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Weakly Connected k-Rainbow Domination in Graphs

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Abstract. Let G be a simple and connected graph, and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2, 3, \ldots, k\}$, i.e., $f : V(G) \to \mathcal{P}(\{1, 2, 3, \ldots, k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \varnothing$, we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, 3, \ldots, k\}$, then f is called a k-rainbow dominating function (kRDF) of G. A kRDF $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ is said to be a *weakly connected k-rainbow dominating function* (WCkRDF) if the set $S = \{v \in V(G) : f(v) \neq \varnothing\}$ is weakly connected dominating. The *weight* $w(f)$ of f is defined as $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *weakly connected k-rainbow domination number* of G, denoted by $\gamma_{rk}^{wc}(G)$ is the minimum weight of WCkRDF. A weakly connected k-rainbow dominating function of G with weight $\gamma_{rk}^{wc}(G)$, i.e., $\omega(f) = \gamma_{rk}^{wc}(G)$ is referred to as a γ_{rk}^{wc}-function of G. In this paper, we initiate the study of the weakly connected k-rainbow domination parameter. First, we establish fundamental properties and bounds for weakly connected k-rainbow domination. Then, we determine the weakly connected k-rainbow domination number for various classes of graphs. Furthermore, we characterize the weakly connected k-rainbow dominating function under the join of graphs and determine the weakly connected k-rainbow domination number for this binary operation.

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1. Introduction

Graph domination provides a foundational framework for various applications, where the vertices in a dominating set represent service providers or essential resources that ensure accessibility to every vertex in the network, as described by T. W. Haynes et al. [1]. In 1987, Hedetniemi [2] introduced the concept of dominating functions, offering an analytical approach to studying this discrete structure. This framework established connections between domination, graph labelings, and colorings, leading to the development of new domination function parameters. Two decades later, in 2008, Brešar et al. [3] introduced the concept of rainbow domination, which extends the notion of domination by incorporating multiple service types, each represented by a distinct color. The goal of rainbow domination is to assign services so that any vertex not directly receiving a service has access to all service types within its neighborhood. In the following years, this concept gained significant attention from researchers, leading to numerous studies exploring its properties and applications, as discussed in [4–11].

In 1997, J. E. Dunbar et al. [12] introduced the concept of a weakly connected dominating set in a connected graph and examined the weakly connected domination number along with related parameters. Further insights into weakly connected domination can be found in [13–19]. Rainbow domination extends to weakly connected domination by ensuring resource distribution while maintaining a weakly connected dominating set. This integration is crucial for networks that require both connectivity and service diversity, leading to new parameters and optimization techniques in graph theory.

In this paper, we initiate the study of the weakly connected k-rainbow domination parameter. First, we establish fundamental properties and bounds for weakly connected k-rainbow domination. Then, we determine the weakly connected k-rainbow domination number for various classes of graphs. Furthermore, we characterize the weakly connected k-rainbow dominating function under the join of graphs and determine the weakly connected k-rainbow domination number for this binary operation.

2. Terminology and Notation

For general graph theory terminology, we adhere to the definitions provided by Harary in [20]. Let $G = (V, E)$ be a simple and connected graph, where $V = V(G)$ represents the *vertex set* and $E = E(G)$ denotes the *edge set* of G. The number of edges incident to a vertex v is called its *degree*, denoted as $\deg(v)$. The *maximum degree* of G, represented by $\Delta(G)$, is given by $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$. The *open neighborhood* of a vertex u, denoted by $N_G(u)$, is the set of all vertices adjacent to u, i.e., $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of u is defined as $N_G[u] = N_G(u) \cup \{u\}$. Similarly, for a subset $S \subseteq V(G)$, the *closed neighborhood* of S is given by $N_G[S] = \bigcup_{v \in S} N_G[v]$. The *subgraph weakly induced* by a set $S \subseteq V(G)$, as defined by E. P. Sandueta et al. [18], is the graph $\langle S\rangle_w = (N_G[S], E_w(S))$, where the edge set is given by $E_w(S) = \{uv \in E(G) : u \in S \text{ or } v \in S\}$.

A set $S \subseteq V(G)$ is said to be a *dominating set* of G if $N_G[S] = V(G)$. A dominating set S is called a *minimal dominating set* if no proper subset $S' \subset S$ is itself a dominating set. The *domination number*, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set S with $|S| = \gamma(G)$ is referred to as a γ-*set* of G.

