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Convergence Analysis of Multi-Step Collocation Method to First-Order Volterra Integro-Differential Equation with Non-Vanishing Delay

Ahmed Ali Eashel[1], Saeed Pishbin[1,*], Parviz Darania[1]

[1] *Department of Mathematics, Faculty of Science, Urmia University, P.O.Box 165, Urmia-Iran*

Abstract. Generally, solutions to functional equations involving non-vanishing delays tend to exhibit lower regularity compared to those of smooth functions. In this context, we examine a first-order Volterra integro-differential equation (VIDE) with a non-vanishing delay, delving into the characteristics of its solutions. To enhance the accuracy of traditional one-step collocation methods [1], we employ multi-step collocation techniques to obtain numerical solutions for the VIDE with non-vanishing delay. The global convergence properties of the multi-step numerical approach are scrutinized using the Peano Kernel Theorem. Subsequently, for comparative analysis, we utilize a one-step collocation method to numerically solve this equation, showcasing the effectiveness and precision of the multi-step collocation method.

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Key Words and Phrases: Volterra integro-differential equation, Delay integro-differential equation, Multi-step collocation methods, Convergence analysis

1. Introduction

Here, we deal with the approximate solution of the delay first-order Volterra integro-differential equation of the form

$$x'(t) = c_1(t)x(t)+c_2(t)x(\tau(t))+f(t)+\int_{t_0}^{t} K(t,s,x(s))\mathrm{d}s+\int_{t_0}^{\tau(t)} \hat{K}(t,s,x(s))\mathrm{d}s,\ t\in J=[t_0,T], \tag{1}$$

where

$$x(t)=\zeta(t), \quad t\in[\tau(t_0),t_0],$$

$x(t)$ is the unknown solution and $c_1, c_2, K, \hat{K}$ are given functions. Also $\tau(t)$ is delay (or lag) function which will be defined completely in section 2.

*Corresponding author.
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Email addresses: a.alieashel@gmail.com (A. Ali Eashel), s.pishbin@urmia.ac.ir (S. Pishbin), p.darania@urmia.ac.ir (P. Darania)

Volterra integro-differential equations (VIDEs) may be viewed as ordinary differential equations affected by a memory term arising from an integral operator which exhibit a common characteristic, when the initial data was smooth, it resulted in smooth solutions. In general this statement does not hold for an equation involving a non-vanishing delay. In such cases, the presence of delays gives rise to primary points of discontinuity, resulting in a solution that is less regular compared to the initially supplied smooth functions.

The initial-value problem for Volterra integro-differential equation arise in some mathematical modelling processes in biological and physical phenomena, such as fluid dynamics, viscoelasticity in materials with memory, such as population dynamics [2–6]. Also, VIDEs with non-vanishing delay have been used in describing some phenomena of population growth the transmission of an epidemic with the influx of immigrants into the population, incorporating scenarios into models like the predator-prey model. For instance, in [7] , we can find the following delay integro-differential equations (IDEs) related to population dynamics:

$$z'(t) = cz(t)\Big(1 - \frac{1}{L}\int_{-\tau}^{0} z(t+s)\mathrm{d}\sigma(s)\Big),$$

where L is the environmental carrying capacity and c is the intrinsic rate. Also, VIDEs with non-vanishing delay have applications in economics and Belairs model to life span (See examples on pages 212-214 in [1] for more details).

For the numerical soltion of Volterra integro-differential equations, we can find several numerical methods in the literature, i.e. rationalized Haar functions [8], Galerkin and collocation type methods [1, 9–12], Runge-Kutta methods [1, 9], Lagrange polynomial method [13] and multi-step methods [14]. In [14], the authors, have analyzed multi-step collocation methods for Volterra IDEs and derived order of convergence of the proposed method. The study involved an examination of numerical stability analysis, and various classes of methods that are A_0-stable were presented. In [15], the authors considered the Chebyshev collocation method to solve the IDEs and developed this method for the system of Volterra-Fredholm IDEs. Also, polynomial approximation based on Taylor expansion [16] has been developed for the Volterra-Fredholm and VIDEs in real application. Integro-differential equations involving convolution integrals as a generalization of the fractional differential equations has been solved by numerical method in [17] and has been shown that the proposed numerical scheme is stable. The authors, [18] consider a graded mesh refinement algorithm for solving time-delayed parabolic partial differential equations with a small diffusion parameter. Also, in [19] a class of boundary layer originated singularly perturbed parabolic reaction-diffusion problems with large time delay have been studied. A nonlinear system of singularly perturbed delay differential equation whose each component of the solution has multiple layers has been investigated in [20]. Delay integral equations (IEs) have been solved approximately by many authors (see, e.g.,[1, 21–28]). Recently, some authors are interested in working on the numerical solution of delay IDEs. In [29], the authors used multi-step methods to find a approximate solution of singularly perturbed delay VIDEs. One- step polynomial collocation method [1] has been applied to find numerical solution of delay VIDEs by Brunner. He performed the convergence analysis of the collocation method using Peano kernel theorem and investigated super convergence analysis of the numerical approximation in detail. The delay VIDEs have been achieved through the utilization of the two-point multi-step block (2PBM) [30] method which was formulated by Taylor expansion. The authors developed the 2PBM method by considering the predictor-corrector formulae. Also, in [31] an effective numerical method has been applied to solve VIDEs including neutral terms with variable delays by fundamental matrices of Laguerre polynomials. Here, we consider multi-step collocation methods to delay equation (1) and try to increase the order of convergence in comparing of the one-step collocation methods in [1]. The paper is organized as follows:

Section 2 is dedicated to proposing the structure of solutions for VIDEs with non-vanishing delays and subsequently outlining the application of the multi-step polynomial collocation method by employing well-known interpolation polynomials. In Section 3, we construct and analyze the convergence of numerical solutions, taking into consideration Peano's Theorem for interpolation. This is succeeded by the discussion of two test problems in Section 4 to validate the theoretical results. Finally, in Section 5, we conclude the paper and suggest potential future avenues for research, which are currently less explored.

2. Numerical method for the delay VIDE

In this section, firstly we will state structure of the solution of VIDEs with non-vanishing delays (1) and then consider the multi-step collocation method to solve this equation numerically.

2.1. Structure of the solution of VIDEs with non-vanishing delays

Let $\tau(t) = t - \alpha(t)$ be strictly increasing on J with $\alpha(t) \geq \alpha_0 > 0$ for $t \in J$ and $\alpha(t) \in C^{\nu}(J)$ for some $\nu \geq 0$. The presence of a non-vanishing delay $\tau(t)$ gives rise to the primary discontinuity points denoted as ς_i so that they are obtained from the following formula

$$\tau(\varsigma_i) = \varsigma_{i-1}, \qquad i \geq 1, \quad \varsigma_0 = t_0,$$

and

$$\varsigma_\mu - \varsigma_{\mu-1} = \alpha(\varsigma_\mu) \geq \alpha_0 > 0, \qquad \text{for all} \quad \mu \geq 0.$$

Now, we write the equation (1) in the local form

$$x'(t) = c_1(t)x(t) + q_{1,i}(t) + \int_{\varsigma_i}^{t} K(t,s,x(s))\mathrm{d}s, \quad t \in [\varsigma_i, \varsigma_{i+1}], \tag{2}$$

where

$$q_{1,i}(t) = c_2(t)x(\tau(t)) + f(t) + \int_{t_0}^{\varsigma_i} K(t,s,x(s))\mathrm{d}s + \int_{t_0}^{\tau(t)} \hat{K}(t,s,x(s))\mathrm{d}s, \quad i \geq 1.$$

For $t \in (\varsigma_0, \varsigma_1]$, we derive

$$q_{1,0}(t) = f(t) + c_2(t)x(\tau(t)) - \int_{\tau(t)}^{t_0} \hat{K}(t,s,\zeta(s))\mathrm{d}s.$$

and

$$\lim_{t\to t_0^-} x'(t) = \zeta'(t_0),$$

$$\lim_{t\to t_0^+} x'(t) = c_1(t_0)x(t_0) + q_{1,0}(t_0),$$

then, x' has a discontinuity at $t = t_0$. Assume that $c_1, c_2, f \in C(J)$, $K(.,.) \in C(D \times R)$ and $\hat{K}(.,.) \in C(D_\tau \times R)$, with $D = \{(t,s) : t_0 \le s \le t \le T\}$, $D_\tau = \{(t,s) : \tau(t_0) \le s \le \tau(t), t \in J\}$. For $t \in (\varsigma_1, \varsigma_2]$, we have

$$x'(\varsigma_1^-) = c_1(\varsigma_1)x(\varsigma_1) + q_{1,0}(\varsigma_1^-) + \int_{\varsigma_0}^{\varsigma_1} K(\varsigma_1, s, x(s))\mathrm{d}s,$$

$$x'(\varsigma_1^+) = c_1(\varsigma_1)x(\varsigma_1) + q_{1,1}(\varsigma_1^+).$$

Then

$$x'(\varsigma_1^-) - x'(\varsigma_1^+) = 0,$$

and for $t \in [\varsigma_i, \varsigma_{i+1}], i \ge 2$, this continuity is maintained for the other points of ς_i as well.

Now, using these arguments and a similar process in the proof of Theorem 2.2 from [32], we consider the following theorem which gives the relevant conditions for the investigation of the unique solution of the equation (1):

Theorem 1. *Assume that $\tau(t) = t - \alpha(t)$ be strictly increasing on J with $\alpha(t) \ge \alpha_0 > 0$ for $t \in J$ and $\alpha(t) \in C^\nu(J)$ for some $\nu \ge 0$. Also*

1. *$c_1, c_2, f \in C(J)$ and $C_1 = \max_{t\in J} |c_1(t)|$.*
2. *$K(.,.) \in C(D \times R)$ and $\hat{K}(.,.) \in C(D_\tau \times R)$ with $D = \{(t,s) : t_0 \le s \le t \le T\}$, $D_\tau = \{(t,s) : \tau(t_0) \le s \le \tau(t), t \in J\}$.*
3. *K satisfies the Lipschitz condition*

$$|K(t,s,x) - K(t,s,y)| \le L|x - y| \quad \forall (t,s) \in D, x, y \in R.$$

So it can be said for each initial function $\zeta(t) \in C[\tau(t_0), t_0]$ the equation (1) has a unique solution $x \in C(J) \cap C^1(t_0, T]$. Also, in general, at $t = t_0$ its derivative is discontinuous (but bounded):

$$\lim_{t\to t_0^-} x'(t) \ne \lim_{t\to t_0^+} x'(t).$$

Proof. We should consider local form of the equation (1) by (2). For $\mu = 0$, we have $t \in [\varsigma_0, \varsigma_1]$ and derive

$$x'(t) = c_1(t)x(t) + q_{1,0}(t) + \int_{t_0}^{t} K(t,s,x(s))\mathrm{d}s, \quad x(t_0) = \zeta(t_0). \tag{3}$$

It is equivalent to a nonlinear VIE of the second kind as:

$$x(t) = \hat{q}_{1,0}(t) + \int_{t_0}^{t} c_1(s)x(s)\mathrm{d}s + \int_{t_0}^{t}\int_{s}^{t} K(\eta, s, x(s))\mathrm{d}\eta\mathrm{d}s, \tag{4}$$

where $\hat{q}_{1,0}(t) = \zeta(t_0) + \int_{t_0}^{t} q_{1,0}(s)\mathrm{d}s$.

