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Parametric Estimation for a New Pareto-Type Model Based on Constant-Stress Partially Accelerated Censoring Data

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Abstract. In Low failure or High longevity, accelerated life tests (ALTs) are used to speed up tests. This paper investigates precise estimation issues related to point and interval estimations for a new Pareto-type (NPT) distribution. The data are collected under constant stress partially (ALTs) concerning Type-I generalized hybrid censored samples (Type-I GHCS). The point maximum likelihood estimates (MLEs) of the model parameters and accelerated factor are obtained with the help of the expectation–maximization (EM) algorithm. Also, Bayes estimates under various loss functions with the help of the Metropolis-Hastings (MH) algorithm method are constructed. The asymptotic confidence intervals, bootstrap confidence interval, and highest posterior density (HPD) credible intervals are derived. The performance of different estimators is compared with the help of a Monte Carlo simulation study. Finally, we analyze a real data set to show the applicability of the model considered.

2020 Mathematics Subject Classifications: 62F10, 62F12, 62F15, 62F40, 62N02

Key Words and Phrases: A new Pareto-type distribution, Accelerated life tests model, Type-I generalized hybrid censoring scheme, EM algorithm, Maximum likelihood estimation, Bayesian estimation

1. Introduction

There are many real-life scenarios where data demand a probability distribution with decreasing and/or upside down bathtub (unimodal) shaped failure rate functions. The probability that an event occurs at a given moment, given that it has not yet occurred, decreases over time a phenomenon known as a decreasing failure rate. Patients undergoing

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heart transplants, for example, face an increasing hazard of death in the first few days after the transplant while the body adjusts to the new organ. As the patient improves, the hazard rate reduces Ref.[1]. The upside-bathtub-shaped failure rate function would be appropriate in this case. To represent a model of failure rate with a relatively high rate of failure in the middle of the predicted lifetime, an unimodal hazard rate function is used. When product failures are driven by fatigue and corrosion, the related failure rates have unimodal forms Ref.[2]. In addition, in some medical scenarios, such as breast cancer and infection with novel viruses, the hazard rate is unimodal, as demonstrated by Ref.[3] and Ref.[4]. The NPT distribution exhibits both a decreasing and an upside-down bathtub (unimodal) shaped failure rate function Ref.[5]. NPT distribution is considered a generalization of the well-known Pareto distribution. When the "probability" or fraction of the population that owns a small amount of wealth per person is relatively high, and then steadily decreases as wealth increases, as in Ref.[6], experimenter seeks a probability distribution describing this case, the NPT model is an excellent alternative to the most common model Pareto distribution. Because it is used to analyze various income and reliability data, this distribution might be a versatile model. It is used in actuarial sciences, reliability, finance, and climatology to describe the occurrence of extreme weather events. In light of these characteristics, multiple authors adopted this model in data modeling, particularly censored observations. As a result, our motivation to validate this distribution throughout this article stems from its practical usefulness in the various areas outlined above, as well as the supporting references. Among these references, [7] proposed the comparisons methods of estimation for the model, Ref.[8] studied the ML, Bayes estimation and prediction of the model under progressive type-II censoring. Also, Ref.[9] derived simpler expressions for many relevant economic inequality and risk indices using the incomplete beta function, Ref.[10] presented a generalized of the NPT distribution. Recently, Ref.[11] developed inference based on NPT records with applications to precipitation and Covid-19 data. The NPT distribution, denoted by NPT(θ, σ) is specified by the following Probability Density Function (PDF), Cumulative Distribution Function (CDF), Survival Function (SF), and Hazard Rate function (HRF) given by

$$\begin{aligned} f(x) &= \frac{2\theta\sigma^{\theta}x^{\theta-1}}{(x^{\theta}+\sigma^{\theta})^{2}} \;\; ; \quad x \geq \sigma, \;\; \theta, \sigma > 0 \\ F(x) &= 1 - \frac{2\sigma^{\theta}}{x^{\theta}+\sigma^{\theta}} \\ S(x) &= \frac{2\sigma^{\theta}}{x^{\theta}+\sigma^{\theta}} \\ h(x) &= \frac{\theta x^{\theta-1}}{x^{\theta}+\sigma^{\theta}}. \end{aligned} \tag{1}$$

