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An Introduction to Mixed $\mathcal{H}\big(\theta(\mu,\nu)\big)$-Open Sets Generated by Hereditary Classes in Generalized Topological Spaces

Fahad Alsharari[1,*], Abdo Qahis[2]

[1] *Department of Mathematics, College of Science, Jouf University, Sakaka 72311, Saudi Arabia*
[2] *Department of Mathematics, College of Science and Arts, Najran University, Najran, Saudi Arabia*

Abstract. In [1], Kim and Min introduced the operation γ_* and $\mathcal{H}(\theta)$-open sets within the context of generalized topological spaces, utilizing a hereditary class $\mathcal{H}$. In this study, we extend these concepts by employing two generalized topologies, μ and ν, along with a hereditary class $\mathcal{H}$. Specifically, we introduce and investigate the mixed operation $\gamma_*(\mu,\nu)$ (denoted briefly as $\gamma_*(\mu,\nu)$) and the mixed $\mathcal{H}(\theta(\mu,\nu))$-open sets (denoted as $\mathcal{H}(\theta(\mu,\nu))$-open sets). We explore the interrelationships between $\gamma_*(\mu,\nu)$, γ_*, and the μ-closure, as well as the connections between $\mathcal{H}(\theta(\mu,\nu))$-open sets, $\theta(\mu,\nu)$-open sets, and μ-open sets. Additionally, we define the concepts of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular open sets. Finally, we examine properties and characterizations of $\mathcal{H}(\theta(\mu,\nu))$-open sets in terms of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular open sets.

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1. Introduction

Á. Császár formulated the idea of generalized topology and generalized open sets in [2], along with the notion of θ-open sets and their properties. For further details, one can refer to [3–5]. In [6], he also introduced the concept of hereditary classes in generalized topological spaces. Specifically, a subset $\mathcal{H} \subseteq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of a non-empty set X) is termed a hereditary class on X if it satisfies the condition that for any $A \subseteq B$ and $B \in \mathcal{H}$, it follows that $A \in \mathcal{H}$. Building on these foundations of generalized topology and hereditary classes, authors in [1] introduced the concepts of $\mathcal{H}(\theta)$-open sets and the operator γ_*.

*Corresponding author.
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Email addresses: f.alsharari@ju.edu.sa (F. Alsharari), cahis82@gmail.com (A. Qahis)

In this study, we extend these ideas by examining two generalized topologies, denoted as μ and ν, within the framework of a hereditary class $\mathcal{H}$. We introduce and investigate the concepts of the mixed operator $\gamma_*(\mu,\nu)$ (denoted briefly as $\gamma_*(\mu,\nu)$) and mixed $\mathcal{H}(\theta(\mu,\nu))$-open sets (referred to as $\mathcal{H}(\theta(\mu,\nu))$-open sets). Our exploration includes a detailed study of their properties and the relationships between $\gamma_*(\mu,\nu)$, the operator γ_*, and the μ-closure. In addition, we examine the interconnections between sets that are $\mathcal{H}(\theta(\mu,\nu))$-open, $\theta(\mu,\nu)$-open, and μ-open. We also present various properties and characterizations of these concepts in terms of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular sets.

2. Preliminaries

Let $X \neq \emptyset$ and let $\mathcal{P}(X)$ be its power set. A family $\mu \subseteq \mathcal{P}(X)$ is called a **generalized topology** (GT) on X if:

$$\emptyset \in \mu,$$
$$\bigcup_{i\in I} U_i \in \mu \quad \text{for any collection } \{U_i\}_{i\in I} \subseteq \mu.$$

This concept was first introduced by Á. Császár in [2]. A pair (X,μ) is then referred to as a **generalized topological space** (GTS) on X. The elements of μ are called μ-**open** sets, while their complements are called μ-**closed** sets. The union of all elements of μ is denoted by $\mathcal{M}_\mu$, as stated in [7]. A GTS (X,μ) is is said to be **strong** [8] if $X \in \mu$. For a subset A of a GTS (X,μ), the μ-**closure** of A, denoted $c_\mu(A)$, is defined as the intersection of all μ-closed sets that contain A. The μ-**interior** of A, denoted $i_\mu(A)$, is the union of all μ-open sets that are contained within A (see [2, 7]).