We rewrite equation (4) in operator form

$$x(t) = \hat{q}_{1,0}(t) + \mathcal{V}(x)(t), \tag{5}$$

where

$$\mathcal{V}(x)(t) = \int_{t_0}^{t} c_1(s)x(s)\mathrm{d}s + \int_{t_0}^{t}\int_{s}^{t} K(\eta, s, x(s))\mathrm{d}\eta\mathrm{d}s.$$

We choose a positive constant $\delta_0 > 0$ and define $A_1 = C([t_0, t_0 + \delta_0])$. Then A_1 is a Banach space equipped with the maximum norm. We will prove that the operator $\mathcal{V}$ restricted to A_1 is a contraction operator. For $x, y \in A_1$

$$\begin{aligned} |\mathcal{V}(x) - \mathcal{V}(y)| &\leq \int_{t_0}^{t} |c_1(s)|\ |x(s) - y(s)|\mathrm{d}s + \int_{t_0}^{t}\int_{s}^{t} |K(\eta, s, x(s)) - K(\eta, s, y(s))|\mathrm{d}\eta\mathrm{d}s \\ &\leq C_1\delta_0\|x - y\|_\infty + \tfrac{L\delta_0^2}{2}\|x - y\|_\infty \leq \|x - y\|_\infty\beta(\delta_0 + \delta_0^2), \end{aligned} \tag{6}$$

where $\beta = \max\{C_1, \frac{L}{2}\}$. We know that $\beta > 0$, then, for $0 < \delta_0 < \frac{-1+\sqrt{1+\frac{4}{\beta}}}{2}$, $\mathcal{V}$ is a contraction map on the Banach space A_1, and has a unique fixed point $x_1 \in A_1$. In the sequel, we study the case $t \in [t_0 + \delta_0, t_0 + 2\delta_0]$ if $t_0 + 2\delta_0 < \varsigma_1$. By defining

$$A_2 = \{x \in C([t_0, t_0 + 2\delta_0]), \quad and \quad x(t) = x_1(t), \quad for \quad t \in [t_0, t_0 + \delta_0]\},$$

we will show that $\mathcal{V}$ is also a contraction map on A_2:

$$\begin{aligned} |\mathcal{V}(x) - \mathcal{V}(y)| &\leq \int_{t_0}^{t} |c_1(s)|\ |x(s) - y(s)|\mathrm{d}s + \int_{t_0}^{t}\int_{s}^{t} |K(\eta, s, x(s)) - K(\eta, s, y(s))|\mathrm{d}\eta\mathrm{d}s \\ &\leq \int_{t_0}^{t} |c_1(s)|\ |x(s) - y(s)|\mathrm{d}s + \int_{t_0}^{t} (t-s)L|x(s) - y(s)|\mathrm{d}s \\ \leq &\int_{t_0}^{t_0+\delta_0} |c_1(s)|\ |x_1(s) - x_1(s)|\mathrm{d}s + \int_{t_0}^{t_0+\delta_0} (t-s)L|x_1(s) - x_1(s)|\mathrm{d}s \\ &+ \int_{t_0+\delta_0}^{t} |c_1(s)|\ |x(s) - y(s)|\mathrm{d}s + \int_{t_0+\delta_0}^{t} (t-s)L|x(s) - y(s)|\mathrm{d}s \\ &\leq C_1\delta_0\|x - y\|_\infty + \tfrac{L\delta_0^2}{2}\|x - y\|_\infty \leq \|x - y\|_\infty\beta(\delta_0 + \delta_0^2). \end{aligned} \tag{7}$$

For $0 < \delta_0 < \frac{-1+\sqrt{1+\frac{4}{\beta}}}{2}$, it also follows from the Banach fixed point theorem that $\mathcal{V}$ has a unique fixed point $x_2 \in A_2$, which satisfies $x_2(t) = x_1(t)$ for $t \in [t_0, t_0 + \delta_0]$. Therefore, we obtain a unique solution to VIDE (3) on the interval $[t_0, t_0 + 2\delta_0]$. In the sequel, we proceed similarly to the strategy considered in the proof of the Theorem 2.2 from [32] and complete the proof.

Uniqueness: Suppose that (4) possesses two continuous solutions x and z on the interval $[\varsigma_0, \varsigma_1]$. Hence, by the Lipschitz condition:

$$\begin{aligned} |x(t) - z(t)| &\leq \int_{t_0}^{t} |c_1(s)|\ |x(s) - z(s)|\mathrm{d}s + \int_{t_0}^{t}\int_{s}^{t} |K(\eta, s, x(s)) - K(\eta, s, z(s))|\mathrm{d}\eta\mathrm{d}s \\ &\leq (C_1 + L(\varsigma_1 - \varsigma_0)) \int_{t_0}^{t} |x(s) - z(s)|\mathrm{d}s. \end{aligned} \tag{8}$$

It follows from the classical Gronwall lemma [1]

$$|x(t) - z(t)| \leq 0 \times Exp[(C_1 + L(\varsigma_1 - \varsigma_0))(t - t_0)] = 0, \quad t \in [\varsigma_0, \varsigma_1]. \tag{9}$$

The continuity of x and z then implies that $x(t) = z(t)$ for all $t \in [\varsigma_0, \varsigma_1]$. For $\mu \geq 1$, the above discussion ($\mu = 0$) is readily adapted to establish the (local) existence and uniqueness of a solution.

If the data in the delay VIDE (1) are smooth functions, the corresponding solution will essentially inherit this smoothness, except at the primary discontinuity points ς_μ. Considering similar strategy considered in the Theorem 2.3 from [32] for the local form (2), we derive a regularity result for the solution of the equation (1) as:

Theorem 2. *Assume that $\tau(t) = t - \alpha(t)$ be strictly increasing on J with $\alpha(t) \geq \alpha_0 > 0$ for $t \in J$ and $\alpha(t) \in C^\nu(J)$ for some $\nu \geq d$. Also*

1. *$c_1, c_2, f \in C^d(J)$ and $\zeta(t) \in C^d[\tau(t_0), t_0]$.*
2. *$K(.,.) \in C^d(D \times R)$ and $\hat{K}(.,.) \in C^d(D_\tau \times R)$.*
3. *K satisfies the Lipschitz condition*

$$|K(t, s, x) - K(t, s, y)| \leq L|x - y| \quad \forall (t, s) \in D, x, y \in R.$$

The unique solution of the equation (1) is $(d+1)$-times continuously differentiable on each left-open macro-interval $(\varsigma_\mu, \varsigma_{\mu+1}]$ for each $\mu = 0, 1, \ldots, M$ and has a bounded first derivative on J. Also, in general, at $t = \varsigma_\mu, (\mu = 0, 1, \ldots, \min\{d, M\})$, we have:

$$\lim_{t \to \varsigma_\mu^-} x^{(\mu)}(t) = \lim_{t \to \varsigma_\mu^+} x^{(\mu)}(t),$$

while the $(\mu + 1)$st derivative of x is in general not continuous at $t = \varsigma_\mu$. If $\min\{d, M\} = d < M$, the solution possesses a continuous $(d + 1)$st derivative on $[\varsigma_\mu, T]$.

Remark 1. *If the data in the delay VIDE* (1) *are smooth functions with the degree of smoothness d, the corresponding solution will essentially inherit this smoothness on each left-open macro-interval* $(\varsigma_\mu, \varsigma_{\mu+1}]$. *Also, for* $t = \varsigma_\mu(\mu = 0, 1, \cdots, \min\{d, M\})$, *the μ st derivative of the solution is continuous at this points. Since solutions of this problem generally suffer from a loss of regularity at the primary discontinuity points ς_μ, the mesh* $l_h = \{t_n \ : \ t_0 < t_1 < \cdots < t_N\}$ *underlying the collocation space will have to include these points if the collocation solution is to attain its optimal global or local order of convergence. Thus, we shall employ meshes of the form*

$$L_h := \bigcup_{\mu=0}^{M} l_h^{(\mu)}, \qquad l_h^{(\mu)} := \{t_n^{(\mu)} \ : \ \varsigma_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_{N_\mu}^{(\mu)} = \varsigma_{\mu+1}\}.$$

Such a mesh is called a constrained mesh (with respect to $\tau(t)$) for J. We will refer to L_h as the macro-mesh and call the $l_h^{(\mu)}$ the underlying local meshes. Maybe we can consider the other ideas [33–35] to overcome the problem of low smoothness of the solution.

2.2. Multi-step method

In this subsection, we apply the multi-step collocation method to solve the equation (1). Let T in $J = [t_0, T]$ is defined so that

$$T = \varsigma_{M+1}, \quad \text{for some} \quad M \geq 1,$$

$$L_h := \bigcup_{\mu=0}^{M} l_h^{(\mu)}, \qquad l_h^{(\mu)} := \{t_n^{(\mu)} \ : \ \varsigma_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_{N_\mu}^{(\mu)} = \varsigma_{\mu+1}\},$$

$$h_n^{(\mu)} = t_{n+1}^{(\mu)} - t_n^{(\mu)}, \ \mu = 0, \ldots, M, \ (M \geq 1) \text{ and for } 0 \leq n \leq N-1,$$

$$Y_h^{(\mu)} = \{t_{n,i}^{(\mu)} = t_n^{(\mu)} + s_i h_n^{(\mu)} : 0 < s_1 < \cdots < s_m \leq 1\},$$

where $\{s_i\}$ are collocation parameters. We consider w' as an approximate solution of x' in $[t_n^{(\mu)}, t_{n+1}^{(\mu)}]$ by

$$w'(t_n^{(\mu)} + z h_n^{(\mu)}) = \sum_{k=0}^{r-1} P_k(z) x'^{(\mu)}_{n-k} + \sum_{j=1}^{m} Q_j(z) W_{n,j}^{(\mu)}, \ z \in (0,1], \ W_{n,j}^{(\mu)} = w'(t_{n,j}^{(\mu)}), \ n \geq r-1, \tag{10}$$

where $x'^{(\mu)}_{n-k} = w'(t_{n-k}^{(\mu)})$ and

$$\begin{aligned} P_k(z) &= \Big(\prod_{i=1}^{m} \frac{z - s_i}{-k - s_i} \Big) \Big(\prod_{i=0, i \neq k}^{r-1} \frac{z + i}{-k + i} \Big), \\ Q_j(z) &= \Big(\prod_{i=0}^{r-1} \frac{z + i}{s_j + i} \Big) \Big(\prod_{i=1, i \neq j}^{m} \frac{z - s_i}{s_j - s_i} \Big). \end{aligned} \tag{11}$$

Now, setting $x_n^{(\mu)} = w(t_n^{(\mu)})$ and

$$\alpha_k(z) = \int_0^z P_k(s)ds,\ (k = 0, \cdots, r-1), \quad \beta_j(z) = \int_0^z Q_j(s)ds,\ (j = 1, \cdots, m),$$

we obtain from (10)

$$w(t_n^{(\mu)} + zh_n^{(\mu)}) = x_n^{(\mu)} + h_n^{(\mu)} \sum_{k=0}^{r-1} \alpha_k(z) x'^{(\mu)}_{n-k} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(z) W_{n,j}^{(\mu)}, \tag{12}$$

and

$$x'^{(\mu)}_{n+1} = \sum_{k=0}^{r-1} P_k(1) x'^{(\mu)}_{n-k} + \sum_{j=1}^{m} Q_j(1) W_{n,j}^{(\mu)},$$

$$x_{n+1}^{(\mu)} = x_n^{(\mu)} + h_n^{(\mu)} \sum_{k=0}^{r-1} \alpha_k(1) x'^{(\mu)}_{n-k} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(1) W_{n,j}^{(\mu)}, \quad n \geq r-1.$$

In $[t_n^{(\mu)}, t_{n+1}^{(\mu)}]$, $0 \leq n < r-1$, the primary values $x'^{(\mu)}_0, x'^{(\mu)}_1, x'^{(\mu)}_2, \ldots, x'^{(\mu)}_{r-1}$ and $x_{r-1}^{(\mu)}$ may be obtained using the appropriate methods (see one- step collocation method in chapter 3 in [1]). Also, when $t = t_0$, we have $x'(t_0) = c_1(t_0)x(t_0) + q_{1,0}(t_0)$ where $x(t_0) = \zeta(t_0)$. In addition, w as an approximate solution should satisfy the following collocation equation

$$\begin{aligned} w'(t_{n,i}^{(\mu)}) = \;& c_1(t_{n,i}^{(\mu)}) w(t_{n,i}^{(\mu)}) + c_2(t_{n,i}^{(\mu)}) w(\tau(t_{n,i}^{(\mu)})) + f(t_{n,i}^{(\mu)}) + \int_{t_0}^{t_{n,i}^{(\mu)}} K(t_{n,i}^{(\mu)}, s, w(s)) \mathrm{d}s \\ & + \int_{t_0}^{\tau(t_{n,i}^{(\mu)})} \hat{K}(t_{n,i}^{(\mu)}, s, w(s)) \mathrm{d}s. \end{aligned} \tag{13}$$

Let $\tau(t)$ be linear. Inserting (10),(12) into (13) and using appropriate change of variables for each sub interval $[t_n^{(\mu)}, t_{n+1}^{(\mu)}]$, we have the following non-linear system in two cases:

