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Optimal Placement of PMUs using k-Power Domination Number in Grid Networks

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Abstract. The problem of monitoring an electric power system by placing Phasor Measurement Units in the system as possible is closely related to the well-known power domination problem in graphs. In this paper, we compute, 2-power domination number for fully connected cubic networks and also developed k-power domination algorithm for specific networks and evaluates its performance across these networks. Furthermore, the algorithm's outputs are used for various other applications such as fault detection and localization, state identification, accuracy improvement, and cyber security enhancement, thereby making it a valuable tool for utilities and grid operators.

2020 Mathematics Subject Classifications: 05C69, 90B10, 94C15, 68M10, 05C85

Key Words and Phrases: Optimal PMUs Placement, Dominating Set, Facility Location Problem, Electrical Power System, Power Grid Monitoring

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1. Introduction

Optimal PMUs Placement(OPP) is viewed as an optimization problem. The task is to find the minimum possible set of PMUs to monitor the entire electric power system. An electrical line that connects a pair of electric nodes is represented by each edge in a graphical representation $G(V, E)$ of the electric power network. Every vertex of the network symbolizes an electric node. PMU placement with a view to measuring the state variable for the vertex at which it is place is the goal. It also helps to ensure that the system is running efficiently and that energy is being used in a cost-effective manner,

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which is calculated by a number of crucial unknowns including the phase angle of machine at generators and the magnitude of the voltage at loads. The concept of k-PD in graphs encapsulates the challenge of achieving comprehensive system monitoring with as few PMUs as possible. Despite recent cost reductions, comprehensive PMU deployment across all electric grid nodes is hampered by several challenges. As per industry protocols, PMUs are routinely placed in substations during refurbishment. However, strategic placement of PMUs at key buses is recommended to ensure system observability.

Power domination algorithms are used to place PMUs in electrical grid networks in the best possible way to monitor the power system. A subset $S \subseteq V$ in a graph is said to be a dominating set G if every vertex in V is either in S or is adjacent to some vertices in S [11]. In 2002, Hayens et al. [11] considered this problem as the *power domination problem* in graphs which is a variation of the *domination problem.* An electric power network is designed by a graph where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes. In 2012, Paul Dorbec et al.[8] presented the idea of k-power domination problem which is a generalization of power domination problem in graphs. The notation for the k-power dominating set is $\gamma_{p,k}(G)$. More extensive conditions are applied to the monitoring of the complete graph G. In order to achieve full observability of the electric power system k-power domination algorithm is employed. This approach not only enhances monitoring efficiency but also offers a cost-effective solution by minimizing the total number of PMUs required, thus overcoming the logistical, financial, and architectural challenges inherent in large-scale PMU deployment.

1.1. Preliminaries and Definitions

A *path*, is a linear network whose nodes are arranged as $v_1, v_2, \ldots, v_r$, with edges of the form $\{v_j, v_{j+1}\}$ where $j = 1, 2, \ldots, r-1$. A *complete network* K_n, in which each pair of unique nodes is connected by a single line. A *cycle* C_n, for$n \geq 3$, contains n edges and n vertices, with the edges forming a loop connecting the vertices in sequence from $v_1 to v_n$ and back to v_1. A *star network*, $K_{1,r}$, is a tree with r nodes, exacctly one node of degree $r-1$, while the other $r-1$ nodes of degree 1. A *friendship graph*, is denoted by F_r, is formed by connecting r number of the cycles C_3 with a central node [10]. *Binary tree* is a tree, each node can have up to two children, typically called the left and right children. The *ladder* is a Cartesian product of $P_r \times P_2$, denoted by $L_{r,1}$. [1].

Cartesian Product of $G \times H$. The following conditions holds:

- In the *Cartesian product* $G \times H$, the vertices (s, t) and (s', t') are neighbours only if $s = t'$ and t is a neighbour to t' in H, or $t = t'$ and u is a neighbour to s' in G.
- The vertex set of $G \times H$ is given by $V(G) \times V(H)$.

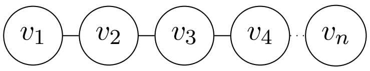
A graph $G = (V, E)$ is defined as a nonempty set of vertices $V = V(G)$ together with a set of edges $E = E(G)$ joining certain pairs of vertices. Vertices u and v are said to be adjacent if u and v are the end vertices of an edge in G. For $u \in V$, the set of all vertices adjacent to u are said to be in the neighbours of u and is denoted by $N(u)$. Then, the closed neighbourhood of u is defined as $N[u] = N(u) \cup \{u\}$.

Dominating Set: A vertex v is represents to as a dominating set over a vertex u if u and v are neighboring vertices in G. If every vertex not in S is adjacent to at least one vertex in S, then set $S \subseteq V(G)$ is called a dominating set. The least size of such a set S is called the domination number of G, denoted by $\gamma(G)$.

