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Hyperstability of Functional Equation Deriving from Quintic Mapping in Banach Spaces by Fixed Point Method

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Abstract. In this work, we examine the hyperstability of the quintic functional equation $\phi(u+3v)-5\phi(u+2v)-\phi(u-2v)+10\phi(u+v)+5\phi(u-v)-10\phi(u)-120\phi(v)=0$, in Banach spaces by means of Brzdęk's fixed point theorem.

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1. Introduction and preliminaries

One of the most important areas of mathematical research, which has its origins in problems relating to applied mathematics, is the investigation of stability issues for functional equations. Ulam [28] stated the following as the first query pertaining to the stability of homomorphisms.

Let $\mathcal{U}$ be a group and $\mathcal{V}$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, is there a $\delta > 0$ such that if a function $\phi : \mathcal{U} \to \mathcal{V}$ fulfills

$$d(\phi(uv), \phi(u)\phi(v)) < \delta,$$

for all $u, v \in \mathcal{U}$, then there is a homomorphism $\Phi : \mathcal{U} \to \mathcal{V}$ with

$$d(\phi(u), \Phi(u)) < \epsilon,$$

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for all $u \in \mathcal{U}$?

Hyers provided the first partial answer to Ulam's concern regarding the Cauchy equation in Banach spaces in [19]. Later, Aoki was the first to generalize Hyers' findings and not until much later by Rassias [24] and Găvruţa [18]. Since then, several functional equations' stability issues have been thoroughly researched (see [7, 8, 20, 25]). If any function f approximates (in some sense) the solution to the functional equation, then the functional equation is said to be hyperstable. It appears that the first hyperstability finding, which dealt with ring homomorphisms, was published in [6]. Hyperstability, however, is mentioned for the first time in [21].

Brzdęk investigated the hyperstability results for the Cauchy equation (see [9–11]). The hyperstability of the parametric basic equation of information was addressed by Gselmann in [17]. Bahyrycz and Piszczek reported the Jensen functional equation's hyperstability in [4]. For a certain class of complete metric spaces, Brzdęk and Ciepliński [13] demonstrated a simple fixed point theorem, namely, complete non-Archimedean metric spaces, p-adic strings, and superstrings that are related to several quantum physics-related phenomena. They also demonstrated that how effective and practical this theorem is for demonstrating the Hyers-Ulam stability of a huge class of functional equations in a single variable.

The fixed point theorem [12, Theorem 1] was restated in 2-Banach spaces by El-Fassi [15] in 2017, and a radical quartic functional equation was introduced and examined its Ulam stability in 2-Banach spaces by fixed-point approache. In [5], Bounader examined the hyperstability of the quartic functional equation in Banach spaces. In [2], Aribou *et al.* presented the hyperstability results of a cubic- quartic functional equation in ultrametric Banach spaces. And also, in 2020, Sayar and Bergam [27], examined stability and hyperstability for the quadratic functional equation in 2-Banach space by Brzdęk fixed-point theorem.

Motivated by the above results on the hyperstability of additive functional equations, quadratic functional equations, cubic functional equations and quartic functional equations, in the current work, we try to examine the hyperstability of the following quintic functional equation

$$\phi(u+3v) - 5\phi(u+2v) - \phi(u-2v) + 10\phi(u+v) + 5\phi(u-v) - 10\phi(u) - 120\phi(v) = 0,$$

in Banach spaces by means of Brzdęk's fixed point approach.

A concept employed by Brzdęk in [9–11] and later by Piszczek [23] served as the inspiration for the manner of the primary results' confirmation. Its foundation is a fixed-point theorem for functional spaces discovered by Brzdęk (see [12], Theorem 1]). Most frequently, the superstability and hyperstability, which also accepts bounded functions-are concerned. Numerous articles have been written on this topic and we refer to [1, 3, 4, 14, 16, 17, 21, 22, 26, 29].

Throughout the paper, $\mathbb{N}$ denote the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}_m$ denote the set of all natural numbers greater than or equal to m, for every $m \in \mathbb{N}$ and $\mathbb{R}_+$ denote the set of all positive real numbers. We use the notation $\mathcal{U}_0$ for the set $\mathcal{U} \backslash \{0\}$.

Theorem 1. *[13] Let $\mathcal{U}$ be a non-empty set, $(\mathcal{V}, d)$ be a complete metric space, and $\Upsilon : \mathcal{V}^{\mathcal{U}} \to \mathcal{V}^{\mathcal{U}}$ fulfill the hypothesis*

$$\lim_{m\to+\infty} \Upsilon\delta_m = 0,$$

for $\{\delta_m\}_{m\in\mathbb{N}}$ in $\mathcal{V}^{\mathcal{U}}$ with

$$\lim_{m\to+\infty} \delta_m = 0.$$

Suppose that an operator $\Phi : \mathcal{V}^{\mathcal{U}} \to \mathcal{V}^{\mathcal{U}}$ fulfills

$$d\left(\Phi\psi(u), \Phi\varsigma(u)\right) \leq \Upsilon\left(\Delta(\psi,\varsigma)\right)), \quad \psi,\varsigma \in \mathcal{V}^{\mathcal{U}},$$

for all $u \in \mathcal{U}$, where a mapping $\Delta : \mathcal{V}^{\mathcal{U}} \times \mathcal{V}^{\mathcal{U}} \to \mathbb{R}_+^{\mathcal{U}}$ is defined by

$$\Delta\left(\psi,\varsigma\right)(u) := d(\psi(u),\varsigma(u)), \quad \psi,\varsigma \in \mathcal{V}^{\mathcal{U}},\ u \in \mathcal{U}.$$

If there is a mapping $\vartheta : \mathcal{U} \to \mathbb{R}_+$ and $\zeta : \mathcal{U} \to \mathcal{V}$ fulfilling

$$d\left(\Phi\psi(u), \Phi\varsigma(u)\right) \leq \psi(u)$$

and

$$\vartheta^*(u) := \sum_{m\in\mathbb{N}_0} \left(\Upsilon^m\vartheta\right)(u) < \infty$$

for all $u \in \mathcal{U}$, then the limit

$$\lim_{m\to+\infty} \left(\Phi^m\zeta\right)(u)$$

exists for each $u \in \mathcal{U}$. Furthermore, the mapping $\chi \in \mathcal{V}^{\mathcal{U}}$, defined by

$$\chi(u) := \lim_{m\to+\infty} \left(\Phi^m\zeta\right)(u)$$

is a fixed point of Φ with

$$d\left(\zeta(u), \chi(u)\right) \leq \vartheta^*(u)$$

for all $u \in \mathcal{U}$.