I) For $\mu = 0$, we have

$$\begin{aligned} W_{n,i}^{(0)} = \;& c_1(t_{n,i}^{(0)}) \Big(x_n^{(0)} + h_n^{(0)} \sum_{k=0}^{r-1} \alpha_k(s_i) x'^{(0)}_{n-k} + h_n^{(0)} \sum_{j=1}^{m} \beta_j(s_i) W_{n,j}^{(0)} \Big) + f(t_{n,i}^{(0)}) \\ & + \sum_{l=0}^{r-2} h_l^{(0)} \int_0^1 K\Big(t_{n,i}^{(0)}, t_l^{(0)} + sh_l^{(0)}, w(t_l^{(0)} + sh_l^{(0)})\Big) \mathrm{d}s \\ & + \sum_{l=r-1}^{n-1} h_l^{(0)} \int_0^1 K\Big(t_{n,i}^{(0)}, t_l^{(0)} + sh_l^{(0)}, x_l^{(0)} + h_l^{(0)} \sum_{k=0}^{r-1} \alpha_k(s) x'^{(0)}_{l-k} + h_l^{(0)} \sum_{j=1}^{m} \beta_j(s) W_{l,j}^{(0)} \Big) \mathrm{d}s \\ & + h_n^{(0)} \int_0^{s_i} K\Big(t_{n,i}^{(0)}, t_n^{(0)} + sh_n^{(0)}), x_n^{(0)} + h_n^{(0)} \sum_{k=0}^{r-1} \alpha_k(s) x'^{(0)}_{n-k} + h_n^{(0)} \sum_{j=1}^{m} \beta_j(s) W_{n,j}^{(0)} \Big) \mathrm{d}s \\ & + c_2(t_{n,i}^{(0)}) \zeta(\tau(t_{n,i}^{(0)})) + \int_{t_0}^{\tau(t_{n,i}^{(0)})} \hat{K}(t_{n,i}^{(0)}, s, \zeta(s)) \mathrm{d}s. \end{aligned}$$

II) For $\mu \geq 1$

$$
\begin{aligned}
W_{n,i}^{(\mu)} = \; & c_1(t_{n,i}^{(\mu)})\Big(x_n^{(\mu)} + h_n^{(\mu)}\sum_{k=0}^{r-1}\alpha_k(s_i)x_{n-k}^{\prime(\mu)} + h_n^{(\mu)}\sum_{j=1}^{m}\beta_j(s_i)W_{n,j}^{(\mu)}\Big) + f(t_{n,i}^{(\mu)}) \\
& + \sum_{\eta=0}^{\mu-1}\sum_{l=0}^{r-2} h_l^{(\eta)}\int_0^1 K\Big(t_{n,i}^{(\mu)}, t_l^{(\eta)} + sh_l^{(\eta)}, w(t_l^{(\eta)} + sh_l^{(\eta)})\Big)\mathrm{d}s \\
& + \sum_{\eta=0}^{\mu-1}\sum_{l=r-1}^{N_\mu-1} h_l^{(\eta)}\int_0^1 K\Big(t_{n,i}^{(\mu)}, t_l^{(\eta)} + sh_l^{(\eta)}, x_l^{(\eta)} + h_l^{(\eta)}\sum_{k=0}^{r-1}\alpha_k(s)x_{l-k}^{\prime(\eta)} + h_l^{(\eta)}\sum_{j=1}^{m}\beta_j(s)W_{l,j}^{(\eta)}\Big)\mathrm{d}s \\
& + \sum_{l=0}^{r-2} h_l^{(\mu)}\int_0^1 K\Big(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)}, w(t_l^{(\mu)} + sh_l^{(\mu)})\Big)\mathrm{d}s \\
& + \sum_{l=r-1}^{n-1} h_l^{(\mu)}\int_0^1 K\Big(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)}, x_l^{(\mu)} + h_l^{(\mu)}\sum_{k=0}^{r-1}\alpha_k(s)x_{l-k}^{\prime(\mu)} + h_l^{(\mu)}\sum_{j=1}^{m}\beta_j(s)W_{l,j}^{(\mu)}\Big)\mathrm{d}s \\
& + h_n^{(\mu)}\int_0^{s_i} K\Big(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)}), x_n^{(\mu)} + h_n^{(\mu)}\sum_{k=0}^{r-1}\alpha_k(s)x_{n-k}^{\prime(\mu)} + h_n^{(\mu)}\sum_{j=1}^{m}\beta_j(s)W_{n,j}^{(\mu)}\Big)\mathrm{d}s \\
& + c_2(t_{n,i}^{(\mu)})\Big(x_n^{(\mu-1)} + h_n^{(\mu-1)}\sum_{k=0}^{r-1}\alpha_k(s_i)x_{n-k}^{\prime(\mu-1)} + h_n^{(\mu-1)}\sum_{j=1}^{m}\beta_j(s_i)W_{n,j}^{(\mu-1)}\Big) \\
& + \sum_{\eta=0}^{\mu-2}\sum_{l=0}^{r-2} h_l^{(\eta)}\int_0^1 \hat{K}\Big(t_{n,i}^{(\mu)}, t_l^{(\eta)} + sh_l^{(\eta)}, w(t_l^{(\eta)} + sh_l^{(\eta)})\Big)\mathrm{d}s \\
& + \sum_{\eta=0}^{\mu-2}\sum_{l=r-1}^{N_\mu-1} h_l^{(\eta)}\int_0^1 \hat{K}\Big(t_{n,i}^{(\mu)}, t_l^{(\eta)} + sh_l^{(\eta)}, x_l^{(\eta)} + h_l^{(\eta)}\sum_{k=0}^{r-1}\alpha_k(s)x_{l-k}^{\prime(\eta)} + h_l^{(\eta)}\sum_{j=1}^{m}\beta_j(s)W_{l,j}^{(\eta)}\Big)\mathrm{d}s \\
& + \sum_{l=0}^{r-2} h_l^{(\mu-1)}\int_0^1 \hat{K}\Big(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)}, w(t_l^{(\mu-1)} + sh_l^{(\mu-1)})\Big)\mathrm{d}s \\
& + \sum_{l=r-1}^{n-1} h_l^{(\mu-1)}\int_0^1 \hat{K}\Big(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)}, x_l^{(\mu-1)} + h_l^{(\mu-1)}\sum_{k=0}^{r-1}\alpha_k(s)x_{l-k}^{\prime(\mu-1)} \\
& + h_l^{(\mu-1)}\sum_{j=1}^{m}\beta_j(s)W_{l,j}^{(\mu-1)}\Big)\mathrm{d}s \\
& + h_n^{(\mu-1)}\int_0^{s_i} \hat{K}\Big(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)}), x_n^{(\mu-1)} + h_n^{(\mu-1)}\sum_{k=0}^{r-1}\alpha_k(s)x_{n-k}^{\prime(\mu-1)} \\
& + h_n^{(\mu-1)}\sum_{j=1}^{m}\beta_j(s)W_{n,j}^{(\mu-1)}\Big)\mathrm{d}s.
\end{aligned}
$$

We can get $\boldsymbol{W}_n^{(\mu)}$ by solving the non-linear system. Substituting it into (12), the approximate solution of (1) can be obtained.

Remark 2. *If you consider linear case of the equation* (1) *as:*

$$x'(t) = c_1(t)x(t)+c_2(t)x(\tau(t))+f(t)+\int_{t_0}^{t} K(t,s)x(s)\mathrm{d}s+\int_{t_0}^{\tau(t)} \hat{K}(t,s)x(s)\mathrm{d}s,\ t \in J = [t_0, T], \tag{14}$$

where

$$x(t) = \zeta(t), \quad t \in [\tau(t_0), t_0].$$

Then we have the following linear algebraic system for $\mu = 0$ *and* $\mu \geq 1$ *as:*

I) For $\mu = 0$*, we have*

$$\begin{aligned} W_{n,i}^{(0)} = \;& c_1(t_{n,i}^{(0)})\Big(x_n^{(0)} + h_n^{(0)}\sum_{k=0}^{r-1}\alpha_k(s_i)x_{n-k}^{\prime(0)} + h_n^{(0)}\sum_{j=1}^{m}\beta_j(s_i)W_{n,j}^{(0)}\Big) + f(t_{n,i}^{(0)}) \\ & +\sum_{l=0}^{r-2} h_l^{(0)}\int_0^1 K(t_{n,i}^{(0)}, t_l^{(0)} + sh_l^{(0)})w(t_l^{(0)} + sh_l^{(0)})\mathrm{d}s \\ & +\sum_{l=r-1}^{n-1} h_l^{(0)}\int_0^1 K(t_{n,i}^{(0)}, t_l^{(0)} + sh_l^{(0)})\Big(x_l^{(0)} + h_l^{(0)}\sum_{k=0}^{r-1}\alpha_k(s)x_{l-k}^{\prime(0)} + h_l^{(0)}\sum_{j=1}^{m}\beta_j(s)W_{l,j}^{(0)}\Big)\mathrm{d}s \\ & +h_n^{(0)}\int_0^{s_i} K(t_{n,i}^{(0)}, t_n^{(0)} + sh_n^{(0)})\Big(x_n^{(0)} + h_n^{(0)}\sum_{k=0}^{r-1}\alpha_k(s)x_{n-k}^{\prime(0)} + h_n^{(0)}\sum_{j=1}^{m}\beta_j(s)W_{n,j}^{(0)}\Big)\mathrm{d}s \\ & +c_2(t_{n,i}^{(0)})\zeta(\tau(t_{n,i}^{(0)})) + \int_{t_0}^{\tau(t_{n,i}^{(0)})} \hat{K}(t_{n,i}^{(0)}, s)\zeta(s)\mathrm{d}s. \end{aligned}$$

II) For $\mu \geq 1$

$$
\begin{aligned}
\Big(I_m - h_n^{(\mu)}(\boldsymbol{C}_{1n}^{(\mu)}\boldsymbol{\beta} + h_n^{(\mu)}\boldsymbol{D}_n^{n,\mu})\Big)\boldsymbol{W}_n^{(\mu)} = \;& x_n^{(\mu)}\boldsymbol{C}_{1n}^{(\mu)} + h_n^{(\mu)}\boldsymbol{C}_{1n}^{(\mu)}\boldsymbol{\alpha}\boldsymbol{X}_n^{(\mu)} + \boldsymbol{F}_n^{(\mu)} \\
& + \sum_{\eta=0}^{\mu-1}\sum_{l=0}^{r-2} h_l^{(\eta)}\boldsymbol{Z}_n^{l,\eta} + \sum_{\eta=0}^{\mu-1}\sum_{l=r-1}^{N_\mu-1} h_l^{(\eta)}(x_l^{(\eta)}\boldsymbol{V}_n^{l,\eta} \\
& + h_l^{(\eta)}\boldsymbol{G}_n^{l,\eta}\boldsymbol{X}_l^{(\eta)} + h_l^{(\eta)}\boldsymbol{D}_n^{l,\eta}\boldsymbol{W}_l^{(\eta)}) \\
& + \sum_{l=0}^{r-2} h_l^{(\mu)}\boldsymbol{Z}_n^{l,\mu} + \sum_{l=r-1}^{n-1} h_l^{(\mu)}(x_l^{(\mu)}\boldsymbol{V}_n^{l,\mu} + h_l^{(\mu)}\boldsymbol{G}_n^{l,\mu}\boldsymbol{X}_l^{(\mu)} \\
& + h_l^{(\mu)}\boldsymbol{D}_n^{l,\mu}\boldsymbol{W}_l^{(\mu)}) + h_n^{(\mu)}(x_n^{(\mu)}\boldsymbol{V}_n^{n,\mu} + h_n^{(\mu)}\boldsymbol{G}_n^{n,\mu}\boldsymbol{X}_n^{(\mu)}) \\
& + x_n^{(\mu-1)}\boldsymbol{C}_{2n}^{(\mu-1)} + h_n^{(\mu-1)}\boldsymbol{C}_{2n}^{(\mu-1)}\boldsymbol{\alpha}\boldsymbol{X}_n^{(\mu-1)} + h_n^{(\mu-1)}\boldsymbol{C}_{2n}^{(\mu-1)}\boldsymbol{\beta} \\
& + \sum_{\eta=0}^{\mu-2}\sum_{l=0}^{r-2} h_l^{(\eta)}\hat{\boldsymbol{Z}}_n^{l,\eta} + \sum_{l=0}^{r-2} h_l^{(\mu-1)}\hat{\boldsymbol{Z}}_n^{l,\mu-1} \\
& + \sum_{\eta=0}^{\mu-2}\sum_{l=r-1}^{N_\mu-1} h_l^{(\eta)}(x_l^{(\eta)}\hat{\boldsymbol{V}}_n^{l,\eta} + h_l^{(\eta)}\hat{\boldsymbol{G}}_n^{l,\eta}\boldsymbol{X}_l^{(\eta)} + h_l^{(\eta)}\hat{\boldsymbol{D}}_n^{l,\eta}\boldsymbol{W}_l^{(\eta)}) \\
& + \sum_{l=r-1}^{n-1} h_l^{(\mu-1)}(x_l^{(\mu-1)}\hat{\boldsymbol{V}}_n^{l,\mu-1} + h_l^{(\mu-1)}\hat{\boldsymbol{G}}_n^{l,\mu-1}\boldsymbol{X}_l^{(\mu-1)} \\
& + h_l^{(\mu-1)}\hat{\boldsymbol{D}}_n^{l,\mu-1}\boldsymbol{W}_l^{(\mu-1)}) + h_n^{(\mu-1)}(x_n^{(\mu-1)}\hat{\boldsymbol{V}}_n^{n,\mu-1} \\
& + h_n^{(\mu-1)}\hat{\boldsymbol{G}}_n^{n,\mu-1}\boldsymbol{X}_n^{(\mu-1)} + h_n^{(\mu-1)}\hat{\boldsymbol{D}}_n^{n,\mu-1}),
\end{aligned} \tag{15}
$$