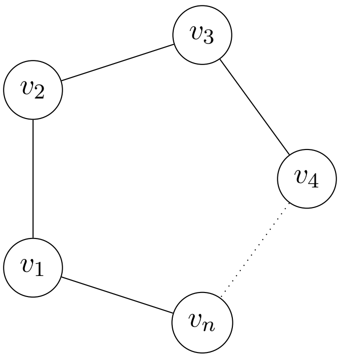
***k*-Power Dominating Set(k-PDS)**: Given an integer $k \geq 0$ and a graph $G(V, E)$, a set $S \subseteq V(G)$ is called a k-PDS if it monitors all vertices in G through inductive step. Define the sets $M^i(S)$ of vertices monitored by S at level i as follows:

1. $M^0(S) = N[S]$
2. $M^{i+1}(S) = M^i(S) \cup \{N(v) : \exists v \in M^i(S), |N(v) \setminus M^i(S)| \leq k\}$.

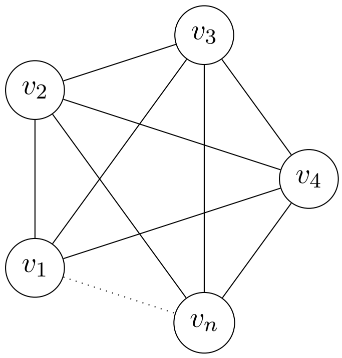
If $M^\infty(S) = V(G)$, then the set S is a k-PDS . The $\gamma_{p,k}(G)$, is the minimum size of a k-PDS of G.



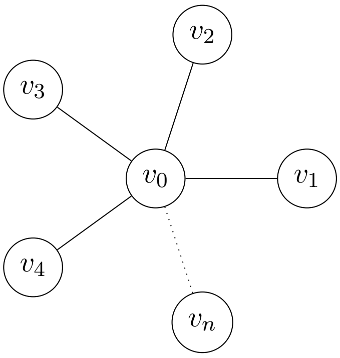
(a) Path Graph, P_n



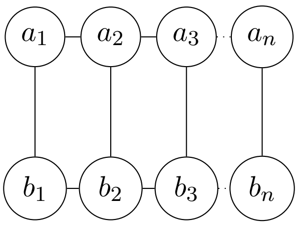
(b) Cycle Graph, C_n



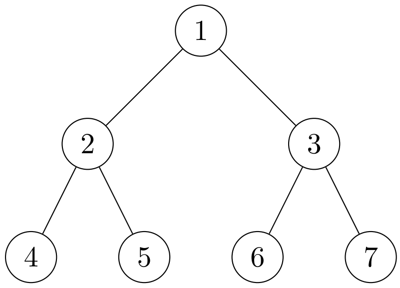
(c) Complete Graph, K_n



(d) Star Graph, $K_{1,n}$



(e) Ladder Graph, $L_{n,1}$



(f) Binary Tree, T_n

The zero forcing process can be treated as a coloring process on the vertices of the graph. If vertex x is colored red and exactly one neighbor y of x is green, then change the color of y to red and we say that x forces y. **A zero forcing set** for G is a subset

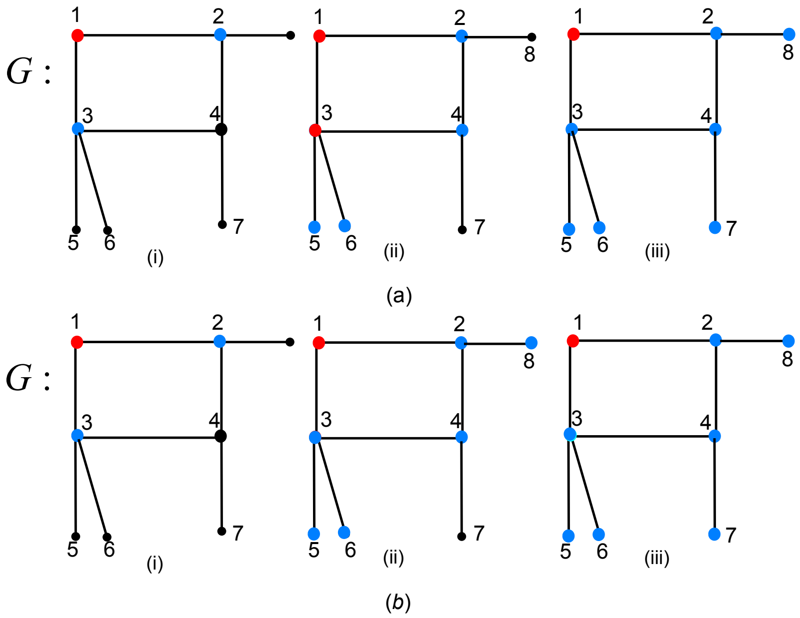


Figure 2: (a) Vertex in red is a power dominating set in G with $\gamma_p(G) = 2$ (b) Vertex in red is a 2-power dominating set in G with $\gamma_p(G) = 1$

of vertices H such that if initially the vertices in H are colored red and the remaining vertices are colored green, then repeated application of the above process can color all vertices of G red. This concept helps to identify how control can propagate through a network. The cardinality of a minimum zero forcing set of G is represented by $\zeta(G)$ [3].
In Figure 1, we illustrate as follows: For the graph G, in Figure $1(a)$, the vertex labeled 1 marked in red, is a power dominating set. At the first time step, vertex 1 monitors vertex 2 and vertex 3, further propagation not possible from vertices labeled 2 and 3. Now include one more vertex in a power dominating set in vertex labeled as 3, then at the second time step, vertex 3 monitors vertices 5 and 6, vertex 2 monitors vertex 8, and at the third time step, vertex 4 monitors vertex 7.
For the same graph G, in Figure $1(b)$, we determine a 2-power dominating set S as follows: the vertex labeled 1 marked in red is a 2-power dominating set. At the first time step, vertex 1 monitors vertex 2 and vertex 3. At the second time step, vertex 3 monitors vertices 5 and 6, vertex 2 monitors vertex 8. Finally, at the third time step, vertex 4 monitors vertex 7, completing the monitoring process in fewer steps. This demonstrates the efficiency of a 2-power dominating set in reducing the number of monitored steps.

