s})}{2} \mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega; p) - \frac{1}{2(\mathring{s} - æ)^{\beta-1}}$$
$$\times \left[\left(\mathfrak{J}^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega', §) + \left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega', §) \right]$$
$$\leq \frac{(\mathring{s} - æ)}{2} \left(\left|§'(æ)\right| + \left|§'(\mathring{s})\right|\right) \int_0^1 \sum_{n=0}^{+\infty} \left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)} \frac{w^n}{n!}\right|$$
$$\times \left|\frac{(1-t_o)^{\beta+\alpha n-1} - (t_o)^{\beta+\alpha n-1}}{h(t_o)}\right| \mathrm{d}t_o.$$

Theorem 3. *Consider the function $§ : I = [æ, æ + \zeta(\mathring{s}, æ)] \longrightarrow (0, +\infty)$, where $I \in \mathbb{R}$, and assume it is differentiable on I. Additionally, let $|§'|^q$ be an h-Godunova-Levin preinvex function on I, with $p > 1$ and $q = \frac{p}{p-1}$; then.*

$$\left| \frac{§(æ) + §(æ + \zeta(\mathring{s}, æ))}{2} \mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega; p) - \frac{1}{2\zeta(\mathring{s}, æ)^{\beta-1}} \right.$$
$$\left. \times \left[\left(\Im^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega', §) + \left(\Im^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega', §) \right] \right|$$

$$\leq \frac{\zeta(\mathring{s},æ)}{2}\left(\left|§'(æ)\right|^q+\left|§'(\mathring{s})\right|^q\right)^{1/q}$$
$$\times\left(\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|^p \mathrm{d}t_o\right)^{1/p}$$
$$\times\left(\int_0^1\frac{1}{h(t_o)}\mathrm{d}t_o\right)^{1/q}.$$

Proof. Using Lemma 1, we have:

$$\left|\frac{§(æ)+§(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\right.$$
$$\times\left[\left(\Im^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',§)+\left(\Im^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega',§)\right]|$$
$$=\left|\frac{\zeta(\mathring{s},æ)}{2}I\right|$$
$$\leq\frac{\zeta(\mathring{s},æ)}{2}\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|$$
$$\times\left|§'(æ+t_o\zeta(\mathring{s},æ))\right|\mathrm{d}t_o.$$

Using Holder integral inequality, we have

$$\leq\frac{\zeta(\mathring{s},æ)}{2}\left(\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|^p\mathrm{d}t_o\right)^{1/p}$$
$$\times\left(\int_0^1\left|§'(æ+t_o\zeta(\mathring{s},æ))\right|^q\mathrm{d}t_o\right)^{1/q}. \tag{11}$$

Since $(1/p)+(1/q)=1$*, and because* $|§'|^q$ *is an (h-GL) preinvex function, we obtain:*

$$\int_0^1\left|§'(æ+t_o\zeta(\mathring{s},æ))\right|^q\mathrm{d}t_o\leq\int_0^1\left(\frac{|§'(æ)|^q}{h(t_o)}+\frac{|§'(\mathring{s})|^q}{h(1-t_o)}\right)\mathrm{d}t_o$$
$$\leq\left(\left|§'(æ)\right|^q+\left|§'(\mathring{s})\right|^q\right)\int_0^1\frac{1}{h(t_o)}\mathrm{d}t_o. \tag{12}$$

Using (12) *in* (11)*, we have the required result.*

Theorem 4. *With the assumptions of Theorem 3, we get the following inequality related to the Hermite-Hadamard inequality:*

$$\left|\frac{§(æ)+§(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\right.$$
$$\times\left[\left(\Im^{æ^+}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',§)+\left(\Im^{(æ+\zeta(\mathring{s},æ))^-}_{æ,\beta-1}\right)(\omega',§)\right]$$
$$\leq\frac{\zeta(\mathring{s},æ)}{2^{1/q}}\left(\left|§'(æ)\right|^q+\left|§'(\mathring{s})\right|^q\right)^{1/q}\left[\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\left(\frac{1}{2}\right)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega\left(\frac{1}{2}\right)^{\alpha};p\right)\right]^{1-(1/q)}$$

$$\times\left(\int_0^1 \frac{\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|}{h(t_o)}\ \mathrm{d}t_o\right)^{\frac{1}{q}}.$$

where $\beta,\alpha\in\mathbb{R}^{+}$.