Now, considering a hereditary class $\mathcal{H}$, an operator $()^* : \mathcal{P}(X) \to \mathcal{P}(X)$ was introduced in [3]. Specifically, $c^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined using $()^*$ by $c^*(A) = A \cup A^*$, where $A^* = \{x \in X \mid A \cap M \notin \mathcal{H}, \forall M \in \mu, x \in M\}$. Here, $x \notin A^*$ if and only if there exists $M \in \mu$ such that $x \in M$ and $M \cap A \in \mathcal{H}$.

Recalling definitions and notations from [3], let μ be a GT on X and $\mathcal{P}(X)$ be the power set of X. A collection $\theta \subseteq \mathcal{P}(X)$ is defined as follows: $A \in \theta$ if for each $x \in A$, there exists $M \in \mu$ containing x such that $M \subseteq c_\mu(M) \subseteq A$. The family θ is a GT on X included in μ, and the elements of θ are $\theta(\mu)$-**open** sets, with complements called $\theta(\mu)$-**closed** sets. For $A \subseteq X$, the operation $\gamma_\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined in [3], by $\gamma_\theta(A) = \{x \in X \mid c_\mu(M) \cap A \neq \emptyset, \forall M \in \mu, x \in M\}$.

Kim and Min in [5], extended the study to θ-open using a hereditary class $\mathcal{H}$: a collection $\mathcal{H}(\theta) \subseteq \mathcal{P}(X)$ is defined such that $A \in \mathcal{H}(\theta)$ if for each $x \in A$, there exists $M \in \mu$ containing x with $M \subseteq c^*_\mu(M) \subseteq A$. The family $\mathcal{H}(\theta)$ is a GT on X included in μ, with elements termed $\mathcal{H}(\theta)$-**open** sets and their complements $\mathcal{H}(\theta)$-**closed** sets. Additionally, the operation $\gamma_* : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined in [5] as $\gamma_*(A) = \{x \in X \mid c^*_\mu(M) \cap A \neq \emptyset, \forall M \in \mu, x \in M\}$.

In [4], Á. Császár and Makai Jr. introduced $\theta(\nu_1,\nu_2)$-open sets as a means of combining two generalized topologies (GTs), ν_1 and ν_2, on a set X. A subset $A \subseteq X$ belongs to

$\theta(\nu_1, \nu_2)$ if, for every $x \in A$, there exists $M \in \nu_1$ such that $x \in M \subseteq c_{\nu_2}(M) \subseteq A$. Moreover, the family $\theta(\nu_1, \nu_2)$ itself forms a GT contained within ν_1 on X. The sets in $\theta(\nu_1, \nu_2)$ are called $\theta(\nu_1, \nu_2)$-open sets, while their complements are referred to as $\theta(\nu_1, \nu_2)$-closed sets.

Subsequently, in [9], Abdo Qahis and Awn Alqahtani introduced a modification of this concept, defining the class of $\tilde{\theta}(\nu_1, \nu_2)$-open sets. A subset $A \subseteq X$ is said to be mixed $\tilde{\theta}(\nu_1, \nu_2)$-open (or simply $\tilde{\theta}(\nu_1, \nu_2)$-open) if, for every $x \in A$, there exists $M \in \nu_1$ such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$.

In conclusion of this section, we review the following important facts due to their significance to the content of our paper.

Theorem 1. *[6] Let μ be a GTS on X and $\mathcal{H}$ a hereditary class on X. Then $A^* \subseteq c^*_\mu(A) \subseteq c_\mu(A)$ for any $A \subseteq X$.*

In [10], the authors introduced the operator $i^* : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $i^*(A) = X \setminus c^*(X \setminus A)$ for $A \subseteq X$.

Theorem 2. *[10] Let μ be a GT on X and $\mathcal{H}$ a hereditary class. Then for $A \subseteq X$,*

(i) $c^*_\mu(A) = X \setminus i^*_\mu(X \setminus A)$.

(ii) $i_\mu(A) \subseteq i^*_\mu(A) \subseteq A$.

