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On Length and Mean Fuzzy Ideals of Sheffer Stroke Hilbert Algebras

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Abstract. This paper presents a detailed exploration of Sheffer stroke Hilbert algebras, introducing the innovative concepts of length fuzzy ideals and mean fuzzy ideals within an interval-valued fuzzy framework. These new constructs extend classical ideal theory by incorporating fuzzy logic, providing precise mathematical tools to analyze and measure membership gradations. Specifically, the study establishes critical relationships between length fuzzy ideals and mean fuzzy ideals, their hierarchical subsets, and their implications for algebraic consistency and computational logic. Key findings demonstrate that length fuzzy ideals align closely with interval-valued fuzzy subsets, while mean fuzzy ideals offer a unique averaging perspective for understanding ideal structures. These contributions significantly advance the field of fuzzy algebra, offering theoretical insights and potential applications in computational logic, uncertainty modeling, and algorithmic design.

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1. Introduction

Hilbert algebras, often referred to as implicative algebras, are algebraic structures that extend the classical operations of logic. These algebras are typically defined by a set of axioms involving a binary operation, the Sheffer stroke, which is a generalization of the NAND operation in propositional logic. The study of Hilbert algebras is integral to understanding non-classical logics, modal logics, and lattice theory, offering essential insights into the foundational structure of logical systems [3].

The Sheffer stroke is a fundamental element in both classical and non-classical logic due to its property as a functionally complete operation [13]. This means it can operate by

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itself without requiring any other logical operators to form a comprehensive logical system. In simpler terms, every logical axiom can be restated using just the Sheffer stroke. This capability greatly simplifies the manipulation and control of various properties within the logical system it creates. Moreover, it is noteworthy that the axioms of Boolean algebra, which correspond to classical propositional logic, can be entirely represented using the Sheffer stroke. This highlights the Sheffer stroke's foundational importance and its versatility within both logical and algebraic systems.

The Sheffer stroke has been utilized in various algebraic structures, such as Boolean algebras, basic algebras, MV-algebras, BCK-algebras, MTL-algebras and ortholattices, among others, and is also explored within fuzzy contexts (see [1, 4–7, 9–12]). In 2021, Oner et al. [6] extended the Sheffer stroke to Hilbert algebras, defining the Sheffer stroke Hilbert algebra and studying its various properties. In [5], they introduced the concepts of a deductive system and filter for Sheffer stroke Hilbert algebras and explored their fuzzification. Additionally, Oner et al. [6] presented the idea of an ideal in Sheffer stroke Hilbert algebras and analyzed its properties.

The field of fuzzy logic, introduced by [15], broadens classical logic by incorporating truth values that range continuously between 0 and 1, rather than being restricted to binary true/false values. This flexibility makes fuzzy logic particularly useful in scenarios involving uncertainty and imprecision. Integrating fuzzy logic with Hilbert algebras results in the concept of fuzzy ideals, where the elements of an ideal can have varying degrees of membership rather than being limited to crisp values. This extension offers a more refined approach to analyzing the algebraic properties of Hilbert algebras [2].

A recent innovation in the theory of fuzzy ideals is the introduction of length-fuzzy ideals. This concept enhances the classical definition of an ideal in Sheffer stroke Hilbert algebras by associating a fuzzy function that measures the "length" or degree of membership of elements within an ideal. This new perspective provides a more nuanced understanding of the structure and behavior of these algebras, enriching the classical theory with elements of fuzzy logic [8]. The application of length-fuzzy ideals allows for a more refined analysis of ideals with fuzzy characteristics, enabling better modeling of systems with inherent uncertainty or imprecision. By using fuzzy functions to measure membership degrees, this approach is applicable in decision-making processes under ambiguity, the design of algorithms for complex computations, and the study of structures in systems with incomplete or vague data. Integrating fuzzy logic into classical theory not only deepens its theoretical base but also extends its applicability to fields such as computer science, engineering, and areas involving uncertain or imprecise information processing.

This paper examines the properties and characteristics of length-fuzzy ideals in Sheffer stroke Hilbert algebras. By investigating these properties, the goal is to provide fresh perspectives on the theoretical foundation of Hilbert algebras and their potential applications in fields such as logic, computer science, and beyond. The concepts of length fuzzy ideals and mean fuzzy ideals are introduced in the context of Sheffer stroke Hilbert algebras, and their properties are analyzed. The paper further explores the relationships between length fuzzy ideals (and mean fuzzy ideals) and traditional ideals. Additionally, it discusses how length fuzzy ideals (and mean fuzzy ideals) are related to upper and lower level subsets based on the length (or mean) of a fuzzy structure within Sheffer stroke Hilbert algebras.

2. Preliminaries

Sheffer stroke Hilbert algebras represent an important algebraic system in the study of logic and lattice theory. These algebras are characterized by the inclusion of the Sheffer stroke (NAND) operation, a fundamental logical connectives in Boolean algebra. By extending classical Hilbert algebras with this operation, Sheffer stroke Hilbert algebras provide a powerful framework for investigating logical structures, with applications in fuzzy logic, decision-making, and computational theory. Their study enhances both the theoretical understanding of algebraic systems and their practical applications in modeling uncertainty and imprecision.

