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Eccentricity-Based Energies of Non-Commuting Graph for Dihedral Groups

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Abstract. Spectral graph theory is a research topic that combines algebra and graph theory, with the intersection representing a graph as a matrix. The eigenvalues of the matrix give the value of graph energy. This research focuses on the non-commuting graph for dihedral groups corresponding to eccentricity-based matrices including eccentricity, sum eccentricity, and average degree eccentricity matrices.

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1. Introduction

Spectral graph theory is a combined research topic between algebra and graph theory with the intersection being a matrix representation of a graph. Originally, the adjacency matrix was the first representation of a graph. The research has extended to the degree-based and distance-based matrices, and recently, eccentricity-based matrices have been developed. Wang, et al. [1] defined the eccentricity matrix and is inspired by the idea of Randic [2]. Later, Mahato [3] continued to discuss this type of matrix and presented the spectra perspectives in 2020. Meanwhile, the sum eccentricity was introduced by Sowaity

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and Sharada [4] and the average degree eccentricity matrix was pioneered by Mathad et al. [5].

The matrix of a graph is a square matrix whose size depends on the order of the graph. Therefore, we can calculate the eigenvalues of a matrix, which are hereinafter referred to as the eigenvalues of the corresponding graph. The sum of absolute eigenvalues is the energy of a graph defined by Gutman [6] in 1978. Moreover, the graph energy value has been discussed in [7] and [8].

The graph energy can further be associated with the graph defined on the group including the non-commuting graph. It is shown in [9] who discussed the Wiener-Hosoya energy, and for Sombor energy can be found in [10]. The algebraic discussion also can be found in [11, 12]. Therefore, this research aims to analyze the non-commuting graph energy associated with the eccentricity-based matrices and dihedral groups as its vertex set.

2. Preliminaries

In this section, we recall the fundamental definitions and theorems useful for our main results. We start with the definition of the non-commuting graph.

Definition 1. *[13] Let G be a finite group. The non-commuting graph of G is denoted by Ω_G, in which the vertex set is $G\backslash Z(G)$, where $Z(G)$ is the center of G, and two distinct vertices u and v are joined by an edge whenever $uv \neq vu$.*

Throughout this paper, we denote the non-commuting graph for dihedral groups of order $2n$, D_{2n}, as $\Omega_{D_{2n}}$, where $n \geq 3$. The vertex set and edge set of $\Omega_{D_{2n}}$ are denoted by $V(\Omega_{D_{2n}})$ and $L(\Omega_{D_{2n}})$, respectively. Vertex $x \in V(\Omega_{D_{2n}})$ is adjacent to $y \in V(\Omega_{D_{2n}})$ if and only if edge $xy \in L(\Omega_{D_{2n}})$. The distance between both vertices in $\Omega_{D_{2n}}$ is denoted by d_{xy} and the degree of x is denoted by $d(x)$. The eccentricity of x is given by $e(x) = max\{d_{xy}|y \in V(\Omega_{D_{2n}})\}$.

The construction of the graph matrices of $\Omega_{D_{2n}}$ is based on the definition of eccentricity, sum-eccentricity, and average degree-eccentricity matrices as presented below:

Definition 2. *[1] The eccentricity matrix of $\Omega_{D_{2n}}$ is $E(\Omega_{D_{2n}}) = [\epsilon_{ij}]$ in which (i,j)-th entry is*

$$\epsilon_{ij} = \begin{cases} d_{x_ix_j}, & \text{if } d_{x_ix_j} = min\{e(x_i), e(x_j)\} \\ 0, & \text{if } d_{x_ix_j} < min\{e(x_i), e(x_j)\}. \end{cases}$$

Definition 3. *[4] The sum eccentricity matrix of $\Omega_{D_{2n}}$ is $SE(\Omega_{D_{2n}}) = [s_{ij}]$ in which (i,j)-th entry is*

$$s_{ij} = \begin{cases} e(x_i) + e(x_j), & \text{if } x_ix_j \in L(\Omega_{D_{2n}}) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4. *[5] The average degree eccentricity matrix of $\Omega_{D_{2n}}$ is $ADE(\Omega_{D_{2n}}) = [a_{ij}]$ whose (i,j)-th entry is*

$$a_{ij} = \begin{cases} \frac{1}{4}(d(x_i) + d(x_j) + e(x_i) + e(x_j)), & \text{if } x_ix_j \in L(\Omega_{D_{2n}}) \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $E(\Omega_{D_{2n}})$ is defined by

$$P_{E(\Omega_{D_{2n}})}(\mu) = \left|\mu I_n - E(\Omega_{D_{2n}})\right|, \tag{1}$$

where I_n is an $n \times n$ identity matrix.

