EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5735
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Note on the Affine Group Representations

Amjad Saleh Alghamdi[1,*], Taghreed Alqurashi[2]

[1] *Department of Mathematics, Umm Al-Qura University, Makkah, Saudi Arabia*
[2] *Department of Mathematics, Al-Baha University, Albaha, Saudi Arabia*

Abstract. The representation of the affine group is a crucial topic in harmonic analysis. In this paper, we use the wavelet transform to investigate the intertwining operator. This approach helps to compare different unitary representations of the affine group.

2020 Mathematics Subject Classifications: 22D10, 43A32, 22E46, 42C40, 44A10

Key Words and Phrases: Affine group, intertwining operator, unitary representations, wavelet transform, Poisson integral, Laplace transform

1. Introduction

The affine group is a non-commutative, locally compact Lie group of the smallest dimensionality, and it is often used to build wavelets. Gelfand and Naimark[7] initially introduced the unitary representations of the affine group. The induced representations of the affine group from a complex character was explained in [4, 5]. Moreover, in [4] they described the intertwining operators related to Hilbert spaces in terms of representations of the affine group. In this paper, we will illustrate the intertwining operators between all three affine representations as follows:

- the Poisson integral between the quasi-regular representation on $H_2(\mathbb{R})$ and the left regular representation on $L_2(\mathrm{Aff}, d\nu)$.
- the Laplace transform between the co-adjoint representation on $L_2(\mathbb{R}_+)$ and the left regular representation on $L_2(\mathrm{Aff}, d\nu)$.
- the Fourier transform between the quasi-regular representation on $H_2(\mathbb{R})$ and the co-adjoint representation on $L_2(\mathbb{R}_+)$.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5735

Email addresses: asmghamdi@uqu.edu.sa (A. Alghamdi), talqorashi@bu.edu.sa (T. Alqurashi)

2. The Affine Group

An element of the affine group Aff [4, 5] is denoted by (a, b), where $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$. The group operation on Aff is defined by

$$(a, b) * (a', b') = (aa', ab' + b), \tag{1}$$

where $e = (1, 0)$ is the identity element and the inverse of (a, b) is given by $(a, b)^{-1} = (a^{-1}, -ba^{-1})$. We can decompose the affine group as a semi-direct product $\text{Aff} = A \ltimes N$. The subgroup N, defined by $\{(1, b) : b \in \mathbb{R}\}$, is a normal subgroup that can be identified with $\mathbb{R}$ by mapping $(1, b) \leftrightarrow b$. Moreover, the subgroup $A = \{(a, 0) : a > 0\}$ is identified with $\mathbb{R}_+$ where $(a, 0) \leftrightarrow a$, [8].

The affine group is locally compact. Thus, it has a left Haar measure, which is given as follows:

$$d\nu(a, b) = a^{-2} da db, \tag{2}$$

and it is left invariant measure that is $d\nu((a', b') * (a, b)) = d\nu(a, b)$. In addition, we can obtain a right Haar measure

$$d\mu(a, b) = a^{-1} da db,$$

which is right invariant. Therefore, the affine group is non-unimodular , and the modular function of the group is given by $\triangle(a, b) = a^{-1}$ [8]. The measure on the subgroup A is the Haar measure $\frac{da}{a}$, and on the subgroup N is the Lebesgue measure db.

3. Unitary representations of the affine Group

The affine group has three unitary representations[4, 5] given in the following:

- Left regular representation

$$\big[\Lambda(a, b)F\big](x, y) := F((a, b)^{-1} * (x, y)) = F\left(\frac{x}{a}, \frac{y-b}{a}\right), \tag{3}$$

  where $(x, y) \in \text{Aff}$.

- Co-adjoint representation on the half real lines

$$[\rho^{\pm}(a, b)g](x) = \sqrt{a} e^{2\pi i b x} g(ax), \tag{4}$$

  where $g \in L_2(\mathbb{R}_{\pm}, da)$.

- Quasi-regular representation on the real line. The Hilbert space $L_2(\mathbb{R})$ with respect to π contains precisely two closed proper invariant subspaces $H_2(\mathbb{R})$ and $H_2^{\perp}(\mathbb{R})$ such that

$$L_2(\mathbb{R}) = H_2(\mathbb{R}) \oplus H_2^{\perp}(\mathbb{R}).$$

Therefore, the quasi-regular representation π is decomposed into two irreducible representations. That is

$$\pi(a,b) = \pi^+(a,b) \oplus \pi^-(a,b).$$

The operator of representation $\pi^+ : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ is given by the following:

$$[\pi^+(a,b)f](x) = \left(\frac{1}{a}\right)^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right), \tag{5}$$

where $f \in H_2(\mathbb{R})$. Also, for the representation $\pi^- : H_2^\perp(\mathbb{R}) \to H_2^\perp(\mathbb{R})$, the operator is given by (5) where $f \in H_2^\perp(\mathbb{R})$.