Proof. According to Lemma 1, we have

$$\left|\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)-\frac{1}{2\zeta(\mathring{s},æ)^{\beta-1}}\right.$$
$$\left.\times\left[\left(\Im^{æ^{+}}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)+\left(\Im^{(æ+\zeta(\mathring{s},æ))^{-}}_{æ,\beta-1}\right)(\omega',\S)\right]\right|$$
$$=\left|\frac{\zeta(\mathring{s},æ)}{2}I\right|$$
$$\leq\frac{\zeta(\mathring{s},æ)}{2}\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|\left|\S'(æ+t_o\zeta(\mathring{s},æ))\right|\mathrm{d}t_o$$

Applying the power mean inequality, we derive:

$$\left|\frac{\S(æ)+\S(æ+\zeta(\mathring{s},æ))}{2}\mathfrak{J}^{\alpha}_{\beta,\gamma}(\omega;p)\right.$$
$$\left.-\frac{1}{2\zeta(\mathring{s},æ)^{\beta}}\left[\left(\mathfrak{J}^{æ^{+}}_{æ+\zeta(\mathring{s},æ),\beta-1}\right)(\omega',\S)+\left(\mathfrak{J}^{(æ+\zeta(\mathring{s},æ))^{-}}_{æ,\beta-1}\right)(\omega',\S)\right]\right|$$
$$\leq\frac{\zeta(\mathring{s},æ)}{2}\left(\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|\mathrm{d}t_o\right)^{1-(1/q)}$$
$$\times\left(\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta+1,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta+1,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|\right.$$
$$\left.\times\left|\S'(æ+t_o\zeta(\mathring{s},æ))\right|^{q}\ \mathrm{d}t_o\right)^{1/q}.$$

Since $|\S'|^{q}$ *is (h-GL) preinvex, we have*

$$\int_0^1\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|\left|\S'(æ+t_o\zeta(\mathring{s},æ))\right|^{q}\ \mathrm{d}t_o$$
$$\leq\int_0^1\frac{\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|}{h(t_o)}\left|\S'(æ)\right|^{q}\ \mathrm{d}t_o$$
$$+\int_0^1\frac{\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|}{h(1-t_o)}\left|\S'(\mathring{s})\right|^{q}\ \mathrm{d}t_o$$

$$=\left(\left|\S'(æ)\right|^{q}+\left|\S'(\mathring{s})\right|^{q}\right)\int_0^1\frac{\left|(1-t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|}{h(t_o)}\ \mathrm{d}t_o.$$

Now consider,

$$
\begin{aligned}
&\int_0^1 \left|(1-t_o)^{\beta}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(1-t_o)^{\alpha};p\right)-(t_o)^{\beta}\mathfrak{J}^{\alpha}_{\beta,\gamma}\left(\omega(t_o)^{\alpha};p\right)\right|\ \mathrm{d}t_o\\
&=\sum_{n=0}^{+\infty}\left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)}\frac{w^n}{n!}\right|\int_0^1\left|(1-t_o)^{\beta+\alpha\mathring{s}-1}-(t_o)^{\beta+\alpha\mathring{s}-1}\right|\mathrm{d}t_o\\
&=\sum_{n=0}^{+\infty}\left|\frac{(\gamma)_n}{\Gamma(\beta n+\alpha)}\frac{w^n}{n!}\right|\\
&\times\left[\int_0^{1/2}\left|(1-t_o)^{\beta+\alpha\mathring{s}-1}-(t_o)^{\beta+\alpha\mathring{s}-1}\right|\mathrm{d}t_o+\int_{1/2}^{1}\left|(1-t_o)^{\beta+\alpha\mathring{s}-1}-(t_o)^{\beta+\alpha\mathring{s}-1}\right|\mathrm{d}t_o\right]\\
&=2\left[\mathfrak{J}^{\alpha}_{\beta+1,\gamma}(\omega;p)-\left(\frac{1}{2}\right)^{\beta-1}\mathfrak{J}^{\alpha}_{\beta+1,\gamma}\left(\omega\left(\frac{1}{2}\right)^{\alpha};p\right)\right].
\end{aligned}
$$

5. Conclusion

In this work, we discussed the refinements of some well known inequalities for different convexity through prabhaker fractional operators. Using the Prabhakar fractional integral operators, Hermite-Hadamard fractional inequalities and trapezoidal inequalities for h Godunova Levin convex and preinvex functions are developed. To obtained some other well known inequalities, and presented in the form of corollaries, which shows the straightened of our main results. Various fractional versions of other recognized inequalities can be derived for h-Godunova-Levin convex and preinvex functions, contributing to significant advancements in the theory of fractional inequalities.

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