Lemma 1. *[11] Let μ and ν be two GTs on a nonempty set X and $A \subseteq X$. Then the following statements hold:*

(i) $x \in i_{\theta(\mu,\nu)}(A)$ if and only if there exists a μ-open set M containing x such that $M \subseteq c_\nu(M) \subseteq A$.

(ii) If A is ν-open in X, then $\gamma_{\theta(\mu,\nu)}(A) = c_\mu(A)$.

Definition 1. *[1] Let μ be GT on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. Then (X, μ) is $\mathcal{H}$-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c^*(V) \subseteq U$.*

Theorem 3. *[11] Let μ and ν be measures on a nonempty set X. Then X is (μ, ν)-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c_\nu(V) \subseteq U$.*

3. Properties of the Mixed Operator $\gamma_*(\mu, \nu)$

In [4], Császár and Makai Jr introduced an operation $\gamma_{\theta(\mu,\nu)} : \mathcal{P}(X) \to \mathcal{P}(X)$, utilizing two generalized topologies μ and ν on X. According to their definition, $x \in \gamma_{\theta(\mu,\nu)}(A)$ if and only if $c_\nu(M) \cap A \neq \emptyset$ for every μ-open set M containing x. If $x \notin \mathcal{M}_\mu$, then by definition $x \in \gamma_{\theta(\mu,\nu)}(A)$. Additionally, $x \notin \gamma_{\theta(\mu,\nu)}(A)$ if and only if there exists $M \in \mu$ with $x \in M$ such that $c_\nu(M) \cap A = \emptyset$.

Definition 2. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. An operation $\gamma_*(\mu,\nu) : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: for every $A \subseteq X$,*

$$\gamma_*(\mu,\nu)(A) = \{x \in X : c_\nu^*(M) \cap A \neq \emptyset,\ \forall M \in \mu \text{ and } x \in M\}.$$

If $x \notin \mathcal{M}_\mu$, then by definition $x \in \gamma_*(\mu,\nu)(A)$.

According to Definition 2, $x \notin \gamma_*(\mu,\nu)(A)$ if and only if there exists $M \in \mu$ and $x \in M$ such that $c_\nu^*(M) \cap A = \emptyset$.

The following is an immediate consequence that can be obviously obtained.

Corollary 1. *Let μ and ν be two GT's on a nonempty set X such that $\mu = \nu$, and let $\mathcal{H}$ be a hereditary class on X. For any $A \subseteq X$, the following statements hold:*

(i) $\gamma_*(\mu,\nu)(A) = \gamma_*(A)$.

(ii) *If $\mathcal{H} = \{\emptyset\}$, then* $\gamma_*(\mu,\nu)(A) = \gamma_*(A) = \gamma_\theta(A)$.

Theorem 4. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then for any $A \subseteq X$, we have $\gamma_*(\mu,\nu)(A) \subseteq \gamma_{\theta(\mu,\nu)}(A)$.*

Proof. Let $x \in \gamma_*(\mu,\nu)(A)$. For each μ-open set M containing x, we have $c_\nu^*(M) \cap A \neq \emptyset$. Since $c_\nu^*(M) \subseteq c_\nu(M)$, it follows that $c_\nu(M) \cap A \neq \emptyset$. Therefore, $x \in \gamma_{\theta(\mu,\nu)}(A)$, and so $\gamma_*(\mu,\nu)(A) \subseteq \gamma_{\theta(\mu,\nu)}(A)$.

The following example demonstrates that, in general, $\gamma_*(\mu,\nu)(A) \neq \gamma_\theta(\mu,\nu)(A)$.

Example 1. *Let $X = \{a,b,c,d\}$. Consider two generalized topologies: $\mu = \{\emptyset, \{b,d\}\}$ and $\nu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$ on X.*

For a set $A = \{a,c\}$, we have $c_\nu(\{b,d\}) = X$, $\mathcal{M}_\mu = \{b,d\}$, and $c_\nu(\{b,d\}) \cap A \neq \emptyset$. Thus, $\gamma_{\theta(\mu,\nu)}(A) = X$. Since $\mathcal{M}_\mu = \{b,d\}$, it is clear that $a,c \in \gamma_*(\mu,\nu)(A)$. Noting that $\{b,d\}^*(\mathcal{H},\nu) = \{d\}$, we find $c_\nu^*(\{b,d\}) \cap A = \emptyset$, hence $b,d \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) = \{a,c\}$ and $\gamma_*(\mu,\nu)(A) \subset \gamma_{\theta(\mu,\nu)}(A)$.

