**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2025, Vol. 18, Issue 2, Article Number 5750
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Flux at Infinity of Subharmonic Functions on $\mathbb{R}^2$

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**Abstract.** For a $C^2$-function $f(x)$ on a bounded domain $\omega$ in $\mathbb{R}^2$ the flux is defined by means of outer normal derivative of $f$. In this paper, we introduce the notion of $flux(f)$ for any real-valued function on $\mathbb{R}^2$. We define flux on bounded domain $\omega$ and take limits when $\omega$ grows into $\mathbb{R}^2$ and the limit is defined as "at infinity", the $flux(f)$ at infinity denoted as $flux_\infty f$. This limit $flux_\infty f$ may or may not be finite. The related development is carried out by employing the notion of inversion on $\mathbb{R}^2$ and the fact that a harmonic function defined outside a compact set in $\mathbb{R}^2$ is the difference of two subharmonic functions on $\mathbb{R}^2$ that are harmonic outside a compact set.

**2020 Mathematics Subject Classifications**: 31A05, 31A10

**Key Words and Phrases**: Subharmonic Function, Harmonic Function, Flux

## 1. Introduction

In the Euclidean plane $\mathbb{R}^2$, let $\omega$ be a bounded domain with smooth boundary and $f(x)$ be a $C^2-$function defined on a neighbourhood of $\overline{\omega}$. Then the exit flux (the flux acting outwards from a closed surface) of $f$ from $\omega$ is defined as $\int\limits_{\partial\omega} \frac{\partial f}{\partial n^+} ds$ where $\frac{\partial}{\partial n^+}$ is the outward normal derivative at boundary points [1]. By using Green's theorem, the function $f(x)$ is harmonic on $\omega$ if and only if the exit flux of $f$ from $\omega$ is 0. There is also a theorem in (Brelot [2]), by using a series representation for harmonic functions defined outside a compact set in $\mathbb{R}^2$, it states that given a harmonic function $h$ outside a compact set in $\mathbb{R}^2$, there exists a harmonic function $H$ on $\mathbb{R}^2$ such that $|h - H|$ is bounded if and only if the flux at infinity of $h$ is 0. Now, what is the relation between these two notions of flux in $\mathbb{R}^2$? In this article, we find an answer to this question by using more generally, subharmonic function on $\mathbb{R}^2$ which is locally Lebesgue integrable functions with a certain mean value property.

First we extend the definition of flux to a subharmonic function defined on a bounded domain in $\mathbb{R}^2$ by using the distributions. Then we introduce the notion of flux at infinity for a subharmonic function defined outside a compact set in $\mathbb{R}^2$. For this we require the

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5750

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measure associated with a subharmonic function in a local Riesz representation as an integral, (Helms [3], Ransford [4]). We prove that for a subharmonic function $s(x)$ defined on $\mathbb{R}^2$, the flux at infinity of $s$ is finite if and only if the total measure associated with $s$ in $\mathbb{R}^2$ is finite or equivalently if and only if $s(x)$ has a harmonic majorant outside a compact set.

As an aside, we have also that if $s(x)$ is a subharmonic function outside a compact set in $\mathbb{R}^2$, then $s(x) = v(x) - \alpha_s \log|x|$ where $v(x)$ is subharmonic on $\mathbb{R}^2$, $\alpha_s \geq 0$ a constant.

We define the flux at infinity of $s(x)$ as $[flux_\infty s(x)] =$ [total measure associated with v(x) on $\mathbb{R}^2$] $- \alpha_s$; this $[flux_\infty s(x)]$ is independent of the representation of $s(x) = v(x) - \alpha_s \log|x|$, which may or may not be finite. We deduce that if $h(x)$ is a function harmonic outside a compact set then $h(x) = H(x) + \alpha_h \log|x| + b(x)$, where $H(x)$ is harmonic on $\mathbb{R}^2$, and $\alpha_h$ is uniquely determined constant and $b(x)$ is bounded harmonic tending to 0 at infinity. We conclude with the result that if the subharmonic function $s(x)$ outside a compact set in $\mathbb{R}^2$ has a harmonic majorant, then the flux at infinity of $s(x)$ is $\alpha_h$ where $h(x)$ is the least harmonic majorant of $s(x)$ outside a compact set.