4. Intertwining Operators

In this section, we study the intertwining operators between the unitary representations of the affine group by using the wavelet transform and the induced wavelet transform.

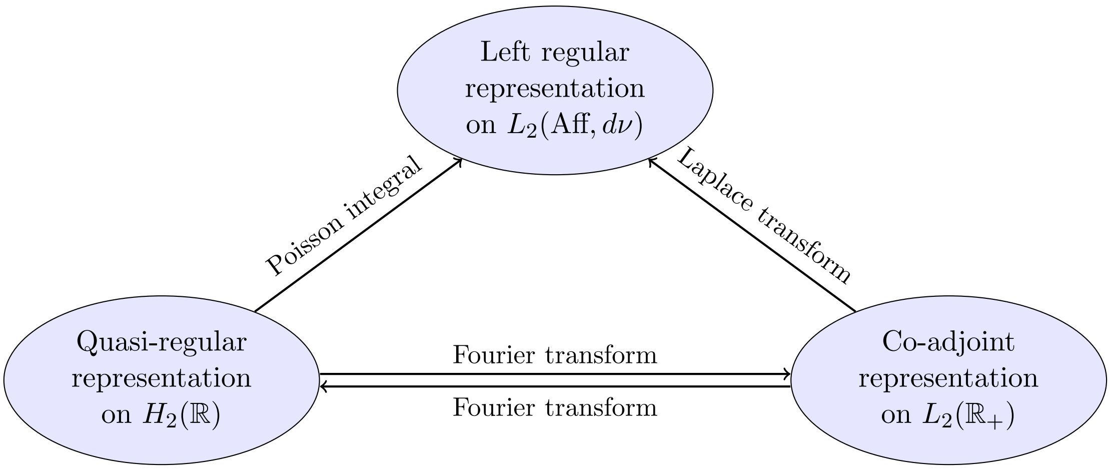

Figure 1: Intertwining operators between affine group representations

4.1. Wavelet Transform

Definition 1. *[12] Let V be a Hilbert space with an inner product $\langle .,. \rangle$ and ρ be a unitary representation of a group G in the space V. Let $F : V \to \mathbb{C}$ be the functional $\upsilon \mapsto \langle \upsilon, \upsilon_0 \rangle$ defined by a vector $\upsilon_0 \in V$. The vector υ_0 is called the mother wavelet. We define a wavelet transform $\mathcal{W}$ acting from V to the space $L_2(G,\mathbb{C})$ of $\mathbb{C}$-valued functions on G by the formula:*

$$\mathcal{W} : \upsilon \mapsto \tilde{\upsilon}(g) = \langle \rho(g^{-1})\upsilon, \upsilon_0 \rangle = \langle \upsilon, \rho(g)\upsilon_0 \rangle, \quad \upsilon \in V, g \in G. \tag{6}$$

The collection of the vectors $\upsilon_g = \rho(g)\upsilon_0$ is called wavelets.

Theorem 1. *[12] The wavelet transform* (6) *intertwines the unitary representation* ρ *and the left regular representation on* $L_2(G, \mathbb{C})$*:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

Corollary 1. *The Poisson integral for* $a \in \mathbb{R}_+$ *and* $b \in \mathbb{R}$ *is given by*

$$[P\varphi](a,b) = \frac{1}{2}\int_{\mathbb{R}} \frac{a}{(x-b)^2 + a^2}\varphi(x)dx. \tag{7}$$

It is the wavelet transform that intertwines the quasi-regular representations $\pi^{\pm}$ (5) *with the left regular representation* $\Lambda(a,b)$ *given by* (3).