The upcoming fixed point theorem, which corresponds to Theorem 1 in complete normed space, is then discussed. This outcome is an important factor in the formulation of stability findings.

Theorem 2. *Let $\mathcal{U}$ be a nonempty set, $(\mathcal{V}, \|\cdot\|)$ be a Banach space and let $\phi_1, \phi_2, \cdots, \phi_l : \mathcal{U} \to \mathcal{U}$ be mappings and $L_1, \cdots, L_l : X \to \mathbb{R}_+$be functions. Suppose that $\Phi : \mathcal{V}^{\mathcal{U}} \to \mathcal{V}^{\mathcal{U}}$ and two operators $\Upsilon : \mathbb{R}_+^{\mathcal{U}\times\mathcal{U}} \to \mathbb{R}_+^{\mathcal{U}\times\mathcal{U}}$ fulfill the conditions:*

$$\|\Phi\psi(u) - \Phi\varsigma(u)\| \leq \sum_{i=1}^{l} L_i(u) \left\|\psi\left(\phi_i(u)\right) - \varsigma\left(\phi_i(u)\right)\right\|$$

for all $\psi, \varsigma \in \mathcal{V}^{\mathcal{U}}$, $u \in \mathcal{U}$ *and*

$$\Upsilon\delta(u) := \sum_{i=1}^{l} L_i(u)\delta\left(\phi_i(u)\right), \quad \delta \in \mathbb{R}_+^{\mathcal{U}\times\mathcal{U}}, \quad u \in \mathcal{U}.$$

If there exist mappings $\vartheta : \mathcal{U} \times \mathcal{U} \to \mathbb{R}_+$ *and* $\zeta : \mathcal{U} \to \mathcal{V}$ *satisfying*

$$\|\Phi\zeta(u) - \zeta(u)\| \leq \vartheta(u)$$

and

$$\vartheta^*(u) := \sum_{m=0}^{\infty} \left(\Upsilon^m \vartheta\right)(u) < \infty$$

for all $u \in \mathcal{U}$, *then the limit*

$$\lim_{m\to+\infty} \left(\Phi^m \zeta\right)(u) \tag{1}$$

exists for every $u \in \mathcal{U}$. *Furthermore, the mapping* $\chi : \mathcal{U} \to \mathcal{V}$ *defined by*

$$\chi := \lim_{m\to+\infty} \left(\Phi^m \zeta\right)(u)$$

is a fixed point of Φ *with*

$$\|\zeta(u) - \chi(u)\| \leq \vartheta^*(u)$$

for all $u \in \mathcal{U}$.

For our notational handiness, we use the abbreviation

$$\begin{aligned} D\phi(u,v) &= \phi(u+3v) - 5\phi(2v+u) - \phi(u-2v) + 10\phi(v+u) \\ &\quad +5\phi(u-v) - 10\phi(u) - 120\phi(v). \end{aligned} \tag{2}$$

2. Main results

In this section, we demonstrate various hyperstability and stability of (2) in Banach spaces utilizing Theorem 2. Suppose that $\mathcal{U}$ is a normed space along with $\mathcal{U}_0 = \mathcal{U}\backslash\{0\}$ and $(\mathcal{V}, \|\cdot\|)$ is a Banach space.

Theorem 3. *Let* $\tau_1, \tau_2 : \mathcal{U}_0 \times \mathcal{U}_0 \to \mathbb{R}_+$ *be two functions such that*

$$W := \{m \in \mathbb{N} : \alpha_m < 1\} = \varnothing,$$

where

$$\begin{aligned} \alpha_m \quad := \quad & \frac{1}{120}\eta_1(m+3)\eta_2(m+3) + \frac{1}{120}\eta_1(m-2)\eta_2(m-2) + \frac{1}{24}\eta_1(m+2)\eta_2(m+2) \\ & + \frac{1}{12}\eta_1(m+1)\eta_2(m+1) + \frac{1}{12}\eta_1(m)\eta_2(m) + \frac{1}{24}\eta_1(m-1)\eta_2(m-1) \end{aligned}$$

and

$$\eta_i(m) := \inf \{l \in \mathbb{R}_+ : \tau_i(mu) \leq l\tau_i(u), \quad u \in \mathcal{U}_0\}$$

for all $m \in \mathbb{N}$*, where* $i = 1, 2$*. Assume that* $\phi : \mathcal{U} \to \mathcal{V}$ *fulfills*

$$\|D\phi(u, v)\| \leq \tau_1(u)\tau_2(v), \quad u, v \in \mathcal{U}_0 \tag{3}$$

such that $3v + u \neq 0$*,* $2v + u \neq 0$*,* $u - 2v \neq 0$*,* $u - v \neq 0$ *and* $v + u \neq 0$*. Then there is only one quintic mapping* $H : \mathcal{U} \to \mathcal{V}$ *fulfilling*

$$\|\phi(u) - H(u)\| \leq \eta_0\tau_1(u)\tau_2(u)$$

for all $u \in \mathcal{U}_0$*, where*

$$\eta_0 := \inf_{m \in W} \left\{ \frac{\eta_1(m)}{120(1 - \alpha_m)} \right\}.$$

Proof. Replacing (u, v) by (nu, u) in (3), we obtain

$$\begin{aligned} \Bigg\| \frac{1}{120}\phi((3+n)u) - \frac{1}{24}\phi((2+n)u) - \frac{1}{120}\phi((n-2)u) + \frac{1}{12}\phi((1+n)u) \\ + \frac{1}{24}\phi((n-1)u) - \frac{1}{12}\phi(nu) - \phi(u) \Bigg\| \leq \frac{1}{120}\tau_1(nu)\tau_2(u) \end{aligned} \tag{4}$$

for all $u \in \mathcal{U}_0$ and all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define the operator $\Phi_n : \mathcal{V}^{\mathcal{U}_0} \to \mathcal{V}^{\mathcal{U}_0}$ by

$$\begin{aligned} \Phi_n\psi(u) \quad := \quad & \frac{1}{120}\psi((3+n)u) - \frac{1}{24}\psi((2+n)u) - \frac{1}{120}\psi((n-2)u) \\ & + \frac{1}{12}\psi((1+n)u) + \frac{1}{24}\psi((n-1)u) - \frac{1}{12}\psi(nu) \end{aligned} \tag{5}$$