Corollary 2. *Let μ and ν be two GT's on a nonempty set X and let $\mathcal{H}$ be a hereditary class on X. If $\mathcal{H} = \{\emptyset\}$, then $\gamma_*(\mu,\nu)(A) = \gamma_{\theta(\mu,\nu)}(A)$ for any $A \subseteq X$.*

Theorem 5. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. For any subsets A and B of X, the following properties hold:*

(i) $\gamma_*(\mu,\nu)(\emptyset) = \emptyset$.

(ii) *If $A \subseteq B$, then* $\gamma_*(\mu,\nu)(A) \subseteq \gamma_*(\mu,\nu)(B)$.

(iii) $A \subseteq c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$.

Proof. (1) and (2) are obvious.

(3) For $x \in c_\mu(A)$ and any μ-open set M containing x, we have $M \cap A \neq \emptyset$. Consequently, $c_\nu^*(M) \cap A \neq \emptyset$. Therefore, $x \in \gamma_*(\mu,\nu)(A)$, implying that $c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$.

The following example shows that, in general, $c_\mu(A) \neq \gamma_*(\mu,\nu)(A)$.

Example 2. *Let $X = \{a, b, c, d\}$. Consider two generalized topologies: $\mu = \{\emptyset, \{a\}\}$, $\nu = \big\{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\big\}$ on X, and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$.*

For a set $A = \{b, c, d\}$, since A is μ-closed, $c_\mu(A) = A$. Given $\mathcal{M}_\mu = \{a\}$, it follows by the definition of the operator $\gamma_*(\mu, \nu)$ that $X - \mathcal{M}_\mu = \{b, c, d\} \subseteq \gamma_*(\mu, \nu)(A)$.

Next, we show that $a \in \gamma_*(\mu, \nu)(A)$. Since $M = \{a\} \in \mu$ and $\{a\}^*(\mathcal{H}, \nu) = \{a, d\}$, we have $c^*_\nu(M) \cap A \neq \emptyset$. Thus, $a \in \gamma_*(\mu, \nu)(A)$, implying $c_\mu(A) \subset \gamma_*(\mu, \nu)(A) = X$. Thus $c_\mu(A) \neq \gamma_*(\mu, \nu)(A)$.

The following Corollary follows immediately from Theorem 5(iii), and Theorem 1.

Corollary 3. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. For $A \subseteq X$, $A^* \subseteq c^*_\mu(A) \subseteq \gamma_*(\mu, \nu)(A)$.*

Theorem 6. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then $\gamma_*(\mu, \nu)(A)$ is μ-closed.*

Proof. Let $x \in X - \gamma_*(\mu, \nu)(A)$. This means there exists $M_x \in \mu$ such that $c^*_\nu(M_x) \cap A = \emptyset$. Since $M_x \subseteq c^*_\nu(M_x)$, it follows that $M_x \cap A = \emptyset$. Therefore, every $y \in M_x$ implies $y \in X - \gamma_*(\mu, \nu)(A)$, implying $X - \gamma_*(\mu, \nu)(A) = \bigcup_{x \in X - \gamma_*(\mu,\nu)(A)} M_x$. Thus, $X - \gamma_*(\mu, \nu)(A)$ is μ-open, hence $\gamma_*(\mu, \nu)(A)$ is μ-closed.

Theorem 7. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. If A is ν-open in X, then $\gamma_*(\mu, \nu)(A) = c_\mu(A)$.*

Proof. From (iii) of Theorem 5, we have $c_\mu(A) \subseteq \gamma_*(\mu, \nu)(A)$.

For the converse inclusion, suppose $x \in \gamma_*(\mu, \nu)(A)$. For each $M \in \mu$ such that $x \in M$ and $c^*_\nu(M) \cap A \neq \emptyset$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, it follows that $c_\nu(M) \cap A \neq \emptyset$. Thus, there exists $y \in c_\nu(M) \cap A$. Since A is ν-open and contains y, we have $M \cap A \neq \emptyset$, implying $x \in c_\mu(A)$. Therefore, $\gamma_*(\mu, \nu)(A) \subseteq c_\mu(A)$. Combining this with the earlier inclusion, we conclude $\gamma_*(\mu, \nu)(A) = c_\mu(A)$.