Definition 1. *[13] The operation $|$ in a groupoid $A = (A, |)$ is referred to as the Sheffer stroke or Sheffer operation if it satisfies the following condition: for all $\mathfrak{c}, \mathfrak{b}, \mathfrak{d} \in A$,*

$$(S1)\ \mathfrak{c}|\mathfrak{b} = \mathfrak{b}|\mathfrak{c},$$
$$(S2)\ (\mathfrak{c}|\mathfrak{c})|(\mathfrak{c}|\mathfrak{b}) = \mathfrak{b},$$
$$(S3)\ \mathfrak{c}|((\mathfrak{b}|\mathfrak{d})|(\mathfrak{b}|\mathfrak{d})) = ((\mathfrak{c}|\mathfrak{b})|(\mathfrak{c}|\mathfrak{b}))|\mathfrak{b},$$
$$(S4)\ (\mathfrak{c}|((\mathfrak{c}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{c}|((\mathfrak{c}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{c}.$$

To improve the clarity of this manuscript on Sheffer stroke Hilbert algebras, we will adopt the following notation throughout:

$$\mathfrak{p}|(\mathfrak{q}|\mathfrak{q}) = \mathfrak{p}^{\mathfrak{q}}.$$

Definition 2. *[6] A Sheffer stroke Hilbert algebra (abbreviated SHA) refers to a groupoid $A = (A, |, 0)$ equipped with a Sheffer stroke operation $|$ and 0 is the fixed element in A, and it must satisfy the following conditions: for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in A$,*

(1) $(\mathfrak{p}|(\mathfrak{q}^{\mathfrak{r}}|\mathfrak{p}^{\mathfrak{q}}))|((\mathfrak{p}^{\mathfrak{q}})^{(\mathfrak{p}^{\mathfrak{r}})}|(\mathfrak{p}^{\mathfrak{q}})^{(\mathfrak{p}^{\mathfrak{r}})}) = \mathfrak{p}^{\mathfrak{p}}$,

(2) $\mathfrak{p}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}} \Rightarrow \mathfrak{p} = \mathfrak{q}$.

Proposition 1. *[6] Let $A = (A, |, 0)$ be an SHA. Then the binary relation $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p}^{\mathfrak{q}} = 1$ is a partial order on A.*

Definition 3. *[6] Let $A = (A, |, 0)$ be an SHA. A nonempty subset G of A is called an ideal of A if for all $\mathfrak{p}, \mathfrak{q} \in A$,*

(1) $0 \in G$,

(2) $\mathfrak{p}^{\mathfrak{q}} \in G$ *and* $\mathfrak{q} \in G \Rightarrow \mathfrak{p} \in G$.

3. Length fuzzy ideals of Sheffer stroke Hilbert algebras

This paper introduces the concept of length fuzzy ideals in Sheffer stroke Hilbert algebras and examines their associated properties. It establishes the connections between length fuzzy ideals and conventional ideals. Furthermore, it explores the relationships between length fuzzy ideals and the upper and lower level subsets of the length in an interval-valued fuzzy structure within Sheffer stroke Hilbert algebras.

From now on, unless stated otherwise, we denote an SHA by $A = (A, |, 0)$.

Definition 4. *A fuzzy structure (A, f) of A is defined as:*

(1) *a fuzzy ideal of A of type 1 (simply a 1-fuzzy ideal of A) if*

$$(\forall \mathfrak{p} \in A)(f(0) \geq f(\mathfrak{p})), \tag{1}$$
$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(f(\mathfrak{p}) \geq \min\{f(\mathfrak{p}^{\mathfrak{q}}), f(\mathfrak{q})\}). \tag{2}$$

(2) *a fuzzy ideal of A of type 2 (simply a 2-fuzzy ideal of A) if*

$$(\forall \mathfrak{p} \in A)(f(0) \leq f(\mathfrak{p})), \tag{3}$$
$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(f(\mathfrak{p}) \leq \min\{f(\mathfrak{p}^{\mathfrak{q}}), f(\mathfrak{q})\}). \tag{4}$$

(3) *a fuzzy ideal of A of type 3 (simply a 3-fuzzy ideal of A) if*

$$(\forall \mathfrak{p} \in A)(f(0) \geq f(\mathfrak{p})), \tag{5}$$
$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(f(\mathfrak{p}) \geq \max\{f(\mathfrak{p}^{\mathfrak{q}}), f(\mathfrak{q})\}). \tag{6}$$

(4) *a fuzzy ideal of A of type 4 (simply a 4-fuzzy ideal of A) if*

$$(\forall \mathfrak{p} \in A)(f(0) \leq f(\mathfrak{p})), \tag{7}$$
$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(f(\mathfrak{p}) \leq \max\{f(\mathfrak{p}^{\mathfrak{q}}), f(\mathfrak{q})\}). \tag{8}$$

Definition 5. *[14] Given an interval-valued fuzzy structure $(A, \widetilde{f})$ over A, we define a fuzzy structure $(A, \widetilde{f}_l)$ on A as follows:*

$$\widetilde{f}_l : A \to [0, 1]; \mathfrak{p} \mapsto \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}),$$

which is referred to as the length of $\widetilde{f}$.

Definition 6. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is referred to as a length 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) ideal of A if the fuzzy structure $(A, \widetilde{f}_l)$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) ideal of A.*

Proposition 2. *Given an interval-valued fuzzy structure $(A, \widetilde{f})$ on A, the following statements hold.*

(1) *If* $(A, \widetilde{f})$ *is a length* k*-fuzzy ideal of* A *for* $k \in \{1, 3\}$*, then*

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\mathfrak{p} \leq \mathfrak{p} \Rightarrow \widetilde{f}_l(\mathfrak{p}) \geq \widetilde{f}_l(\mathfrak{p})).$$

(2) *If* $(A, \widetilde{f})$ *is a length* k*-fuzzy ideal of* A *for* $k \in \{2, 4\}$*, then*

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\mathfrak{p} \leq \mathfrak{q} \Rightarrow \widetilde{f}_l(\mathfrak{p}) \leq \widetilde{f}_l(\mathfrak{q})).$$