Furthermore, the eigenvalues of $\Omega_{D_{2n}}$ are the roots of $P_{E(\Omega_{D_{2n}})}(\mu) = 0$. The eccentricity energy definition is based on the eigenvalues of $\Omega_{D_{2n}}$ [6] as

$$\varepsilon_E(\Omega_{D_{2n}}) = \sum_{i=1}^{n} |\mu_i|.$$

The eccentricity spectral radius of $\Omega_{D_{2n}}$ [14] is

$$\rho_E(\Omega_{D_{2n}}) = max\{|\mu_i| : i = 1, 2, \ldots, n\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $E(\Omega_{D_{2n}})$. Similarly, one can apply the notation for SE and ADE-matrices in the same manner.

The energy value of $\Omega_{D_{2n}}$ is classified as hyperenergetic if the energy of $\Omega_{D_{2n}}$ is greater than $4(n-1)$ for odd n (or $2(2n-3)$ for even n)[15].

Let $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1}\rangle$. We denote $\Omega_1 = \{a^p : 1 \le p \le n\} \backslash Z(D_{2n})$ and $\Omega_2 = \{a^p b : 1 \le p \le n\}$, where $Z(D_{2n})$ is the center of D_{2n}.

We need the following results to determine the entries of the matrix of $\Omega_{D_{2n}}$.

Theorem 1. *[16] In $\Omega_{D_{2n}}$, the distance between x_i and x_j in $V(\Omega_{D_{2n}})$ is*

(i) for odd n, $d_{x_i x_j} = \begin{cases} 2, & \text{if } x_i, x_j \in \Omega_1 \\ 1, & \text{otherwise,} \end{cases}$ *, and*

(ii) for the even n, $d_{x_i x_j} = \begin{cases} 2, & \text{if } x_i, x_j \in \Omega_1 \\ 2, & x_i \in \Omega_2, x_j \in \left\{a^{\frac{n}{2}+i} b\right\} \\ 1, & \text{otherwise.} \end{cases}$*, for $i = 1, 2, \ldots, n$*

Theorem 2. *[17] In $\Omega_{D_{2n}}$,*

(i) the degree of a^i on $\Omega_{D_{2n}}$ is $d_{a^i} = n$, and

(ii) the degree of $a^i b$ on $\Omega_{D_{2n}}$ is $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

The eccentricity of every vertex in $\Omega_{D_{2n}}$ can be found in [17] as follows.

Theorem 3. *[17] In $\Omega_{D_{2n}}$, the eccentricity of $x \in V(\Omega_{D_{2n}})$ is*

(i) for odd n, $e(x) = \begin{cases} 2, & \text{if } x \in \Omega_1 \\ 1, & \text{if } x \in \Omega_2 \end{cases}$ *and*

(ii) for even n, $e(x) = 2$.

The following theorems simplify the process of formulating the characteristic formula.

Theorem 4. *[18] If*

$$T = \begin{bmatrix} a(J-I)_{n-2} & cJ_{(n-2)\times\frac{n}{2}} & cJ_{(n-2)\times\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} \end{bmatrix},$$

then for real numbers a, b, c, d, the characteristic polynomial of T is

$$\begin{aligned} P_T(\mu) = & (\mu+a)^{n-3}(\mu-b+2d)^{\frac{n}{2}-1}(\mu+b)^{\frac{n}{2}} \\ & \left(\mu^2-(b+(n-2)d+a(n-3))\mu+a(n-3)(b+(n-2)d)-n(n-2)c^2\right). \end{aligned}$$

Lemma 1. *[19] Let a, b, c, and d be real numbers. Then the determinant of*

$$\begin{vmatrix} (\mu+a)I_{n_1}-aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\mu+b)I_{n_2}-bJ_{n_2} \end{vmatrix}$$

can be simplified as

$$(\mu+a)^{n_1-1}(\mu+b)^{n_2-1}\left((\mu-(n_1-1)a)(\mu-(n_2-1)b)-n_1n_2cd\right),$$

where $1 \le n_1, n_2 \le n$ and $n_1+n_2 = n$.