A set $S \subseteq V(G)$ is said to be a *weakly connected dominating set* in G if S is dominating and the subgraph $\langle S\rangle_w$ weakly induced by S is connected. The *weakly connected domination number*, denoted by $\gamma_w(G)$ is the minimum cardinality among all weakly connected dominating sets. A weakly connected dominating set S with $|S| = \gamma_w(G)$ is referred to as a γ_w-*set* of G, as defined by J. E. Dunbar et al. in [12].

A function $f : V(G) \to \mathcal{P}(\{1, 2, 3, \ldots, k\})$ assigns to each vertex of a graph G a set of colors chosen from the set $\{1, 2, 3, \ldots, k\}$. If, for every vertex $v \in V(G)$ such that $f(v) = \varnothing$, we have $\bigcup_{u\in N_G(v)} f(u) = \{1, 2, 3, \ldots, k\}$, then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight* $\omega(f)$ of f is defined as $\omega(f) = \sum_{v\in V(G)}|f(v)|$. The *$k$-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF. A k-rainbow dominating function of G with weight $\gamma_{rk}(G)$, i.e., $\omega(f) = \gamma_{rk}(G)$, is referred to as a γ_{rk}-function of G, as defined by B. Brešar et al. in [3].

A kRDF $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ is said to be a *weakly connected k-rainbow dominating function* (WCkRDF) if the set $S = \{v \in V(G) : f(v) \neq \varnothing\}$ is weakly connected dominating. The *weight* $\omega(f)$ of f is defined as $\omega(f) = \sum_{v\in V(G)}|f(v)|$. The *weakly connected k-rainbow domination number* of G, denoted by $\gamma^{wc}_{rk}(G)$, is the minimum weight of a WCkRDF. A weakly connected k-rainbow dominating function of G with weight $\gamma^{wc}_{rk}(G)$, i.e., $\omega(f) = \gamma^{wc}_{rk}(G)$, is referred to as a γ^{wc}_{rk}-function of G. Clearly, when $k = 1$, the weakly connected 1-rainbow domination number $\gamma^{wc}_{r1}(G)$ is equivalent to the classical weakly connected domination number $\gamma_w(G)$ of G.

For any graph G and a γ^{wc}_{rk}-function f of G, set $V^f_i = \{x \in V(G) : |f(x)| = i\}$ for $i \in \{1, 2, \ldots, k\}$.

3. Preliminary Results

We begin this section by presenting some properties and bounds, and then we determine the weakly connected k-rainbow domination number of G.

Remark 1. *Every weakly connected k-rainbow dominating function f of G is also a k-rainbow dominating function of G. In particular,*

$$\gamma_{rk}(G) \leq \gamma^{wc}_{rk}(G).$$

Theorem 1. *Let G be a connected graph with $\Delta(G) \leq k$ for some positive integer $k \geq 2$. Then there exists a γ^{wc}_{rk}-function f of G such that $|f(v)| < k$ for every vertex $v \in V(G)$.*

Proof. Let g be a γ_{rk}^{wc}-function of G such that $|V_k^g|$ is as small as possible. We claim that $|f(v)| < k$ for every vertex $v \in V(G)$ as desired. Assume, to the contrary, that there exists a vertex $v \in V(G)$ such that $|g(v)| = k$. Since $|g(v)| = k$, by the definition there exists a vertex $x_1 \in N_G(v)$ such that $g(x_1) = \varnothing$. Let there exist r vertices $x_1, x_2, \ldots, x_r \in N_G(v)$ such that $g(x_i) = \varnothing$ for $i \in \{1, 2, \ldots, r\}$, where $r \leq \Delta(G) \leq k$. Now, define a new function $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(x_i) = \{i\}$ for $i \in \{1, 2, \ldots, r-1\}$, $f(x_r) = \{r, r+1, \ldots, k\}$, $f(v) = \varnothing$, and $f(u) = g(u)$ for all other vertices $u \in V(G)$. Clearly, f is a weakly connected k-rainbow dominating function of G of weight $\omega(g)$, contradicting the choice of g. Thus $|f(v)| < k$ for every vertex $v \in V(G)$, and the proof is complete. □