where

$$
\boldsymbol{\beta} = \begin{pmatrix} \beta_j(s_i) \\ i,j = 1,\cdots,m \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \alpha_k(s_i) \\ i, = 1,\dots,m, \\ k = 0,\dots,r-1, \end{pmatrix},
$$

$$
\boldsymbol{W}_l^{(.)} = \Big(W_{l,1}^{(.)},\dots,W_{l,m}^{(.)}\Big)^T, \quad \boldsymbol{C}_{1n}^{(.)} = diag\Big(c_1(t_{n,1}^{(.)}),\dots,c_1(t_{n,m}^{(.)})\Big),
$$

$$
\boldsymbol{C}_{2n}^{(.)} = diag\Big(c_2(t_{n,1}^{(.)}),\dots,c_2(t_{n,m}^{(.)})\Big),
$$

$$
\big(\boldsymbol{D}_n^{l,.}\big)_{i,j} = \begin{cases} \displaystyle\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_j(s)\mathrm{d}s, & l = r-1,\dots,n-1, \\ \displaystyle\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_j(s)\mathrm{d}s, & l = n, \\ \qquad i,j = 1,\dots,m, & \end{cases}
$$

$$(\boldsymbol{G}_n^{l,\cdot})_{i,k} = \begin{cases} \displaystyle\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})\alpha_k(s)\mathrm{d}s, \quad l = r-1, \cdots, n-1, \\ \displaystyle\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})\alpha_k(s)\mathrm{d}s, \quad l = n, \\ i = 1, \ldots, m, \quad k = 0, \ldots, r-1, \end{cases}$$

$$\boldsymbol{X}_l^{(\cdot)} = (x_l'^{(\cdot)}, \cdots, x_{l-r+1}'^{(\cdot)})^T, \quad \boldsymbol{F}_n^{(\mu)} = (f(t_{n,1}^{(\mu)}), \cdots, f(t_{n,m}^{(\mu)}))^T,$$

$$\boldsymbol{Z}_n^{l,\cdot} = \Big(\int_0^1 K(t_{n,1}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})w(t_l^{(\cdot)} + sh_l^{(\cdot)})\mathrm{d}s, \ldots, \int_0^1 K(t_{n,m}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})w(t_l^{(\cdot)} + sh_l^{(\cdot)})\mathrm{d}s\Big)^T,$$

$$\boldsymbol{V}_n^{l,\cdot} = \begin{cases} \Big(\displaystyle\int_0^1 K(t_{n,1}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})\mathrm{d}s, \ldots, \int_0^1 K(t_{n,m}^{(\mu)}, t_l^{(\cdot)} + sh_l^{(\cdot)})\mathrm{d}s\Big)^T, \quad l = r-1, \ldots, n-1, \\ \Big(\displaystyle\int_0^{s_1} K(t_{n,1}^{(\mu)}, t_n^{(\cdot)} + sh_n^{(\cdot)})\mathrm{d}s, \ldots, \int_0^{s_m} K(t_{n,m}^{(\mu)}, t_n^{(\cdot)} + sh_n^{(\cdot)})\mathrm{d}s\Big)^T, \quad l = n. \end{cases}$$

Also the matrices $\hat{\boldsymbol{D}}_n^{l,\cdot}, \hat{\boldsymbol{G}}_n^{l,\cdot}, \hat{\boldsymbol{Z}}_n^{l,\cdot}$ *and* $\hat{\boldsymbol{V}}_n^{l,\cdot}$ *are defined similarly to the above matrices, only instead of* K*, we put* $\hat{K}$*. By solving the linear system obtained above, we can get* $\boldsymbol{W}_n^{(\mu)}$ *and substituting it into (12), the approximate solution of (14) can be achieved.*

3. Convergence analysis

In this section, we consider convergence analysis of the proposed numerical method for the linear case (14) and at the end of this section, we will explain how to extended it for the non-linear case.

Remark 3. *In [14], the authors applied multi-step collocation methods for classical VIDEs and in the Theorem 3.1, showed that the order of convergence of this method is* $m+r-1$*. Note that in this paper, they approximated exact solution instead of the derivative of the exact solution. If they approximate the derivative of the exact solution and by integrating get the approximate of the exact solution then they could archive the order of convergence* $m+r$*.*

Theorem 3. *Assume that for* $d \geq m+r$*,* $c_\vartheta \in C^d(I)$*,* $\vartheta = 1, 2, K \in C^d(D)$ $\hat{K} \in C^d(D_\tau)$ *and* $\zeta \in C^{d+1}([\tau(t_0), t_0])$*. Let* $\tau(t) = t - \alpha(t)$*, be strictly increasing on* J *with* $\alpha(t) \geq \alpha_0 > 0$ *for* $t \in J$ *and* $\alpha(t) \in C^d(J)$*. Also, for*

$$\boldsymbol{P} = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline P_{r-1}(1) & P_{r-2}(1), \cdots, P_0(1), \end{array}\right],$$

the spectral radius of the matrix, denoted by $\rho(\boldsymbol{P})$*, is less than 1 and the starting errors are* $\|\varepsilon\|_{\infty,[t_0^{(\mu)},t_{r-1}^{(\mu)}]} = \mathcal{O}(h^{(\mu)})^p$*. Then the estimates*

$$\|e^{(\gamma)}\|_\infty = \|x^{(\gamma)} - w^{(\gamma)}\|_\infty \leq C_\gamma h^p, \quad \gamma = 0, 1, \tag{16}$$

hold for any collocation parameters $\{s_i\}$ *in* $[0,1]$*, and* $h = \max_{l,\nu} h_l^{(\nu)}$*,* $p = m + r$*.*

Proof. Assume that $e := x - w$ show the collocation error for the approximate solution w in (12) which satisfies in the following equation

$$e'(t) = c_1(t)e(t) + c_2(t)e(\tau(t)) + \delta(t) + \int_{t_0}^{t} K(t,s)e(s)\mathrm{d}s + \int_{t_0}^{\tau(t)} \hat{K}(t,s)e(s)\mathrm{d}s, \quad t \in J, \tag{17}$$

$$e(t) = 0, \qquad t \in [\tau(t_0), t_0].$$

Also,

$$\delta(t) = 0, \qquad t \in Y_h = \bigcup_{\mu=0}^{M} Y_h^{(\mu)}.$$

Considering (17), for $t \in I^{(\mu)} = (\varsigma_\mu, \varsigma_{\mu+1}]$, we have

$$e'(t) = c_1(t)e(t) + \mathcal{G}_\mu(t) + \delta(t) + \int_{\varsigma_\mu}^{t} K(t,s)e(s)\mathrm{d}s, \quad t \in I^{(\mu)}, \tag{18}$$

where

$$\mathcal{G}_\mu(t) = c_2(t)e(\tau(t)) + \int_{t_0}^{\varsigma_\mu} K(t,s)e(s)\mathrm{d}s + \int_{t_0}^{\tau(t)} \hat{K}(t,s)e(s)\mathrm{d}s. \tag{19}$$

For $\mu = 0$, it follows that

$$e'(t) = c_1(t)e(t) + \delta(t) + \int_{t_0}^{t} K(t,s)e(s)\mathrm{d}s, \quad t \in I^{(0)}. \tag{20}$$

Using this remark on $I^{(0)}$, we can obtain error bound as:

$$\| e^{(\nu)} \|_\infty \leq C_\nu (h^{(0)})^p, \quad \nu = 0, 1. \tag{21}$$

and $e^{(\nu)}(\varsigma_1) = \mathcal{O}((h_l^{(0)})^p)$.

Now, on the interval $I^{(\mu)}$, $1 \leq \mu \leq M$, the collocation error equation (17) in $t = t_{n,i}^{(\mu)}$, satisfies the equation

$$\begin{aligned} e'(t_{n,i}^{(\mu)}) = \; & c_1(t_{n,i}^{(\mu)})e(t_{n,i}^{(\mu)}) + c_2(t_{n,i}^{(\mu)})e(\tau(t_{n,i}^{(\mu)})) + \delta(t_{n,i}^{(\mu)}) \\ & + \int_{t_0}^{t_{n,i}^{(\mu)}} K(t_{n,i}^{(\mu)},s)e(s)\mathrm{d}s + \int_{t_0}^{\tau(t_{n,i}^{(\mu)})} \hat{K}(t_{n,i}^{(\mu)},s)e(s)\mathrm{d}s, \quad t \in I^{(\mu)}, \end{aligned} \tag{22}$$

after some computation equation (22) reduce the following form

$$
\begin{aligned}
e'(t_{n,i}^{(\mu)}) = \quad & \delta_h(t_{n,i}^{(\mu)}) + c_1(t_{n,i}^{(\mu)})e(t_{n,i}^{(\mu)}) + c_2(t_{n,i}^{(\mu)})e(t_{n,i}^{(\mu-1)}) \\
& + \sum_{v=0}^{\mu-1} \int_{\varsigma_v}^{\varsigma_{v+1}} K(t_{n,i}^{(\mu)}, s)e(s)\mathrm{d}s \\
& + \sum_{l=1}^{r-1} h_l^{(\mu)} \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)})e(t_l^{(\mu)} + sh_l^{(\mu)})\mathrm{d}s \\
& + \sum_{l=r}^{n-1} h_l^{(\mu)} \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)})e(t_l^{(\mu)} + sh_l^{(\mu)})\mathrm{d}s \\
& + h_n^{(\mu)} \int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})e(t_n^{(\mu)} + sh_n^{(\mu)})\mathrm{d}s \\
& + \sum_{v=0}^{\mu-2} \int_{\varsigma_v}^{\varsigma_{v+1}} \hat{K}(t_{n,i}^{(\mu)}, s)e(s)\mathrm{d}s \\
& + \sum_{l=1}^{r-1} h_l^{(\mu-1)} \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)})e(t_l^{(\mu-1)} + sh_l^{(\mu-1)})\mathrm{d}s \\
& + \sum_{l=r}^{n-1} h_l^{(\mu-1)} \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)})e(t_l^{(\mu-1)} + sh_l^{(\mu-1)})\mathrm{d}s \\
& + h_n^{(\mu-1)} \int_0^{\tilde{s}_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})e(t_n^{(\mu-1)} + sh_n^{(\mu-1)})\mathrm{d}s.
\end{aligned} \tag{23}
$$

By the hypothesis on the starting error it follows that

$$
\varepsilon(t_l^{(\mu)} + vh_l^{(\mu)}) = (h_l^{(\mu)})^{m+r} q_l(s), \quad l = 0, \ldots, r-2, \quad s \in (0, 1], \tag{24}
$$

with $\|q_l\|_\infty \le C_1$ independent of $h_l^{(\mu)}$.