In the Brelot's axiomatic potential theory without positive potentials, a superharmonic function $s(x)$ on the harmonic space $X$ is termed as admissible in (Anandam [5, 6]), if it has a harmonic minorant outside a compact set. Therein is a definition of flux at infinity of a harmonic function outside a compact set in $X$. But only with the three axioms of (Brelot [7]), it is not possible to introduce a comprehensive comparative study of the total measure of a superharmonic function and its flux at infinity on a harmonic space (that is locally compact) without positive potentials. See also the paper (Bajunaid et. al. [8]).

If $f(x)$ is a $C^2$-function defined on a bounded domain $\bar{\omega}$ with smooth boundary, then the $flux(f)$ is defined by means of the outer normal derivatives of $f$. In this note, we introduce the notion of the linear function $flux(f)$ for any real-valued function $f$ defined on $\mathbb{R}^2$. Since any real-valued function $f$ on $\mathbb{R}^2$ is the difference of two subharmonic functions, it is enough to define $flux(s)$ for any subharmonic functions $s$ on $\mathbb{R}^2$ with associated Radon measure $\mu$ in a local Riesz representation as a sum of an integral with respect to $\mu$ and a harmonic function.

If $u$ is a $C^2$-function on $\mathcal{D} = \{x : |x| \leq r\}$ in $\mathbb{R}^2$, then $\int\limits_{|x|<r} \Delta u(x)dx = \int\limits_{|z|=r} \frac{\partial u}{\partial n^+}(z)d\sigma$ which is the flux($u$) on $\mathcal{D}$. If $s$ is any subharmonic function on $\mathbb{R}^2$, then there exists an increasing sequence of $C^2$-subharmonic functions tending to $s$. Thus, we can write $\int\limits_{|x|<r} \Delta s(x)dx = \int\limits_{|z|=r} \frac{\partial s}{\partial n^+}(z)d\sigma$. Since $\Delta s \geq 0$ then $\Delta s$ defines a Radon measure $\mu$ on $\mathbb{R}^2$ and we define flux($s$) $= \mu(\mathbb{R}^2) = \lim\limits_{r\to\infty} \int \frac{\partial s}{\partial n^+} d\sigma$ at infinity. Thus the $flux(s)$ at infinity may or may not be finite.

By using inversion, we show that $flux(s)$ is finite if and only if it has a harmonic majorant $h$ outside a compact set. Any such harmonic function outside a compact set has a unique representation $h(x) = H(x) + \alpha \log|x| + b(x)$ where $H(x)$ is harmonic on $\mathbb{R}^2$, $\alpha$ is a constant and $b(x) \to 0$ at infinity. Finally, we prove that if $h(x)$ is the least harmonic majorant of $s(x)$ outside a compact set, then flux($s$) $= \alpha$.

## 2. Preliminaries

An upper and semi-continuous function $u$, $-\infty \leq u(x) < \infty$, $u \not\equiv -\infty$ on $\mathbb{R}^2$ is said to be subharmonic if for any $x = re^{i\theta}$, where $R$ is the distance of $\partial\omega$ from the origin.

$$u(x) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta - \phi + r^2} u(Re^{i\theta}) d\phi$$

Suppose $v(x)$ is a locally Lebesgue integrable function in $\mathbb{R}^2$ such that $\Delta v \geq 0$, in the sense of distributions, then there exists a unique subharmonic $u(x)$ such that $v(x) = u(x)$ almost everywhere. For any locally integrable function $f(x)$, there exists a function $\phi(x)$ such that $\Delta\phi(x) = f(x)$, (Brelot [2]); write $\Delta s_1(x) = g^+(x)$ and $\Delta s_2(x) = g^-(x)$ where $g(x) = \Delta f(x)$, then $s_1(x)$ and $s_2(x)$ are subharmonic on $\mathbb{R}^2$ and $\Delta[f - (s_1 - s_2)] = 0$ so that $f - (s_1 - s_2)$ is a harmonic function $h$ almost everywhere. Thus any locally integrable function is the difference of two subharmonic functions almost everywhere.