Theorem 2. *Let G be a connected graph. Then $\gamma_{r2}^{wc}(G) \geq \gamma_w(G)$.*

Proof. Let f be a γ_{r2}^{wc}-function of G. By definition the set $V(G) \setminus V_0^f$ is a weakly connected dominating set of G and thus

$$\gamma_{r2}^{wc}(G) = \sum_{v \in V(G) \setminus V_0^f} |f(v)| \geq \sum_{v \in V(G) \setminus V_0^f} 1 = |V(G) \setminus V_0^f| \geq \gamma_w(G),$$

as desired. □

The graph G illustrated in Figure 1 demonstrates that the bound in Theorem 2 is sharp.

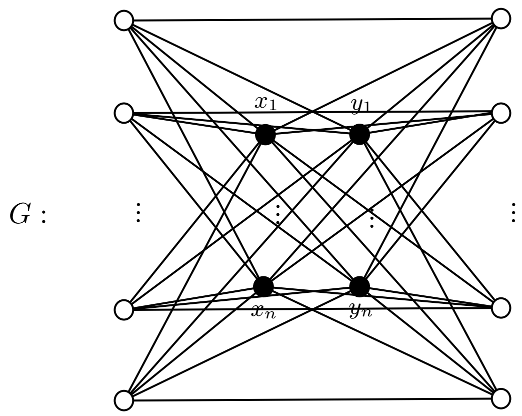

Figure 1: A graph G attaining the bound in Theorem 2

The exact values of the k-rainbow domination number for $k \in \{2, 3\}$ of paths and cycles determined in [6] and [21] as follows.

Theorem 3. *(i)* $\gamma_{r2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ *and* $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

(ii) For $n \geq 5$, $\gamma_{r3}(P_n) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1 & \text{if} \quad n \equiv 0 \pmod 4 \\ \lceil \frac{3n}{4} \rceil & \text{if} \quad n \equiv 1, 2, 3 \pmod 4). \end{cases}$

(iii) For $n \geq 5$, $\gamma_{r3}(C_n) = \lceil \frac{3n}{4} \rceil$.

Using Theorem 3 and Remark 1, we will determine the weakly connected k-rainbow domination number for $k \in \{2,3\}$ for paths P_n and cycle C_n.

Proposition 1. *Let n be a positive integer. Then*

(i) for all $n \geq 1$, $\gamma_{r2}^{wc}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.

(ii) for all $n \geq 1$, $$\gamma_{r3}^{wc}(P_n) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1 & \text{if } n \equiv 0 \pmod 4 \\ \lceil \frac{3n}{4} \rceil & \text{if } n \equiv 1,2,3 \pmod 4. \end{cases}$$

(iii) for all $n \geq 3$, $\gamma_{r2}^{wc}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$

(iv) for all $n \geq 3$, $\gamma_{r3}^{wc}(C_n) = \lceil \frac{3n}{4} \rceil$.

Proof. Let $P_n = [v_1, v_2, \ldots, v_{n-1}, v_n]$ be a path of order n and $C_n = [v_1, v_2, \ldots, v_{n-1}, v_n, v_1]$ be a cycle of order n.

We first establish the result (i) and (ii) for paths. For $k \in \{2,3\}$, define the function $f^k : V(P_n) \to \mathcal{P}(\{1,2,\ldots,k\})$ as follows:

(a) If $n \equiv 1 \pmod 4$, then let $f^k(v_{4i+1}) = \{1\}$ for $0 \leq i \leq \frac{n-1}{4}$, $f^k(v_{4i+3}) = \{2,k\}$ for $0 \leq i \leq \frac{n-5}{4}$, and $f^k(x) = \varnothing$ otherwise.

(b) If $n \equiv 2 \pmod 4$, then let $f^k(v_n) = \{1\}$, $f^k(v_{4i+1}) = \{1\}$ for $0 \leq i \leq \frac{n-2}{4}$, $f^k(v_{4i+3}) = \{2,k\}$ for $0 \leq i \leq \frac{n-6}{4}$, and $f^k(x) = \varnothing$ otherwise.

(c) If $n \equiv 3 \pmod 4$, then let $f^k(v_{4i+1}) = \{1\}$, $f^k(v_{4i+3}) = \{2,k\}$ for $0 \leq i \leq \frac{n-3}{4}$ and $f^k(x) = \varnothing$ otherwise.