for all $\psi \in \mathcal{V}^{\mathcal{U}_0}$ and all $u \in \mathcal{U}_0$. Moreover, putting

$$\vartheta_n(u) := \frac{1}{120}\tau_1(nu)\tau_2(u) \tag{6}$$

for all $u \in \mathcal{U}_0$, and observe that

$$\vartheta_n(u) = \frac{1}{120}\tau_1(nu)\tau_2(u) \leq \frac{1}{120}\eta_1(n)\tau_1(u)\tau_2(u) \tag{7}$$

for all $u \in \mathcal{U}_0$ and all $n \in \mathbb{N}$. Using the conditions (5) and (7) in (4), we get

$$\|\phi(u) - \Phi_n\phi(u)\| \leq \vartheta_n(u)$$

for all $u \in \mathcal{U}_0$. Moreover, for any $u \in \mathcal{U}_0$ and every $\psi, \varsigma \in \mathcal{V}^{\mathcal{U}_0}$, we obtain

$$\|\Phi_n\psi(u) - \Phi_n\varsigma(u)\| \quad = \quad \Bigg\| \frac{1}{120}\psi((3+n)u) - \frac{1}{24}\psi((2+n)u) - \frac{1}{120}\psi((n-2)u)$$

$$
\begin{aligned}
& +\frac{1}{12}\psi((1+n)u)+\frac{1}{24}\psi((n-1)u)-\frac{1}{12}\psi(nu) \\
& -\frac{1}{120}\varsigma((3+n)u)+\frac{1}{24}\varsigma((2+n)u)+\frac{1}{120}\varsigma((n-2)u) \\
& -\frac{1}{12}\varsigma((1+n)u)-\frac{1}{24}\varsigma((n-1)u)+\frac{1}{12}\varsigma(nu)\Big\| \\
\leq\ & \frac{1}{120}\Big\|(\psi-\varsigma)((3+n)u)\Big\|+\frac{1}{24}\Big\|(\psi-\varsigma)((2+n)u)\Big\| \\
& +\frac{1}{120}\Big\|(\psi-\varsigma)((n-2)u)\Big\|+\frac{1}{12}\Big\|(\psi-\varsigma)((1+n)u)\Big\| \\
& +\frac{1}{24}\Big\|(\psi-\varsigma)((n-1)u)\Big\|+\frac{1}{12}\Big\|(\psi-\varsigma)(nu)\Big\|.
\end{aligned}
$$

This brings us to define the operator $\Upsilon_n : \mathbb{R}_+^{\mathcal{U}_0\times\mathcal{U}_0} \to \mathbb{R}_+^{\mathcal{U}_0\times\mathcal{U}_0}$ by

$$
\begin{aligned}
\Upsilon_n\delta(u) \ :=\ & \frac{1}{120}\delta((3+n)u)+\frac{1}{24}\delta((2+n)u)+\frac{1}{120}\delta((n-2)u) \\
& +\frac{1}{12}\delta((1+n)u)+\frac{1}{24}\delta((n-1)u)+\frac{1}{12}\delta(nu)
\end{aligned}
$$

for all $u \in \mathcal{U}_0$ and all $\delta \in \mathbb{R}_+^{\mathcal{U}_0\times\mathcal{U}_0}$. For every $n \in \mathbb{N}$, the operator previously defined has the form specified in (1) with $\phi_1(u) = (3+n)u$, $L_1(u) = \frac{1}{120}$; $\phi_2(u) = (2+n)u$, $L_2(u) = \frac{1}{24}$; $\phi_3(u) = (n-2)u$, $L_3(u) = \frac{1}{120}$; $\phi_4(u) = (1+n)u$, $L_4(u) = \frac{1}{12}$; $\phi_5(u) = (n-1)u$, $L_5(u) = \frac{1}{24}$; $\phi_6(u) = nu$, $L_6(u) = \frac{1}{12}$ for all $u \in \mathcal{U}_0$.

By induction, we will prove that for all $u \in \mathcal{U}_0$, $m \in \mathcal{N}_0$, and $n \in W$, we have

$$
\left(\Upsilon_n^m \vartheta_n\right)(u) \leq \frac{1}{120}\eta_1(n)\alpha_n^m \tau_1(u)\tau_2(u). \tag{8}
$$

We may deduce the inequality (8) holds for $m = 0$ from (6) and (7). Following that, we suppose that (8) is true for $m = k$, where $k \in \mathbb{N}$. Then

$$
\begin{aligned}
\left(\Upsilon_n^{k+1}\vartheta_n\right)(u) \ =\ & \Upsilon_n\left(\left(\Upsilon_n^k\vartheta_n\right)(u)\right) \\
=\ & \frac{1}{120}\left(\Upsilon_n^n\vartheta_n\right)((3+n)x)+\frac{1}{24}\left(\Upsilon_n^n\vartheta_n\right)((2+n)u)+\frac{1}{120}\left(\Upsilon_n^n\vartheta_n\right)((n-2)u) \\
& +\frac{1}{12}\left(\Upsilon_n^n\vartheta_n\right)((1+n)u)+\frac{1}{24}\left(\Upsilon_n^n\vartheta_n\right)((n-1)u)+\frac{1}{12}\left(\Upsilon_n^n\vartheta_n\right)(nu) \\
\leq\ & \frac{1}{120}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1((3+n)u)\tau_2((3+n)u)\right) \\
& +\frac{1}{24}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1((2+n)u)\tau_2((2+n)u)\right) \\
& +\frac{1}{120}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1((n-2)u)\tau_2((n-2)u)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{12}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1((1+n)u)\tau_2((1+n)u)\right)\\
&+\frac{1}{24}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1((n-1)u)\tau_2((n-1)u)\right)\\
&+\frac{1}{12}\left(\frac{1}{120}\eta_1(n)\alpha_n^k\tau_1(nu)\tau_2(nu)\right)\\
\leq\;& \frac{1}{120}\eta_1(n)\alpha_n^{k+1}\tau_1(u)\tau_2(u)
\end{aligned}
$$

for all $u \in \mathcal{U}_0$ and all $n \in W$. Thus, for $m = k+1$, the inequality (8) holds. Since the inequality (8) holds for all $m \in \mathbb{N}_0$, we may obtain this conclusion. Thus we obtain

$$
\begin{aligned}
\vartheta_n^*(u) &= \sum_{m=0}^{\infty}\left(\Upsilon_n^m\vartheta_n\right)(u)\\
&\leq \sum_{0\leq n\leq\infty}\frac{1}{120}\eta_1(n)\alpha_n^m\tau_1(u)\tau_2(u)\\
&\leq \frac{\eta_1(n)}{120(1-\alpha_n)}\tau_1(u)\tau_2(u) < \infty
\end{aligned}
$$

for all $u \in \mathcal{U}_0$ and $n \in W$. Therefore, according to Theorem 2, we obtain the limit mapping

$$
H_n(u) := \lim_{m\to+\infty}\left(\Phi_n^m\phi\right)(u)
$$

exists for each $u \in \mathcal{U}_0$ and $n \in W$, and

$$
\|\phi(u) - H_n(u)\| \leq \frac{\eta_1(n)\tau_1(u)\tau_2(u)}{120(1-\alpha_n)} \tag{9}
$$

for all $u \in \mathcal{U}_0$ and $n \in W$.