2. Literature review

The PD problem is NP-complete . Even when restricted to chordal and bipartite graphs or even split graphs, which are a subclass of chordal graphs and it has been shown to be NP-complete. But in the case of interval graphs, Liao and Lee presented a linear method for this problem, assuming that the graph's interval ordering is known [15]. In the absence of interval order, they provided an $O(nlogn)$ algorithm, which they demonstrated to be asymptotically optimal. For trees and, more broadly, graphs with bounded treewidth, more effective techniques have been proposed [12]. The PDN of G, $\gamma_p(G) \geq 1$ [11]. Dorfling and Henning [9] determined the PDN and minimal power dominating sets for grid graphs. Dorbec et al. calculated $\gamma_p(G)$ in [3] when G is the lexicographic product of any two path graphs. Subsequently, Barrera and Ferrero found numerous instances in which their upper bounds for $\gamma_p(G)$ correspond with the power domination number whether G is a cylinder, a torus, or a generalised Petersen graph [2]. More generally, Zhao, Kang, and G.J. Chang provided upper bounds for $\gamma_p(G)$ for an arbitrary graph G [22]. Additional upper bounds have been provided for claw-free graphs [22] and block graphs [21].

Chang et al. [5] present the k-PD, which is a generalization of Power domination(PD) in graphs. The k-PDN is an NP-complete problem as demonstrated by the computation of the problem's complexity in their work. [5] demonstrates the sharpness of the bound and shows that for any connected graph G of rank n, $\gamma_k^P(G) < \frac{n}{k+2}$. Additionally, it is demonstrated that for any $(k+2)$-regular graph of rank n, $\gamma_k^P(G) < \frac{n}{k+3}$, and the graphs that meet this last restriction are described. The concept of propagation radius is presented by Dorbec et al. [7], defining it as the least number of steps required for the propagation of k PD sets. The propagation radius for a significant range of parameters in Sierpinski graphs S_n is specifically determined in the paper. However, challenges are faced in establishing concise formulas, particularly for Sierpinski graphs S_n^p, where $2k+2 \geq p \geq k+1+\sqrt{k}+1$ and $n \geq 3$.

The computational complexity of failed power domination has been addressed by Glasser et al. [10], highlighting the NP-hardness of power domination computation. The paper focuses on identifying graphs where every vertex acts as its own power dominating set (PDS), contributing a list of such graphs. However, while offering this valuable insight, it fails to explore specific traits and properties of graphs with a PD number of 0, which remains an uncharted domain.

Significant progress has been made by Brimkov et al. [4] in exploring connected PD within graphs, with the NP-hardness of determining the domination number in connected graphs being firmly established. NP-completeness of certain graphs has been proven, leading to various structural discoveries. Various structural insights have been provided by their research through the proof of the NP-completeness of specific graphs. Moreover, a formula for the bi-connected components' numerical values and the graph's connected PD number have been determined. Despite these successes, attempts to determine the PD of IEEE Bus 300 have run into problems since the calculations have exceeded timeout limits, revealing problems with the scalability of the computations and their practical use.

The employment of bounded-tree width dynamic programs to solve the power dominating set has been the main focus of Guo et al. [17]. [17] presents a linear-time technique that is streamlined for determining the PD set in trees. Furthermore, it has been shown that the PD set, as defined by $|P|$, is not more accurately measurable than the dominating set and is W[2]-hard. However, there are still a lot of topics that require additional study. In line with variants of domination and power domination with applications in fuzzy graphs, domination in vague graphs[16], survey on domination in vague graphs with application in transferring cancer patient [14], topological indices in fuzzy graphs [13], vague graphs with novel application[19, 20], abelian covers of the Wreath graph W (3, 2) and the Foster graph F26A[6].

3. Fully Connected Cubic networks (FCCNs)

A fully connected cubic network is a recursive network and is defined as follows[18]:

Let $Z_7 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and for $a \in Z_7$, let $a^s = aa \ldots a(s\ times),\ s \geq 1$. For $r \geq 1$. An r-level $FCCN$, denoted by $FCCN_r,\ r \geq 1$, is a graph defined inductively as follows:

1. $FCCN_1$ is a network with $V(FCCN_1) = Z_7$ and $E(FCCN_1) = \{(0,1), (0,2), (1,3), (2,3), (4,5), (4,6), (5,7), (6,7), (0,4), (1,5), (2,6), (3,7)\}$.