For a subharmonic function $u(x)$, $\Delta u \geq 0$ in the sense of distributions, therefore $\int_{2\pi} d\mu(x) = \Delta u(x)$ is a Radon measure $\mu$ on $\mathbb{R}^2$; furthermore, for any bounded domain $\omega$, $u(x) = \int_{y\in\omega} \log|x - y| d\mu(y) + h(x)$ where $h(x)$ is a harmonic function on $\omega$. The total measure $\mu(\mathbb{R}^2)$ associated with $u$ may or may not be finite; $\mu(\mathbb{R}^2) = 0$ if and only if $u(x)$ is a harmonic function.

If $f(x)$ is a $C^2-$function defined on a neighbourhood of $\overline{\omega}$ where $\omega$ is a bounded domain with smooth boundary $\partial\omega$, then by Green's theorem $\int_\omega \Delta f(x)dx = \int_{\partial\omega} \frac{\partial f}{\partial n^+} ds$, where $\frac{\partial}{\partial n^+}$ is the outward normal derivative. In particular, for a $C^2-$subharmonic function $u$ in $\mathbb{R}^2$, the total measure of $u$ is $\lim_{\omega \longrightarrow \mathbb{R}^2} \int_{\partial\omega} \frac{\partial u}{\partial n^+} ds = \lim_{m\longrightarrow\infty} \int_{\partial\omega_m} \frac{\partial u}{\partial n^+} ds$.

Suppose now that $s(x)$ is a subharmonic function on $\mathbb{R}^2$. Then (see Brelot [2]) there exists an increasing sequence $\{s_n\}$ of $C^2-$functions tending to $s$ on $\mathbb{R}^2$. Then $\Delta s_n \longrightarrow \Delta s$ so that the total measure associated with $s$ equals to $\lim_{n\longrightarrow\infty}$ total measure of $s_n$. Remark that this limit does not depend on the particular sequence $\{s_n\}$ of $C^2-$subharmonic functions that increases to $s$.

## 3. Inversion and associated measure of a subharmonic function

Let $s(x)$ be subharmonic on $\mathbb{R}^2$. Then $\Delta s(x) \geq 0$ in the sense of distributions [9, 10], and hence a Radon measure $\mu(x)$, $d\mu(x) = \frac{1}{2\pi}\Delta s(x)dx$ such that locally (i.e. in a neighbourhood $N$ of a point $x$) $s(x) = \int_N \log|x - y|d\mu(y) +$ a harmonic function on $N$.

**Definition 1.** *(Inversion) [11] A map* $x \to x^* = \frac{x}{|x|^2}$ *if* $x \neq 0, \infty$; $0 \to \infty$ *and* $\infty \to 0$ *is known as an inversion (see for example: Axler et al. pp 59–61).*
*If* $u(x)$ *is a subharmonic (respectively harmonic) function defined outside a compact set*

*then* $u(x^*) = u\left(\frac{x}{|x|^2}\right)$ *is a subharmonic (respectively harmonic) function in a neighbourhood of* $0$*, excluding* $0$*. For a set* $E$ *in* $\mathbb{R}^2$ *write* $E^* = \{x^* : x \in E\}$*.*

**Theorem 1.** *Let* $u(x)$ *be a subharmonic function defined outside a compact set* $K$ *with associated measure* $\lambda$*. Then* $u(x)$ *has a harmonic majorant* $h(x)$ *outside a compact set* $K$ *if and only if* $\int\limits_{\mathbb{R}^2\setminus K} d\lambda(x) < \infty$*.*