Now, we will prove that H_n fulfills (2). It is enough to prove the following inequality

$$
\|D(\Phi_n^m\phi)(u,v)\| \leq \alpha_n^m\tau_1(u)\tau_2(v), \tag{10}
$$

for all $u, v \in \mathcal{U}_0$ and $n \in W$. Consider $k \in \mathbb{N}$ and suppose that (10) holds for $m = k$. Then, for each $u, v \in \mathcal{U}_0$ and $n \in W$, we get

$$
\begin{aligned}
\left\|D(\Phi_n^{k+1}\phi)(u,v)\right\| \quad &\leq \frac{1}{120}\alpha_n^k\tau_1((3+n)u)\tau_2((3+n)v) + \frac{1}{24}\alpha_n^k\tau_1((2+n)u)\tau_2((2+n)v)\\
&+\frac{1}{120}\alpha_n^k\tau_1((n-2)u)\tau_2((n-2)v) + \frac{1}{12}\alpha_n^k\tau_1((1+n)u)\tau_2((1+n)v)\\
&+\frac{1}{24}\alpha_n^k\tau_1((n-1)u)\tau_2((n-1)v) + \frac{1}{12}\alpha_n^k\tau_1((n)u)\tau_2((n)v)\\
&\leq \alpha_n^{k+1}\tau_1(u)\tau_2(v).
\end{aligned}
$$

By induction, we need to prove that (10) holds for all $u, v \in \mathcal{U}_0$, $m \in \mathbb{N}_0$, and $n \in W$. Taking the limit $m \to \infty$ in (10), we get

$$H_n(u+3v) - 5H_n(u+2v) - H_n(u-2v) + 10H_n(u+v)$$
$$+5H_n(u-v) - 10H_n(u) - 120H_n(v) = 0$$

for all $u, v \in \mathcal{U}_0$ such that $3v+u \neq 0$, $2v+u \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $v+u \neq 0$, $m \in \mathbb{N}_0$, and $n \in W$. This implies that we can be defined $H : \mathcal{U} \to \mathcal{V}$ which fulfills

$$\begin{aligned} H(u) \quad := \quad & \frac{1}{120}H((n+3)u) - \frac{1}{120}H((n-2)u) - \frac{1}{24}H((n+2)u) \\ & +\frac{1}{12}H((n+1)u) - \frac{1}{12}H(nu) + \frac{1}{24}H((n-1)u), \end{aligned} \tag{11}$$

for all $u \in \mathcal{U}_0$ and all $n \in W$.

Next, we need to show that each quintic mapping $H : \mathcal{U} \to \mathcal{V}$ fulfills the inequality

$$\|\phi(u) - H(u)\| \leq L\tau_1(u)\tau_2(u) \tag{12}$$

for all $u \in \mathcal{U}_0$, with $0 < L$, is equal to H_n for every $n \in W$. As a result, we set $n_0 \in W$ and $H : \mathcal{U} \to \mathcal{V}$ fulfilling (12). From (9), for every $u \in \mathcal{U}_0$, we have

$$\begin{aligned} \|H(u) - H_{n_0}(u)\| \quad & \leq \quad \|H(u) - \phi(u)\| + \|\phi(u) - H_{n_0}(u)\| \\ & \leq \quad L\tau_1(u)\tau_2(u) + \vartheta^*_{n_0}(u) \\ & \leq \quad L_0\tau_1(u)\tau_2(u) \sum_{m=0}^{\infty} \alpha^m_{n_0}, \end{aligned} \tag{13}$$

where $L_0 := (1-\alpha_{n_0})L + \frac{1}{120}\eta_1(n_0) > 0$ and we exclude the case that $\tau_1(u) \equiv 0$ or $\tau_2(u) \equiv 0$ which is trivial. From the observation, the functions H and H_{n_0} are the solutions to the functional equation (11) for every $n \in W$.

Next, we prove that, for every $j \in \mathbb{N}_0$, we obtain

$$\|H(u) - H_{n_0}(u)\| \leq L_0\tau_1(u)\tau_2(u) \sum_{m=j}^{\infty} \alpha^{\infty}_{n_0} \tag{14}$$

for all $u \in \mathcal{U}_0$. The inequality (13) is valid for the case $j = 0$. The next step is to correct $k \in \mathbb{N}$ and suppose that (14) is true for $j = k$. In the sense of (13), for every $u \in \mathcal{U}_0$, we obtain

$$\begin{aligned} \|H(u) - H_{n_0}(u)\| \quad \leq \quad & \frac{1}{120}L_0\tau_1((3+n_0)u)\tau_2((3+n_0)u)) \sum_{m=k}^{\infty} \alpha^m_{n_0} \\ & +\frac{1}{24}L_0\tau_1((2+n_0)u)\tau_2((2+n_0)u)) \sum_{m=k}^{\infty} \alpha^m_{n_0} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{120}L_0\tau_1((-2+n_0)u)\tau_2((-2+n_0)u))\sum_{m=k}^{\infty}\alpha_{n_0}^m\\
&+\frac{1}{12}L_0\tau_1((1+n_0)u)\tau_2((1+n_0)u))\sum_{m=k}^{\infty}\alpha_{n_0}^m\\
&+\frac{1}{24}L_0\tau_1((-1+n_0)u)\tau_2((-1+n_0)u))\sum_{m=k}^{\infty}\alpha_{n_0}^m\\
&+\frac{1}{12}L_0\tau_1((n_0)u)\tau_2((n_0)u))\sum_{m=k}^{\infty}\alpha_{n_0}^m\\
\leq\ & L_0\alpha_{n_0}\tau_1(u)\tau_2(u)\sum_{m=k}^{\infty}\alpha_{n_0}^m\\
\leq\ & L_0\tau_1(u)\tau_2(u)\sum_{m=k+1}^{\infty}\alpha_{n_0}^m.
\end{aligned}
$$