3. Main Results

In this section, we find the eccentricity-based energies of $\Omega_{D_{2n}}$.

3.1. Eccentricity Energy

This part aims to determine the energy formula of $\Omega_{D_{2n}}$ associated with the eccentricity matrix.

Theorem 5. *In $\Omega_{D_{2n}}$, the eccentricity energy of $\Omega_{D_{2n}}$ is*

$$\varepsilon_E(\Omega_{D_{2n}}) = \begin{cases} 2(3n-5), & \text{if } n \text{ is odd} \\ 6(n-2), & \text{if } n \text{ is even} \end{cases}.$$

Proof.

(i) Let n be odd. According to Theorem 1 (i) and Definition 2, we can construct the eccentricity matrix of $\Omega_{D_{2n}}$. The matrix size is $(2n-1)\times(2n-1)$ excluding one center's element of D_{2n}. The entries of $E(\Omega_{D_{2n}}) = [\epsilon_{ij}]$ are

(a) for $1 \le i, j \le n-1$ and $i \ne j$, $\epsilon_{ij} = 2$ since $d_{x_ix_j} = min\{e(x_i), e(x_j)\} = 2$;

(b) for $1 \le i \le n-1$ and $j = n, n+1, \ldots, 2n-1$ or vice versa, $\epsilon_{ij} = 1$ since $d_{x_ix_j} = min\{e(x_i), e(x_j)\} = 1$;

(c) for $n \leq i, j \leq 2n-1$, $\epsilon_{ij} = 1$ since $d_{x_i x_j} = min\{e(x_i), e(x_j)\} = 1$;

(d) for $i = j$, $\epsilon_{ij} = 0$.

Then $E(\Omega_{D_{2n}})$ is as follows:

$$E(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 2 & \dots & 2 & 1 & 1 & \dots & 1 \\ 2 & 0 & \dots & 2 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 \end{pmatrix} \end{array},$$

and the characteristic formula of $E(\Omega_{D_{2n}})$,

$$P_{E(\Omega_{D_{2n}})}(\mu) = \begin{vmatrix} (\mu+2)I_{n-1} - 2J_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times(n-1)} & (\mu+1)I_n - J_n \end{vmatrix}.$$

By Lemma 1, with $a = 2$, $b = c = d = 1$, $n_1 = n-1$ and $n_2 = n$, we obtain

$$P_{E(\Omega_{D_{2n}})}(\mu) = (\mu+2)^{n-2}(\mu+1)^{n-1}\left(\mu^2 - (3n-5)\mu + (n-1)(n-4)\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = -2$ of multiplicity $n-2$, $\mu_2 = -1$ of multiplicity $n-1$, and $\mu_{3,4} = \frac{1}{2}\left(3n - 5 \pm \sqrt{5n^2 - 10n + 9}\right)$. The eccentricity spectral radius of $\Omega_{D_{2n}}$ is

$$\rho_E(\Omega_{D_{2n}}) = \frac{1}{2}\left(3n - 5 + \sqrt{5n^2 - 10n + 9}\right).$$

The eccentricity energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned} \varepsilon_E(\Omega_{D_{2n}}) =& (n-2)|-2| + (n-1)|-1| + \left|\frac{1}{2}\left(3n - 5 \pm \sqrt{5n^2 - 10n + 9}\right)\right| \\ =& 2(3n-5) \end{aligned}$$

(ii) Let n be even. Based on Theorem 1 (ii) and Definition 2, $E(\Omega_{D_{2n}}) = [\epsilon_{ij}]$ is $(2n-2) \times (2n-2)$ excluding two center's elements of D_{2n}. The entries of $E(\Omega_{D_{2n}})$ are