*Proof.* Write $z = x^*$. Then $v(z) = u(z) - h(z)$ is an upper bounded subharmonic function on $0 < |z| < \epsilon$; hence extends as a subharmonic function $v(z)$ on $|z| < \epsilon$ whose associated measure is denoted by $\sigma(x)$ so that $v(z) = \int\limits_{|y|<\epsilon} \log|z-y|d\sigma(y) +$ a harmonic function on $|z| < \epsilon$. Note that $\int\limits_{|y|\leq\frac{\epsilon}{2}} d\sigma(y)$ is finite and $\sigma = \lambda$ on any compact set in $0 < |z| < \frac{\epsilon}{2}$.
Hence $\lambda(E)$ where $E = (z : 0 < |z| < \frac{\epsilon}{2})$ is finite. Take an inversion to conclude that $\lambda(E^*)$ is finite leading to the conclusion $\lambda(\mathbb{R}^2\setminus K)$ is finite. Note that $E^*$ is a neighbourhood of the point at infiniy.
<u>Converse:</u> If $\int\limits_{\mathbb{R}^2\setminus K} d\lambda(x) < \infty$, define a Radon measure $\nu$ on $N = \{z : |z| < \epsilon\}$ by taking for any compact set $A \in \mathbb{R}^2$, $\nu(A) = \lambda[(A\cap N)^*]$ where $(A \cap N)^*$ is the inversion set of $(A \cap N)$. Note $\nu$ is a Radon measure on $\mathbb{R}^2$ with compact harmonic support taking $\nu(\{0\}) = 0$.

Write $z = x^*$, $x$ in a neighbourhood of infinity. Then $s(x) = \int\limits_{\mathbb{R}^2} \log|x-y|d\nu(y)$ is subharmonic on $\mathbb{R}^2$. Hence $\exists$ a harmonic function $h(x) \geq s(x)$ on $N$. Note that $s(z)$ on $N\setminus\{0\}$ is the same as $u(x^*) = u(z)$ up to an additive harmonic function $v(z)$. Also note that on $N\setminus\{0\}$, the harmonic functions $h(z)$ and $v(z)$ can also be written as $h(x^*)$ and $v(x^*)$ so that we have $h(x^*) \geq u(x^*) + v(x^*)$ on $N\setminus\{0\}$. An inversion shows that $h(x) \geq u(x) + v(x)$. Thus $u(x) \leq h(x) - v(x)$ near infinity. That is $u(x)$ is majorised by a harmonic function near infinity if the associated measure $\lambda$ of $u$ is such that $\int\limits_{\mathbb{R}^2\setminus K} d\lambda(x) < \infty$.

**Corollary 1.** *Let* $s(x)$ *be a subharmonic function on* $\mathbb{R}^2$ *with associated measure* $\mu(x)$*. Then* $s(x)$ *has a harmonic majorant outside a compact set if and only if* $\int\limits_X d\mu(x) < \infty$*.*

*Proof.* Follows from the theorem 1

## 4. Flux at infinity

Suppose $u(x)$ is a subharmonic function defined outside a compact set. Then there exists a subharmonic function $s(x)$ that is harmonic on a neighbourhood of 0 and a constant $\alpha \geq 0$ such that $u(x) = s(x) - \alpha\log|x|$ outside a disc (Anandam [12]).

**Definition 2.** *Let $u(x)$ be a subharmonic function defined outside of a compact set. Let $u(x) = s(x) - \alpha \log|x|$. Then define the $flux_\infty u$ = flux at infinity of $u(x)$ = [Total measure of $s(x)$]$-\alpha$.*

**Remark 1.** *If $u(x) = s_1(x) - \alpha_1 \log|x|$ is another such decomposition outside a compact set, then $s(x) - \alpha \log|x| = s_1(x) - \alpha_1 \log|x|$ outside a compact set. Since $v(x) = s(x) + \alpha_1 \log|x| = s_1(x) + \alpha \log|x|$ outside a compact set represents two forms of the same subharmonic function $v(x)$, [total measure of $v(x)$] can be calculated by using the normal derivative form in the Green's theorem.*
*Hence,*

$$\text{total measure of } s_1 + \alpha_2 = \text{total measure of } s_2 + \alpha_1,$$

*so that*

$$\text{total measure of } s_1 - \alpha_1 = \text{total measure of } s_2 - \alpha_2$$

**Remark 2.** *If $s(x)$ is a subharmonic function on $\mathbb{R}^2$, then total measure of $s(x) = flux_\infty s(x)$*

**Theorem 2.** *Let $h(x)$ be a harmonic function defined outside a compact set in $\mathbb{R}^2$. Then $h(x) = H(x) + \alpha \log|x| + b(x)$ where $H(x)$ is harmonic on $\mathbb{R}^2$, $\alpha$ - a constant, and $b(x)$ bounded harmonic tending to 0 at infinity. This representation is unique and $[flux_\infty h(x)] = \alpha$.*

*Proof.* Write $h(x) = s(x) - \beta \log|x|$, $\beta \geq 0$ and $s(x)$ subharmonic on $\mathbb{R}^2$. The function $s(x)$ should be harmonic outside a compact set $K$. Hence $s(x) = \int\limits_K \log|x-y| d\mu(y) +$ a harmonic function $H(x)$ on $\mathbb{R}^2$.