2. When $r \geq 2,\ FCCN_r$ is constructed from eight node-distinct copies of $FCCN_{r-1}$ by connecting additional 28 edges. Particularly, if, for $0 \leq t \leq 7$, we let $tFCCN_{r-1}$ denote a copy of $FCCN_{r-1}$ with each node being prefixed with t, then $FCCN_r$ is defined by

$V(FCCN_r) = \bigcup_{t=0}^{7} V(tFCCN_{r-1})$,

$E(FCCN_r) = (\bigcup_{t=0}^{7} E(tFCCN_{r-1})) \cup \{(ab^{r-1}, ba^{r-1}) | 0 \leq a < b \leq 7\}$.

For $0 \leq t \leq 7,\ tFCCN_{r-1}$ is called an $(r-1)$-level sub-$FCCN$ of $FCCN_r$, or simply a sub-$FCCN$ of $FCCN_r$, if there is no ambiguity.

3. Given an $FCCN_r,\ r \geq 2$, a boundary node is a node of the form t^r. An intercubic edge is an edge of the form (ab^{r-1}, ba^{r-1}). In essence, each node of an $FCCN$ has four links, with each boundary node having one $I \backslash O$ channel link that is not counted in the node degree. Obviously, $tFCCN_{r-1}$ has 7 intercubic vertices and 1 boundary vertex for $0 \leq t \leq 7$ and $r \geq 2$ [18]. See Figure 3.

4. 2-PD in Fully Connected Cubic Network(FCCN)

In this section, we find the k-PDN, $k \geq 2$ for $FCCN_r,\ r \geq 2$. The power domination process on a graph G is choosing a set $S \subseteq V(G)$ and applying the zero forcing process to the closed neighborhood $N[S]$ of S. The set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set for G [3].

Rao et al.[22] proved that the bound is tight for fully connected cubic networks. In 2015 Ferrero et al. [3] proved the following theorem which shows the relationship between zero forcing set and power dominating set.

Theorem 1. [3] *Let G be a graph. Then* $\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil \leq \gamma_p(G)$ *and this bound is tight.*

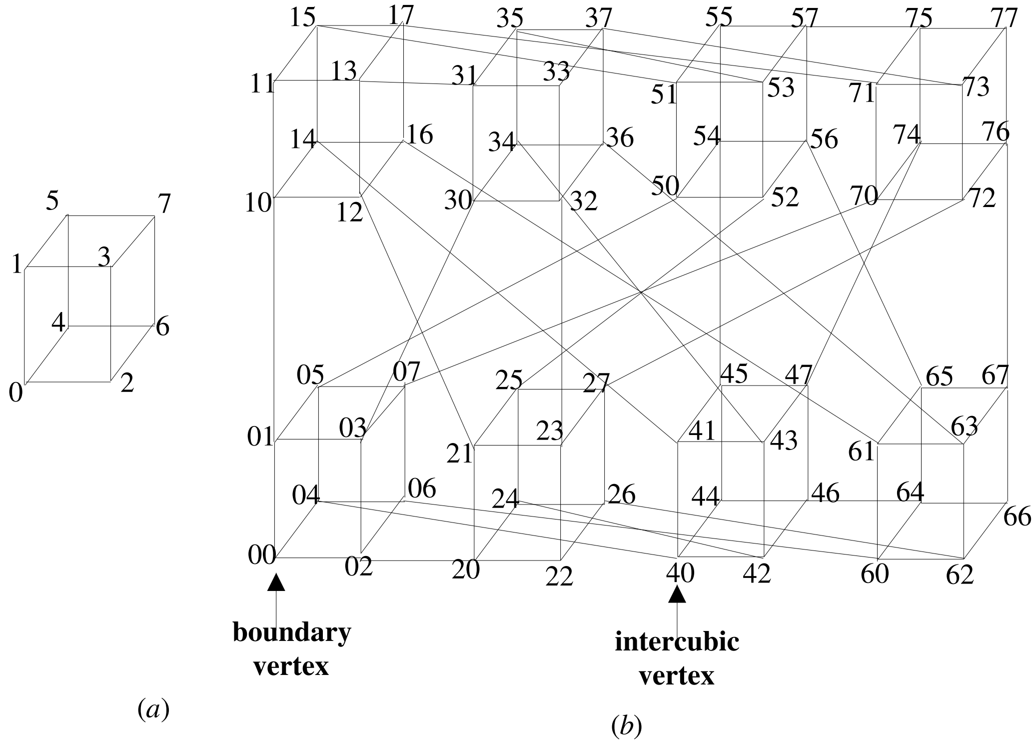


Figure 3: Fully connected cubic networks (a) $FCCN_1$ (b) $FCCN_2$ with $00, 11, \ldots, 77$ as boundary vertices

In 2022 Anitha et al. [18] proved the following theorem which shows the relationship between zero forcing set and power dominating set described in Theorem 2.

Theorem 2. [18] *For the FCCN* $H_r \cong FCCN_r,\ r \geq 2$, *we have* $\gamma_p(H_r) = 2^{3r-4}$.

Theorem 3. [18] *For the FCCN* $H_r \cong FCCN_r,\ r \geq 2$, *we have* $\zeta(H_r) = 2^{3r-2}$.

In this section we solve 2-power domination number for $H_n, n \geq 2$

Lemma 4. *For* $FCCN_2$, *we have* $\gamma_{p,2}(FCCN_2) \geq 2$.