(a) for $i, j = 1, 2, \dots, n-2$ and $i \neq j$, $\epsilon_{ij} = 2$ since $d_{x_i x_j} = min\{e(x_i), e(x_j)\} = 2$;

(b) for $i = 1, 2, \dots, n-2$ and $j = n-1, n, n+1, \dots, 2n-2$ or vice versa, $\epsilon_{ij} = 0$ since $d_{x_i x_j} = 1 < min\{e(x_i), e(x_j)\} = 2$;

(c) for $i = n - 2 + p$ and $j = n - 2 + \frac{n}{2} + p$ or vice versa where $p = 1, 2, \ldots, \frac{n}{2}$, $\epsilon_{ij} = 2$ since $d_{x_i x_j} = min\{e(x_i), e(x_j)\} = 2$;

(d) for $i, j = n - 1, n, n + 1, \ldots, 2n - 2$, $\epsilon_{ij} = 1$ except ($i = n - 2 + p$ and $j = n - 2 + \frac{n}{2} + p$ for $p = 1, 2, \ldots, \frac{n}{2}$) or vice versa, and $i \neq j$, $\epsilon_{ij} = 0$ since $d_{x_i x_j} 1 < min\{e(x_i), e(x_j)\} = 2$;

(e) for $i = j$, $\epsilon_{ij} = 0$.

This implies that $E(\Omega_{D_{2n}})$ is

$$
\begin{array}{c|ccccccccccccc}
 & a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \ldots & a^{n-1}b \\
\hline
a & 0 & 2 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
a^2 & 2 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1} & 2 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
b & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 0 \\
ab & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{\frac{n}{2}-1}b & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 \\
a^{\frac{n}{2}}b & 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
a^{\frac{n}{2}+1}b & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1}b & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0
\end{array}.
$$

Based on Theorem 4 with $a = b = 2$, $c = d = 0$, then we have

$$P_{E(\Omega_{D_{2n}})}(\lambda) = (\lambda + 2)^{\frac{3(n-2)}{2}} (\lambda - 2)^{\frac{n}{2}-1} \left(\lambda^2 - 2(n-2)\lambda + 4(n-3)\right).$$

The roots of $P_{E(\Omega_{D_{2n}})}(\lambda)$ give the eigenvalues of $\Omega_{D_{2n}}$. Therefore, the eccentricity energy of $\Omega_{D_{2n}}$ is

$$\varepsilon_E(\Omega_{D_{2n}}) = \left(\frac{3(n-2)}{2}\right)|-2| + \left(\frac{n}{2} - 1\right)|2| + |n - 2 \pm (n-4)| = 6(n-2).$$

3.2. Sum Eccentricity Energy

This part focuses on $\Omega_{D_{2n}}$'s sum eccentricity matrix for odd and even n.

Theorem 6. *In $\Omega_{D_{2n}}$, the sum eccentricity spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_{SE}(\Omega_{D_{2n}}) = \begin{cases} n - 1 + \sqrt{(n-1)(10n-1)}, & \text{if } n \text{ is odd} \\ 2\left(n - 2 + \sqrt{(n-2)(5n-2)}\right), & \text{if } n \text{ is even} \end{cases}.$$

Proof.

(i) Let n be odd. According to Theorem 3 and Definition 3, we can construct the sum eccentricity matrix of $\Omega_{D_{2n}}$. The entries of $SE(\Omega_{D_{2n}}) = [s_{ij}]$ are

(a) for $1 \leq i, j \leq n-1$ and $i \neq j$, then $s_{ij} = 0$;

(b) for $1 \leq i \leq n-1$ and $j = n, n+1, \ldots, 2n-1$ or vice versa and $i \neq j$, then $s_{ij} = 2+1 = 3$ since $d_{x_i x_j} = min\{e(x_i), e(x_j)\} = 1$;

(c) for $i, j = n, n+1, \ldots, 2n-1$ and $i \neq j$, then $s_{ij} = 1+1 = 2$;