The following Corollary follows from Lemma 1(ii) and Theorem 7.

Corollary 4. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. If $A \in \nu$, then*

$$\gamma_*(\mu, \nu)(A) = c_\mu(A) = \gamma_{\theta(\mu,\nu)}(A).$$

4. $\mathcal{H}\big(\theta(\mu, \nu)\big)$-Open Sets

Definition 3. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. We define the collection $\mathcal{H}\big(\theta(\mu, \nu)\big) \subseteq \mathcal{P}(X)$ such that $A \in \mathcal{H}\big(\theta(\mu, \nu)\big)$ if and only if for each $x \in A$, there exists $M \in \mu$ such that $x \in M \subseteq c^*_\nu(M) \subseteq A$.*

The elements of $\mathcal{H}\big(\theta(\mu, \nu)\big)$ are called mixed $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open (briefly, $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open) , and their complements are called mixed $\mathcal{H}\big(\theta(\mu, \nu)\big)$-closed (briefly, $\mathcal{H}\big(\theta(\mu, \nu)\big)$-closed).

Remark 1. *Consider μ and ν to be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If $\mu = \nu$, then $\mathcal{H}\big(\theta(\mu,\nu)\big) = \mathcal{H}(\theta)$.*

Theorem 8. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then $\theta(\mu,\nu) \subseteq \mathcal{H}\big(\theta(\mu,\nu)\big) \subseteq \mu$.*

Proof. To show that $\theta(\mu,\nu) \subseteq \mathcal{H}\big(\theta(\mu,\nu)\big)$, let $A \in \theta(\mu,\nu)$ and $x \in A$. Then there exists $M \in \mu$ such that $x \in M \subseteq c_\nu(M) \subseteq A$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, we have $x \in M \subseteq c^*_\nu(M) \subseteq A$. Therefore, A is an $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open set.

Next, to show that $\mathcal{H}\big(\theta(\mu,\nu)\big) \subseteq \mu$, suppose $A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ and let $x \in A$. Then there exists a μ-open set M_x such that $x \in M_x \subseteq c^*_\nu(M_x) \subseteq A$. Therefore, $A = \bigcup_{x\in A} M_x \in \mu$.

Remark 2. *Based on Theorem 8, we can illustrate the following diagram.*

$$\begin{array}{ccc} \mathcal{K}(\theta(\mu,\nu)) & \longrightarrow & \mu\text{-open} \\ \uparrow & & \downarrow \\ \theta(\mu,\nu) & \longrightarrow & \mu^*\text{-open} \end{array}$$

The following example demonstrates that the above implications are not reversible in general.

Example 3. *Let $X = \{a,b,c,d\}$. Consider two generalized topologies $\mu = \big\{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\big\}$ and $\nu = \big\{\emptyset, \{b,d\}\big\}$ and a hereditary class $\mathcal{H} = \big\{\emptyset, \{b\}\big\}$. Note that:*

(i) For a set $A = \{a,b,c\}$, we have $M_a = M_b = \{a,b\} \in \mu$ and $M_c = \{b,c\} \in \mu$. Then $M^*_a(\mathcal{H},\nu) = M^*_b(\mathcal{H},\nu) = M^\star_c(\mathcal{H},\nu) = \{a,c\}$ and $c^*_v(M_a) = c^*_v(M_b) = c^*_v(M_c) = \{a,b,c\} \subseteq A$;

(ii) Since $c_\nu(\{a,b\}) = c_\nu(\{b,c\}) = c_\nu(\{a,b,c\}) = X \nsubseteq A$, then $\theta(\mu,\nu) = \{\emptyset\}$.

(iii) From (1) and (2), we show that the set A is $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open but it is not $\theta(\mu,\nu)$-open. Also, it is easy to check that $B = \{a,b\}$ is μ-open but it is not $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open.