Thus the condition (14) is valid for $j = 1 + k$. As a result, we may say that the inequality (14) is true for all $j \in \mathbb{N}_0$. Taking the limit $j \to \infty$ in inequality (14), we obtain

$$H = H_{n_0}. \tag{15}$$

Also, in view of (9), we have

$$\|\phi(u) - H_{n_0}(u)\| \leq \frac{\eta_1(n)\tau_1(u)\tau_2(u)}{120(1-\alpha_n)},$$

for all $u \in \mathcal{U}_0$ and all $n \in W$. This implies the condition (3) with $H = H_{n_0}$ and (15) shows the uniqueness of H.

Theorem 4. *Let $\tau : \mathcal{U}_0 \times \mathcal{U}_0 \to \mathbb{R}_+$ be a function such that*

$$W := \{m \in \mathbb{N} : \alpha_m < 1\} = \varnothing,$$

where

$$
\begin{aligned}
\alpha_m \quad := \quad & \frac{1}{120}\eta(3+m) + \frac{1}{24}\eta(2+m) + \frac{1}{120}\eta(-2+m)\\
& +\frac{1}{12}\eta(1+m) + \frac{1}{24}\eta(-1+m) + \frac{1}{12}\eta(m)
\end{aligned}
$$

and

$$\eta(m) := \inf\left\{l \in \mathbb{R}_+ : \tau(mu) \leq l\tau(u), \quad u \in \mathcal{U}_0\right\}, \tag{16}$$

for every $m \in \mathbb{N}$. Assume that $\phi : \mathcal{U} \to \mathcal{V}$ fulfills

$$\|D\phi(u,v)\| \leq \tau(u) + \tau(v) \tag{17}$$

for all $u, v \in \mathcal{U}_0$*, such that* $3v+u \neq 0$*,* $2v+u \neq 0$*,* $u-2v \neq 0$*,* $u-v \neq 0$ *and* $v+u \neq 0$*. Then there is only one quintic mapping* $H : \mathcal{U} \to \mathcal{V}$ *satisfying*

$$\|H(u) - \phi(u)\| \leq \eta_0 \tau(u), \tag{18}$$

for all $u \in \mathcal{U}_0$*, where*

$$\eta_0 := \inf_{m \in W} \left\{ \frac{1+\eta(m)}{120(1-\alpha_m)} \right\}.$$

Proof. Replacing (u, v) by (nu, u) in (17), we have

$$\begin{aligned} &\left\| \frac{1}{120}\phi((3+n)u) - \frac{1}{24}\phi((2+n)u) - \frac{1}{120}\phi((-2+n)u) + \frac{1}{12}\phi((1+n)u) \right. \\ &\left. + \frac{1}{24}\phi((-1+n)u) - \frac{1}{12}\phi(nu) - \phi(u) \right\| \leq \frac{1}{120}\left(\tau_1(nu) + \tau_2(u)\right) \end{aligned} \tag{19}$$

for all $u \in \mathcal{U}_0$ and all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define the operator $\Phi_n : \mathcal{V}^{\mathcal{U}_0} \to \mathcal{V}^{\mathcal{U}_0}$ by

$$\begin{aligned} \Phi_n \psi(u) \ := \ & \frac{1}{120}\psi((n+3)u) - \frac{1}{24}\psi((n+2)u) - \frac{1}{120}\psi((n-2)u) \\ & + \frac{1}{12}\psi((n+1)u) + \frac{1}{24}\psi((n-1)u) - \frac{1}{12}\psi(nu) \end{aligned}$$

for all $\psi \in \mathcal{V}^{\mathcal{U}_0}$ and all $u \in \mathcal{U}_0$. Moreover, put

$$\vartheta_n(u) := \frac{1}{120}\left(\tau(nu) + \tau(u)\right), \tag{20}$$

for all $u \in \mathcal{U}_0$, and observe that

$$\vartheta_n(u) = \frac{1}{120}\left(\tau(nu) + \tau(u)\right) \leq \frac{1}{120}\left(\eta(n) + 1\right)\tau(u) \tag{21}$$

for all $u \in \mathcal{U}_0$ and all $n \in \mathbb{N}$. The inequality (19), then has the following form

$$\|\phi(u) - \Phi_n \phi(u)\| \leq \vartheta_n(u)$$

for all $u \in \mathcal{U}_0$. Moreover, for any $u \in \mathcal{U}_0$ and every $\psi, \varsigma \in \mathcal{V}^{\mathcal{U}_0}$, we obtain

$$\begin{aligned} \|\Phi_n\psi(u) - \Phi_n\varsigma(u)\| \ \leq \ & \frac{1}{120}\left\|(\psi-\varsigma)((n+3)u)\right\| + \frac{1}{24}\left\|(\psi-\varsigma)((n+2)u)\right\| \\ & + \frac{1}{120}\left\|(\psi-\varsigma)((n-2)u)\right\| + \frac{1}{12}\left\|(\psi-\varsigma)((n+1)u)\right\| \\ & + \frac{1}{24}\left\|(\psi-\varsigma)((n-1)u)\right\| + \frac{1}{12}\left\|(\psi-\varsigma)(nu)\right\|. \end{aligned}$$