Recall now the analogous error equations for equation (17), for e and e', they are, respectively,

$$
e(t_n^{(\mu)}+zh_n^{(\mu)}) = e(t_n^{(\mu)})+h_n^{(\mu)} \sum_{k=0}^{r-1} \alpha_k(z)e'^{(\mu)}_{n-k}+h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(z)E_{n,j}^{(\mu)}+(h_n^{(\mu)})^{p+1} R_{m+r,n}^{(\mu)}(z), \tag{25}
$$

and

$$
e'(t_n^{(\mu)} + zh_n^{(\mu)}) = \sum_{k=0}^{r-1} P_k(z)e'^{(\mu)}_{n-k} + \sum_{j=1}^{m} Q_j(z)E_{n,j}^{(\mu)} + (h_n^{(\mu)})^p R'^{(\mu)}_{m+r,n}(z), \quad z \in (0, 1], \tag{26}
$$

where

$$
R'^{(\mu)}_{d,n}(s) = \int_{1-r}^{1} K_{d,r}(s, z)e'^{(d)}(t_n^{(\mu)} + zh_n^{(\mu)})dz,
$$

$$K_{d,r}(s,z) = \frac{1}{(d-1)!}\left\{(s-z)_+^{d-1} - \sum_{k=0}^{r-1} P_k(s)(-k-z)_+^{d-1} - \sum_{j=1}^{m} Q_j(s)(s_j-z)_+^{d-1}\right\},$$

with $E_{n,i}^{(\mu)} = X_{n,i}^{(\mu)} - W_{n,i}^{(\mu)}$.

Substituting the equations (25) and (26) in the equation (23), we get

$$\begin{aligned}
E_{n,i}^{(\mu)} = \;& c_1(t_{n,i}^{(\mu)})e(t_n^{(\mu)}) + c_1(t_{n,i}^{(\mu)})h_n^{(\mu)}\sum_{k=0}^{r-1}\alpha_k(s_i)e_{n-k}^{\prime(\mu)} + c_1(t_{n,i}^{(\mu)})h_n^{(\mu)}\sum_{j=1}^{m}\beta_j(s_i)E_{n,j}^{(\mu)} \\
& + c_2(t_{n,i}^{(\mu)})e(t_n^{(\mu-1)}) + c_2(t_{n,i}^{(\mu)})h_n^{(\mu)}\sum_{k=0}^{r-1}\alpha_k(s_i)e_{n-k}^{\prime(\mu-1)} + c_2(t_{n,i}^{(\mu)})h_n^{(\mu)}\sum_{j=1}^{m}\beta_j(s_i)E_{n,j}^{(\mu-1)} \\
& + c_1(t_{n,i}^{(\mu)})(h_n^{(\mu)})^{p+1}R_{m+r,n}^{(\mu)}(s_i) + c_2(t_{n,i}^{(\mu)})(h_n^{(\mu-1)})^{p+1}R_{m+r,n}^{(\mu-1)}(s_i) \\
& + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1}(h_l^{(v)})^{p+2}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(v)} + sh_l^{(v)})R_{m+r,l}^{(v)}(s)\mathrm{ds} \\
& + \sum_{l=r}^{n-1}(h_l^{(\mu)})^{p+2}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)})R_{m+r,l}^{(\mu)}(s)\mathrm{ds} \\
& + (h_n^{(\mu)})^{p+2}\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})R_{m+r,n}^{(\mu)}(s)\mathrm{ds} \\
& + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2}(h_l^{(v)})^{p+2}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(v)} + sh_l^{(v)})R_{m+r,l}^{(v)}(s)\mathrm{ds} \\
& + \sum_{l=r}^{n-1}(h_l^{(\mu-1)})^{p+2}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)})R_{m+r,l}^{(\mu-1)}(s)\mathrm{ds} \\
& + (h_n^{(\mu-1)})^{p+2}\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})R_{m+r,n}^{(\mu-1)}(s)\mathrm{ds} \\
& + \sum_{l=1}^{r-1}(h_l^{(\mu)})^{p+1}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + sh_l^{(\mu)}q_l(s)\mathrm{ds} \\
& + \sum_{l=1}^{r-1}(h_l^{(\mu-1)})^{p+1}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + sh_l^{(\mu-1)})q_l(s)\mathrm{ds}
\end{aligned} \tag{27}$$

$$
\begin{aligned}
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1} h_l^{(v)}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})\mathrm{d}s e(t_l^{(v)})\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1}\sum_{k=0}^{r-1} h_l^{(v)}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})h_l^{(v)}\alpha_k(s)\mathrm{d}s e_{l-k}^{\prime(v)}\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1} h_l^{(v)}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})h_l^{(v)}\sum_{j=1}^{m}\beta_j(s)\mathrm{d}s E_{l,j}^{(v)}\\
&+\sum_{l=r}^{n-1} h_l^{(\mu)}\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)}+sh_l^{(\mu)})\mathrm{d}s e(t_l^{(\mu)})\\
&+\sum_{l=r}^{n-1}\sum_{k=0}^{r-1}(h_l^{(\mu)})^2\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)}+sh_l^{(\mu)})\alpha_k(s)\mathrm{d}s e_{l-k}^{\prime(\mu)}\\
&+\sum_{l=r}^{n-1}\sum_{j=1}^{m}(h_l^{(\mu)})^2\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)}+sh_l^{(\mu)})\beta_j(s)\mathrm{d}s E_{l,j}^{(\mu)}\\
&+h_n^{(\mu)}\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)}+sh_n^{(\mu)})\mathrm{d}s e(t_n^{(\mu)})\\
&+(h_n^{(\mu)})^2\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)}+sh_n^{(\mu)})\sum_{k=0}^{r-1}\alpha_k(s)\mathrm{d}s e_{n-k}^{\prime(\mu)}\\
&+(h_n^{(\mu)})^2\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)}+sh_n^{(\mu)})\sum_{j=1}^{m}\beta_j(s)\mathrm{d}s E_{n,j}^{(\mu)}\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2} h_l^{(v)}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})\mathrm{d}s e(t_l^{(v)})\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2}(h_l^{(v)})^2\sum_{k=0}^{r-1}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})\alpha_k(s)\mathrm{d}s e_{l-k}^{\prime(v)}\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2}(h_l^{(v)})^2\sum_{j=1}^{m}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(v)}+sh_l^{(v)})\beta_j(s)\mathrm{d}s E_{l,j}^{(v)}\\
&+\sum_{l=r}^{n-1} h_l^{(\mu-1)}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)}+sh_l^{(\mu-1)})\mathrm{d}s e(t_l^{(\mu-1)})\\
&+\sum_{l=r}^{n-1}(h_l^{(\mu-1)})^2\sum_{k=0}^{r-1}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)}+sh_l^{(\mu-1)})\alpha_k(s)\mathrm{d}s e_{l-k}^{\prime(\mu-1)}\\
&+\sum_{l=r}^{n-1}(h_l^{(\mu-1)})^2\sum_{j=1}^{m}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)}+sh_l^{(\mu-1)})\beta_j(s)\mathrm{d}s E_{l,j}^{(\mu-1)}\\
&+h_n^{(\mu-1)}\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)}+sh_n^{(\mu-1)})\mathrm{d}s e(t_n^{(\mu-1)})\\
&+(h_n^{(\mu-1)})^2\sum_{k=0}^{r-1}\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)}+sh_n^{(\mu-1)})\alpha_k(s)\mathrm{d}s e_{n-k}^{\prime(\mu-1)}\\
&+(h_n^{(\mu-1)})^2\sum_{j=1}^{m}\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)}+sh_n^{(\mu-1)})\beta_j(s)\mathrm{d}s E_{n,j}^{(\mu-1)}.
\end{aligned}
$$

Now, we define the following matrices and vectors:

$$\boldsymbol{\varepsilon}_n^{(\mu)} = \left(e_n'^{(\mu)}, ..., e_{n-r+1}'^{(\mu)}\right)^T, \quad \boldsymbol{E}_n^{(\mu)} = \left(E_{n,1}^{(\mu)}, ..., E_{n,m}^{(\mu)}\right)^T,$$

$$\boldsymbol{D}_n^{(.),l} = \begin{cases} \left(\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_1(s)\mathrm{ds}, ..., \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_m(s)\mathrm{ds}\right), & l = r-1, \dots, n-1, \\ \left(\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_1(s)\mathrm{ds}, ..., \int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_m(s)\mathrm{ds}\right), & l = n. \end{cases}$$

$$\tilde{\boldsymbol{D}}_n^{(.),l} = \begin{cases} \left(\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_1(s)\mathrm{ds}, ..., \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_m(s)\mathrm{ds}\right), & l = r-1, \dots, n-1, \\ \left(\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_1(s)\mathrm{ds}, ..., \int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\beta_m(s)\mathrm{ds}\right), & l = n. \end{cases}$$

$$\boldsymbol{G}_n^{(.),l} = \begin{cases} \left(\int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_0(s)\mathrm{ds}, ..., \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_{r-1}(s)\mathrm{ds}\right), & l = r-1, \dots, n-1, \\ \left(\int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(\mu)})\alpha_0(s)\mathrm{ds}, ..., \int_0^{s_i} K(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_{r-1}(s)\mathrm{ds}\right), & l = n. \end{cases}$$

$$\tilde{\boldsymbol{G}}_n^{(.),l} = \begin{cases} \left(\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_0(s)\mathrm{ds}, ..., \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_{r-1}(s)\mathrm{ds}\right), & l = r-1, \dots, n-1, \\ \left(\int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(\mu)})\alpha_0(s)\mathrm{ds}, ..., \int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_l^{(.)} + sh_l^{(.)})\alpha_{r-1}(s)\mathrm{ds}\right), & l = n. \end{cases}$$

$$\boldsymbol{H}_{n,\kappa,\varpi}^{(.)} = \left(h_\kappa^{(.)}\Big(\int_0^1 K(t_{n,i}^{(\mu)}, t_\kappa^{(.)} + sh_\kappa^{(.)})\mathrm{ds}, ..., h_\varpi^{(.)}\int_0^1 K(t_{n,i}^{(\mu)}, t_\varpi^{(.)} + sh_\varpi^{(.)})\mathrm{ds}\right),$$

$$\tilde{\boldsymbol{H}}_{n,\kappa,\varpi}^{(.)} = \left(h_\kappa^{(.)}\Big(\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_\kappa^{(.)} + sh_\kappa^{(.)})\mathrm{ds}, ..., h_\varpi^{(.)}\int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_\varpi^{(.)} + sh_\varpi^{(.)})\mathrm{ds}\right),$$

$$\boldsymbol{L}_{1,n}^{(.)} = \mathrm{diag}\left(\int_0^{s_1} K(t_{n,1}^{(\mu)}, t_n^{(.)} + sh_n^{(.)})\mathrm{ds}, ..., \int_0^{s_m} K(t_{n,m}^{(\mu)}, t_n^{(.)} + sh_n^{(.)})\mathrm{ds}\right),$$

$$\tilde{\boldsymbol{L}}_{1,n}^{(.)} = \mathrm{diag}\left(\int_0^{s_1} \hat{K}(t_{n,1}^{(\mu)}, t_n^{(.)} + sh_n^{(.)})\mathrm{ds}, ..., \int_0^{s_m} \hat{K}(t_{n,m}^{(\mu)}, t_n^{(.)} + sh_n^{(.)})\mathrm{ds}\right),$$

$$\boldsymbol{\alpha} = \left(\alpha_0(c_i), ..., \alpha_{r-1}(c_i)\right), \quad \boldsymbol{\beta} = \left(\beta_1(c_i), ..., \beta_m(c_i)\right), \quad \boldsymbol{\Psi}_{m,n,l}^{(\mu,\nu),(p+1,p+2)} = (\Psi_{m,n,l}^{(\mu,\nu),(p+1,p+2)})_i. \tag{28}$$

where for $i = 1, \cdots, m$, we have

$$
\begin{aligned}
(\Psi^{(\mu,\nu),(p+1,p+2)}_{m,n,l})_i = \; & (h_n^{(\mu)})^{p+1} c_1(t_{n,i}^{(\mu)}) R_{m+1,n}^{(\mu)}(s_i) \\
& + (h_n^{(\mu-1)})^{p+1} c_2(t_{n,i}^{(\mu)}) R_{m+1,n}^{(\mu-1)}(s_i) \\
& + \sum_{l=0}^{N_\mu - 1} \sum_{v=0}^{\mu-1} (h_l^{(v)})^{p+2} \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(v)} + s h_l^{(v)}) R_{m+1,l}^{(v)}(s) \mathrm{ds} \\
& + \sum_{l=r}^{n-1} (h_l^{(\mu)})^{p+2} \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + s h_l^{(\mu)}) R_{m+1,l}^{(\mu)}(s) \mathrm{ds} \\
& + (h_n^{(\mu)})^{p+2} \int_0^{s_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) R_{m+1,n}^{(\mu)}(s) \mathrm{ds} \\
& + \sum_{l=0}^{N_\mu - 1} \sum_{v=0}^{\mu-2} (h_l^{(v)})^{p+2} \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(v)} + s h_l^{(v)}) R_{m+1,l}^{(v)}(s) \mathrm{ds} \qquad (29) \\
& + \sum_{l=r}^{n-1} (h_l^{(\mu-1)})^{p+2} \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + s h_l^{(\mu-1)}) + R_{m+1,l}^{(\mu-1)}(s) \mathrm{ds} \\
& + (h_n^{(\mu-1)})^{p+2} \int_0^{s_i} \hat{K}(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) R_{m+1,n}^{(\mu-1)}(s) \mathrm{ds} \\
& + \sum_{l=1}^{r-1} (h_l^{(\mu)})^{p+1} \int_0^1 K(t_{n,i}^{(\mu)}, t_l^{(\mu)} + s h_l^{(\mu)} q_l(s) \mathrm{ds} \\
& + \sum_{l=1}^{r-1} (h_l^{(\mu-1)})^{p+1} \int_0^1 \hat{K}(t_{n,i}^{(\mu)}, t_l^{(\mu-1)} + s h_l^{(\mu-1)}) q_l(s) \mathrm{ds}.
\end{aligned}
$$