Proof. Let $\mathfrak{p}, \mathfrak{q} \in A$ be such that $\mathfrak{p} \leq \mathfrak{q}$. If $(A, \widetilde{f})$ is a length k-fuzzy ideal of A for $k \in \{1, 3\}$, then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\geq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_l(0), \widetilde{f}_l(\mathfrak{q})\} \\ &= \widetilde{f}_l(\mathfrak{q}) \end{aligned}$$

and

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\leq \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_l(0), \widetilde{f}_l(\mathfrak{q})\} \\ &= \widetilde{f}_l(\mathfrak{q}). \end{aligned}$$

If $(A, \widetilde{f})$ is a length k-fuzzy ideal of A for $k \in \{2, 4\}$, then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\geq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_l(0), \widetilde{f}_l(\mathfrak{q})\} \\ &= \widetilde{f}_l(\mathfrak{q}) \end{aligned}$$

and

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\leq \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_l(0), \widetilde{f}_l(\mathfrak{q})\} \\ &= \widetilde{f}_l(\mathfrak{q}). \end{aligned}$$

Theorem 1. *For any interval-valued fuzzy structure* $(A, \widetilde{f})$ *on* A*, the following assertions are true:*

(1) *Every length* 3*-fuzzy ideal of* A *is also a length* 1*-fuzzy ideal of* A*.*

(2) *Every length* 2*-fuzzy ideal of* A *is also a length* 4*-fuzzy ideal of* A*.*

Proof. (1) Let $(A, \widetilde{f})$ be a length 3-fuzzy ideal of A and $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\geq \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &\geq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.

(2) Let $(A, \widetilde{f})$ be a length 2-fuzzy ideal of A and $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &\leq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \\ &\leq \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.

Theorem 2. *Given an ideal S of A and $B_1, B_2 \in P([0,1])$, let $(A, \widetilde{f})$ be an interval-valued fuzzy structure over A given by*

$$\widetilde{f} : A \to P([0,1]); \mathfrak{p} \mapsto \begin{cases} B_2 & \textit{if } \mathfrak{p} \in S, \\ B_1 & \textit{otherwise.} \end{cases}$$

(1) *If $B_1 \subset B_2$, then $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.*

(2) *If $B_2 \subset B_1$, then $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.*

Proof. If $\mathfrak{p} \in S$, then $\widetilde{f}(\mathfrak{p}) = B_2$ and so

$$\widetilde{f}_l(\mathfrak{p}) = \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) = \sup \widetilde{f}(\mathfrak{p}) - \inf \widetilde{f}(\mathfrak{p}) = \sup B_2 - \inf B_2.$$

If $\mathfrak{p} \notin S$, then $\widetilde{f}(\mathfrak{p}) = B_1$ and so

$$\widetilde{f}_l(\mathfrak{p}) = \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) = \sup \widetilde{f}(\mathfrak{p}) - \inf \widetilde{f}(\mathfrak{p}) = \sup B_1 - \inf B_1.$$

(1) Assume that $B_1 \subset B_2$. Then $\sup B_2 - \inf B_2 \geq \sup B_1 - \inf B_1$. Since $0 \in I$, $\widetilde{f}_l(0) = \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) = \sup B_2 - \inf B_2 \geq \widetilde{f}_l(\mathfrak{p})$ for all $\mathfrak{p} \in A$.

Case 1: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \in S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_2 - \inf B_2$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_2 - \inf B_2$. Thus, $\min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_2 - \inf B_2$. Since S is an ideal of A, $\mathfrak{p} \in S$ and so $\widetilde{f}_l(\mathfrak{p}) = \sup B_2 - \inf B_2$. Thus, $\widetilde{f}_l(\mathfrak{p}) = \sup B_2 - \inf B_2 = \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 2: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \notin S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_1 - \inf B_1$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_1 - \inf B_1$, so $\min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 3: Let $\mathfrak{p}^{\mathfrak{q}} \notin S$ and $\mathfrak{q} \in S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_1 - \inf B_1$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_2 - \inf B_2$, so $\min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 4: Let $\mathfrak{p}^{\mathfrak{q}} \in S$ and $\mathfrak{q} \notin S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_2 - \inf B_2$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_1 - \inf B_1$, so $\min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Hence, $\widetilde{f}_l$ is a 1-fuzzy ideal of A and so $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.

(2) Assume that $B_2 \subset B_1$. Then $\sup B_2 - \inf B_2 \leq \sup B_1 - \inf B_1$. Since $0 \in I$, $\widetilde{f}_l(0) = \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) = \sup B_2 - \inf B_2 \leq \widetilde{f}_l(\mathfrak{p})$ for all $\mathfrak{p} \in A$.

Case 1: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \in S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_2 - \inf B_2$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_2 - \inf B_2$. Thus, $\max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_2 - \inf B_2$. Since S is an ideal of A, $x \in S$ and so $\widetilde{f}_l(\mathfrak{p}) = \sup B_2 - \inf B_2$. Thus, $\widetilde{f}_l(\mathfrak{p}) = \sup B_2 - \inf B_2 = \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 2: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \notin S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_1 - \inf B_1$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_1 - \inf B_1$, so $\max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 3: Let $\mathfrak{p}^{\mathfrak{q}} \notin S$ and $\mathfrak{q} \in S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}})) = \sup B_1 - \inf B_1$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_2 - \inf B_2$, so $\max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Case 4: Let $\mathfrak{p}^{\mathfrak{q}} \in S$ and $\mathfrak{q} \notin S$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) = \sup B_2 - \inf B_2$ and $\widetilde{f}_l(\mathfrak{q}) = \sup B_1 - \inf B_1$, so $\max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} = \sup B_1 - \inf B_1$. Thus, $\widetilde{f}_l(\mathfrak{p}) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$.

Hence, $\widetilde{f}_l$ is a 4-fuzzy ideal of A and so $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.