Different potential scenarios concerning the reliability and quality of items face the challenge of lacking adequate knowledge about the failure of such products under normal operating conditions. To address this, experimenters uses accelerated life testing (ALT) or partially accelerated life testing (PALT) to quickly gather sufficient failure data and understand the relationship between failures and external stress factors. These tests can significantly save time, labor, resources, and costs. In ALTs, all units are subjected to stress levels higher than usual to induce failures more quickly. In contrast, PALTs involve testing units under both normal and elevated stress conditions in order to collect more failure data within a constrained time frame without subjecting all units to high stress. PALTs are particularly beneficial in scenarios where time and cost are pressing concerns. The information obtained from such tests can be utilized to estimate the failure behavior of the units under normal conditions. Ref.[12] performed a proper statistical modeling of PALTs, where the authors considered the tempered random variable model for PALTs. In contrast, with a step stress ALT model, the stress level changes at specific times or when a designated number of units fail. As a result, the stress on all the products increases gradually according to some type of rule. Many authors have extensively researched the step stress models. For recent literature on this topic, Refs. [13],[14], and [15]. To better extrapolate the lifetime under normal use conditions, partially accelerated life test (PALT) was developed. It divides all products into groups and test each in normal and accelerated environments. Two sorts of prevalent PALTs are known as the constant stress partially accelerated life tests (CSPALTs) and the step stress partially accelerated life tests (SSPALTs). CSPALTs set different groups of units under use and stress condition respectively. By comparison, SSPALTs start with normal conditions and switch to accelerated conditions if the unit does not cease to function before a prefixed point in time τ. The specific illustration of the differences between the two PALTs are displayed in Fig.(1).