Proof. $FCCN_2$ contains eight node distinct copies of $FCCN_1$, say $0FCCN_1, 1FCCN_1, \ldots, 7FCCN_1$. Let S be a 2-PD set of G.
We claim that $|S| \geq 2$. A node v in G is of the following two categories:
(i) v is adjacent to only nodes of degree 4.
(ii) v is adjacent to a node of degree 3.
Suppose $|S| = 1$. Then the only node in S cannot be a node of category (i) . Therefore, let $S = \{u\}$ where u is a boundary node. The neighbours of u are x, y, z, say, in some $FCCN_1$, each of which is adjacent to 2 vertices of the same $FCCN_1$ and one node in three disjoint copies of $FCCN_1$. See Figure 3(b). None of these nodes is adjacent to

unmonitored nodes of degree at most 2. Hence $|S| \neq 1$.

Lemma 5. *Let S be a 2-PD set $FCCN_3$. Then $|V(FCCN_3) \cap S| \geq 9$.*

Proof. Let S be a 2-PD set of $FCCN_3$. We claim that $|S| \geq 9$. Suppose not, $FCCN_3$ is composed of eight copies of $FCCN_2$, denoted by $0FCCN_2, 1FCCN_2, \ldots, 7FCCN_2$. Even if, all $iFCCN_2,\ 0 \leq i \leq 7$ contains exactly one node from 2-PD set. A node $v \in S$ in G is of the following three categories:
(i) v is of degree 3.
(ii) v is of degree 4 not adjacent to a node of degree 3.
(iii) v is adjacent to a node of degree 3.
Suppose $|S| = 8$. Let S can be a node of category (i) and category (ii). Therefore, let $S = \{u_i : deg(u_i) = 3 \ or \ deg(u_i) = 4\}$ where u_i is a boundary node in each $FCCN_2$ or not adjacent to a boundary node. The neighbours of u_i are x, y, z, say, in some $FCCN_1$, each of which is adjacent to 2 nodes of the same $FCCN_1$ and one node in three disjoint copies of $FCCN_1$. Further propagation cannot be done. See Figure 1(b). None of these vertices is adjacent to unmonitored vertices of degree at most 2. Hence $|S| \neq 8$.
Suppose $S = \{u_i : deg(u_i) = 4\}$ can be a node of category (iii) in some $FCCN_1$. Then each boundary node is adjacent to two nodes, then the boundary nodes can monitor their neighbouring nodes. Then nodes in $M^1(S)$ is adjacent to two nodes in same $FCCN_1$ and one node in three disjoint copies of $FCCN_1$. Further propagation cannot be done. Hence $|S| \neq 8$. Thus $|S| = 9$. Therefore, $\gamma_{p,2}(FCCN_3) = 9$.

Lemma 6. *For $FCCN$, $n \geq 3$, we have $\gamma_{p,2}(FCCN_r) \geq 9 \times 8^{r-3}$.*

Proof. We prove the result by induction on r. We consider the case when $r = 3$. H_3 has eight node disjoint copies of $FCCN_2$, say $FCCN_2, 1FCCN_2, \ldots, 7FCCN_2$. Let S be a 2-power dominating set of $FCCN_r$. We claim that $|S| \geq 9$. By Lemma 5, there are at least one vertex in each copy of $FCCN_2$ and one more additional vertex to monitor $FCCN_3$. Hence $|S| \geq 9$. Therefore, $\gamma_{p,2}(FCCN_3) \geq 9$.

Assume the result is true for $r = k,\ r \geq 3$. That is, $\gamma_{p,2}(FCCN_k) \geq 9 \times 8^{k-3}$. Consider the case when $r = k+1$. Let S be a 2-power dominating set H_{k+1}. We have to prove that $\gamma_{p,2}(FCCN_{k+1}) \geq 9 \times 8^{k-2}$. Suppose not, let $|S| < 9 \times 8^{k-2}$. By definition, there are 8^{k-2} vertex disjoint copies of $FCCN_3$ in $FCCN_{k+1}$. With the removal of one node from S in a copy of $FCCN_2$, The neighbours of some u are x, y, z, say, in some $FCCN_1$, each of which is adjacent to 2 nodes of the same $FCCN_1$ and one node in three disjoint copies of $FCCN_1$. Further propagation cannot be done. a contradiction. Thus $|S| \geq 9 \times 8^{k-2}$. Therefore, $\gamma_{p,2}(FCCN_{k+1}) \geq 9 \times 8^{k-2}$. Hence the proof.

The Algorithm given below computes the 2-PD for FCCNs $FCCN_r$.

2-PD Algorithm:

Input: FCCN $FCCN_r,\ r \geq 3$, with radix-lexicographic ordering.

Algorithm: (i) Select $S_3 = \{001, 013\} \cup \bigcup\limits_{t=1}^{7} t10$ in $FCCN_3$.

(ii) Let $S_4 = \bigcup\limits_{t=0}^{7} tS_3$ in $FCCN_4$.

(iii) Inductively select $S_r = \bigcup\limits_{t=0}^{7} tS_{r-1}$ in $FCCN_r$.