Proof. We will prove it for the representation π^+ (5) and the result is valid for the representation π^-. Let the fiducial operator $F : H_2(\mathbb{R}) \to \mathbb{C}$ be the functional $\varphi \mapsto \langle \varphi, \varphi_0 \rangle$, and the mother wavelet be the conjugate Poisson kernel $\varphi_0 = \frac{-x}{\pi(1+x^2)}$. Then, $\bar{\varphi_0} = \frac{1}{\pi(1+x^2)}$ is the Poisson kernel. Hence, the wavelet transform that intertwines the quasi-regular representation with the left regular representation is given as follows:

$$\begin{aligned}
[\mathcal{W}\pi^+(a,b)\varphi](x) &= \langle \varphi, \pi^+(a,b)\varphi_0 \rangle \\
&= \int_{\mathbb{R}} \varphi(x) \frac{1}{\sqrt{a}} \bar{\varphi_0}\left(\frac{x-b}{a}\right) dx \\
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \varphi(x) \frac{1}{\pi(1 + (\frac{x-b}{a})^2)} dx \\
&= \frac{1}{\sqrt{a}\pi} \int_{\mathbb{R}} \varphi(x) \frac{a^2}{a^2 + (x-b)^2} dx \\
&= \frac{\sqrt{a}}{\pi} \int_{\mathbb{R}} \varphi(x) \frac{a}{a^2 + (x-b)^2} dx \\
&= \frac{2\sqrt{a}}{\pi}[P\varphi](a,b).
\end{aligned}$$

Corollary 2. *The Laplace transform*

$$F(a+ib) = \int_{\mathbb{R}_+} f(t)e^{-2\pi(a+ib)x}dx, \quad a+ib \in \mathbb{C},$$

is the wavelet transform that intertwines the co-adjoint representation of the affine group $\rho^{\pm}_{\chi}$, *with the left regular representation* $\Lambda(a,b)$ *given by* (3).

Proof. It is enough to prove the corollary for the representation ρ^+(4). The result works for ρ^-. Let the fiducial operator $F : L_2(\mathbb{R}_+) \to \mathbb{C}$ be the functional $f \mapsto \langle f, f_0 \rangle$, and the mother wavelet be $f_0(\lambda) = e^{2\pi\zeta}$. Then, the wavelet transform is given as follows:

$$\begin{aligned}
[\mathcal{W}_{f_0}\rho^+_\chi(a,b)f](\zeta) &= \langle f, \rho^+(a,b)f_0\rangle \\
&= \int_{\mathbb{R}_+} f(\zeta)\overline{\rho^+(a,b)f_0(\zeta)}d\zeta \\
&= \int_{\mathbb{R}_+} f(\zeta)\sqrt{a}e^{-2\pi i b\zeta}\overline{f_0(a\zeta)}d\zeta \\
&= \sqrt{a}\int_{\mathbb{R}_+} f(\zeta)e^{-2i\pi b\zeta}e^{-2\pi a\zeta}d\zeta \\
&= \sqrt{a}\int_{\mathbb{R}_+} f(\lambda)e^{-2\pi(a+ib)\zeta}d\zeta = \sqrt{a}F(a+ib).
\end{aligned}$$

4.2. Induced wavelet Transform

In this subsection, we will study the wavelet transform that produces functions on a homogeneous space rather than the entire group.

Definition 2. *[9] Let H be a closed subgroup of the group G, and $\mathcal{H}$ is a Hilbert space. For some character χ of H where $h \in H$ and ρ is a unitary representation of the group G in the space $\mathcal{H}$ there is a mother wavelet $\upsilon_0 \in \mathcal{H}$ such that:*

$$\rho(h)\upsilon_0 = \chi(h)\upsilon_0. \tag{8}$$

Then, for any continuous section $s : G/H \to G$ the induced wavelet transform is given as follows:

$$\mathcal{W}_{\upsilon_0} : \upsilon \mapsto \tilde{\upsilon}(x) = \langle \upsilon, \rho(s(x))\upsilon_0\rangle, \quad \textit{where} \quad x \in G/H \tag{9}$$

intertwines ρ with the representation ρ_χ in a certain function space on the homogeneous space G/H induced by the character χ of H.

Corollary 3. *The induced wavelet transform that intertwines respectively the quasi-regular representations π^+ and π^-(5) with the co-adjoint representation ρ^+ and ρ^- (4) is the Fourier transform*

$$[\mathcal{F}f](\lambda) = \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\lambda}dx, \quad \lambda \in \mathbb{R}. \tag{10}$$