Definition 7. *Let (A, f) be a fuzzy structure in A. For any $\mathfrak{t} \in [0, 1]$, the sets*

$$U(f, \mathfrak{t}) = \{\mathfrak{p} \in A : f(\mathfrak{p}) \geq \mathfrak{t}\},$$

$$L(f, \mathfrak{t}) = \{\mathfrak{p} \in A : f(\mathfrak{p}) \leq \mathfrak{t}\},$$

are called upper $\mathfrak{t}$-level subset and lower $\mathfrak{t}$-level subset of f, respectively.

Theorem 3. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 1-fuzzy ideal of A if and only if the set $U(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$.*

Proof. Assume that an interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 1-fuzzy ideal of A and let $\mathfrak{t} \in [0, 1]$ be such that $U(\widetilde{f}, \mathfrak{t})$ is nonempty. Obviously, $0 \in U(\widetilde{f}, \mathfrak{t})$. Let $\mathfrak{p}, \mathfrak{q} \in A$ be such that $\mathfrak{p}^{\mathfrak{q}} \in U(\widetilde{f}, \mathfrak{t})$ and $\mathfrak{q} \in U(\widetilde{f}, \mathfrak{t})$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) \geq \mathfrak{t}$ and $\widetilde{f}_l(\mathfrak{q}) \geq \mathfrak{t}$, which imply from (2) that $\widetilde{f}_l(\mathfrak{p}) \geq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \geq t$. Hence, $\mathfrak{p} \in U(\widetilde{f}, \mathfrak{t})$, and therefore $U(\widetilde{f}, \mathfrak{t})$ is an ideal of A.

Conversely, suppose that $U(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$. If $\widetilde{f}_l(0) < \widetilde{f}_l(\mathfrak{k})$ for some $\mathfrak{k} \in A$, then $\mathfrak{k} \in U(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$ and hence $U(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$ is an ideal of A. Thus, $0 \in U(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$, and so $\widetilde{f}_l(0) \geq \widetilde{f}_l(\mathfrak{k})$. This is a contradiction, and thus $\widetilde{f}_l(0) \geq \widetilde{f}_l(\mathfrak{p})$ for all $\mathfrak{p} \in A$. Assume that there exist $\mathfrak{k}, \mathfrak{l} \in A$ such that $\widetilde{f}_l(\mathfrak{k}) < \min\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$. Taking $\mathfrak{t} = \min\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$ implies that $\mathfrak{k} \in U(\widetilde{f}_l, \mathfrak{t})$. Since $U(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A, $a \in U(\widetilde{f}_l, \mathfrak{t})$. Hence, $\widetilde{f}_l(\mathfrak{k}) \geq \mathfrak{t} = \min\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$, which is a contradiction. Hence, $\widetilde{f}_l(\mathfrak{p}) \geq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$ for all $\mathfrak{p}, \mathfrak{q} \in A$. Therefore, $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.

Corollary 1. *If $(A, \widetilde{f})$ is a length 3-fuzzy ideal of A, then the set $U(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$.*

Proof. It is straightforward by Theorems 1 and 3.

Theorem 4. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 4-fuzzy ideal of A if and only if the set $L(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $L(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$.*

Proof. Assume that an interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 4-fuzzy ideal of A and let $\mathfrak{t} \in [0, 1]$ be such that $L(\widetilde{f}, \mathfrak{t})$ is nonempty. Obviously, $0 \in L(\widetilde{f}, \mathfrak{t})$. Let $\mathfrak{p}, \mathfrak{q} \in A$ be such that $\mathfrak{p}^{\mathfrak{q}} \in L(\widetilde{f}, \mathfrak{t})$ and $\mathfrak{q} \in L(\widetilde{f}, \mathfrak{t})$. Then $\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}) \leq \mathfrak{t}$ and $\widetilde{f}_l(\mathfrak{q}) \leq \mathfrak{t}$, which imply from (8) that $\widetilde{f}_l(\mathfrak{p}) \leq \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\} \leq \mathfrak{t}$. Hence, $\mathfrak{p} \in L(\widetilde{f}, \mathfrak{t})$, and therefore $L(\widetilde{f}, \mathfrak{t})$ is an ideal of A.

Conversely, suppose that $L(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $L(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$. If $\widetilde{f}_l(0) > \widetilde{f}_l(\mathfrak{k})$ for some $\mathfrak{k} \in A$, then $\mathfrak{k} \in L(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$ and hence $L(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$ is an ideal of A. Thus, $0 \in L(\widetilde{f}_l, \widetilde{f}_l(\mathfrak{k}))$, and so $\widetilde{f}_l(0) \leq \widetilde{f}_l(\mathfrak{k})$. This is a contradiction, and thus $\widetilde{f}_l(0) \leq \widetilde{f}_l(\mathfrak{p})$ for all $\mathfrak{p} \in A$. Assume that there exist $\mathfrak{k}, \mathfrak{l} \in A$ such that $\widetilde{f}_l(\mathfrak{k}) > \max\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$. Taking $\mathfrak{t} = \max\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$ implies that $\mathfrak{k} \in L(\widetilde{f}_l, \mathfrak{t})$. Since $L(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A, $\mathfrak{k} \in L(\widetilde{f}_l, \mathfrak{t})$. Hence, $\widetilde{f}_l(\mathfrak{k}) \leq \mathfrak{t} = \max\{\widetilde{f}_l(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_l(\mathfrak{l})\}$, which is a contradiction. Hence, $\widetilde{f}_l(\mathfrak{p}) \leq \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}$ for all $\mathfrak{p}, \mathfrak{q} \in A$. Therefore, $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.

Corollary 2. *If $(A, \widetilde{f})$ is a length 2-fuzzy ideal of A, then the set $L(\widetilde{f}_l, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $L(\widetilde{f}_l, \mathfrak{t}) \neq \emptyset$.*

Proof. It is straightforward by Theorems 1 and 4.