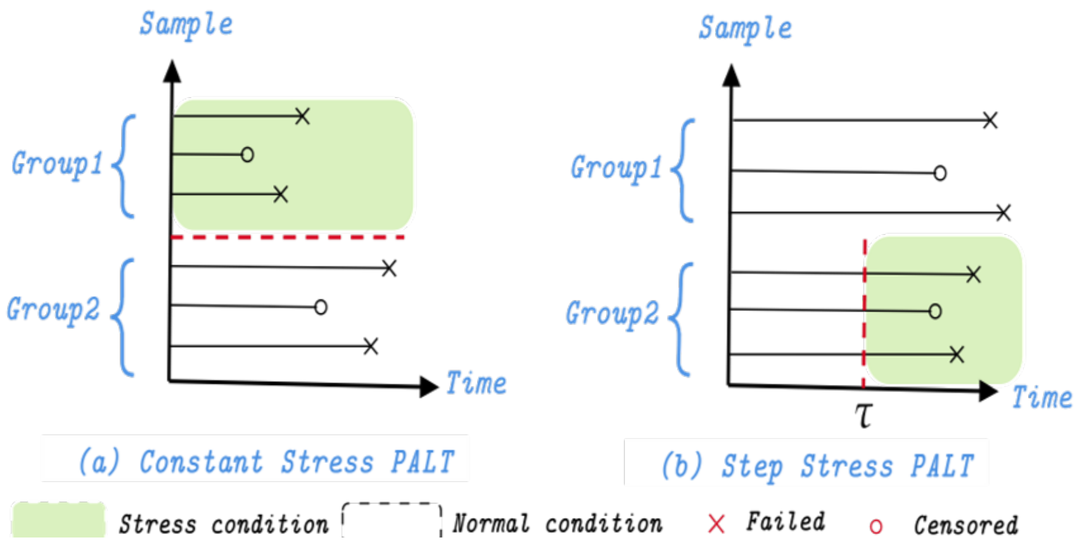

Figure 1: A schematic diagram of (a) CSPALTs and (b) SSPALTs

Our primary focus in this paper lies within the realm of CSPALTs. Numerous research studies have also given attention to CSPALTs, as illustrated by, for instance, Refs.[16],[17],[18],[19],[20], and [21]. Even though the major goal of partial ALT is to reduce the testing period of the experiment, the experimenter spends a significant amount of time waiting for all test units to reach the point of failure. To overcome this problem, censoring schemes are employed to terminate life tests. Type-I and Type-II censoring are commonly regarded as two main scheme for conducting lifetime experiments, in which we terminate the experiments at a specific time point or when a given amount of failures occur. Ref.[22] merged the two censoring techniques mentioned earlier and introduced the initial hybrid censoring scheme, referred to as the Type-I hybrid censoring scheme (Type-I HCS). Within the Type-I HCS , the life testing experiment concludes either at the designated time T or when a predetermined count of m items encounter failure. Therefore, the cessation moment for Type-I HCS is given by $T^* = \min(X_{m:n}, T)$, where $X_{m:n}$ represents the duration until m out of n testing items experience failure. Many researchers have worked on Type-I HCS, including Ref.[23]. One downside of Type-I HCS is that a small number of failures may occur till after a set duration T^*. Hence, drawing statistical conclusions within this type of scheme might prove to be unworkable. To address this drawback and enhance the effectiveness of estimators within life-testing experiments, while also ensuring a specific count of failures before the experiment concludes, and concurrently reducing testing time and costs associated with unit failures, a generalized hybrid Type-I censoring scheme was introduced by Ref.[24]. The Type-I Generalized Hybrid Censoring scheme (Type-I GHCS) approach ensures a minimum threshold of failures, which helps alleviate the shortfall present in Type-I HCS. We suppose, $\{X_{1:n}, \ldots, X_{n:n}\}$ constitute a sequence of ordered observations concerning the lifetimes of failures within a set of size n. The values of l, m, and T are defined in advance, where $l < m < n$ represents the smallest acceptable number of failures fixed before the experiment, m represents the intended quantity of failures, and T represents a certain time instant. These three censoring models mentioned previously can be stated as follows;

- Type-I CS: terminate at T.
- Type-I HCS: terminate at $T^* = \min(X_{m,n}, T)$.
- Type-I GHCS: terminate at $T^* = \max(X_{l:n}, \min(X_{m:n}, T))$.

This article focuses on Type-I GHCS, which is classified into three categories such as;

- $\{X_{1:n} < \cdots < X_{l:n}\}$, *When*, $X_{l:n} > T$.
- $\{X_{1:n} < \cdots < X_{d^*:n}\}$, *When*, $X_{l:n} < T < X_{m:n}$.
- $\{X_{1:n} < \cdots < X_{m:n}\}$, *When*, $X_{m:n} < T$.

Additional literature on Type-I GHCS has recently been offered, including Refs.[25],[26],[27], [28] and, [29]. The purpose of this article is parameters estimation of the NPT distribution based on a partially constant-stress ALT under Type-I GHCS. Maximum likelihood (ML) and Bayesian methods will be used to estimate the parameters of the NPT distribution. The Newton-Raphson (NR) and EM algorithms are developed and described in detail when constructing maximum likelihood estimates (MLEs) and their corresponding confidence intervals. When the Bayes approach is applied, the posterior distribution and the corresponding Bayes estimate often require computing integrals, which can be difficult, especially when using complex high- or low-dimensional models. We applied the MCMC method to estimate the unknown parameters under squared error loss function with dependent priors for the parameters. The Gibbs within Metropolis samples can be used to produce samples from posterior distributions using various MCMC algorithms. Extensive computer studies can be used to compare the performance of Bayes estimators against that of classical MLEs. Using the NR, EM, and MCMC algorithms, approximate 95% confidence intervals for unknown parameters can be generated. We will also compare them based on their average lengths and coverage probability. This article is structured as follows: In Section (2), the proposed model and its fundamental assumptions are discussed. Section (2), we display the point estimate with, maximum likelihood estimation by using Newton-Raphson and expectation maximization method and Bayesian estimates with the help of MCMC method. Section (2) deals with interval estimates. Approximate confidence intervals, bootstrap confidence intervals, and credible intervals for the unknown parameters are also constructed. The findings of the simulation investigation are outlined in Section (3). A real-life example and the simulated example are presented and discussed in Section (3). Finally, some concluding remarks are given in Section (4).

2. Methodology

In this particular instance, unit lifetime has NPT distribution in the model that is being developed. The point estimates of Parameters are created using the Bayesian, EM algorithm and MLE approaches. Additionally, interval estimators are developed using bootstrap methods, HPD credible intervals, and the asymptotic property of MLEs.