Since the integral equals $\mu(K) \log|x| + b(x)$ outside a compact set where $b(x) \longrightarrow 0$ at infinity, we have

$$\begin{aligned} h(x) &= H(x) + [\mu(K) - \beta] \log|x| + b(x) \\ &= H(x) + \alpha \log|x| + b(x), \text{writing } \alpha = \mu(K) - \beta \end{aligned}$$

Uniqueness of the representation: Suppose $h(x) = H_1(x) + \alpha_1 \log|x| + b_1(x)$, then $H(x) + \alpha \log|x| + b(x) = H_1(x) + \alpha_1 \log|x| + b_1(x)$ outside a compact set.

Now for large $R$, take the integral $\frac{1}{2\pi R}\int\limits_0^{2\pi}$ on $|x| = R$, to see

$$H(0) - H_1(0) = (\alpha_1 - \alpha) \log R + [\text{a bounded function}]$$

. This is possible for all large $R$ if and only if $\alpha_1 = \alpha$ in which case $H(x) - H_1(x)$ is bounded, hence a constant tending to 0. Therefore $H_1 = H$, proving the uniqueness of the representation.
flux at infinity of h: Since $h(x) = s(x) - \beta \log|x|$, then $flux_\infty h$ = [total measure of s]$-\beta = \|\mu\| - \beta = \alpha$

**Corollary 2.** *Let $h$ be harmonic outside a compact set. Then there exists a harmonic function $H$ on $\mathbb{R}^2$ such that $|h - H|$ is bounded near infinity if and only if $[flux_\infty h] = 0$.*

*Proof.* From the above theorem $h(x)$ is represented as

$$h(x) = H(x) + \alpha \log |x| + b(x), \tag{1}$$

where $H(x)$ is harmonic on $\mathbb{R}^2$, $b(x)$ is bounded harmonic tending to 0 at infinity and $\alpha$ is a constant. Let $flux_\infty h = 0$. By (1) we get $|h - H| = b(x)$ at infinity. This shows that $|h - H|$ is bounded at infinity. Conversely, if $|h - H|$ is bounded at infinity then $flux_\infty h = 0$.

**Proposition 1.** *If $b(x)$ is a bounded harmonic function outside a compact set then $[flux_\infty b(x)] = 0$.*

*Proof.* Write $b(x) = s(x) - \alpha \log |x|$. Then $s(x)$ is harmonic near infinity so that $s(x) = \int \log |x - y| d\mu|y| + H(x)$, where $H(x)$ is harmonic on $\mathbb{R}^2$.

Now $\int \log |x - y| d\mu(y) - \|\mu\| \log |x| = b_1(x)$ which is harmonic tending to 0 at infinity. Then $b(x) = (\|\mu\| - \alpha) \log |x| + H(x) + b_1(x)$ near infinity.

Since $(b - b_1)$ is bounded, if $\|\mu\| - \alpha = \beta \neq 0$ then $H(x)$ is bounded on one side and hence a constant so that $\beta \log |x|$ should be bounded near infinity, not possible. Hence $\|\mu\| - \alpha = 0$, so that $flux_\infty b(x)] = \|\mu\| - \alpha = 0$

**Proposition 2.** *Let $f(x)$ be a real-valued function is on $\mathbb{R}^2$, such that $\int |f(x)|dx < \infty$. There exists two admissible subharmonic functions $s_1$ and $s_2$ on $\mathbb{R}^2$ such that, if $s = s_1 - s_2$, then $\Delta s(x) = f(x)$ in the sense of distributions.*

*Proof.* Let $\Delta s_1 = f^+$ and $\Delta s_2 = f^-$. Then $s_1$ and $s_2$ are subharmonic on $\mathbb{R}^2$. Since $\int f^+(x)dx < \infty$ and $\int f^-(x)dx < \infty$, then $s_1$ and $s_2$ are admissible. Hence the proposition is proved.