(d) for $i = j$, $s_{ij} = 0$

Then $SE(\Omega_{D_{2n}})$ is as follows:

$$SE(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 & 3 & 3 & \ldots & 3 \\ 0 & 0 & \ldots & 0 & 3 & 3 & \ldots & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 3 & 3 & \ldots & 3 \\ 3 & 3 & \ldots & 3 & 0 & 2 & \ldots & 2 \\ 3 & 3 & \ldots & 3 & 2 & 0 & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 3 & \ldots & 3 & 2 & 2 & \ldots & 0 \end{pmatrix} \end{array}.$$

SE-matrix of $\Omega_{D_{2n}}$ can be written as given below

$$SE(\Omega_{D_{2n}}) = \begin{pmatrix} 0_{n-1} & 3J_{(n-1)\times n} \\ 3J_{n\times(n-1)} & 2(J-I)_n \end{pmatrix},$$

and the characteristic formula of $SE(\Omega_{D_{2n}})$,

$$P_{SE(\Omega_{D_{2n}})}(\mu) = \begin{vmatrix} \mu I_{n-1} & -3J_{(n-1)\times n} \\ -3J_{n\times(n-1)} & (\mu+2)I_n - 2J_n \end{vmatrix}.$$

By Lemma 1, with $a = 0$, $b = 2$, $c = d = 3$, $n_1 = n-1$ and $n_2 = n$, we obtain

$$P_{SE(\Omega_{D_{2n}})}(\mu) = \mu^{n-2}(\mu+2)^{n-1}\left(\mu^2 - 2(n-1)\mu - 9n(n-1)\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = 0$ of multiplicity $n-2$, $\mu_2 = -2$ of multiplicity $n-1$, and $\mu_{3,4} = n-1 \pm \sqrt{(n-1)(10n-1)}$. Therefore, the SE-spectral radius of $\Omega_{D_{2n}}$ is

$$\rho_{SE}(\Omega_{D_{2n}}) = n-1+\sqrt{(n-1)(10n-1)}.$$

(ii) Let n be even. The entries of $SE(\Omega_{D_{2n}}) = [s_{ij}]$ are

(a) for $1 \leq i, j \leq n-2$ and $i \neq i$, $s_{ij} = 0$;

(b) for $1 \leq i \leq n-2$, $n-1 \leq j \leq 2n-2$ or vice versa, $s_{ij} = 2+2 = 4$;

(c) for $n-1 \leq i \leq n+\frac{n}{2}-2$ and $n+\frac{n}{2}-1 \leq j \leq 2n-2$ where $j \neq n-2+\frac{n}{2}+i$ or vice versa, $s_{ij} = 2+2 = 4$;

(d) for $n-1 \leq i,j \leq n+\frac{n}{2}-2$, $n+\frac{n}{2}-2 \leq i,j \leq 2n-2$, and $i \neq j$, $s_{ij} = 2+2 = 4$;

(e) for $i = j$, $j = n-2+\frac{n}{2}+i$, $i = n-2+\frac{n}{2}+j$, $s_{ij} = 0$.

Hence, the matrix construction is as follows.

$$SE(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & \dots & a^{n-1} & b & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & 4 & \dots & 4 & 4 & \dots & 4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 4 & \dots & 4 & 4 & \dots & 4 \\ 4 & \dots & 4 & 0 & \dots & 4 & 0 & \dots & 4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 4 & \dots & 4 & 4 & \dots & 0 & 4 & \dots & 0 \\ 4 & \dots & 4 & 0 & \dots & 4 & 0 & \dots & 4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 4 & \dots & 4 & 4 & \dots & 0 & 4 & \dots & 0 \end{pmatrix} \end{array}.$$

In other words, $SE(\Omega_{D_{2n}})$ is as follows:

$$SE(\Omega_{D_{2n}}) = \begin{pmatrix} 0_{n-2} & 4J_{(n-2)\times\frac{n}{2}} & 4J_{(n-2)\times\frac{n}{2}} \\ 4J_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \\ 4J_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \end{pmatrix}.$$