2.1. Modeling

Suppose in a partially constant-stress ALT model under Type-I GHCS, n indicates that items are separated into two groups with sizes n_1 and $n_2 = n - n_1$. The normal condition is assigned to items n_1, and the stress condition is assigned to items n_2. The hazard failure rate of an item under the stress condition is given by $h_2(X) = \lambda h_1(X)$, where $h_1(X)$ is the FR function under normal conditions, and $\lambda > 1$ is the acceleration factor. Suppose that the lifetime of an item follows the two-parameter NPT distribution. As a result of Eq.(1), the PDF, CDF, and FR function of the lifetime X under accelerated conditions are given respectively by;

$$\begin{aligned} f_2(x) &= \frac{2^{\lambda}\theta\lambda\sigma^{\theta\lambda}x^{\theta-1}}{(x^{\theta}+\sigma^{\theta})^{1+\lambda}} \;\; ; \quad x \geq \sigma \\ F_2(x) &= 1 - \frac{2^{\lambda}\sigma^{\theta\lambda}}{(x^{\theta}+\sigma^{\theta})^{\lambda}} \\ h_2(x) &= \frac{\theta\lambda x^{\theta-1}}{x^{\theta}+\sigma^{\theta}}. \end{aligned} \tag{2}$$

Let, $X_j = \{X_{j1}, \ldots, X_{jdj}\}, j = 1, 2$, and $X = \{X_1, X_2\}$, be the set of data under normal and accelerate conditions, respectively. Also, suppose fixed integers $1 < l_j < m_j < n_j$ and fixed time $T \in (0, \infty)$. If the l_j^{th} failure occurs before time T, terminate the experiment at $\mathbf{min}\{X_{m_j:n_j}, T\}$. If the l_j^{th} failure occurs after time T, terminate the experiment at $X_{l_j:n_j}$. Under this setting, the experimenter would ideally like to observe m_j failures but is willing to accept a bare minimum of l_j failures. The realization of the failure time sample can be presented by $X_{(j1)} \leq X_{(j2)} \leq \cdots \leq X_{(jD_j)}$, D_j represents the number of failures before time point T, where;

$$D_j = \begin{cases} l_j & if \quad X_{l_j} > T \\ d_j^* & if \quad X_{l_j} < T < X_{m_j} \\ m_j & if \quad X_{m_j} < T \end{cases}$$

Moreover, let C_j denote the recorded lifetimes for all survived components (terminated time), then C_j can also be expressed as;

$$C_j = \begin{cases} X_{l_j} & if \quad X_{l_j} > T \\ T & if \quad X_{l_j} < T < X_{m_j} \\ X_{m_j} & if \quad X_{m_j} < T \end{cases}$$

For given type-I GHC sample under partially constant-stress ALT, $X_j = \{X_{j1}, \ldots, X_{jdj}\}$, $j = 1, 2$ the joint likelihood function can be written as

$$L(\Phi|X) \quad = \quad \prod_{j=1}^{2} \frac{n_j!}{(n_j - D_j)!}(1 - F_j(C_j|\Phi))^{-(D_j - n_j)} e^{\sum_{i=1}^{D_j} \log(f_j(x_i|\Phi))}, \tag{3}$$

where $\Phi = \{\theta, \sigma, \lambda\}$.

2.2. Point Estimation

In this subsection, we discuss ML and Bayes point estimations under accelerated type-I GHC sample from NTP distribution.In the ML approach, we estimate the parameters using the Newton-Raphson and EM algorithms. In the Bayesian approach, we use the MCMC method.