By substituting these matrices and vectors into the equation (28), the matrix form of the equation (27), can be written as follows:

$$
\begin{aligned}
&\left[\boldsymbol{I} - h_n^{(\mu)}\left(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\beta} + h_n^{(\mu)}\boldsymbol{D}_n^{(\mu),n}\right) \quad -h_n^{(\mu)}\left(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\alpha} + h_n^{(\mu)}\boldsymbol{G}_n^{(\mu),n}\right)\right]\begin{bmatrix}\boldsymbol{E}_n^{(\mu)}\\ \boldsymbol{\varepsilon}_n^{(\mu)}\end{bmatrix}\\
&= h_n^{(\mu)}\left[\boldsymbol{C}_{2,n}^{(\mu)}\boldsymbol{\beta} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_n^{(\mu-1)}\\ \boldsymbol{\varepsilon}_n^{(\mu-1)}\end{bmatrix} + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1}(h_l^{(v)})^2\left[\boldsymbol{D}_n^{(\nu),l} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_l^{(v)}\\ \boldsymbol{\varepsilon}_l^{(v)}\end{bmatrix}\\
&+\sum_{l=r}^{n-1}(h_l^{(\mu)})^2\left[\boldsymbol{D}_n^{(\mu),l} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_l^{(\mu)}\\ \boldsymbol{\varepsilon}_l^{(\mu)}\end{bmatrix} + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2}(h_l^{(v)})^2\left[\tilde{\boldsymbol{D}}_n^{(\nu),l} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_l^{(v)}\\ \boldsymbol{\varepsilon}_l^{(v)}\end{bmatrix}\\
&+\sum_{l=r}^{n-1}(h_l^{(\mu-1)})^2\left[\tilde{\boldsymbol{D}}_n^{(\mu-1),l} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_l^{(\mu-1)}\\ \boldsymbol{\varepsilon}_l^{(\mu-1)}\end{bmatrix} + (h_n^{(\mu-1)})^2\left[\tilde{\boldsymbol{D}}_n^{(\mu-1),n} \quad \boldsymbol{0}\right]\begin{bmatrix}\boldsymbol{E}_n^{(\mu-1)}\\ \boldsymbol{\varepsilon}_n^{(\mu-1)}\end{bmatrix} \qquad (30)\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1}(h_l^{(v)})^2\left[\boldsymbol{0} \quad \boldsymbol{G}_n^{(v),l}\right]\begin{bmatrix}\boldsymbol{E}_l^{(v)}\\ \varepsilon_l^{(v)}\end{bmatrix} + \sum_{l=r}^{n-1}(h_l^{(\mu)})^2\left[\boldsymbol{0} \quad \boldsymbol{G}_n^{(\mu),l}\right]\begin{bmatrix}\boldsymbol{E}_l^{(\mu)}\\ \varepsilon_l^{(\mu)}\end{bmatrix}\\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2}(h_l^{(v)})^2\left[\boldsymbol{0} \quad \tilde{\boldsymbol{G}}_n^{(v),l}\right]\begin{bmatrix}\boldsymbol{E}_l^{(v)}\\ \varepsilon_l^{(v)}\end{bmatrix} + \sum_{l=r}^{n-1}(h_l^{(\mu-1)})^2\left[\boldsymbol{0} \quad \tilde{\boldsymbol{G}}_n^{(\mu-1),l}\right]\begin{bmatrix}\boldsymbol{E}_l^{(\mu-1)}\\ \varepsilon_l^{(\mu-1)}\end{bmatrix}\\
&+(h_n^{(\mu-1)})^2\left[\boldsymbol{0} \quad \tilde{\boldsymbol{G}}_n^{(\mu-1),n}\right]\begin{bmatrix}\boldsymbol{E}_n^{(\mu-1)}\\ \varepsilon_n^{(\mu-1)}\end{bmatrix} + h_n^{(\mu)}\left[\boldsymbol{0} \quad \boldsymbol{C}_{2,n}^{(\mu)}\tilde{\boldsymbol{\alpha}}\right]\begin{bmatrix}\boldsymbol{E}_n^{(\mu-1)}\\ \varepsilon_n^{(\mu-1)}\end{bmatrix}\\
&+\boldsymbol{F}^{(v)} + \boldsymbol{\Psi}_{m,n,l}^{(\mu,\nu),(p+1,p+2)},
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{F}^{(v)} = \;& \sum_{v=0}^{\mu-1}\boldsymbol{H}_{n,0,N_\mu-1}^{(v)}\boldsymbol{\epsilon}_{0,N_\mu-1}^{(v)} + \boldsymbol{H}_{n,r,n-1}^{(\mu)}\boldsymbol{\epsilon}_{r,n-1}^{(\mu)}\\
&+\sum_{v=0}^{\mu-2}\tilde{\boldsymbol{H}}_{n,0,N_\mu-1}^{(v)}\boldsymbol{\epsilon}_{0,N_\mu-1}^{(v)} + h_n^{(\mu)}\boldsymbol{L}_{1,n}^{(\mu)}e(t_n^{(\mu)}) + \tilde{\boldsymbol{H}}_{n,r,n-1}^{(\mu-1)}\boldsymbol{\epsilon}_{r,n-1}^{(\mu-1)} \qquad (31)\\
&+h_n^{(\mu-1)}\tilde{\boldsymbol{L}}_{1,n}^{(\mu-1)}e(t_n^{(\mu-1)}) + \boldsymbol{C}_{1,n}^{(\mu)}e(t_n^{(\mu)}) + \boldsymbol{C}_{2,n}^{(\mu)}e(t_n^{(\mu-1)}).
\end{aligned}
$$

Also, equation (26) with $z = 1$, leads to

$$
\boldsymbol{\varepsilon}_l^{(.)} = \boldsymbol{P}\boldsymbol{\varepsilon}_{l-1}^{(.)} + \boldsymbol{Q}\boldsymbol{E}_{l-1}^{(.)} + \mathcal{O}(h^{(.)})^p, \qquad (32)
$$

where

$$
\boldsymbol{P} = \left[\begin{array}{c|c}\mathbf{0}_{r-1,1} & \mathbf{I}_{r-1}\\ \hline P_{r-1}(1) & P_{r-2}(1),\cdots,P_0(1),\end{array}\right], \quad \boldsymbol{Q} = \left[\begin{array}{ccc} & \mathbf{0}_{r-1,m} & \\ \hline Q_1(1) & \cdots & Q_m(1)\end{array}\right].
$$

The solution of the difference equation (32) is

$$\boldsymbol{\varepsilon}_l^{(.)} = \boldsymbol{P}^{l-r+1}\boldsymbol{\varepsilon}_{r-1}^{(.)} + \sum_{j=r-1}^{l-1} \boldsymbol{P}^{l-j-1}\boldsymbol{Q}E_j^{(.)} + \mathcal{O}(h^{(.)})^p. \tag{33}$$

Now, from equations (30) and (32), we get

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{I} - h_n^{(\mu)}\Big(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\beta} + h_n^{(\mu)}\boldsymbol{D}_n^{(\mu),n}\Big) & -h_n^{(\mu)}\Big(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\alpha} + h_n^{(\mu)}\boldsymbol{G}_n^{(\mu),n}\Big) \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_n^{(\mu)} \\ \boldsymbol{\varepsilon}_n^{(\mu)} \end{bmatrix} \\
&= \begin{bmatrix} (h_{n-1}^{(\mu)})^2\boldsymbol{D}_n^{(\mu),n-1} & \boldsymbol{0} \\ \boldsymbol{Q} & \boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{n-1}^{(\mu)} \\ \boldsymbol{\varepsilon}_{n-1}^{(\mu)} \end{bmatrix} \\
&+h_n^{(\mu)} \begin{bmatrix} \boldsymbol{C}_{2,n}^{(\mu)}\boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_n^{(\mu-1)} \\ \boldsymbol{\varepsilon}_n^{(\mu-1)} \end{bmatrix} + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1} (h_l^{(v)})^2 \begin{bmatrix} \boldsymbol{D}_n^{(\nu),l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(v)} \\ \boldsymbol{\varepsilon}_l^{(v)} \end{bmatrix} \\
&+\sum_{l=r}^{n-2} (h_l^{(\mu)})^2 \begin{bmatrix} \boldsymbol{D}_n^{(\mu),l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(\mu)} \\ \boldsymbol{\varepsilon}_l^{(\mu)} \end{bmatrix} + \sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2} (h_l^{(v)})^2 \begin{bmatrix} \tilde{\boldsymbol{D}}_n^{(\nu),l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(v)} \\ \boldsymbol{\varepsilon}_l^{(v)} \end{bmatrix} \\
&+\sum_{l=r}^{n-1} (h_l^{(\mu-1)})^2 \begin{bmatrix} \tilde{\boldsymbol{D}}_n^{(\mu-1),l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(\mu-1)} \\ \boldsymbol{\varepsilon}_l^{(\mu-1)} \end{bmatrix} + (h_n^{(\mu-1)})^2 \begin{bmatrix} \tilde{\boldsymbol{D}}_n^{(\mu-1),n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_n^{(\mu-1)} \\ \boldsymbol{\varepsilon}_n^{(\mu-1)} \end{bmatrix} \\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-1} (h_l^{(v)})^2 \begin{bmatrix} \boldsymbol{0} & \boldsymbol{G}_n^{(v),l} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(v)} \\ \varepsilon_l^{(v)} \end{bmatrix} + \sum_{l=r}^{n-1} (h_l^{(\mu)})^2 \begin{bmatrix} \boldsymbol{0} & \boldsymbol{G}_n^{(\mu),l} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(\mu)} \\ \varepsilon_l^{(\mu)} \end{bmatrix} \\
&+\sum_{l=0}^{N_\mu-1}\sum_{v=0}^{\mu-2} (h_l^{(v)})^2 \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{G}}_n^{(v),l} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(v)} \\ \varepsilon_l^{(v)} \end{bmatrix} + \sum_{l=r}^{n-1} (h_l^{(\mu-1)})^2 \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{G}}_n^{(\mu-1),l} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_l^{(\mu-1)} \\ \varepsilon_l^{(\mu-1)} \end{bmatrix} \\
&+(h_n^{(\mu-1)})^2 \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{G}}_n^{(\mu-1),n} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_n^{(\mu-1)} \\ \varepsilon_n^{(\mu-1)} \end{bmatrix} + h_n^{(\mu)} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{C}_{2,n}^{(\mu)}\tilde{\boldsymbol{\alpha}} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_n^{(\mu-1)} \\ \varepsilon_n^{(\mu-1)} \end{bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{F}^{(v)} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Psi}_{m,n,l}^{(\mu,\nu),(p+1,p+2)} \\ \mathcal{O}(h_n^{(\mu)})^p \end{bmatrix},
\end{aligned} \tag{34}$$

where expression (25), with $z = 1$, leads to

$$\begin{aligned}
e(t_n^{(.)}) &= e(t_{n-1}^{(.)} + h_n^{(.)}) \\
&= e(t_{n-1}^{(.)}) + h_{n-1}^{(.)}\sum_{k=0}^{r-1}\alpha_k(1)e'^{(.)}_{n-1-k} + h_{n-1}^{(.)}\sum_{j=1}^{m}\beta_j(1)E_{n-1,j}^{(.)} + (h_{n-1}^{(.)})^{p+1}R_{m+r,n-1}^{(.)}(1).
\end{aligned}$$

Note that, in the equation (34), the matrix

$$\begin{bmatrix} \boldsymbol{I} - h_n^{(\mu)}\Big(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\beta} + h_n^{(\mu)}\boldsymbol{D}_n^{(\mu),n}\Big) & -h_n^{(\mu)}\Big(\boldsymbol{C}_{1,n}^{(\mu)}\boldsymbol{\alpha} + h_n^{(\mu)}\boldsymbol{G}_n^{(\mu),n}\Big) \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \qquad \mu = 0, 1, ...,$$

coincides with Theorem 4.5.3 in [1] and Theorem 5.1 in [36] and then this matrix is invertible, so it's inverse is uniformly bounded.