(d) If $n \equiv 0 \pmod 4$, then let $f^k(v_n) = \{1\}$, $f^k(v_{4i+1}) = \{1\}$, $f^k(v_{4i+3}) = \{2,k\}$ for $0 \leq i \leq \frac{n-4}{4}$ and $f^k(x) = \varnothing$ otherwise.

In all cases, f^k is a WCkRDF of P_n of weight $\gamma_{rk}(P_n)$ and thus $\gamma_{rk}^{wc}(P_n) \leq \gamma_{rk}(P_n)$ for $k \in \{2,3\}$. By Remark 1, we obtain $\gamma_{rk}^{wc}(P_n) = \gamma_{rk}(P_n)$ for $k \in \{2,3\}$, and Theorem 3-(i,ii) leads to the desired values.

Now, we prove (iii) and (iv) for cycles simultaneously. Let $k \in \{2,3\}$. If $n \equiv 0 \pmod 4$, then define the function $f^k : V(C_n) \to \mathcal{P}(\{1,2,\ldots,k\})$ by $f^k(v_{4i+1}) = \{1\}$, $f^k(v_{4i+3}) = \{2,k\}$ for $0 \leq i \leq \frac{n-4}{4}$, and $f^k(x) = \varnothing$ otherwise. If $n \not\equiv 0 \pmod 4$, then let f^k be the function defined in the above item (i) and (ii) depending on n. In all cases, f^k is a WC2RDF of C_n of weight $\gamma_{rk}(C_n)$, and thus $\gamma_{rk}^{wc}(C_n) \leq \gamma_{rk}(C_n)$ for $k \in \{2,3\}$. Now, Remark 1 leads to $\gamma_{rk}^{wc}(C_n) = \gamma_{rk}(C_n)$ for $k \in \{2,3\}$, and (iii) and (iv) follow from Theorem 3-(i,iii). □

Proposition 2. *Let k be a positive integer and G be a connected graph of order n. Then*

$$\min\{n, k\} \leq \gamma_{rk}^{wc}(G) \leq n.$$

In particular, if $n \leq k$, then $\gamma_{rk}^{wc}(G) = n$.

Proof. Suppose that f is a γ_{rk}^{wc}-function of G. If $V_0^f = \varnothing$, then we have $\gamma_{rk}^{wc}(G) = \sum_{v \in V(G)} |f(v)| \geq \sum_{v \in V(G)} 1 = n$. Assume that $V_0^f \neq \varnothing$ and $v \in V_0^f$. Then we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, 3, \ldots, k\}$, and thus $\gamma_{rk}^{wc}(G) = \sum_{u \in V(G)} |f(u)| \geq \sum_{u \in N_G(v)} |f(u)| \geq k$. Combining the above inequalities, we get $\min\{n, k\} \leq \gamma_{rk}^{wc}(G)$.

For the upper bound, consider the function $g : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(v) = \{1\}$ for all $v \in V(G)$. Then g is a k-rainbow dominating function. Let $S = \{v \in V(G) : g(v) \neq \varnothing\}$. Observe that $S = V(G)$ since $g(v) = \{1\}$ for all $v \in V(G)$. Thus, S is a weakly connected dominating set of G. It would imply that g is a weakly connected k-rainbow dominating function. Thus, $\gamma_{rk}^{wc}(G) \leq n$. Clearly, if $n \leq k$, then $\gamma_{rk}^{wc}(G) = n$. □

Shao et al. in [21] proved the next result.

Theorem 4. *For positive integers n and $k \geq 2$, let G be a connected graph of order $n \geq k$ with $k > \Delta(G)^2$. Then $\gamma_{rk}(G) = n$.*

The next results are direct consequences of Theorem 4 and Remark 1.

Corollary 1. *For a positive integer n and $k \geq 2$, let G be a connected graph of order $n \geq k$ with $k > \Delta(G)^2$. Then $\gamma_{rk}^{wc}(G) = n$.*

Corollary 2. *For positive integers n and $k \geq 5$, $\gamma_{rk}^{wc}(P_n) = \gamma_{rk}^{wc}(C_n) = n$.*

The following result shows that the difference $\gamma_{r2}^{wc}(G) - \gamma_{r2}(G)$ can be arbitrarily large. A caterpillar $C(n; d_1, \ldots, d_n)$ is defined as a tree in which removal of all its leaves yields a path $P_n = [x_1, x_2, \ldots, x_n]$ and that d_i is the number of leaf neighbors of x_i for each i. The path $P_n = [x_1, x_2, \ldots, x_n]$ is called the backbone of the caterpillar.