**Theorem 3.** *Suppose $s(x)$ is a subharmonic function on $\mathbb{R}^2$, having a harmonic majorant outside a compact set $K$. Then the flux of $s$ at infinity=flux at infinity of the least harmonic majorant of $s$ outside $K$.*

*Proof.* Suppose $s(x)$ has a harmonic majorant on $|x| > r$. Take an increasing sequence of numbers $r_n$ such that $r < r_n$ for all $n$ and $r_n \longrightarrow \infty$. Let $s_n$ be the function on $|x| > r$ which is the Dirichlet solution on $r < |x| < r_n$ with boundary values $s(x)$ and extended by $s(x)$ on $|x| > r_n$.
Then note:

(a) $s_n$ tends to the least harmonic majorant $h$ of $s$ on $|x| > r$;

(b) for every $n$, flux of $s_n$ at infinity = flux of $s$ at infinity;

(c) flux of $h(x)$ at infinity = $\lim_{n \longrightarrow \infty}$ flux at infinity of $s_n(x)$.

Consequently, the flux of $s(x)$ at infinity = flux of the least harmonic majorant $h(x)$ of $s(x)$.

## 5. Total measures of subharmonic functions and least harmonic majorants

**Proposition 3.** *If $h_1$, $h_2$ are two harmonic functions outside of a compact set in $\mathbb{R}^2$, such that $h_1 \geq h_2$. Let $h_i = H_i + \alpha_i \log|x| + b_i(x)$, for $i = 1, 2, \cdots$, be the unique representations as above. Then $\alpha_1 \geq \alpha_2$.*

*Proof.* For $H_1(x) + \alpha_1 \log|x| + b_1(x) \geq H_2(x) + \alpha_2 \log|x| + b_2(x)$ near infinity. Then for large $r$, $\int\limits_{|x|=r} H_1(x) + \alpha_1 \log|x| + b_1(x)dx \geq \int\limits_{|x|=r} H_2(x) + \alpha_2 \log|x| + b_2(x)dx$.

Then $rH_1(0) + 2\pi^r \alpha_1 \log r + \int\limits_{|x|=r} b_1(x)dx \geq rH_2(0) + 2\pi^r \alpha_2 \log r + \int\limits_{|x|=r} b_2(x)dx$. Now, if $|b_i(x)| \leq m_i$, then $|\int\limits_{|x|=r} b_i(x)dx \leq m_i r 2\pi$. Hence, from the above, after dividing by $r$, we get the inequality

$$2\pi\alpha_1 \geq 2\pi\alpha_2 \log r + (\text{a bounded function in r}).$$

Allow $r \longrightarrow \infty$ to conclude that $\alpha_1 \geq \alpha_2$.

If $u$ is a $C^2$-function and $\omega$ is a bounded domain in $\mathbb{R}^2$ $\int\limits_{\omega} \Delta u(x)d(x) = \int\limits_{\partial\omega} \frac{\partial u}{\partial n^+} ds$, in the classical sense. When $v$ is a subharmonic function on $\mathbb{R}^2$, there exists an increasing sequence of $C^2$-subharmonic functions tending to $v$ [2]. In classical sense, where limits are not permissible, the continuity of subharmonic function (upper continuous function) is not always true, therefore the distribution sense is adopted to show the continuity of the function, as limits exists in distribution case [9].
Consequently in the sense of distributions

$$\int\limits_{\omega} \Delta v(x)d(x) = \int\limits_{\partial\omega} \frac{\partial v}{\partial n^+} ds.$$

**Theorem 4.** *Let $v(x)$ be a subharmonic function on $\mathbb{R}^2$ having a harmonic majorant near infinity. Let $h(x) = \alpha \log|x| + b(x) +$ (a harmonic function $\mathbb{R}^2$) be the least harmonic majorant of $v(x)$ outside a compact set. Then the total measure associated with $v(x)$ equals $\alpha$.*