Based on Theorem 4 with $a = b = 0$, $c = d = 4$, then

$$P_{SE(\Omega_{D_{2n}})}(\mu) = \mu^{\frac{3(n-2)}{2}} (\mu+8)^{\frac{n}{2}-1} \left(\mu^2 - 4(n-2)\mu - 16n(n-2)\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = 0$ of multiplicity $\frac{3(n-2)}{2}$, $\mu_2 = -8$ of multiplicity $\frac{n}{2}-1$, and $\mu_{3,4} = 2\left(n-2 \pm \sqrt{(n-2)(5n-2)}\right)$. Therefore, the SE-spectral radius of $\Omega_{D_{2n}}$ is

$$\rho_{SE}(\Omega_{D_{2n}}) = 2\left(n-2+\sqrt{(n-2)(5n-2)}\right).$$

Theorem 7. *In $\Omega_{D_{2n}}$, the sum eccentricity energy of $\Omega_{D_{2n}}$ is*

$$\varepsilon_{SE}(\Omega_{D_{2n}}) = \begin{cases} 2\left(n-1+\sqrt{(n-1)(10n-1)}\right), & \text{if } n \text{ is odd} \\ 4\left(n-2+\sqrt{(n-2)(5n-2)}\right), & \text{if } n \text{ is even} \end{cases}.$$

Proof.

(i) Let n be odd. According to Theorem 6, SE-energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned}\varepsilon_{SE}(\Omega_{D_{2n}}) &= (n-2)\,|0| + (n-1)\,|-2| + \left|n-1 \pm \sqrt{(n-1)(10n-1)}\right| \\ &= 2\left(n-1+\sqrt{(n-1)(10n-1)}\right).\end{aligned}$$

(ii) Let n be even. According to Theorem 6, the SE-energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned}\varepsilon_{SE}(\Omega_{D_{2n}}) &= \left(\frac{3(n-2)}{2}\right)|0| + \left(\frac{n}{2}-1\right)|-8| + \left|2(n-2) \pm 2\sqrt{(n-2)(5n-2)}\right| \\ &= 4\left(n-2+\sqrt{(n-2)(5n-2)}\right).\end{aligned}$$

3.3. Average Degree-Eccentricity Matrix

Next, we show the energy of $\Omega_{D_{2n}}$ concerning the average degree-eccentricity matrix for odd and even n.

Theorem 8. *In $\Omega_{D_{2n}}$, the average degree-eccentricity spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_{ADE}(\Omega_{D_{2n}}) = \begin{cases} \frac{1}{2}\left((n-1)\left(n-\frac{1}{2}\right) + \sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2 + \frac{1}{4}n(n-1)(3n+1)^2}\right), & \text{if } n \text{ is odd} \\ \frac{1}{2}\left((n-2)(n-1) + \sqrt{(n-2)^2(n-1)^2 + \frac{9}{4}n^3(n-2)}\right), & \text{if } n \text{ is even.} \end{cases}$$

Proof.

(i) Let n be odd. Based on Theorems 2 and 3, and Definition 4, we have the average degree-eccentricity matrix with entries of $[a_{ij}]$ are

(a) for $i, j = 1, 2, \ldots, n-1$ and $i \neq j$, then $a_{ij} = 0$;

(b) for $i = 1, 2, \ldots, n-1$ and $j = n, n+1, \ldots, 2n-1$ or vice versa and $i \neq j$, then $a_{ij} = \frac{1}{4}(n + 2(n-1) + 2 + 1) = \frac{3n+1}{4}$;

(c) for $i, j = n, n+1, \ldots, 2n-1$ and $i \neq j$, then $a_{ij} = \frac{1}{4}(2(n-1) + 2(n-1) + 1 + 1) = n - \frac{1}{2}$;

(d) for $i = j$, $a_{ij} = 0$

Then $ADE(\Omega_{D_{2n}})$ is as follows:

$$ADE(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} \\ 0 & 0 & \dots & 0 & \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} \\ \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} & 0 & n-\frac{1}{2} & \dots & n-\frac{1}{2} \\ \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} & n-\frac{1}{2} & 0 & \dots & n-\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{3n+1}{4} & \frac{3n+1}{4} & \dots & \frac{3n+1}{4} & n-\frac{1}{2} & n-\frac{1}{2} & \dots & 0 \end{pmatrix} \end{array}.$$