Theorem 5. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, $\widetilde{f}_{\inf}(\mathfrak{p}) = \widetilde{f}_{\inf}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy ideal of A,

$$(\forall \mathfrak{p} \in A)(\widetilde{f}_{\sup}(0) \geq \widetilde{f}_{\sup}(\mathfrak{p})), \tag{9}$$

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\widetilde{f}_{\sup}(\mathfrak{p}) \geq \min\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q})\}). \tag{10}$$

Let $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(0) &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) \\ &\geq \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(0) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_l(\mathfrak{p}). \end{aligned}$$

Let $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(0) \\ &\geq \min\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q})\} - \widetilde{f}_{\inf}(0) \\ &= \min\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(0), \widetilde{f}_{\sup}(\mathfrak{q}) - \widetilde{f}_{\inf}(0)\} \\ &= \min\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q}) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_l)$ is a 1-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.

Theorem 6. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, we have $\widetilde{f}_{\inf}(\mathfrak{p}) = \widetilde{f}_{\inf}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy ideal of A, we have

$$(\forall \mathfrak{p} \in A)(\widetilde{f}_{\sup}(0) \leq \widetilde{f}_{\sup}(\mathfrak{p})), \tag{11}$$

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\widetilde{f}_{\sup}(\mathfrak{p}) \leq \max\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q})\}). \tag{12}$$

Let $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(0) &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) \\ &\leq \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(0) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_l(\mathfrak{p}). \end{aligned}$$

Let $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(0) \\ &\leq \max\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{p})\} - \widetilde{f}_{\inf}(0) \\ &= \max\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(0), \widetilde{f}_{\sup}(\mathfrak{q}) - \widetilde{f}_{\inf}(0)\} \\ &= \max\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q}) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_l)$ is a 4-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.

Corollary 3. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 2-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.*

Proof. It is straightforward by Theorems 1 and 6.

Corollary 4. *For $j \in \{2, 4\}$, every $(2(3), j)$-hyperfuzzy ideal of A is a length 4-fuzzy ideal.*

Proof. It is straightforward by Theorem 6 and Corollary 3.

Theorem 7. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(\mathfrak{p}) = \widetilde{f}_{\sup}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A, we have

$$(\forall \mathfrak{p} \in A)(\widetilde{f}_{\inf}(0) \leq \widetilde{f}_{\inf}(\mathfrak{p})), \tag{13}$$

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\widetilde{f}_{\inf}(\mathfrak{p}) \leq \max\{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\inf}(\mathfrak{q})\}). \tag{14}$$

Let $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(0) &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) \\ &\geq \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_l(\mathfrak{p}). \end{aligned}$$

Let $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &\geq \widetilde{f}_{\sup}(0) - \max\{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q}) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \min\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_l)$ is a 1-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.

Corollary 5. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 2-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 1-fuzzy ideal of A.*

Proof. It is straightforward by Theorems 1 and 7.

Corollary 6. *For $i \in \{2, 4\}$, every $(i, 2(3))$-hyperfuzzy ideal of A is a length 1-fuzzy ideal.*

Proof. It is straightforward by Theorem 7 and Corollary 5.

Theorem 8. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A, then $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(\mathfrak{p}) = \widetilde{f}_{\sup}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A, we have

$$(\forall \mathfrak{p} \in A)(\widetilde{f}_{\inf}(0) \geq \widetilde{f}_{\inf}(\mathfrak{p})), \tag{15}$$

$$(\forall \mathfrak{p}, \mathfrak{q} \in A)(\widetilde{f}_{\inf}(\mathfrak{p}) \geq \min\{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\inf}(\mathfrak{q})\}). \tag{16}$$

Let $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(0) &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(0) \\ &\leq \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_l(\mathfrak{p}). \end{aligned}$$

Let $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned} \widetilde{f}_l(\mathfrak{p}) &= \widetilde{f}_{\sup}(\mathfrak{p}) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}) \\ &\leq \widetilde{f}_{\sup}(0) - \min\{\widetilde{f}_{\mathfrak{p}}\mathrm{inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) - \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_{\sup}(\mathfrak{q}\mathfrak{q}) - \widetilde{f}_{\inf}(\mathfrak{q})\} \\ &= \max\{\widetilde{f}_l(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_l(\mathfrak{q})\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_l)$ is a 4-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a length 4-fuzzy ideal of A.

4. Mean fuzzy ideals of Sheffer stroke Hilbert algebras

In this section, we introduce the concept of the mean of an interval-valued fuzzy structure within Sheffer stroke Hilbert algebras. We also define the notion of mean fuzzy ideals in these algebras and investigate their related properties. Furthermore, we establish the relationships between mean fuzzy ideals and traditional fuzzy ideals.

Definition 8. *[14] Given an interval-valued fuzzy structure $(A, \widetilde{f})$ over A, we define a fuzzy structure $(A, \widetilde{f_m})$ in A as follows:*

$$\widetilde{f_m} : A \to [0,1]; \mathfrak{p} \mapsto \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2},$$

which is called the mean of $\widetilde{f}$.