Theorem 9. *Let μ and ν be two GTs on a nonempty set X, and $\mathcal{H}$ be a hereditary class on X. Then the family $\mathcal{H}\big(\theta(\mu,\nu)\big)$ is a GT contained in μ on X.*

Proof. Firstly, $\emptyset \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ is obvious. Now, let $\{A_\alpha \subseteq X : A_\alpha \in \mathcal{H}\big(\theta(\mu,\nu)\big)\}$ for $\alpha \in \Lambda$. Consider $x \in \cup_\alpha A_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that for some μ-open set M containing x, we have $M \subseteq c^*_\nu(M) \subseteq A_{\alpha_0}$. This implies there exists $x \in M \in \mu$ such that $M \subseteq c^*_\nu(M) \subseteq \cup_\alpha A_\alpha$ and so $\cup_\alpha A_\alpha \in \mathcal{H}\big(\theta(\mu,\nu)\big)$.

Theorem 10. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then, A is $\mathcal{H}(\theta(\mu,\nu))$-closed if and only if $\gamma_*(\mu,\nu)(A) = A$.*

Proof. Let A be $\mathcal{H}(\theta(\mu,\nu))$-closed in X. Since $X-A \in \mathcal{H}(\theta(\mu,\nu))$, for each $x \in X-A$, there exists $M \in \mu$ such that $x \in M \subseteq c^*_\nu(M) \subseteq X-A$. Thus, $c^*_\nu(M) \cap A = \emptyset$, implying $x \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) \subseteq A$, implying that $\gamma_*(\mu,\nu)(A) = A$.

For the reverse inclusion, suppose $\gamma_*(\mu,\nu)(A) = A$ and let $x \in X-A = X-\gamma_*(\mu,\nu)(A)$. Then there exists $M \in \mu$ such that $x \in M$ and $c^*_\nu(M) \cap A = \emptyset$. Hence, $x \in M \subseteq c^*_\nu(M) \subseteq X-A$, showing that $X-A$ is $\mathcal{H}(\theta(\mu,\nu))$-open. Therefore, A is $\mathcal{H}(\theta(\mu,\nu))$-closed.

From Theorem 10 and Theorem 7, the following Corollary is directly obtained.

Corollary 5. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$ be $\mathcal{H}(\theta(\mu,\nu))$-closed. If $A \in \nu$, then A is μ-closed.*

Definition 4. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. The $\mathcal{H}(\theta(\mu,\nu))$-closure of $A \subseteq X$, denoted by $c_{\mathcal{H}\theta(\mu,\nu)}(A)$, is the intersection of all $\mathcal{H}(\theta(\mu,\nu))$-closed sets containing A. The $\mathcal{H}(\theta(\mu,\nu))$-interior of A, denoted by $i_{\mathcal{H}(\theta(\mu,\nu))}(A)$, is the union of all $\mathcal{H}(\theta(\mu,\nu))$-open sets contained in A.*

Theorem 11. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then $\gamma_*(\mu,\nu)(A) \subseteq c_{\mathcal{H}\theta(\mu,\nu)}(A)$.*

Proof. Let $x \notin c_{\mathcal{H}\theta(\mu,\nu)}(A)$. Then there exists an $\mathcal{H}(\theta(\mu,\nu))$-open set W containing x such that $W \cap A = \emptyset$. Since $W \in \mathcal{H}(\theta(\mu,\nu))$, there exists $M \in \mu$ containing x such that $x \in M \subseteq c^*_\nu(M) \subseteq W \subseteq X-A$. This implies $c^*_\nu(M) \cap A = \emptyset$ and hence $x \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) \subseteq c_{\mathcal{H}\theta(\mu,\nu)}(A)$.