Now, with assumption $h = \max_{l,\nu} h_l^{(\nu)}$, we have

$$\| \mathbb{E}^{(1)} \|_\infty \leq C_1 h^p,$$

where C_1 is a constant and

$$\mathbb{E}^{(1)} = \begin{bmatrix} \boldsymbol{E}_n^{(\mu)} \\ \boldsymbol{\varepsilon}_n^{(\mu)} \end{bmatrix}.$$

Hence, this proves Theorem 3, in same manner in Theorems 3.2.3 and 4.5.2, in [1].

Remark 4. *We conclude this section with a comment regarding the extension of the results of Theorem 3 to the non-linear equation* (1)*. Under the assumption of the existence of a (unique) solution $x(t)$ on J, the non-linear analogue of the error equation* (1) *is*

$$\begin{aligned} e'(t) = \; & c_1(t)e(t) + c_2(t)e(\tau(t)) + \delta(t) \\ & + \int_{t_0}^{t} \Big(K(t,s,x(s)) - K(t,s,w(s))\Big)\mathrm{d}s + \int_{t_0}^{\tau(t)} \Big(\hat{K}(t,s,x(s)) - \hat{K}(t,s,w(s))\Big)\mathrm{d}s. \end{aligned} \tag{35}$$

If the partial derivatives $\frac{\partial K}{\partial x}$ and $\frac{\partial \hat{K}}{\partial x}$ are continuous and bounded on the domain of their own definition. Assuring the existence of a unique collocation solution w, then (35) *may again be written in the form* (17)*. The roles of K and $\hat{K}$ are now assumed by*

$$k(t,s) = \frac{\partial K(t,s,Z(s))}{\partial x},$$

and

$$\hat{k}(t,s) = \frac{\partial \hat{K}(t,s,Z(s))}{\partial x},$$

where $Z(s) = \theta x(s) + (1-\theta)w$. Hence, the above proof is easily adapted to deal with the non-linear case (1)*, and so the convergence results of Theorem 3 remain valid for non-linear equations.*

4. Numerical examples

Here, we examine two numerical instances to demonstrate the effectiveness of the suggested method. All calculations were executed using Mathematica® software, Version 11.1. We choose $s_1 = 0.8, s_2 = 1$ and $r = 2$. Since $s_2 = 1$, then we have $\rho(\boldsymbol{P}) = 0 < 1$.

Throughout the subsequent part of this section, all numerical experiments employ $T = 1$, and the initial values are derived from well-established exact solutions. The analysis of the numerical results presented in the Tables reveals that the multi-step method demonstrates greater accuracy compared to the one-step method employed in [1].

In Tables 1,2, we report the maximum of the absolute errors at the grid points for $m = 2$ and $r = 2$. Also, we calculate the order of convergence by

$$p = \log_2\Big(\frac{\|e_N\|_\infty}{\|e_{2N}\|_\infty}\Big),$$

and report it in Table 3. From Theorem 3, we know that for $m, r = 2$, the order of convergence is equal to $p = m + r = 4$ while for one-step collocation methods, the order of convergence is equal to $p = 2$. Additionally, in Figures 1 and 2, we graph the convergence order for both the one-step and multi-step schemes across various values of N. For the one-step method, order of convergence tends to $p = 2$ and for multi-step schemes, that tends to $p = 4$. The observed order of convergence aligns with the theoretical findings outlined in Theorem 3. Obviously, noting Tables 1, 2 and Figures 1, 2, we see that using the multi-step collocation methods can get higher convergence orders than the classical one-step collocation methods when the same number of collocation parameters is used.

Example 1. *Examine VIDEs with non-vanishing delay*

$$\begin{cases} x'(t) = & -tx(t) - (t+1)x(\frac{1}{2}t) + f(t) + \displaystyle\int_{\frac{1}{4}}^{t} \sin(s-t)x^2(s)\mathrm{d}s + \int_{\frac{1}{4}}^{\frac{1}{2}t} t\cos(s)x^2(s)\mathrm{d}s, \quad t \in [\frac{1}{4}, 1], \\ x(t) = & e^{2t}, \qquad t \in [\frac{1}{8}, \frac{1}{4}], \end{cases}$$

and f so that the exact solution is $x(t) = e^{2t}$.

We use FindRoot command in Mathematica software to solve non-linear algebraic equations associated with the nonlinear test problem 1. If you specify only one starting value, FindRoot searches for a solution using Newton methods. If FindRoot does not succeed in finding a solution to the accuracy you specify within MaxIterations steps, it returns the most recent approximation to a solution that it found. You can then apply FindRoot again, with this approximation as a starting point. If we do not choose the starting point correctly, the software warns when running and we can change the starting point.

Example 2. *Examine VIDEs with non-vanishing delay*

$$\begin{cases} x'(t) = & (t^2+2)x(\frac{1}{2} - t) + t + \displaystyle\int_{0}^{t-\frac{1}{2}} (2s + 3t + 1)x(s)\mathrm{d}s, \quad t \in [0, 1], \\ x(t) = & 1, \qquad t \in [-\frac{1}{2}, 0], \end{cases}$$

and the exact solution is

$$x(t) = \begin{cases} 1 + \frac{7t}{4} - \frac{t^2}{4} + \frac{5t^3}{3}, & 0 < t \leq \frac{1}{2}, \\ \frac{6847}{4608} - \frac{59t}{128} + \frac{433t^2}{128} - \frac{299t^3}{144} + \frac{109t^4}{32} - \frac{11t^5}{8} + \frac{43t^6}{72}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Resulting from the delay function$\tau(t) = t - \frac{1}{2}$, we have

$$\varsigma_\mu = \mu\frac{1}{2}, \quad \mu = 0, 1.$$

Also

$$\lim_{t\to 0^-} x'(t) \neq \lim_{t\to 0^+} x'(t), \qquad \lim_{t\to \frac{1}{2}^-} x'(t) = \lim_{t\to \frac{1}{2}^+} x'(t).$$

Table 1: L_∞ errors and CPU time based on seconds for $m, r = 2$ in Example 1.

	$(s_1, s_2) = (0.8, 1)$		$(s_1, s_2) = (0.8, 1)$	
N	One-step method [1]	CPU time(sec)	Multi-step method	CPU time(sec)
4	2.74×10^{-2}	0.391	2.56×10^{-5}	1.64
8	6.94×10^{-3}	0.563	2.14×10^{-6}	2.14
16	1.74×10^{-3}	1.29	1.47×10^{-7}	9.46
32	4.37×10^{-4}	4.12	9.64×10^{-9}	43.6

Table 2: L_∞ errors and CPU time based on seconds for $m, r = 2$ in Example 2.

	$(s_1, s_2) = (0.8, 1)$		$(s_1, s_2) = (0.8, 1)$	
N	One-step method [1]	CPU time(sec)	Multi-step method	CPU time(sec)
4	5.14×10^{-2}	0.469	2.06×10^{-5}	0.672
8	1.24×10^{-3}	0.562	1.74×10^{-6}	2.70
16	3.06×10^{-3}	1.23	1.20×10^{-7}	7.79
32	7.58×10^{-4}	4.36	7.86×10^{-9}	29.9

Table 3: Order of convergence for $m = r = 2$ in Examples 1 and 2.

	Order of convergence		Order of convergence	
N	One-step for Ex. 1	Multi-step for Ex. 1	One-step for 2	Multi-step for Ex. 2
8	1.884	3.545	2.146	3.559
16	1.993	3.838	2.024	3.855
32	1.996	3.929	2.012	3.941
64	1.999	3.989	2.000	3.991

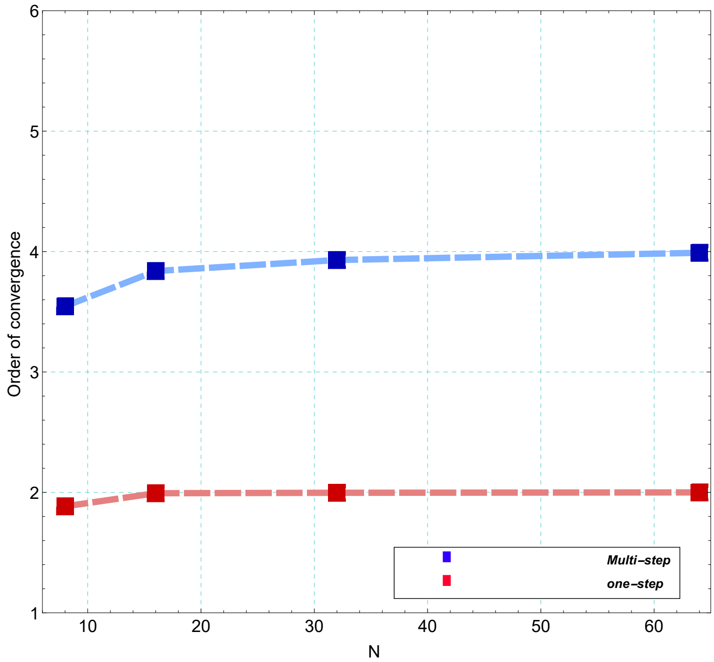

Figure 1: Plot of order of convergence for the one-step and multi-step scheme in Example 1.

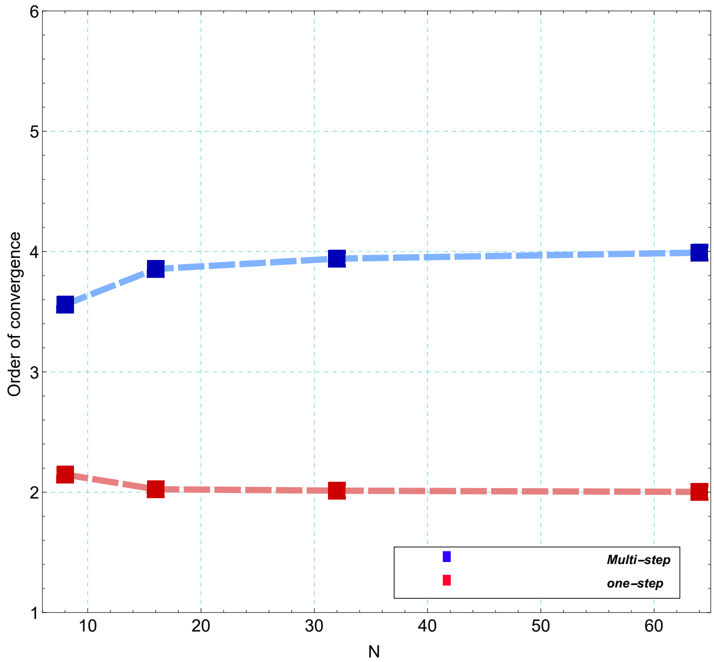

Figure 2: Plot of order of convergence for the one-step and multi-step scheme in Example 2.