This brings us to define the operator $\Upsilon_n : \mathbb{R}_+^{\mathcal{U}_0 \times \mathcal{U}_0} \to \mathbb{R}_+^{\mathcal{U}_0 \times \mathcal{U}_0}$ by

$$\Upsilon_n \delta(u) \ := \ \frac{1}{120}\delta((n+3)u) + \frac{1}{24}\delta((n+2)u) + \frac{1}{120}\delta((n-2)u)$$

$$+\frac{1}{12}\delta((n+1)u)+\frac{1}{24}\delta((n-1)u)+\frac{1}{12}\delta(nu)$$

for all $u\in\mathcal{U}_0$ and all $\delta\in\mathbb{R}_+^{\mathcal{U}_0\times\mathcal{U}_0}$. For every $n\in\mathbb{N}$, the form of the operator previously specified is given in (1) with

$$\begin{aligned}
\phi_1(u) &= (n+3)u, & L_1(u)=\frac{1}{120},\\
\phi_2(u) &= (n+2)u, & L_2(u)=\frac{1}{24},\\
\phi_3(u) &= (n-2)u, & L_3(u)=\frac{1}{120},\\
\phi_4(u) &= (n+1)u, & L_4(u)=\frac{1}{12},\\
\phi_5(u) &= (n-1)u, & L_5(u)=\frac{1}{24},\\
\phi_6(u) &= nu, & L_6(u)=\frac{1}{12}
\end{aligned}$$

for all $u\in\mathcal{U}_0$. By induction, we will verify that for every $u\in\mathcal{U}_0$, $m\in\mathbb{N}_0$, and $n\in W$, we have

$$\left(\Upsilon_n^m\vartheta_n\right)(u)\le\frac{1}{120}\left(\eta(n)+1\right)\alpha_n^m\tau(u). \tag{22}$$

The condition (22) for $m=0$ is derived from (20) and (21).

Suppose that (22) holds for $m=k$. Then

$$\begin{aligned}
\left(\Upsilon_n^{k+1}\vartheta_n\right)(u) &= \Upsilon_n\left(\left(\Upsilon_n^{k}\vartheta_n\right)(u)\right)\\
&= \frac{1}{120}\left(\Upsilon_n^n\vartheta_n\right)((n+3)x)+\frac{1}{24}\left(\Upsilon_n^n\vartheta_n\right)((n+2)u)+\frac{1}{120}\left(\Upsilon_n^n\vartheta_n\right)((n-2)u)\\
&\quad+\frac{1}{12}\left(\Upsilon_n^n\vartheta_n\right)((n+1)u)+\frac{1}{24}\left(\Upsilon_n^n\vartheta_n\right)((n-1)u)+\frac{1}{12}\left(\Upsilon_n^n\vartheta_n\right)(nu)\\
&\le \frac{1}{120}(\eta(n)+1)\alpha_n^{k+1}\tau(u),
\end{aligned}$$

for all $u\in\mathcal{U}_0$ and every $n\in W$. Therefore, for $m=k+1$, the inequality (22) holds. As a result, we may say that the inequality (22) holds for all $m\in\mathbb{N}_0$. Thus we have

$$\begin{aligned}
\vartheta_n^*(u) &= \sum_{m=0}^{\infty}\left(\Upsilon_n^m\vartheta_n\right)(u)\\
&\le \sum_{n=0}^{\infty}\frac{1}{120}\left(\eta(n)+1\right)\alpha_n^m\tau(u)\\
&\le \frac{(\eta(n)+1)}{120(1-\alpha_n)}\tau(u)<\infty
\end{aligned}$$

for all $u \in \mathcal{U}_0$ and $n \in W$. Therefore, according to Theorem 2, we obtain the limit function

$$H_n(u) := \lim_{m\to+\infty} \left(\Phi_n^m \phi\right)(u)$$

exists for every $u \in \mathcal{U}_0$ and every $n \in W$, and

$$\|\phi(u) - H_n(u)\| \leq \frac{(\eta(n)+1)\tau(u)}{120(1-\alpha_n)} \tag{23}$$

for all $u \in \mathcal{U}_0$ and $n \in W$. Now, we want to prove that H_n fulfills (2), it is enough to show the following inequality

$$\|D(\Phi_n^m \phi)(u,v)\| \leq \alpha_n^m \left(\tau(u) + \tau(v)\right) \tag{24}$$

for all $u, v \in \mathcal{U}_0$, $m \in \mathbb{N}_0$, and $n \in W$. Since the condition (17) is all that is required in the situation $m = 0$, assume that $k \in \mathbb{N}$ and suppose that (24) holds for every $m = k$ and $u, v \in \mathcal{U}_0$ and $n \in W$. Then, for each $u, v \in \mathcal{U}_0$ and $n \in W$, we have

$$\left\|D(\Phi_n^{k+1}\phi)(u,v)\right\| \leq \alpha_n^{k+1}\left(\tau(u) + \tau_2(v)\right).$$

By induction, we need to prove that (24) holds for every $u, v \in \mathcal{U}_0$, $m \in \mathbb{N}_0$, and $n \in W$. Taking the limit $m \to \infty$ in (24), we obtain

$$DH_n(u,v) = 0$$

for all $u, v \in \mathcal{U}_0$, $m \in \mathbb{N}_0$, and $n \in W$. This implies that the mapping $H : \mathcal{U} \to \mathcal{V}$ satisfies

$$\begin{aligned} H(u) \quad := \quad & \frac{1}{120}H((n+3)u) - \frac{1}{24}H((n+2)u) - \frac{1}{120}H((n-2)u) \\ & + \frac{1}{12}H((n+1)u) + \frac{1}{24}H((n-1)u) - \frac{1}{12}H(nu) \end{aligned} \tag{25}$$

for all $u \in \mathcal{U}_0$ and all $n \in W$.

Next, we need to show that each quintic mapping $H : \mathcal{U} \to \mathcal{V}$ fulfills the inequality

$$\|\phi(u) - H(u)\| \leq L\tau(u) \tag{26}$$

for all $u \in \mathcal{U}_0$, with some $0 < L$, is equal to H_n for every $n \in W$. As a result, we fix $n_0 \in W$ and $H : \mathcal{U} \to \mathcal{V}$ fulfills (26). From (16), for every $u \in \mathcal{U}_0$, we get

$$\begin{aligned} \|H(u) - H_{n_0}(u)\| \quad & \leq \quad \|H(u) - \phi(u)\| + \|\phi(u) - H_{n_0}(u)\| \\ & \leq \quad L\tau(u) + \vartheta_{n_0}^*(u) \\ & \leq \quad L_0\tau(u)\sum_{m=0}^{\infty}\alpha_{n_0}^m, \end{aligned} \tag{27}$$

where $L_0 := (1-\alpha_{n_0})L + \frac{1}{120}\eta(n_0) > 0$ and we exclude the case that $\tau(u) \equiv 0$ which is trivial. From the observation, the functions H and H_{n_0} are the solutions to the equations (25) for every $n \in W$. Next, we prove that, for every $j \in \mathbb{N}_0$, we obtain

$$\|H(u) - H_{n_0}(u)\| \leq L_0\tau(u)\sum_{m=j}^{\infty}\alpha_{n_0}^{\infty}, \tag{28}$$

for all $u \in \mathcal{U}_0$. The inequality (27) is valid for the case $j = 0$.