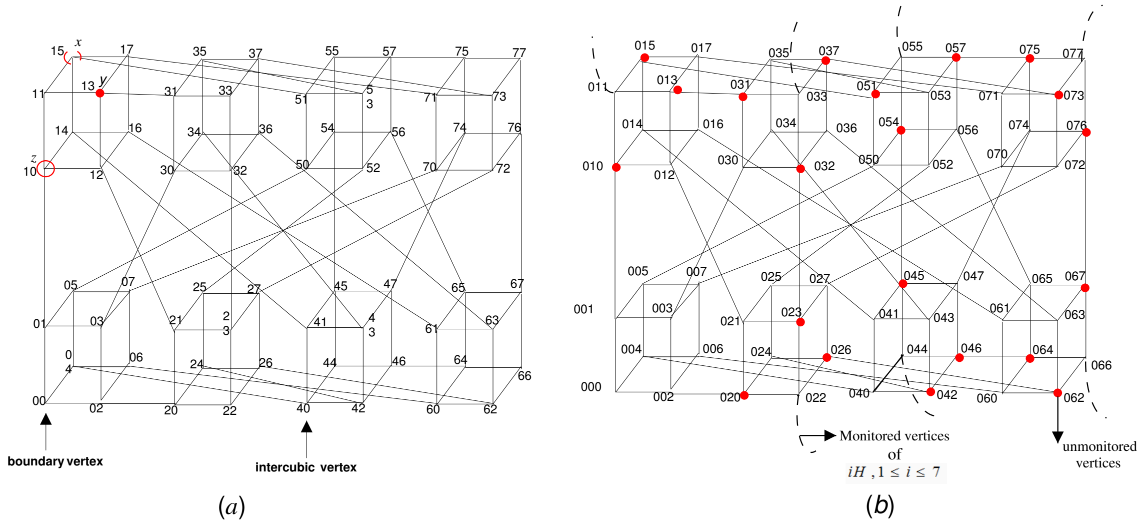


Figure 4: $0^{n-2}FCCN_2$ of $FCCN_r$

Output: $\gamma_{p,2}(FCCN_r) = 9 \times 8^{r-3}$

Proof of Correctness: Let S_r be a 2-PD set of $FCCN_r$ with $|S_r| = 9 \times 8^{r-3}$. Let S_3 be a 2-PD set of $FCCN_3$. Then $M^0(S_3) = N[S_3] = \bigcup\limits_{t=0}^{7} \{000, 005, 003, 010, 013, 031, 011, 017\} \cup \bigcup\limits_{t=1}^{7} \{101, 110, 111, 114, 112\}$. Then $M^0(S_4) = N[S_4] = \bigcup\limits_{t=0}^{7} tM^0(S_3)$. Proceeding inductively, $M^0(S_r) = N[S_r] = \bigcup\limits_{t=0}^{7} tM^0(S_{r-1})$ Then nodes in $M^0(S_r)$ say, $\bigcup\limits_{t=0}^{7} t^{r-2}S''$ where $S'' = \{00, 02, 03, 10, 11, 12, 21, 22, 23, 30, 31, 33\}$ is adjacent to exactly two unmonitored vertices say, $\bigcup\limits_{t=0}^{7} t^{r-2}S'''$ where $S''' = \{04, 06, 07, 14, 15, 16, 25, 26, 27, 34, 35, 37\}$. Now $M^1(S_r) = M^0(S_r) \cup \bigcup\limits_{t=0}^{7} t^{n-2}S'''$. Then for every vertex $v \in M^1(S_r),\ |N[v] \backslash M^1(S_r)| = 2$. rNow $M^2(S_n) = M^1(S_n) \cup \bigcup\limits_{t=0}^{7} t^{n-2}ij,\ 4 \leq i \leq 7, 0 \leq j \leq 3$. Similarly in the next step, $M^3(S_r) = M^2(S_r) \cup \bigcup\limits_{t=0}^{7} t^{r-2}ij,\ 4 \leq i, j \leq 7$. Proceeding inductively,
$M^3(S_r) = V(FCCN_r)$. Hence the proof. Therefore, $\gamma_{p,2}(FCCN_r) \leq 9 \times 8^{r-3}$.

The following result is a consequence of Lemma 6 and by 2-PD Algorithm .

Theorem 7. *For FCCNs* $FCCN_r,\ r \geq 3$, *we have* $\gamma_{p,2}(FCCN_r) = 9 \times 8^{r-3}$.

In 2012, Chang et al. [5] obtained the following results for a connected graph G.

Theorem 8. *If* G *is connected and* $\Delta(G) \leq k+1$, *then* $\gamma_{p,k}(G) = 1$.

The following result is a consequence of Theorem 8

Theorem 9. *For FCCNs* $FCCN_r,\ r \geq 3$, *we have* $\gamma_{p,k}(FCCN_r) = 1,\ \ k \geq 4$.