Proof. For simplicity, it suffices to prove the corollary for the representation π^+(5). The same argument is valid for π^-. Let the mother wavelet be $\psi_0(x) = e^{2\pi i x}$. It is clear that ψ_0 satisfies the following condition:

$$\pi^+(1,b)\psi_0 = \chi(1,b)\psi_0,$$

where $\chi(1,b) = e^{2\pi ib}$ is the character of the subgroup N. Let $\mathsf{s} : \mathbb{R}_+ \to \mathrm{Aff}$ be the continuous section defined as $\mathsf{s}(a) = (a, 0)$ where $a \in \mathbb{R}_+$. Then, for $f \in H_2(\mathbb{R})$, we calculate the induced wavelet transform as follows:

$$\begin{aligned}[\mathcal{W}_{\psi_0} f](\lambda) &= \langle f, \pi^+(s(\lambda))\psi_0\rangle \\ &= \langle f, \pi^+(\lambda, 0)\psi_0\rangle \\ &= \int_{\mathbb{R}} f(x)\overline{\pi^+(\lambda, 0)\psi_0(x)}dx \\ &= \frac{1}{\sqrt{\lambda}}\int_{\mathbb{R}} f(x)\overline{\psi_0}\left(\frac{x}{\lambda}\right)dx \\ &= \frac{1}{\sqrt{\lambda}}\int_{\mathbb{R}} f(x)e^{-2\pi i\frac{x}{\lambda}}dx = \frac{1}{\sqrt{\lambda}}\hat{f}\left(\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{R}_+.\end{aligned}$$

This is the Fourier transform. Next, for the co-adjoint representation ρ^+, the mother wavelet $\psi_0(x) = 1$ satisfies the condition

$$\rho^+(a, 0)\psi_0 = \chi(a, 0)\psi_0,$$

where $\chi(a, 0) = a^{\frac{1}{2}}$ is the character of the subgroup A. Let $\mathsf{s} : \mathbb{R} \to \mathrm{Aff}$ be the continuous section defined as $\mathsf{s}(b) = (1, b)$, where $b \in \mathbb{R}$. Then, for $g \in L_2(\mathbb{R}_+)$ the induced wavelet transform $[\mathcal{W}_{\psi_0} g](\xi) = \langle g, \rho^+(s(\xi))\psi_0\rangle$ where $\xi \in \mathbb{R}$, is the Fourier transform.

5. Conclusion

We demonstrated the intertwining operator of the affine group, which allows a connection between the representations while preserving the action of the affine group. We find that the Laplace transform intertwines the co-adjoint representation with the left-regular representation of the affine group. Moreover, the Poisson integral is the intertwining operator between the quasi-regular and left-regular representations. Finally, the Fourier transform intertwines the co-adjoint with the quasi-regular representation.

References

[1] Syed Ali, Jean-Pierre Antoine, and Jean-Pierre Gazeau. *Coherent states, wavelets, and their generalizations. 2nd updated ed.* Springer New York, NY, 01 2014.

[2] Taghreed Alqurashi and Vladimir V. Kisil. Metamorphism as a covariant transform for the ssr group. *Boletín de la Sociedad Matemática Mexicana*, 29(2), May 2023.

[3] Ingrid Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[4] Abdelhamid S. Elmabrok and Ondrej Hutník. Induced representations of the affine group and intertwining operators: I. Analytical approach. *J. Phys. A*, 45(24):244017, 15, 2012.

[5] G.B. Folland. *A course in abstract harmonic analysis, second edition*. Chapman and Hall/CRC, 02 2015.

[6] Jason Fulman and Robert M. Guralnick. Enumeration of conjugacy classes in affine groups. *Algebra Number Theory*, 18(6):1189–1219, 2024.

[7] I. M. Gelfand and M. A. Naimark. *Unitary representations of the group of linear transformations of the straight line*. Dokl. Akad. Nauk SSSR 55 , 567–570. also p. 18–21 in Gelfand's Collected Papers, vol II, Springer-Verlag, Berlin, 1988, 1947.

[8] Eberhard Kaniuth and Keith F Taylor. *Induced representations of locally compact groups*, volume 197. Cambridge university press, 2013.

[9] V.V. Kisil. *Symmetry, geometry and quantization with hypercomplex numbers*, pages 11–76. Bulgar. Acad. Sci., Sofia, 2017.

[10] Giorgis Petridis, Oliver Roche-Newton, Misha Rudnev, and Audie Warren. An Energy Bound in the Affine Group. *International Mathematics Research Notices*, 2022(2):1154–1172, 2020.

[11] Rendra Soekarta and Miftah Sigit. Implementation of affine group algebra on digital image security. *Mobile and Forensics*, 4:137–146, 2023.

[12] V.V.Kisil. The real and complex techniques in harmonic analysis from the point of view of covariant transform. *Eurasian Math. J.5,95-121*, 2014.

[13] M. W. Wong. *Wavelet transforms and localization operators*, volume 136 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2002.