ADE-matrix of $\Omega_{D_{2n}}$ can be written as given below

$$ADE(\Omega_{D_{2n}}) = \begin{pmatrix} 0_{n-1} & \left(\frac{3n+1}{4}\right) J_{(n-1)\times n} \\ \left(\frac{3n+1}{4}\right) J_{n\times(n-1)} & \left(n-\frac{1}{2}\right)(J-I)_n \end{pmatrix},$$

and the characteristic formula of $ADE(\Omega_{D_{2n}})$,

$$P_{ADE(\Omega_{D_{2n}})}(\mu) = \begin{vmatrix} \mu I_{n-1} & -\left(\frac{3n+1}{4}\right) J_{(n-1)\times n} \\ -\left(\frac{3n+1}{4}\right) J_{n\times(n-1)} & \left(\mu+n-\frac{1}{2}\right) I_n - \left(n-\frac{1}{2}\right) J_n \end{vmatrix}.$$

By Lemma 1, with $a=0$, $b=n-\frac{1}{2}$, $c=d=\frac{3n+1}{4}$, $n_1=n-1$ and $n_2=n$, we obtain

$$P_{ADE(\Omega_{D_{2n}})}(\mu) = \mu^{n-2}\left(\mu+n-\frac{1}{2}\right)^{n-1}\left(\mu^2-(n-1)\left(n-\frac{1}{2}\right)\mu-\frac{1}{16}n(n-1)(3n+1)^2\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1=0$ of multiplicity $n-2$, $\mu_2=\frac{1}{2}-n$ of multiplicity $n-1$, and $\mu_{3,4}=\frac{1}{2}\left((n-1)\left(n-\frac{1}{2}\right)\pm\sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2+\frac{1}{4}n(n-1)(3n+1)^2}\right)$. Therefore, the ADE-spectral radius of $\Omega_{D_{2n}}$ is

$$\rho_{ADE}(\Omega_{D_{2n}}) = \frac{1}{2}\left((n-1)\left(n-\frac{1}{2}\right)+\sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2+\frac{1}{4}n(n-1)(3n+1)^2}\right).$$

(ii) Let n be even. The entries of $ADE(\Omega_{D_{2n}})=[a_{ij}]$ are

(a) for $1\le i,j\le n-2$ and $i\ne i$, $a_{ij}=0$;

(b) for $1\le i\le n-2$, $n-1\le j\le 2n-2$ or vice versa, $a_{ij}=\frac{1}{4}\left(n+2(n-2)+2+2\right)=\frac{3n}{4}$;

(c) for $n-1\le i\le n+\frac{n}{2}-2$ and $n+\frac{n}{2}-1\le j\le 2n-2$ where $j\ne n-2+\frac{n}{2}+i$ or vice versa, $a_{ij}=\frac{1}{4}\left(2(n-2)+2(n-2)+2+2\right)=n-1$;

(d) for $n-1 \leq i,j \leq n+\frac{n}{2}-2$, $n+\frac{n}{2}-2 \leq i,j \leq 2n-2$, and $i \neq j$, $a_{ij} = \frac{1}{4}\left(2(n-2)+2(n-2)+2+2\right) = n-1$;

(e) for $i=j$, $j = n-2+\frac{n}{2}+i$, $i = n-2+\frac{n}{2}+j$, $a_{ij}=0$.

Thus,

$$ADE(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & \dots & a^{n-1} & b & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & \frac{3n}{4} & \dots & \frac{3n}{4} & \frac{3n}{4} & \dots & \frac{3n}{4} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{3n}{4} & \dots & \frac{3n}{4} & \frac{3n}{4} & \dots & \frac{3n}{4} \\ \frac{3n}{4} & \dots & \frac{3n}{4} & 0 & \dots & n-1 & 0 & \dots & n-1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{3n}{4} & \dots & \frac{3n}{4} & n-1 & \dots & 0 & n-1 & \dots & 0 \\ \frac{3n}{4} & \dots & \frac{3n}{4} & 0 & \dots & n-1 & 0 & \dots & n-1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{3n}{4} & \dots & \frac{3n}{4} & n-1 & \dots & 0 & n-1 & \dots & 0 \end{pmatrix} \end{array}.$$