5. k-PD Algorithm

```
Algorithm PDS($G(V,E),k){
    min_set ← V(G)
    min_set_size ← ∞
    vertices ← list of all vertices in G
    for size := 1 to |vertices| {
      for each subset S ⊆ vertices with |S| = size {
        monitored ← S ∪ (⋃_{v∈S} N(v))
        new_monitored ← monitored
        while True {
          to_add ← ∅
          for each vertex v ∈ monitored {
            neighbors ← N(v)
            unmonitored_neighbors ← neighbors \ monitored
            if |unmonitored_neighbors| ≤ k {
              to_add ← to_add∪unmonitored_neighbors            }           }
          if to_add = ∅ {
            break             }
          new_monitored ← new_monitored ∪ to_add
          monitored ← new_monitored
        }
        if |monitored| = |V(G)| {
          if |S| < min_set_size {
            min_set ← S
            min_set_size ← |S|
          }
        }
      }
    }
    return min_set    }
```

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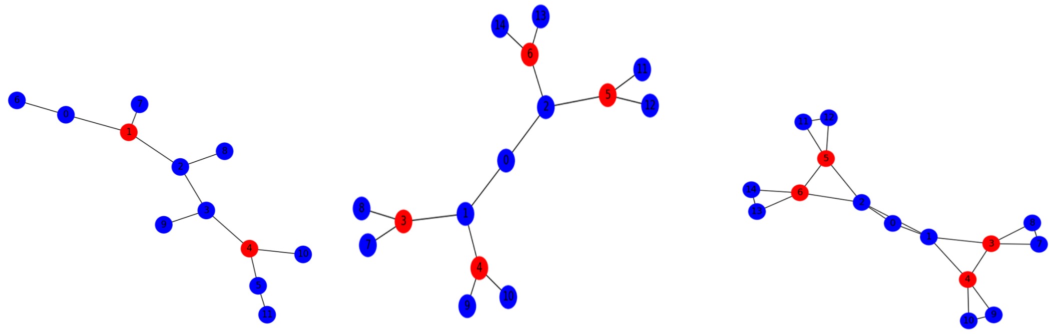
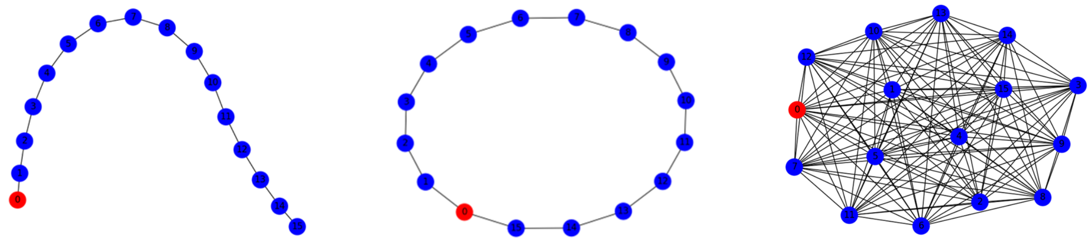
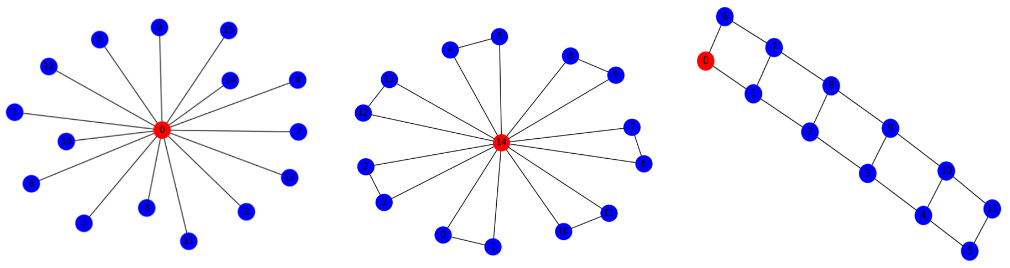


(a) Comb graph P_6^+ (b) Binary tree T_3 (c) Sibling tree ST_3^1

Figure 6: Power Dominating Set in Comb Graph, Binary Tree and Sibling Tree using Power Domination Algorithm



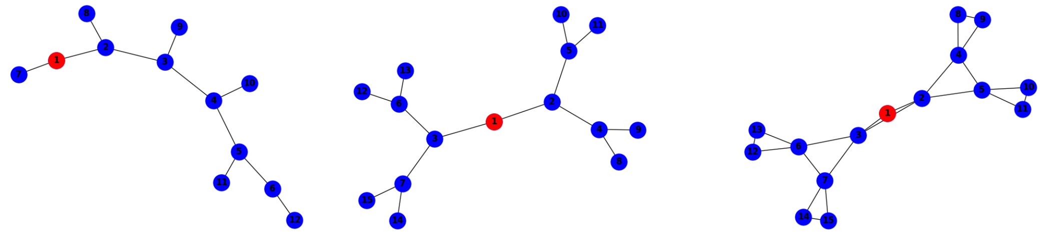
(a) Path graph P_{16} (b) Cycle graph C_{16} (c) Complete graph K_{16}



(d) Star graph $K_{1,15}$ (e) Fan graph F_5 (f) Ladder graph L_6

Figure 5: k-Power Dominating Set in Different Types of Graph Networks using k-Power Domination Algorithm

The PDS algorithm identifies the k-power domination set, depicted by red vertices in the pictures above, as the smallest subset of nodes required for full network observability. The k-power domination set is deliberately placed in each graph to provide optimal coverage, whether along linear lines, across cyclic structures, at central hubs and outlying nodes, or scattered across multiple layers, improving power system reliability and