Definition 9. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is called a mean 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) ideal of A if the fuzzy structure $(A, \widetilde{f_m})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) ideal of A.*

Proposition 3. *If $(A, \widetilde{f})$ is a mean k-fuzzy ideal of A for $k = 1, 3$, then*

$$(\forall \mathfrak{p} \in A)(\widetilde{f_m}(0) \geq \widetilde{f_m}(\mathfrak{p})).$$

Proof. Let $(A, \widetilde{f})$ be a mean k-fuzzy ideal of A for $k = 1, 3$ and $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f_m}(0) &= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(0)}{2} \\ &\geq \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\ &= \widetilde{f_m}(\mathfrak{p}). \end{aligned}$$

Proposition 4. *If $(A, \widetilde{f})$ is a mean k-fuzzy ideal of A for $k = 2, 4$, then*

$$(\forall \mathfrak{p} \in A)(\widetilde{f_m}(0) \leq \widetilde{f_m}(\mathfrak{p})).$$

Proof. Let $(A, \widetilde{f})$ be a mean k-fuzzy ideal of A for $k = 2, 4$ and $\mathfrak{p} \in A$. Then

$$\begin{aligned} \widetilde{f_m}(0) &= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(0)}{2} \\ &\leq \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\ &= \widetilde{f_m}(\mathfrak{p}). \end{aligned}$$

Theorem 9. *Every mean 3-fuzzy ideal of A is a mean 1-fuzzy ideal of A.*

Proof. Let $(A, \widetilde{f})$ be a mean 3-fuzzy ideal of A and $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(\mathfrak{p})}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&\geq \max\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2}\right\} + \max\left\{\frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&\geq \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2}\right\} + \min\left\{\frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) + \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q}) + \widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}$$

Hence, $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.

Theorem 10. *Every mean 2-fuzzy ideal of A is a mean 4-fuzzy ideal of A.*

Proof. Let $(A, \widetilde{f})$ be a mean 2-fuzzy ideal of A and $\mathfrak{p}, \mathfrak{q} \in A$. Then

$$\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(\mathfrak{p})}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&\leq \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2}\right\} + \min\left\{\frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&\leq \max\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2}\right\} + \max\left\{\frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) + \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q}) + \widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}$$

Hence, $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.

Theorem 11. *Mean 2-fuzzy ideal and mean 3-fuzzy ideal of A coincide.*

Proof. It is straightforward by Theorems 9 and 10.

Theorem 12. *Given an ideal S of A and $B_1, B_2 \in P([0,1])$, let $(A, \widetilde{f})$ be an interval-valued fuzzy structure over A given by*

$$\widetilde{f} : A \to P([0,1]); \mathfrak{p} \mapsto \begin{cases} B_2, & \text{if } \mathfrak{p} \in S \\ B_1, & \text{otherwise.} \end{cases}$$

(1) *If* $\sup B_2 \geq \sup B_1$ *and* $\inf B_2 \geq \inf B_1$*, then* $(A, \widetilde{f})$ *is a mean* 1*-fuzzy ideal of* A*.*

(2) *If* $\sup B_2 \leq \sup B_1$ *and* $\inf B_2 \leq \inf B_1$*, then* $(A, \widetilde{f})$ *is a mean* 4*-fuzzy ideal of* A*.*

Proof. If $\mathfrak{p} \in S$, then $\widetilde{f}(\mathfrak{p}) = B_2$ and so

$$\widetilde{f}_m(\mathfrak{p}) = \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} = \frac{\sup \widetilde{f}(\mathfrak{p}) + \inf \widetilde{f}(\mathfrak{p})}{2} = \frac{\sup B_2 + \inf B_2}{2}.$$

If $\mathfrak{p} \notin S$, then $\widetilde{f}(\mathfrak{p}) = B_1$ and so

$$\widetilde{f}_m(\mathfrak{p}) = \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} = \frac{\sup \widetilde{f}(\mathfrak{p}) + \inf \widetilde{f}(\mathfrak{p})}{2} = \frac{\sup B_1 + \inf B_1}{2}.$$

(1) Assume that $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \geq \frac{\sup B_1 + \inf B_1}{2}.$$

Case 1: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \in S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \frac{\sup B_2 + \inf B_2}{2}$ and $f_m(\mathfrak{q}) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is an ideal of A, we have $\mathfrak{p} \in S$ and so $\widetilde{f}_m(\mathfrak{p}) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) = \frac{\sup B_2 + \inf B_2}{2} = \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Case 2: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \notin S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Case 3: Let $\mathfrak{p}^{\mathfrak{q}} \notin S$ and $\mathfrak{q} \in S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \frac{\sup B_2 + \inf B_2}{2}$, so $\min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Case 4: Let $\mathfrak{p}^{\mathfrak{q}} \in S$ and $\mathfrak{q} \notin S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \frac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Hence, $\widetilde{f}_m$ is a 1-fuzzy ideal of A and so $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.

(2) Assume that $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \leq \frac{\sup B_1 + \inf B_1}{2}.$$

Case 1: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \in S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \frac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \frac{\sup B_2 + \inf B_2}{2}$, so $\max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is an ideal of A, we have $\mathfrak{p} \in S$ and so $\widetilde{f}_m(\mathfrak{p}) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) = \frac{\sup B_2 + \inf B_2}{2} = \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Case 2: Let $\mathfrak{p}^{\mathfrak{q}}, \mathfrak{q} \notin S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \dfrac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \dfrac{\sup B_1 + \inf B_1}{2}$, so $\max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \leq \dfrac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Case 3: Let $\mathfrak{p}^{\mathfrak{q}} \notin S$ and $\mathfrak{q} \in S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \dfrac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \dfrac{\sup B_2 + \inf B_2}{2}$, so $\max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \leq \dfrac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}})), \widetilde{f}_m(\mathfrak{q})\}$.

Case 4: Let $\mathfrak{p}^{\mathfrak{q}} \in S$ and $\mathfrak{q} \notin S$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) = \dfrac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(\mathfrak{q}) = \dfrac{\sup B_1 + \inf B_1}{2}$, so $\max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus, $\widetilde{f}_m(\mathfrak{p}) \leq \dfrac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$.

Hence, $\widetilde{f}_m$ is a 4-fuzzy ideal of A and so $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.