In other words, $ADE(\Omega_{D_{2n}})$ is as follows:

$$ADE(\Omega_{D_{2n}}) = \begin{pmatrix} 0_{n-2} & \frac{3n}{4}J_{(n-2)\times\frac{n}{2}} & \frac{3n}{4}J_{(n-2)\times\frac{n}{2}} \\ \frac{3n}{4}J_{\frac{n}{2}\times(n-2)} & (n-1)(J-I)_{\frac{n}{2}} & (n-1)(J-I)_{\frac{n}{2}} \\ \frac{3n}{4}J_{\frac{n}{2}\times(n-2)} & (n-1)(J-I)_{\frac{n}{2}} & (n-1)(J-I)_{\frac{n}{2}} \end{pmatrix}.$$

Based on Theorem 4 with $a=b=0$, $c=\frac{3n}{4}$, $d=n-1$, then

$$P_{ADE(\Omega_{D_{2n}})}(\mu) = \mu^{\frac{3(n-2)}{2}}\left(\mu+2(n-1)\right)^{\frac{n}{2}-1}\left(\mu^2-(n-1)(n-2)\mu-\frac{9n^2}{16}n(n-2)\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = 0$ of multiplicity $\frac{3(n-2)}{2}$, $\mu_2 = -2(n-1)$ of multiplicity $\frac{n}{2}-1$, and
$\mu_{3,4} = \frac{1}{2}\left((n-2)(n-1) \pm \sqrt{(n-2)^2(n-1)^2+\frac{9}{4}n^3(n-2)}\right)$. Therefore, the *ADE*-spectral radius of $\Omega_{D_{2n}}$ is

$$\rho_{ADE}(\Omega_{D_{2n}}) = \frac{1}{2}\left((n-2)(n-1)+\sqrt{(n-2)^2(n-1)^2+\frac{9}{4}n^3(n-2)}\right).$$

Theorem 9. *In $\Omega_{D_{2n}}$, the average degree-eccentricity energy of $\Omega_{D_{2n}}$ is*

$$\varepsilon_{ADE}(\Omega_{D_{2n}}) = \begin{cases} (n-1)\left(n-\frac{1}{2}\right)+\sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2+\frac{1}{4}n(n-1)(3n+1)^2}, & \text{if } n \text{ is odd} \\ (n-2)(n-1)+\sqrt{(n-2)^2(n-1)^2+\frac{9}{4}n^3(n-2)}, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

(i) Let n be odd. Based on Theorems 8, the ADE-energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned}\varepsilon_{ADE}(\Omega_{D_{2n}}) &= (n-2)\left|0\right| + (n-1)\left|\frac{1}{2}-n\right| + \\ &\left|\frac{1}{2}\left((n-1)\left(n-\frac{1}{2}\right) \pm \sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2 + \frac{1}{4}n(n-1)(3n+1)^2}\right)\right| \\ &= (n-1)\left(n-\frac{1}{2}\right) + \sqrt{(n-1)^2\left(n-\frac{1}{2}\right)^2 + \frac{1}{4}n(n-1)(3n+1)^2}.\end{aligned}$$

(ii) Let n be even. Based on Theorem 8, the ADE-energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned}\varepsilon_{ADE}(\Omega_{D_{2n}}) &= \left(\frac{3(n-2)}{2}\right)\left|0\right| + \left(\frac{n}{2}-1\right)\left|-2(n-1)\right| + \\ &\left|\frac{1}{2}\left((n-2)(n-1) \pm \sqrt{(n-2)^2(n-1)^2 + \frac{9}{4}n^3(n-2)}\right)\right| \\ &= (n-2)(n-1) + \sqrt{(n-2)^2(n-1)^2 + \frac{9}{4}n^3(n-2)}.\end{aligned}$$

4. Discussion

We can conclude several interesting statements from the results of the previous section.

Corollary 1. *The eccentricity energy of* $\Omega_{D_{2n}}$ *is always an even integer.*

Corollary 2. *The energy of* $\Omega_{D_{2n}}$ *is never an odd integer associated with the sum eccentricity and average degree eccentricity matrices.*

Corollary 3. $\Omega_{D_{2n}}$ *is hyperenergetic associated with the eccentricity-based matrices.*

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