(a) Comb graph P_6^+ (b) Binary tree T_3 (c) Sibling tree ST_3^1

Figure 7: 2-Power Dominating Set in Comb Graph, Binary Tree and Sibling Tree using 2-Power Domination Algorithm

stability. For binary, sibling, and comb graphs, the power domination algorithm differs significantly from the 2-power domination algorithm, which adds redundancy. However, for other graphs, both k=1 and k=2 yield the same result, demonstrating that the smallest subset is sufficient for comprehensive coverage.
Table 1 shows the execution time to compute the k-PDS for various types of graphs with 16 nodes

Type of Graph	**Execution time**	
	k=1	**k=2**
Path P_n	1.018	1.020
Cycle C_n	0.980	0.945
Ladder L_n	0.866	0.929
Complete graph K_n	0.971	0.950
Star Graph $K_{1,n}$	1.010	1.085
Fan graph F_n	0.131	0.129
Comb graph P_n^+	0.415	0.447
Binary tree T_n	1.85	2.12
Sibling tree ST_n^1	0.477	0.379

Table 1: Execution Time to Compute the k-Power Dominating Set for Graphs with 16 Nodes

5.1. Optimal PMU Placement (OPP)

The proposed k-power domination algorithm is a highly efficient and robust solution for OPP in electrical grid systems. It ensures comprehensive network observability by identifying a minimal subset of nodes where PMUs should be placed, thereby guaranteeing that each node in the grid is monitored either directly or indirectly. This approach not only minimizes the number of PMUs required but also maximizes coverage, enhancing the reliability and stability of the electric grid. Unlike traditional greedy algorithms, which often necessitate more PMUs to achieve full observability and may not always yield minimal sets, the k-power domination algorithm provides a precise and minimal solution. Compared to Integer Linear Programming (ILP) methods, which can be computationally intensive and impractical for large-scale networks, this algorithm is computationally efficient and scalable, making it suitable for various grid topologies and sizes.

Moreover, the k-PD algorithm incorporates the concept of k-redundancy, ensuring that each node is observed by multiple PMUs, which significantly improves fault tolerance and robustness against failures. This redundancy guarantees continuous monitoring and control even in the event of PMU malfunctions. The algorithm's flexibility allows it to adapt to dynamic changes in network topology, such as the addition or removal of nodes, ensuring sustained optimal performance in evolving grid environments.

In terms of cost efficiency, the k-PD algorithm reduces the overall number of PMUs needed, thereby lowering implementation and maintenance costs. By optimizing PMU placement to minimize overlap and maximize observability, it ensures operational efficiency and effective data collection. Its proven effectiveness in various studies and real-world applications highlights its practical applicability, making it a preferred choice for grid operators aiming to enhance monitoring and control of electric grids. Overall, the k-power domination algorithm represents a superior method for PMU placement, addressing the limitations of other approaches and significantly contributing to the reliability, stability, and efficiency of power systems.
The following Table II depicts the cost of placement of PMUs in electric grid networks by implementing k-power domination algorithm.

Type of Network	**Heuristic methods**	**PDS Algorithm**	
		$k = 1$	$k = 2$
Path P_n	300000	50000	50000
Cycle C_n	300000	50000	50000
Ladder L_n	250000	50000	50000
Complete graph K_n	50000	50000	50000
Star Graph $K_{1,n}$	50000	50000	50000
Fan graph F_n	50000	50000	50000
Comb graph P_n^+	300000	100000	50000
Binary tree T_n	350000	200000	50000
Sibling tree ST_n^1	250000	200000	50000

Table 2: Cost of placement of PMUs using Various Strategies(in dollars)

The Table II shows costs associated with placing Phasor Measurement Units (PMUs) using heuristic methods and the Power Dominating Set (PDS) algorithm across various network topologies. It demonstrates that the PDS algorithm is generally more cost-effective than heuristic methods. For Path p_n, Cycle C_n, Ladder L_n, and Comb graph P_n^+, the heuristic method costs higher compared to the PDS algorithm's \$50,000. This trend is similar for Binary Tree (Tn) and Sibling Tree (ST) topologies, where the heuristic methods cost \$350,000 and \$250,000 respectively, while the PDS algorithm costs

\$200,000. For $k = 2$, both methods generally show similar costs, around \$50,000, except for Binary Tree (Tn) and Sibling Tree (ST), where the heuristic methods still have higher costs compared to the PDS algorithm's \$50,000. This analysis indicates that the PDS algorithm effectively reduces the cost of PMU placement, especially for more complex or larger networks, by optimizing the number and placement of PMUs needed for comprehensive system observability.

6. Conclusions

In this paper, the PD algorithm is used to improve grid monitoring and control systems. The scalability of the PD and 2-PD algorithm is tested for various graph structures. Despite challenges with specific configurations such as ladder graphs, the algorithm demonstrates its effectiveness in graphical networks and adaptability to structural context. This adaptability is important for real-world applications, where power grids often have complex and heterogeneous structures. PMUs play a important role by providing accurate and timely measurements of voltage and current. Network operators gain comprehensive, real-time information about system status, allowing them to respond to incidents and errors more effectively by strategically placing PMUs in key locations.

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