Theorem 13. *An interval-valued fuzzy structure $(A, \widetilde{f}_m)$ over A is a mean 1-fuzzy ideal of A if and only if the set $U(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$.*

Proof. Assume that an interval-valued fuzzy structure $(A, \widetilde{f}_m)$ over A is a mean 1-fuzzy ideal of A and let $\mathfrak{t} \in [0, 1]$ be such that $U(\widetilde{f}_m, \mathfrak{t})$ is nonempty. Obviously, $0 \in U(\widetilde{f}_m, \mathfrak{t})$. Let $\mathfrak{p}, \mathfrak{q} \in A$ be such that $\mathfrak{p}^{\mathfrak{q}} \in U(\widetilde{f}_m, \mathfrak{t})$ and $\mathfrak{q} \in U(\widetilde{f}_m, \mathfrak{t})$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) \geq \mathfrak{t}$ and $\widetilde{f}_m(\mathfrak{q}) \geq \mathfrak{t}$, which imply from (2) that $\widetilde{f}_m(\mathfrak{p}) \geq \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} \geq \mathfrak{t}$. Hence, $\mathfrak{p} \in U(\widetilde{f}_m, \mathfrak{t})$, and therefore $U(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A.

Conversely, suppose that $U(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$. If $\widetilde{f}_m(0) < \widetilde{f}_m(\mathfrak{k})$ for some $\mathfrak{k} \in A$, then $\mathfrak{k} \in U(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$ and hence $U(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$ is an ideal of A. Thus, $0 \in U(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$, and so $\widetilde{f}_m(0) \geq \widetilde{f}_m(\mathfrak{k})$. This is a contradiction, and thus $\widetilde{f}_m(0) \geq \widetilde{f}_m(\mathfrak{p})$ for all $\mathfrak{p} \in A$. Assume that there exist $\mathfrak{k}, \mathfrak{l} \in A$ such that $\widetilde{f}_m(\mathfrak{k}) < \min\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_m(\mathfrak{l})\}$. Taking $\mathfrak{t} = \min\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_m(\mathfrak{l})\}$ implies that $\mathfrak{k} \in U(\widetilde{f}, \mathfrak{t})$. Since $U(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A, we have $\mathfrak{k} \in U(\widetilde{f}, \mathfrak{t})$. Hence, $\widetilde{f}_m(\mathfrak{k}) \geq \mathfrak{t} = \min\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_m(\mathfrak{k})\}$, which is a contradiction. Hence, $\widetilde{f}_m(\mathfrak{p}) \geq \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$ for all $\mathfrak{p}, \mathfrak{q} \in A$. Therefore, $(A, \widetilde{f}_m)$ is a mean 1-fuzzy ideal of A.

Corollary 7. *If $(A, \widetilde{f})$ is a mean 3-fuzzy ideal of A, then $U(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $U(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$.*

Proof. It is straightforward by Theorems 9 and 13.

Theorem 14. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a mean 4-fuzzy ideal of A if and only if the set $L(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0, 1]$ with $L(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$.*

Proof. Assume that an interval-valued fuzzy structure $(A, \widetilde{f}_m)$ over A is a mean 4-fuzzy ideal of A and let $\mathfrak{t} \in [0, 1]$ be such that $L(\widetilde{f}_m, \mathfrak{t})$ is nonempty. Obviously, $0 \in L(\widetilde{f}_m, \mathfrak{t})$. Let $\mathfrak{p}, \mathfrak{q} \in A$ be such that $\mathfrak{p}^{\mathfrak{q}} \in L(\widetilde{f}_m, \mathfrak{t})$ and $\mathfrak{q} \in L(\widetilde{f}_m, \mathfrak{t})$. Then $\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}) \leq \mathfrak{t}$ and $\widetilde{f}_m(\mathfrak{q}) \leq \mathfrak{t}$, which imply from (8) that $\widetilde{f}_m(\mathfrak{p}) \leq \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\} \leq \mathfrak{t}$. Hence, $\mathfrak{p} \in L(\widetilde{f}_m, \mathfrak{t})$, and therefore $L(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A.

Conversely, suppose that $L(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0,1]$ with $L(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$. If $\widetilde{f}_m(0) > \widetilde{f}_m(\mathfrak{k})$ for some $\mathfrak{k} \in A$, then $\mathfrak{k} \in L(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$ and hence $L(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$ is an ideal of A. Thus, $0 \in L(\widetilde{f}_m, \widetilde{f}_m(\mathfrak{k}))$, and so $\widetilde{f}_m(0) \leq \widetilde{f}_m(\mathfrak{k})$. This is a contradiction, and thus $\widetilde{f}_m(0) \leq \widetilde{f}_m(\mathfrak{p})$ for all $\mathfrak{p} \in A$. Assume that there exist $\mathfrak{k}, \mathfrak{l} \in A$ such that $\widetilde{f}_m(\mathfrak{k}) > \max\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}})), \widetilde{f}_m(\mathfrak{l})\}$. Taking $t = \max\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_m(\mathfrak{l})\}$ implies that $\mathfrak{k} \in L(\widetilde{f}_m, \mathfrak{t})$. Since $L(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A, we have $\mathfrak{k} \in L(\widetilde{f}_m, \mathfrak{t})$. Hence, $\widetilde{f}_m(\mathfrak{k}) \leq \mathfrak{t} = \max\{\widetilde{f}_m(\mathfrak{k}^{\mathfrak{l}}), \widetilde{f}_m(\mathfrak{l})\}$, which is a contradiction. Hence, $\widetilde{f}_m(\mathfrak{p}) \leq \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}$ for all $\mathfrak{p}, \mathfrak{q} \in A$. Therefore, $(A, \widetilde{f}_m)$ is a mean 4-fuzzy ideal of A.