Theorem 5. *For the integer $k \geq 2$ and every non-negative integer c, there exists a connected graph G such that $\gamma_{r2}^{wc}(G) - \gamma_{r2}(G) = c$.*

Proof. Consider the caterpillar $G = C(3c + 1; 2k, 0, 0, 2k, \ldots, 0, 0, 2k)$ with backbone $P_n = [x_1, x_2, \ldots x_{3c+1}]$ in Figure 2. It is easily seen that the function $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $f(x_{3i+1}) = \{1, 2, \ldots, k\}$ for $0 \leq i \leq c$ and $f(x) = \varnothing$ for other vertices, is the unique γ_{rk}-function of G of weight $k(c + 1)$, and the function $g : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(x_{3i+1}) = \{1, 2, \ldots, k\}$ for $0 \leq i \leq c$, $g(x_{3i}) = \{1\}$ for $1 \leq i \leq c$, and $g(x) = \varnothing$ for other vertices, is a γ_{rk}^{wc}-function of G of weight $k(c+1)+c$. Thus, $\gamma_{r2}^{wc}(G) - \gamma_{r2}(G) = c$ and the proof is complete. □

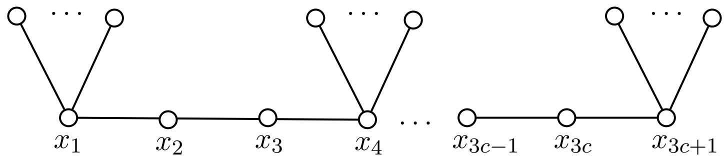

Figure 2: A caterpillar $G = C(3c+1;\ 2k, 0, 0, 2k, \dots, 0, 0, 2k)$

4. Graphs with $\gamma_{rk}^{wc}(G) = k$

In this section, we characterize all graphs G with $\gamma_{rk}^{wc}(G) = k$.

Theorem 6. *Let $k \geq 1$ be an integer, and let G be a connected graph of order $n \geq k$. Then $\gamma_{rk}^{wc}(G) = k$ if and only if $n = k$ or $n > k$ and there exists a set $X = \{x_1, x_2, \dots, x_m\}$ of vertices with $1 \leq |X| \leq k$ such that $(V(G) \setminus X) \subseteq \bigcap_{i=1}^{m} N_G(x_i)$.*

Proof. Let $n = k$ or $n > k$ and there exists a set $X = \{x_1, x_2, \dots, x_m\}$ of vertices with $1 \leq |X| \leq k$ such that $(V(G) \setminus X) \subseteq \bigcap_{i=1}^{m} N_G(x_i)$. By Proposition 2, we have $\gamma_{rk}^{wc}(G) \geq k$. If $n = k$, then obviously $\gamma_{rk}^{wc}(G) = k$. Assume that $n > k$. Then the function $f : V(G) \to \mathcal{P}(\{1, 2, \dots, k\})$ defined by $f(x_i) = \{i\}$ for $1 \leq i \leq m-1$, $f(x_m) = \{t, t+1, \dots, k\}$ and $f(x) = \varnothing$ otherwise, is clearly a WCkRDF of G of weight k, and so $\gamma_{rk}^{wc}(G) \leq k$. Thus, $\gamma_{rk}^{wc}(G) = k$.

Conversely, assume that $\gamma_{rk}^{wc}(G) = k$. Let f be a γ_{rk}^{wc}-function of G. If $V_0^f = \varnothing$, then we have that $n = k$. Assume that $V_0^f \neq \varnothing$ and let $u \in V_0^f$. By definition, $\cup_{i=1}^{m} f(x_i) = \{1, 2, \dots, k\}$. Now let $x_1, x_2, \dots, x_m$ be all vertices in $N_G(u)$ such that $f(x_i) \neq \varnothing$ for $1 \leq i \leq m$. It follows from the condition $\gamma_{rk}^{wc}(G) = k$ that $\cup_{i=1}^{m} |f(x_i)| = k$, $1 \leq i \leq m$, and $(V(G) \setminus X) \subseteq \bigcap_{i=1}^{m} N_G(x_i)$. This completes the proof. □

As a consequence of Proposition 2 and Theorem 6, we have the following.