Next, we set $k \in \mathbb{N}$ and assume (28) is true for $j = k$. In view of (27), for every $u \in \mathcal{U}_0$, we have

$$\begin{aligned}
\|H(u) - H_{n_0}(u)\| &\leq \frac{1}{120}L_0\tau((n_0+3)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} + \frac{1}{24}L_0\tau((n_0+2)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} \\
&\quad + \frac{1}{120}L_0\tau((n_0-2)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} + \frac{1}{12}L_0\tau((n_0+1)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} \\
&\quad + \frac{1}{24}L_0\tau((n_0-1)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} + \frac{1}{12}L_0\tau((n_0)u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m} \\
&\leq L_0\alpha_{n_0}\tau(u)(u)\sum_{m=k}^{\infty}\alpha_{n_0}^{m}.
\end{aligned}$$

Thus

$$\|H(u) - H_{n_0}(u)\| \leq L_0\tau(u)\sum_{m=k+1}^{\infty}\alpha_{n_0}^{m}.$$

So the condition (28) is valid for $j = k+1$. Hence we may infer that for any $j \in \mathbb{N}_0$, the inequality (28) holds. Taking the limit $j \to \infty$ in (28), we obtain

$$H = H_{n_0}. \tag{29}$$

Also, in view of (23), we have

$$\|\phi(u) - H_{n_0}(u)\| \leq \frac{(\eta(n)+1)\tau(u)}{120(1-\alpha_n)}$$

for all $u \in \mathcal{U}_0$ and all $n \in W$. This implies the condition (18) with $H = H_{n_0}$ and (29) confirms the uniqueness of H.

3. Hyperstability

The φ-hyperstability of (2) in Banach spaces is the subject of the following theorem. In particular, we take into account functions $\phi : \mathcal{U} \to \mathcal{V}$ fulfilling (2), i.e.,

$$\|D\phi(u,v)\| \leq \varphi(u,v)$$

for all $u, v \in \mathcal{U}_0$ such that $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$ with a given mapping $\varphi : \mathcal{U}_0 \times \mathcal{U}_0 \to \mathbb{R}_+$.

Next, we find a unique quintic mapping $H : \mathcal{U} \to \mathcal{V}$ which is near to ϕ. After that, assuming some further φ assumptions, we demonstrate that the conditional functional equation (2) belongs to the family of functions $\phi : \mathcal{U} \to \mathcal{V}$ and is φ-hyperstable.

Theorem 5. *Let τ_1, τ_2, α_m and W be as in Theorem 3. Assume that*

$$\begin{cases} \lim_{m\to+\infty} \eta_1(m) = 0, \\ \lim_{m\to+\infty} \eta_1(m)\eta_2(m) = 0, \\ \lim_{m\to+\infty} \eta_1(m-2)\eta_2(m-2) = 0. \end{cases}$$

Then every mapping $\phi : \mathcal{U} \to \mathcal{V}$ fulfilling (3) is a solution of (2) on $\mathcal{U}_0$.

Proof. Suppose that $\phi : \mathcal{U} \to \mathcal{V}$ fulfills (3). By Theorem 3, there is a mapping $H : \mathcal{U} \to \mathcal{V}$ fulfilling (2) and

$$\|\phi(u) - H(u)\| \leq \eta_0 \tau_1(u)\tau_2(u)$$

for all $u \in \mathcal{U}_0$, where

$$\eta_0 := \inf_{m \in W} \left\{ \frac{\eta_1(m)}{120(1-\alpha_m)} \right\}.$$

So, in view of (5), $\eta_0 = 0$. This means that $\phi(u) = H(u)$ for all $u \in \mathcal{U}_0$. So

$$D\phi(u, v) = 0$$

for all $u, v \in \mathcal{U}_0$ such that $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$ which gives that ϕ fulfills (2) on $\mathcal{U}_0$.

Corollary 1. *Let $\epsilon \geq 0$, $c_1, c_2 \in \mathbb{R}$ with $c_1 + c_2 < 0$. Assume that a mapping $\phi : \mathcal{U} \to \mathcal{V}$ fulfills $\phi(0) = 0$ and*

$$\|D\phi(u, v)\| \leq \epsilon \|u\|^{c_1} \|v\|^{c_2} \tag{30}$$

for all $u, v \in \mathcal{U}_0$ such that $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$. Then ϕ is quintic on $\mathcal{U}_0$.

Proof. Theorem 3 is supported by the proof, which defines

$$\tau_1, \tau_2 : \mathcal{U}_0 \times \mathcal{U}_0 \to \mathbb{R}_+ \text{ by } \tau_1(u) = \epsilon_1 \|u\|^{c_1}, \quad \tau_2(v) = \epsilon_2 \|v\|^{c_2}$$

and $\tau_1(0) = \tau_2(0) = 0$ with $\epsilon_1, \epsilon_2 \in \mathbb{R}_+$ and $c_1, c_2 \in \mathbb{R}$ such that $\epsilon_1\epsilon_2 = \epsilon$ and $c_1 + c_2 < 0$.

For each $m \in \mathbb{N}$, we obtain

$$\begin{aligned} \eta_1(m) &= \inf\{l \in \mathbb{R}_+ : \tau_1(mx) \leq \tau_1(u), \quad u \in \mathcal{U}_0\} \\ &= \inf\{l \in \mathbb{R}_+ : \epsilon_1 \|mu\|^{c_1} \leq l\epsilon_1 \|u\|^{c_1}, \quad u \in \mathcal{U}_0\} \end{aligned}$$

$$= \quad m^{c_1}.$$

Similarly, we obtain $\eta_2(m) = m^{c_2}$ for all $m \in \mathbb{N}$.

Now, we can find $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}\alpha_m &= \frac{1}{120}(m+3)^{c_1+c_2} + \frac{1}{24}(m+2)^{c_1+c_2} + \frac{1}{120}(m-2)^{c_1+c_2} \\ &\quad + \frac{1}{12}(m+1)^{c_1+c_2} + \frac{1}{24}(m-1)^{c_1+c_2} + \frac{1}{12}(m)^{c_1+c_2} < 1\end{aligned}$$

for all $m \geq m_0$. According to Theorem 3, there is only one quintic mapping $H : \mathcal{U} \to \mathcal{V}$ satisfying

$$\|\phi(u) - H(u)\| \leq \epsilon\eta_0\tau_1(u)\tau_2(u)$$

for all $m \in \mathcal{U}_0$. Since $c_1 + c_2 < 0$, one of c_1 and c_2 must be negative. Assume that $c_2 < 0$. Then

$$\begin{cases}\lim_{m\to+\infty} \eta_1(m) = \lim_{m\to+\infty} m^{c_1} = 0, \\ \lim_{m\to+\infty} \eta_1(m)\eta_2(m) = \lim_{m\to+\infty} m^{c_1+c_2} = 0, \\ \lim_{m\to+\infty} \eta_1(m-2)\eta_2(m-2) = \lim_{m\to+\infty}(m-2)^{c_1+c_2} = 0.\end{cases}$$

The expected outcomes are thus obtained by Theorem 5.