In this section, by considering $m = r = 2$, the value of n begins at $r-1 = 1$ for the approximate solutions (12). According to the proposed numerical methods, we need starting values $x_0^{\prime(\mu)} = x'(t_0^{(\mu)}), x_1^{\prime(\mu)} = x'(t_1^{(\mu)}) = x'(t_0^{(\mu)} + h_0^{(\mu)})$, $x_1^{(\mu)} = x(t_1^{(\mu)}) = x(t_0^{(\mu)} + h_0^{(\mu)})$ and $w(t_0^{(\mu)} + sh_0^{(\mu)})$ for $\mu = 0, 1$. Every value in the above list was derived from the known exact solutions. We can utilize approximate solutions for the starting values because, as you are aware, we do not have the exact solutions for real problems. The following Remark is taken into consideration for this purpose:

Remark 5. *We can examine the impact of numerical approximations of the initial values using a traditional one-step approach. We consider the polynomial approximation for the equation* (1) *as follows, based on [1]:*

$$w'(t_n^{(\mu)} + zh_n^{(\mu)}) = h_n^{(\mu)} \sum_{j=1}^{m} L_j(z) W_{n,j}^{(\mu)}, \quad n = 0, \dots, N-1, \tag{36}$$

and

$$w(t_n^{(\mu)} + zh_n^{(\mu)}) = x_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(z) W_{n,j}^{(\mu)}, \quad n = 0, \dots, N-1, \tag{37}$$

where $\beta_j(z) = \int_0^z L_j(s)ds$ and $L_j (j = 1, \dots, m)$ represent the Lagrange canonical polynomials. Taking into account Theorem 4.5.2 [1], we have the collocation error related to the collocation solution

$$\|x^{(\nu)} - w^{(\nu)}\|_\infty \le C_\nu h^m, \quad \nu = 0, 1, \tag{38}$$

where the constants C_ν are independent of h.

For $\mu = 0, 1$, the approximation of the starting values $x_1^{\prime(\mu)} = x'(t_0^{(\mu)} + h_0^{(\mu)})$ and $x_1^{(\mu)} = x(t_0^{(\mu)} + h_0^{(\mu)})$ can be obtained from (36) and (37) by setting $z = 1$. It is also possible to obtain $w(t_0^{(\mu)} + sh_0^{(\mu)})$ from (37). We used the multi-step collocation method with $m = 2$ and $r = 2$ and looked at two numerical cases. Next, using the approximation technique with the order $m+r = 4$, we must determine the initial values based on Theorem 3. The starting values are obtained using a numerical technique based on the suggested classical one step methods (37) with the order 4 (see (38)) and collocation parameters $s_i = \frac{1}{5-i}, (i = 1, \dots, 4)$. Table 4 reports the maximum errors for various values of N.

Table 4: L_∞ errors with the approximate starting values and $m = r = 2$.

N	Multi-step method for Example 1	Multi-step method for Example 2
4	9.12×10^{-5}	9.01×10^{-5}
8	8.17×10^{-6}	8.56×10^{-6}
16	7.66×10^{-7}	8.18×10^{-7}
32	1.30×10^{-8}	1.81×10^{-8}

We also solve Example 1 and 2 with $r = 3$, $m = 2$ and report the results in the Table 5.

Table 5: L_∞ errors and CPU time based on seconds for $m = 2$ and $r = 3$ in Examples 1 and 2.

	$(s_1, s_2) = (0.8, 1)$		$(s_1, s_2) = (0.8, 1)$	
N	Multi-step for Example 1	CPU time(sec)	Multi-step for Example 2	CPU time(sec)
4	1.24×10^{-6}	1.90	1.27×10^{-6}	1.26
8	1.09×10^{-7}	2.71	9.96×10^{-8}	2.93
16	4.11×10^{-9}	10.2	4.04×10^{-9}	9.96
32	1.38×10^{-10}	77.9	1.41×10^{-10}	40.6

5. Conclusion

We have shown that the multi-step collocation method offers a reliable and precise numerical technique for estimating solutions to VIDEs with non-zero delay. Finally, to substantiate the theoretical predictions in a practical context, we examined some test problems. These tests served as evidence of the consistency and agreement between the numerical and theoretical analysis. We have considered our proposed methods based on the smoothness of the given function in the Theorems 1 and 2. As future work, for the case the solution derivatives are unbounded especially when either the data is non smooth may be able to use adaptive generated meshes in [37–39]. Also, we will study the proposed multi-step method to solve delay integro-differential-algebraic equations in the following form:

$$A(t)X'(t) = F(t) + B(t)X(t) + \int_0^t K(t, s, X(s))\mathrm{d}s + \int_0^{\tau(t)} \hat{K}(t, s, X(s))\mathrm{d}s, \quad t \in J,$$

subject to $\det A(t) = 0,\ \forall t \in J$. Unlike integro-differential algebraic equations (IDAEs), the solutions of delay integro-differential algebraic equations (DIDAEs) can be included primary discontinuity points. We will try to investigate the structure of the solution of DIDAEs from numerical and theoretical point of view.

References

[1] H. Brunner. *Collocation methods for Volterra integral and related functional differential equations.* Cambridge University Press, Cambridge, 2004.

[2] F. Brauer. On a nonlinear integral equation for population growth problems. *SIAM Journal on Mathematical Analysis*, 6(2):312–317, 1975.

[3] F. Brauer and C. Castillo-Chavez. *Mathematical models in population biology and epidemiology.* Springer, New York, 2001.

[4] B. Chen, J. Hu, and B. K. Ghosh. Finite-time tracking control of heterogeneous multi-AUV systems with partial measurements and intermittent communication. *Science China Information Sciences*, 67(5):152202, 2024.

[5] Y. Kai, J. Ji, and Z. Yin. Study of the generalization of regularized long-wave equation. *Nonlinear Dynamics*, 107(3):2745–2752, 2022.

[6] L. Liu, S. Zhang, L. Zhang, G. Pan, and J. Yu. Multi-UUV maneuvering counter-game for dynamic target scenario based on fractional-order recurrent neural network. *IEEE Transactions on Cybernetics*, 53(6):4015–4028, 2023.

[7] G. A. Bocharov and F. A. Rihan. Numerical modelling in biosciences using delay differential equations. *Journal of Computational and Applied Mathematics*, 125(1-2):183–199, 2000.

[8] K. Maleknejad, F. Mirzaee, and S. Abbasbandy. Solving linear integro-differential equations system by using rationalized Haar functions method. *Applied Mathematics and Computation*, 155(2):317–328, 2004.

[9] H. Brunner. *Volterra integral equations: an introduction to theory and applications*. Cambridge University Press, Cambridge, 2017.

[10] H. Brunner, A. Makroglou, and R. K. Miller. Mixed interpolation collocation methods for first and second order Volterra integro-differential equations with periodic solution. *Applied Numerical Mathematics*, 23(4):381–402, 1997.

[11] M. R. Crisci, E. Russo, and A. Vecchio. Stability of collocation methods for Volterra integro-differential equations. *Journal of Integral Equations and Applications*, 4(4):491–507, 1992.

[12] T. Lin, Y. Lin, M. Rao, and S. Zhang. Petrov–Galerkin methods for linear Volterra integro-differential equations. *SIAM Journal on Numerical Analysis*, 38(3):937–963, 2000.

[13] Y. Jafarzadeh and B. Keramati. Numerical method for a system of integro-differential equations by Lagrange interpolation. *Asian-European Journal of Mathematics*, 9(3):1650058, 2016.

[14] A. Cardone and D. Conte. Multistep collocation methods for Volterra integro-differential equations. *Applied Mathematics and Computation*, 221:770–785, 2013.

[15] A. Akyüz and M. Sezer. Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations. *Journal of the Franklin Institute*, 342(6):688–701, 2005.

[16] A. Karamete and M. Sezer. A Taylor collocation method for the solution of linear integro-differential equations. *International Journal of Computer Mathematics*, 79(9):987–1000, 2002.

[17] J. T. Katsikadelis. Numerical solution of integrodifferential equations with convolution integrals. *Archive of Applied Mechanics*, 89(10):2019–2036, 2019.

[18] K. Kumar, P. C. Podila, P. Das, and H. Ramos. A graded mesh refinement approach for boundary layer originated singularly perturbed time-delayed parabolic convection diffusion problems. *Mathematical Methods in the Applied Sciences*, 44(16):12332–12350, 2021.

[19] S. Saini, P. Das, and S. Kumar. Parameter uniform higher order numerical treatment for singularly perturbed Robin type parabolic reaction diffusion multiple scale problems with large delay in time. *Applied Numerical Mathematics*, 196:1–21, 2024.

[20] P. Das. An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh. *Numerical Algorithms*, 81(2):465–487, 2019.

[21] I. Ali, H. Brunner, and T. Tang. Spectral methods for pantograph-type differential and integral equations with multiple delays. *Frontiers of Mathematics in China*, 4(1):49–61, 2009.

[22] H. Brunner. Collocation and continuous implicit Runge-Kutta methods for a class of delay Volterra integral equations. *Journal of Computational and Applied Mathematics*, 53(1):61–72, 1994.

[23] H. Brunner and Y. Yatsenko. Spline collocation methods for nonlinear Volterra integral equations with unknown delay. *Journal of Computational and Applied Mathematics*, 71(1):67–81, 1996.

[24] M. V. Bulatov, M. N. Machkhina, and V. N. Phat. Existence and uniqueness of solutions to nonlinear integral-algebraic equations with variable limits of integrations. *Communications on Applied Nonlinear Analysis*, 21(1):65–76, 2014.

[25] F. Caliò, E. Marchetti, and R. Pavani. About the deficient spline collocation method for particular differential and integral equations with delay. *Rendiconti del Seminario Matematico*, 61(3):287–300, 2003.

[26] F. Caliò, E. Marchetti, R. Pavani, and G. Micula. About some Volterra problems solved by a particular spline collocation. *Studia Universitatis Babeş-Bolyai Mathematica*, 48(2):45–52, 2003.

[27] M. Khasi, F. Ghoreishi, and M. Hadizadeh. Numerical analysis of a high order method for state-dependent delay integral equations. *Numerical Algorithms*, 66(1):177–201, 2014.

[28] S. Kumar, S. Kumar, and Sumita. A priori and a posteriori error estimation for singularly perturbed delay integro-differential equations. *Numerical Algorithms*, 95(4):1561–1582, 2024.

[29] S. Wu and S. Gan. Errors of linear multistep methods for singularly perturbed Volterra delay integro-differential equations. *Mathematics and Computers in Simulation*, 79(10):3148–3159, 2009.

[30] N. A. Baharum, Z. A. Majid, N. Senu, and H. Rosali. Numerical approach for delay Volterra integro-differential equation. *Sains Malaysiana*, 51(12):4125–4144, 2022.

[31] B. Gürbüz. A numerical scheme for the solution of neutral integro-differential equations including variable delay. *Mathematical Sciences*, 16:13–21, 2022.

[32] Q. Guan, R. Zhang, and Y. Zou. Analysis of collocation methods for nonstandard Volterra integral equations. *IMA Journal of Numerical Analysis*, 32(4):1755–1785, 2012.

[33] M. Chandru, P. Das, and H. Ramos. Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data. *Mathematical Methods in the Applied Sciences*, 41(14):5359–5387, 2018.

[34] P. Das and S. Natesan. Optimal error estimate using mesh equidistribution technique for singularly perturbed system of reaction–diffusion boundary-value problems. *Applied Mathematics and Computation*, 249:265–277, 2014.

[35] S. Saini, P. Das, and S. Kumar. Computational cost reduction for coupled system of multiple scale reaction diffusion problems with mixed type boundary conditions having boundary layers. *Revista de la Real Academia de Ciencias Exactas, Físicas y*

Naturales - Serie A: Matemáticas, 117(2):66, 2023.

[36] P. Darania and S. Pishbin. Multistep collocation methods for integral-algebraic equations with non-vanishing delays. *Mathematics and Computers in Simulation*, 205:33–61, 2023.

[37] P. Das, S. Rana, and H. Ramos. On the approximate solutions of a class of fractional order nonlinear Volterra integro-differential initial value problems and boundary value problems of first kind and their convergence analysis. *Journal of Computational and Applied Mathematics*, 404:113116, 2022.

[38] S. Kumar, Sumita, and J. Vigo-Aguiar. A high order convergent numerical method for singularly perturbed time dependent problems using mesh equidistribution. *Mathematics and Computers in Simulation*, 199:287–306, 2022.

[39] S. Santra, J. Mohapatra, P. Das, and D. Choudhuri. Higher order approximations for fractional order integro-parabolic partial differential equations on an adaptive mesh with error analysis. *Computers & Mathematics with Applications*, 150:87–101, 2023.