Corollary 3. *Let n and k be positive integers. Then*

$$\gamma_{rk}^{wc}(K_n) = \begin{cases} k & \text{if } n \geq k, \\ n & \text{if } n < k. \end{cases}$$

Proposition 3. *Let m, n, and k be positive integers with $k \geq 1$ and $m \leq n$. Then*

$$\gamma_{rk}^{wc}(K_{m,n}) = \begin{cases} m+n & \text{if } m+n \leq k, \\ 2k & \text{if } m \geq 2k, \\ \max\{m, k\} & \text{if } m+n > k \text{ and } m < 2k. \end{cases}$$

Proof. Suppose $K_{m,n}$ is a complete bipartite graph with m, n vertices. Let X and Y be the two partite sets of $K_{m,n}$, where $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. If $m + n \leq k$, then we have $\gamma_{rk}^{wc}(K_{m,n}) = m + n$ by Proposition 2. Hence we assume that $m + n > k$. Consider the following cases:

Case 1. $m \geq 2k$.
Let $f : V(K_{m,n}) \to P(\{1, 2, \ldots, k\})$ be a function defined by $f(x_1) = f(y_1) = \{1, 2, \ldots, k\}$, and $f(v) = \varnothing$ for every vertex $v \in V(K_{m,n}) \setminus \{x_1, y_1\}$. Then $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, \ldots, k\}$ for every $v \in V(K_{m,n}) \setminus \{x_1, y_1\}$. Observe that the subgraph $\langle\{x_1, y_1\}\rangle_w$ induced by $\{x_1, y_1\}$ is connected and $N_{K_{m,n}}[\{x_1, y_1\}] = V(K_{m,n})$. Thus, by definition, f is a weakly connected k-rainbow dominating function of $K_{m,n}$ of weight

$$\omega(f) = \sum_{v \in V(K_{m,n})} |f(v)| = 2k.$$

This implies that $\gamma_{rk}^{wc}(K_{m,n}) \leq 2k$.

Now, let f^* be any γ_{rk}^{wc}-function of $K_{m,n}$. If for every vertex $x \in X$, $f^*(x) \neq \varnothing$ or for every vertex $y \in Y$, $f^*(y) \neq \varnothing$, then clearly $\omega(f^*) \geq m \geq 2k$. Hence, assume that there are two vertices $x \in X$ and $y \in Y$ such that $f^*(x) = \varnothing$ and $f^*(y) = \varnothing$. By definition, we have

$$\begin{aligned}
\omega(f^*) &= \sum_{v \in V(K_{m,n})} |f^*(v)| \\
&= \sum_{v \in X} |f^*(v)| + \sum_{v \in Y} |f^*(v)| \\
&= \sum_{v \in N_{K_{m,n}}(y)} |f^*(v)| + \sum_{v \in N_{K_{m,n}}(x)} |f^*(v)| \\
&\geq 2k.
\end{aligned}$$

Therefore, $\gamma_{rk}^{wc}(K_{m,n}) \geq 2k$, and as a result, we conclude $\gamma_{rk}^{wc}(K_{m,n}) = 2k$.

Case 2. $m + n > k$ and $m < 2k$.
If $m \leq k$, then by Theorem 6, we have $\gamma_{rk}^{wc}(K_{m,n}) = k = \max\{m, k\}$. Assume that $k < m < 2k$. Let $f : V(K_{m,n}) \to \mathcal{P}(\{1, 2, \ldots, k\})$ be a function defined by $f(x_i) = \{i\}$ for $1 \leq i \leq k$ and $f(x_i) = \{1\}$ for each $i \in \{k+1, \ldots, m\}$ and $f(y) = \varnothing$ for each $y \in Y$. As Case 1, we observe that f is a WCkRDF of $K_{m,n}$, and thus $\gamma_{rk}^{wc}(K_{m,n}) \leq m$. Using a similar argument as in Case 1, we can see that $\gamma_{rk}^{wc}(K_{m,n}) = m$. Thus, $\gamma_{rk}^{wc}(K_{m,n}) = \max\{m, k\}$. This completes the proof. □