Example 1. *Let $\mathcal{U}$ be a normed space and $\mathcal{V}$ be a Banach space. Let $\phi : \mathcal{U} \to \mathcal{V}$ be a mapping such that $D\phi(u_0, v_0) \neq 0$ for some $u_0, v_0 \in \mathcal{U}$ and*

$$\|D\phi(u, v)\| \leq c\|u\|^{c_1}\|v\|^{c_2}$$

for all $u, v \in \mathcal{U}_0$ such that $u + 3v \neq 0$, $u + 2v \neq 0$, $u - 2v \neq 0$, $u - v \neq 0$, $u + v \neq 0$, where $\epsilon > 0$ and $c_1, c_2 \in \mathbb{R}$. Assume that the numbers c_1, c_2 satisfy $c_1 + c_2 < 0$. Then the functional equation

$$\begin{aligned}&\phi(u+3v) - 5\phi(u+2v) - \phi(u-2v) + 10\phi(u+v) + 5\phi(u-v) - 10\phi(u) - 120\phi(v) \\ &= 0, \qquad \forall u, v \in \mathcal{U}_0, \qquad (31)\end{aligned}$$

has no solution in the class of functions $\phi : \mathcal{U} \to \mathcal{V}$.

Proof. Suppose that $\phi : \mathcal{U} \to \mathcal{V}$ is a solution of (31). Then (30) holds, and consequently, according to Corollary 1, ϕ is a quintic mapping on U_0, which means that $D\phi(u_0, v_0) = 0$. This is a contradiction.

Corollary 2. *Let $\epsilon \geq 0$, $c \in \mathbb{R}$ with $c < 0$. If a mapping $\phi : \mathcal{U} \to \mathcal{V}$ fulfills $\phi(0) = 0$ and*

$$\|D\phi(u, v)\| \leq \epsilon\left(\|u\|^c + \|v\|^c\right) \qquad (32)$$

for all $u, v \in \mathcal{U}_0$ such that $u + 3v \neq 0$, $u + 2v \neq 0$, $u - 2v \neq 0$, $u - v \neq 0$, $u + v \neq 0$. Then ϕ is quintic on $\mathcal{U}_0$.

Example 2. *Let* $\mathcal{U}_0$ *be a normed space and* $\mathcal{V}$ *be a Banach space. Let* $\phi : \mathcal{U}_0 \to \mathcal{V}$ *be a mapping such that* $D\phi(u_0, v_0) \neq 0$ *for some* $u_0, v_0 \in \mathcal{U}_0$ *and*

$$\|D\phi(u,v)\| \leq c\|u\|^c\|v\|^c$$

for all $u, v \in \mathcal{U}_0$ *such that* $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$, *where* $\epsilon > 0$ *and* $c \in \mathbb{R}$. *Assume that the number* c *satisfies* $c < 0$. *Then the functional equation*

$$\begin{aligned}&\phi(u+3v) - 5\phi(u+2v) - \phi(u-2v) + 10\phi(u+v) + 5\phi(u-v) - 10\phi(u) - 120\phi(v) \\ &= 0, \qquad \forall u, v \in \mathcal{U}_0,\end{aligned} \tag{33}$$

has no solution in the class of functions $\phi : \mathcal{U} \to \mathcal{V}$.

Proof. Suppose that $\phi : \mathcal{U} \to \mathcal{V}$ is a solution of (33). Then (32) holds, and consequently, according to Corollary 2, ϕ is a quintic mapping on U_0, which means that $D\phi(u_0, v_0) = 0$. This is a contradiction.

The findings of hyperstability for inhomogeneous quintic functional equations are demonstrated by the following corollary.

Corollary 3. *Let* $\epsilon, c_1, c_2 \in \mathbb{R}$ *with* $\epsilon \geq 0$ *and* $c_1 + c_2 < 0$. *Assume that mappings* $G : \mathcal{U}^2 \to \mathcal{V}$ *and* $\phi : \mathcal{U} \to \mathcal{V}$ *fulfill* $\phi(0) = 0$ *and*

$$\|D\phi(u,v) - G(u,v)\| \leq \epsilon\|u\|^{c_1}\|v\|^{c_2} \tag{34}$$

for all $u, v \in \mathcal{U}_0$ *such that* $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$. *If the functional equation*

$$D\phi(u,v) = G(u,v)$$

for all $u, v \in \mathcal{U}_0$, *has a solution* $\phi_0 : \mathcal{U} \to \mathcal{V}$ *on* $\mathcal{U}_0$, *then* ϕ *satisfies* (34) *on* $\mathcal{U}_0$.

Proof. From (34), we obtain the mapping $f : \mathcal{U} \to \mathcal{V}$ defined by $f := \phi - \phi_0$ fulfills (32).

The equation (2) on $\mathcal{U}_0$ is therefore implied by Corollary 1, which states that f is a solution. Thus

$$\begin{aligned}D\phi(u,v) - G(u,v) &= D(f+\phi_0)(u,v) - G(u,v) \\ &= 0\end{aligned}$$

for all $u, v \in \mathcal{U}_0$ such that $u+3v \neq 0$, $u+2v \neq 0$, $u-2v \neq 0$, $u-v \neq 0$, $u+v \neq 0$, which means that ϕ is a solution to (3) on $\mathcal{U}_0$.

4. Conclusion

We proved the hyperstability of the quintic functional equation $\phi(u+3v) - 5\phi(u+2v) - \phi(u-2v) + 10\phi(u+v) + 5\phi(u-v) - 10\phi(u) - 120\phi(v) = 0$, in Banach spaces by means of Brzdęk's fixed point theorem.

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Declarations

Availablity of data and materials
Not applicable.
Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.
Conflict of interest
The authors declare that they have no competing interests.

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