Corollary 8. *If $(A, \widetilde{f})$ is a mean 2-fuzzy ideal of A, then $L(\widetilde{f}_m, \mathfrak{t})$ is an ideal of A for all $\mathfrak{t} \in [0,1]$ with $L(\widetilde{f}_m, \mathfrak{t}) \neq \emptyset$.*

Proof. It is straightforward by Theorems 1 and 14.

Theorem 15. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\text{inf}})$ is constant and $(A, \widetilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of A, then $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\text{inf}})$ is constant and $(A, \widetilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\text{inf}})$ is constant, we have $\widetilde{f}_{\text{inf}}(\mathfrak{p}) = \widetilde{f}_{\text{inf}}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\text{sup}})$ is a 1-fuzzy ideal of A, we have $\widetilde{f}_{\text{sup}}(\mathfrak{p}) \geq \min\{\widetilde{f}_{\text{sup}}(\mathfrak{p}), \widetilde{f}_{\text{sup}}(\mathfrak{q})\}$. Thus,

$$
\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\text{sup}}(\mathfrak{p}) + \widetilde{f}_{\text{inf}}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\text{sup}}(\mathfrak{p})}{2} + \frac{\widetilde{f}_{\text{inf}}(0)}{2} \\
&\geq \min\left\{\frac{\widetilde{f}_{\text{sup}}(\mathfrak{p}^{\mathfrak{q}})}{2} + \frac{\widetilde{f}_{\text{inf}}(\mathfrak{q})}{2}\right\} + \frac{\widetilde{f}_{\text{inf}}(0)}{2} \\
&= \min\left\{\frac{\widetilde{f}_{\text{sup}}(\mathfrak{p}^{\mathfrak{q}})}{2} + \frac{\widetilde{f}_{\text{inf}}(0)}{2}, \frac{\widetilde{f}_{\text{sup}}(\mathfrak{q})}{2} + \frac{\widetilde{f}_{\text{inf}}(0)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\text{sup}}(\mathfrak{p}^{\mathfrak{q}})) + \widetilde{f}_{\text{inf}}(\mathfrak{p})}{2}, \frac{\widetilde{f}_{\text{sup}}(\mathfrak{q}) + \widetilde{f}_{\text{inf}}(\mathfrak{q})}{2}\right\} \\
&= \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}
$$

Hence, $(A, \widetilde{f}_m)$ is a 1-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.

Theorem 16. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\text{inf}})$ is constant and $(A, \widetilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of A, then $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\text{inf}})$ is constant and $(A, \widetilde{f}_{\text{sup}})$ is a 4-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\text{inf}})$ is constant, we have $\widetilde{f}_{\inf}(\mathfrak{p}) = \widetilde{f}_{\inf}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy ideal of A, we have $\widetilde{f}_{\sup}(\mathfrak{p}) \leq \max\{\widetilde{f}_{\sup}(\mathfrak{p}), \widetilde{f}_{\sup}(\mathfrak{q})\}$. Thus,

$$\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(\mathfrak{p})}{2} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&\geq \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}}) + \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q}) + \widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 4-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.

Theorem 17. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A, then $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(\mathfrak{p}) = \widetilde{f}_{\sup}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy ideal of A, we have $\widetilde{f}_{\inf}(\mathfrak{p}) \leq \max\{\widetilde{f}_{\inf}(\mathfrak{p}), \widetilde{f}_{\inf}(\mathfrak{q})\}$. Thus,

$$\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&\leq \frac{\widetilde{f}_{\sup}(0)}{2} + \max\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})) + \widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q}) + \widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \max\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 4-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a mean 4-fuzzy ideal of A.

Theorem 18. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A, then $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A. Let $\mathfrak{p}, \mathfrak{q} \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(\mathfrak{p}) = \widetilde{f}_{\sup}(0)$ for all $\mathfrak{p} \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy ideal of A, we obtain $\widetilde{f}_{\inf}(\mathfrak{p}) \geq \min\{\widetilde{f}_{\inf}(\mathfrak{p}), \widetilde{f}_{\inf}(\mathfrak{q})\}$. Thus,

$$\begin{aligned}
\widetilde{f}_m(\mathfrak{p}) &= \frac{\widetilde{f}_{\sup}(\mathfrak{p}) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&= \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(\mathfrak{p})}{2} \\
&\geq \frac{\widetilde{f}_{\sup}(0)}{2} + \min\left\{\frac{\widetilde{f}_{\inf}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\inf}(\mathfrak{q})}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(0)}{2}, \frac{\widetilde{f}_{\sup}(\mathfrak{q})}{2}\right\} \\
&= \min\{\frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\sup}(\mathfrak{p}^{\mathfrak{q}})}{2}, \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\sup}(\mathfrak{q})}{2}\} \\
&= \min\{\widetilde{f}_m(\mathfrak{p}^{\mathfrak{q}}), \widetilde{f}_m(\mathfrak{q})\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 1-fuzzy ideal of A, that is, $(A, \widetilde{f})$ is a mean 1-fuzzy ideal of A.

5. Conclusion

This study extends the theoretical foundation of Sheffer stroke Hilbert algebras by introducing and analyzing the notions of length fuzzy ideals and mean fuzzy ideals within an interval-valued fuzzy structure. By defining these concepts, the research provides a more nuanced understanding of fuzzy logic applications in algebraic structures, emphasizing the relationships between fuzzy ideals and traditional ideals. The characterizations and properties of length fuzzy ideals and mean fuzzy ideals demonstrate their alignment with upper and lower level subsets, offering a framework to explore the gradations of membership functions. Furthermore, the findings highlight the potential of these fuzzy constructs in bridging algebraic theory with practical applications in logic, computer science, and uncertainty modeling. Future studies could investigate the applicability of these ideas in more complex fuzzy systems or extend the analysis to other algebraic frameworks, broadening the impact and utility of these innovative concepts.

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