5. Join of Graphs

Harary [20] defined the *join* of two graphs G and H, denoted by $G + H$, as the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 7. *Let m, n, and k be positive integers such that $\min\{m, n\} \geq k$, and let G and H be graphs of order m and n, respectively. Then $\gamma_{rk}^{wc}(G + H) = k$ if and only if $\gamma_{rk}^{wc}(G) = k$ or $\gamma_{rk}^{wc}(H) = k$.*

Proof. If $\gamma_{rk}^{wc}(G) = k$ (the case $\gamma_{rk}^{wc}(H) = k$ is similar), then by Theorem 6, there exists a set $X \subseteq V(G)$ such that $|X| \leq k$ and each vertex in $V(G) \setminus X$ is adjacent to all vertices in X. By definition of the join of graphs, each vertex in $V(H)$ is adjacent to all vertices in X, and again Theorem 6 leads to $\gamma_{rk}^{wc}(G + H) = k$.

Conversely, assume that $\gamma_{rk}^{wc}(G + H) = k$. Using Theorem 6, it follows that there is a set X of vertices $V(G+H)$ such that $|X| \leq k$ and each vertex in $V(G+H) \setminus X$ is adjacent to all vertices in X. Without loss of generality, we may assume that $X \cap V(G) \neq \varnothing$. Since $|X \cap V(G)| \leq k$ and each vertex in $V(G) \setminus X$ is adjacent to all vertices in X, we deduce from Theorem 6 that $\gamma_{rk}^{wc}(G) = k$. This completes the proof. □

Theorem 8. *Let m, n, and k be positive integers such that $\min\{m, n\} \geq k$, and let G and H be connected graphs of order m and n, respectively. Then*

$$\gamma_{rk}^{wc}(G + H) \leq \min\{2k, \min\{\gamma_{rk}^{wc}(G), \gamma_{rk}^{wc}(H)\}\}.$$

Proof. Let first x, y be two vertices of $G + H$ such that $x \in V(G)$ and $y \in V(H)$, and define the function $f : V(G + H) \to \mathcal{P}(\{1, 2, \dots, k\})$ by $f(x) = f(y) = \{1, 2, \dots, k\}$ and $f(z) = \varnothing$ or all vertices $z \in (V(G) \cup V(H)) \setminus \{x, y\}$. Clearly, f is a WCkRDF of $G + H$ and thus $\gamma_{rk}^{wc}(G + H) \leq 2k$. Now assume, without loss of generality, that $\gamma_{rk}^{wc}(G) = \min\{\gamma_{rk}^{wc}(G), \gamma_{rk}^{wc}(H)\}$ and let f be a γ_{rk}^{wc}-function of G. Since $m \geq k$, we may assume that $\cup_{x \in V(G)} f(x) = \{1, 2, \dots, k\}$. Since each vertex in $V(H)$ is adjacent to all vertices of $V(G)$ in $G + H$, f is a WCkRDF of $G + H$ and thus $\gamma_{rk}^{wc}(G + H) \leq \min\{\gamma_{rk}^{wc}(G), \gamma_{rk}^{wc}(H)\}$. This proves the upper bound. □

Open questions and problems:

We conclude this paper with some open problem and perspective related to our work.

Problem 1. For positive integer $k \geq 3$, characterize the graphs G of order n such that $\gamma_{rk}^{wc}(G) = n$.

Problem 2. For positive integer $k \geq 2$, characterize the graphs G of order n such that $\gamma_{rk}^{wc}(G) = k + \ell$ for some positive integer ℓ.

Problem 3. Determine Nordhaus-Gaddum type results for $\gamma_{rk}^{wc}(G)$.

Problem 4. Design an algorithm for computing the value of $\gamma_{rk}^{wc}(T)$ for any tree T and $k \geq 2$.

6. Conclusion

In this paper, we have introduced and investigated the weakly connected k-rainbow domination parameter in graphs. We established fundamental properties and derived bounds for the weakly connected k-rainbow domination number $\gamma_{rk}^{wc}(G)$. Moreover, we determined the exact values of $\gamma_{rk}^{wc}(G)$ for various graph classes, providing insights into their structural dependencies. Additionally, we examined the weakly connected k-rainbow domination behavior under the join operation of graphs.

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