Definition 5. *Let μ and ν be two GTs on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. A subset $A \subseteq X$ is called $\mathcal{H}(\mu,\nu)$-regular open (briefly, $\mathcal{H}r(\mu,\nu)$-open) if $A = i_\mu(c^*_\nu(A))$. Similarly, A is called $\mathcal{H}(\mu,\nu)$-regular closed (briefly, $\mathcal{H}r(\mu,\nu)$-closed) if $c_\mu(i^*_\nu(A)) = A$.*

Theorem 12. *Let μ and ν be two GTs on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. If $A \in \mathcal{H}(\theta(\mu,\nu))$ and $x \in A$, then there exists a $\mathcal{H}r(\mu,\nu)$-open set U such that $x \in U \subseteq c^*_\nu(U) \subseteq A$.*

Proof. Since $A \in \mathcal{H}(\theta(\mu,\nu))$ and $x \in A$, there exists a μ-open set M such that $x \in M \subseteq c^*_\nu(M) \subseteq A$. Define $U = i_\mu(c^*_\nu(M))$. Then U is $\mathcal{H}r(\mu,\nu)$-open, $M \subseteq U$, and $c^*_\nu(U) = c^*_\nu(i_\mu(c^*_\nu(M))) \subseteq c^*_\nu(M)$. This implies $x \in M \subseteq U \subseteq c^*_\nu(U) \subseteq c^*_\nu(M) \subseteq A$. Thus, we have $x \in U \subseteq c^*_\nu(U) \subseteq A$ for some $\mathcal{H}r(\mu,\nu)$-open set U.

Since every $\mathcal{H}(\mu,\nu)$-regular open set is μ-open in X, the following Corollary is evidently obtained.

Corollary 6. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then, $A \in \mathcal{H}(\theta(\mu,\nu))$ and $x \in A$ if and only if there exists a $\mathcal{H}r(\mu,\nu)$-open set U such that $x \in U \subseteq c^*_\nu(U) \subseteq A$.*

Definition 6. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. A set X is said to be $\mathcal{H}(\mu,\nu)$-regular (or simply $\mathcal{H}(\mu,\nu)$-regular) if for every $x \in X$ and every μ-closed set F with $x \notin F$, there exist sets $U \in \mu$, $V \in \nu^*$ such that:*

$$x \in U, \quad F \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

Theorem 13. *Let μ and ν be GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. Then X is $\mathcal{H}(\mu,\nu)$-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c^*_\nu(V) \subseteq U$.*

Proof. Assume X is $\mathcal{H}(\mu,\nu)$-regular. For $x \in X$ and a μ-open set U containing x, there exist disjoint sets $V \in \mu$ and $W \in \nu^*$ such that $x \in V$, $(X-U) \subseteq W$. Since $V \subseteq X-W$ and $X-W$ is ν^*-closed, we have $c^*_\nu(V) \subseteq X-W$. This implies $c^*_\nu(V) \cap (X-U) \subseteq c^*_\nu(V) \cap W = \emptyset$, hence $x \in V \subseteq c^*_\nu(V) \subseteq U$.

Conversely, suppose F is a μ-closed set and $x \notin F$ for $x \in X$. Since $X-F$ is a μ-open set containing x, by hypothesis, there exists a μ-open set V containing x such that $x \in V$, $V \subseteq c^*_\nu(V) \subseteq X-F$, $c^*_\nu(V) \cap F = \emptyset$, and $F \subseteq X - c^*_\nu(V)$. As $X - c^*_\nu(V) \in \nu^*$ and $V \cap \big(X - c^*_\nu(V)\big) = \emptyset$, it follows that X is $\mathcal{H}(\mu,\nu)$-regular.

Remark 3. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X such that $\mu = \nu$. If X is $\mathcal{H}(\mu,\nu)$-regular, then X is also $\mathcal{H}$-regular.*

Proposition 1. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. If X is (μ,ν)-regular, then X is $\mathcal{H}(\mu,\nu)$-regular.*

Proof. Let X be (μ,ν)-regular. Consider $x \in X$ and an μ-closed set F such that $x \notin F$. Then $X-F$ is a μ-open set containing x. By Theorem 3, there exists a μ-open set V containing x such that

$$x \in V \subseteq c_\nu(V) \subseteq X-F.$$

Since $c^*_\nu(V) \subseteq c_\nu(V)$, it follows that

$$x \in V \subseteq c^*_\nu(V) \subseteq X-F.$$

By Theorem 13, X is $\mathcal{H}(\mu,\nu)$-regular.

Theorem 14. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If X is $\mathcal{H}(\mu,\nu)$-regular, then the following hold:*

(i) For any $A \subseteq X$, $\gamma_*(\mu,\nu)(A) = c_\mu(A)$.