*Proof.* Suppose $v$ has a harmonic majorant $H$ outside a compact set. $v(x) \leq H(x)$ if $|x| \geq m$. Then for large $r > m$ let $h_r$ be the Dirichlet solution in $r < |x| < r + 1$ with boundary values $v(x)$. Write

$$v_r(x) = \begin{cases} h_r(x), & \text{in } \ r < |x| < r + 1 \\ v(x), & \text{otherwise in } \ \mathbb{R}^2 \end{cases}$$

Then $v_r(x)$ is a subharmonic function on $\mathbb{R}^2$ and

$$\int\limits_{|x|<r+2} \Delta v_r(x)dx = \int\limits_{|x|=r+2} \frac{\partial v_r(x)}{\partial n^+} ds = \int\limits_{|x|=r+2} \frac{\partial v(x)}{\partial n^+} ds = \int\limits_{|x|<r+2} \Delta v(x)dx$$

That is, the total measure associated with $v_r$ is the same as the total measure associated with $v$. Note that $v_r(x)$ is a subharmonic function on $\mathbb{R}^2$ increasing with $r$ and $f(x) = \lim_{r\to\infty} v_r(x)$ is subharmonic on $\mathbb{R}^2$ and is a harmonic function $h(x)$ when $|x| > r$; also $h(x)$ is the least harmonic majorant of $v(x)$ on $|x| > r$. Thus

$$f(x) = \begin{cases} v(x), & \text{if } |x| \leq r \text{ is subharmonic on } \mathbb{R}^2 \\ h(x), & \text{if } |x| > r \end{cases}$$

Since $\Delta f(x) = \lim_{r\to\infty} \Delta v_r(x)$ in the sense of distributions, the total measure associated with $f(x)$ is the total measure associated with $v(x)$.
Since near infinity $f(x) = h(x) = \alpha \log|x| + b(x) +$ (a harmonic function on) $\mathbb{R}^2$ total measure associated with $f(x)$ is $\alpha$. Consequently, total measure associated with $v(x)$ equals the constant $\alpha$ in the expression of $h(x)$ which is the least harmonic majorant of $v(x)$ near infinity.

**Corollary 3.** *Let $s_1$ and $s_2$ be two subharmonic functions on $\mathbb{R}^2$, $s_1 \leq s_2$. Then the (total measure associated with $s_1$) $\leq$(total measure associated with $s_2$).*

*Proof.* If (total measure associated with $s_1$) $= \infty$, then $s_1$ does not have a harmonic majorant near infinity consequently, $s_2$ cannot have a harmonic majorant near infinity, so that (total measure associated with $s_2$).

Hence let us assume that the total measure of $s_1$ is finite. In this case, if total measure of $s_2$ is infinite, nothing to prove. So we have to consider only the case when the total measures of $s_1$ and $s_2$ are finite. Then outside a compact set if $h_1$, $h_2$ are the least harmonic majorants of $s_1$, $s_2$, then $h_1 \leq h_2$ so that if we write $h_i = H_i + \alpha_i \log|x| + b_i(x)$, then $\alpha_1 \leq \alpha_2$. Then by the above theorem (total measure associated with $s_1$) $\leq$(total measure associated with $s_2$).

## Conclusion

In complex analysis, the relation between the zeros of an analytic function $f(z)$ and the measure representing the subharmonic function $\log|f(z)|$ as an integral is intriguingly fascinating. In this article, we have attempted to answer the question: if $s(x)$ is a subharmonic function on the complex plane with the associated Radon measure $\mu$ representing $s(x)$ as an integral in a local representation, when will the total measure $\|\mu\|$ be finite?

By introducing the notion of flux at infinity of a subharmonic function defined outside a compact set, we characterise the case where $\|\mu\|$ is finite. We have also mentioned some related researchers on subharmonic functions on locally compact harmonic spaces in the Brelot axiomatic potential without positive potentials.

## Declarations

(i) **Author Contribution** Both the authors contributed equally.

The authors (Amulya Smyrna C. and N. Nathiya) of this manuscript titled "Flux at infinity of subharmonic functions on $\mathbb{R}^2$" have no competing interests to declare that are relevant to the content of this article.

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