(ii) Every μ-open set is $\mathcal{H}(\theta(\mu,\nu))$-open.

Proof. (1) By Theorem 5(iii), we have $c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$. To show the reverse inclusion, let $x \in \gamma_*(\mu,\nu)(A)$ and let U be any μ-open set containing x. From $\mathcal{H}(\mu,\nu)$-regularity, there exists a μ-open set V such that $x \in V \subseteq c^*_\nu(V) \subseteq U$. Since $x \in \gamma_*(\mu,\nu)(A)$, it follows that $c^*_\nu(V) \cap A \neq \emptyset$. Thus, $U \cap A \neq \emptyset$, implying $x \in c_\mu(A)$.

(2) Let M be a μ-open set. From (1), we have $\gamma_*(\mu,\nu)(X-M) = c_\mu(X-M) = X-M$. By Theorem 10, $X-M$ is $\mathcal{H}(\theta(\mu,\nu))$-closed, which means M is $\mathcal{H}(\theta(\mu,\nu))$-open.

The next result follows from Theorem 8 and Theorem 14(ii).

Corollary 7. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If X is $\mathcal{H}(\mu,\nu)$-regular, then $\mu = \mathcal{H}(\theta(\mu,\nu))$.*

Definition 7. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. We define the following notions:*

$$\ell_{\mathcal{H}(\theta(\mu,\nu))(A)} = \{x \in X : c^*_\nu(M) \subseteq A \text{ for some } \mu\text{-open set } M \text{ containing } x\}.$$

$$\ell_{\theta(\mu,\nu))(A)} = \{x \in X : c_\nu(M) \subseteq A \text{ for some } \mu\text{-open set } M \text{ containing } x\}.$$

$$\ell_{\mathcal{H}(\theta)(A)} = \{x \in X : c^*_\mu(M) \subseteq A \text{ for some } \mu\text{-open set } M \text{ containing } x\}.$$

Proposition 2. *For any two GT's ν_1 and ν_2 on a nonempty set X, we have $\ell_{\theta(\nu_1,\nu_2)}(A) \subseteq \ell_{\mathcal{H}(\theta(\mu,\nu))(A)}$ for any $A \subseteq X$.*

Proof. Let $x \in \ell_{\theta(\mu,\nu)}(A)$. Then there exists a μ-open set M containing x such that $c_\nu(M) \subseteq A$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, it follows that $c^*_\nu(M) \subseteq A$. Therefore, $x \in \ell_{\mathcal{H}(\theta(\mu,\nu))(A)}$.

Remark 4. *Let μ and ν be two GTs on a nonempty set X, and let $A \subseteq X$. If $\mu = \nu$, then $\ell_{\mathcal{H}(\theta(\mu,\nu))(A)} = \ell_{\mathcal{H}(\theta)(A)}$.*

Theorem 15. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A \subseteq X$. Then the following properties hold:*

(i) $i_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - c_{\mathcal{H}(\theta(\mu,\nu)}(X-A)$ *and* $c_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - i_{\mathcal{H}(\theta(\mu,\nu)}(X-A)$.

(ii) $\ell_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - \gamma_*(\mu,\nu)(X-A)$ *and* $\gamma_*(\mu,\nu)(A) = X - \ell_{\mathcal{H}(\theta(\mu,\nu)}(X-A)$.

Proof. The proof is obvious.

The following Corollary comes directly from Definition 4 and Definition 7.

Corollary 8. *Let μ and ν be two GTs on a nonempty set X and $A \subseteq X$. Then $i_{\mathcal{H}(\theta(\mu,\nu))}(A)$ if and only if there exists a μ-open set M containing x such that $M \subseteq c^*_\nu(M) \subseteq A$.*

5. Conclusion

This study aimed to introduce and examine the operation $\gamma_*(\mu, \nu)$ and $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open sets within generalized topological spaces. Several significant results regarding these concepts were established. We thoroughly investigated the relationships among $\gamma_*(\mu, \nu)$, γ_*, and the μ-closure, as well as those among $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open sets, $\theta(\mu, \nu)$-open sets, and μ-open sets. Finally, we have derived various properties and characterizations of $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open sets in terms of the concept of $\mathcal{H}(\mu, \nu)$-regularity.

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