2.2.1. ML estimation

The purpose of ML estimation is to identify the values of model parameters that optimize the likelihood function across the whole parameter space. The ML technique is a commonly employed statistical inference technique that may be applied to a diverse range of distributions and models. The ML estimator possesses several key attributes, including consistency, asymptotic efficiency, asymptotic unbiasedness, and asymptotic normality. These quantiles have contributed to the widespread adoption and significance of the MLE as a prominent technique for statistical model fitting. The related PDF and CDF are given in Eqs.(1) and (2). Therefore, the likelihood function is derived as shown below;

$$L(\theta,\sigma,\lambda|X) \propto \ 2^{\lambda n_2}\theta^{D_1+D_2}\lambda^{D_2}\sigma^{\theta(D_1+\lambda D_2)}\left[1+(\frac{C_1}{\sigma})^{\theta}\right]^{(D_1-n_1)}\left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda(D_2-n_2)}$$
$$\times\prod_{j=1}^{2}\prod_{i=1}^{D_j}\frac{x_{ji}^{-(S_j(\lambda)\ \theta+1)}}{\left[1+(\frac{\sigma}{x_{ji}})^{\theta}\right]^{S_j(\lambda)+1}},\ \sigma\ \leq\ \mathbf{min}\{x_{11},x_{21}\}, \tag{4}$$

and

$$S_j(\lambda)=\begin{cases}1, & When, \quad j=1\\ \lambda, & When, \quad j=2.\end{cases}$$

The ML estimators of the parameters θ, σ, and λ are the values that maximize the likelihood function, as shown in Eq.(4).
Maximizing the likelihood function presents challenges, so it is more advantageous to optimize its natural logarithm. The natural logarithm associated with Eq.(4), represented as $\ell(\theta,\sigma,\lambda|X)$, can be expressed as;

$$\ell(\theta,\sigma,\lambda|X)\propto n_2\lambda\log 2+(D_1+D_2)\log\theta+D_2\log\lambda+\theta(D_1+\lambda D_2)\log\sigma$$
$$+(D_1-n_1)\log\left(1+(\frac{C_1}{\sigma})^{\theta}\right)+\lambda(D_2-n_2)\log\left(1+(\frac{C_2}{\sigma})^{\theta}\right)$$
$$-\sum_{j=1}^{2}(S_j(\lambda)\theta+1)\sum_{i=1}^{D_j}\log x_{ji}-\sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\log(1+(\frac{\sigma}{x_{ji}})^{\theta}). \tag{5}$$

The ML estimate of σ is $\hat{\sigma}_{\mathrm{ML}}=x_1$. By calculating the first derivatives of Eq.(5) with respect to θ and λ, and then setting them equal to zero, we obtain the following system of simultaneous equations.

$$\frac{\partial\ell^*(\theta,\lambda|X)}{\partial\theta}=\frac{D_1+D_2}{\theta}+(D_1+\lambda D_2)\log x_1+\frac{(D_1-n_1)(\frac{C_1}{x_1})^{\theta}\log(\frac{C_1}{x_1})}{1+(\frac{C_1}{x_1})^{\theta}}-\sum_{j=1}^{2}S_j(\lambda)\sum_{i=1}^{D_j}\log x_{ji}$$
$$+\frac{\lambda(D_2-n_2)(\frac{C_2}{x_1})^{\theta}\log(\frac{C_2}{x_1})}{1+(\frac{C_2}{x_1})^{\theta}}\ -\sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\frac{(\frac{x_1}{x_{ji}})^{\theta}\log(\frac{x_1}{x_{ji}})}{1+(\frac{x_1}{x_{ji}})^{\theta}}=0, \tag{6}$$

and

$$\frac{\partial\ell^*(\theta,\lambda|X)}{\partial\lambda}=n_2\log 2+\frac{D_2}{\lambda}+D_2\ \theta\ \log x_1+(D_2-n_2)\log\left(1+(\frac{C_2}{x_1})^{\theta}\right)-\theta\sum_{i=1}^{D_2}\log x_{2i}$$
$$-\sum_{i=1}^{D_2}\log(1+(\frac{x_1}{x_{2i}})^{\theta})=0. \tag{7}$$

Using Eq.(7) and for fixed θ, one can obtain the MLE of the unknown parameter λ as a function of the parameter θ as shown below;

$$\hat{\lambda}_{\mathrm{ML}}(\theta) = \frac{D_2}{-n_2 \log 2 - D_2\theta \log x_1 - (D_2 - n_2) \log\left(1 + \left(\frac{C_2}{x_1}\right)^{\theta}\right) + \theta \sum_{i=1}^{D_2} \log x_{2i} + \sum_{i=1}^{D_2} \log\left(1 + \left(\frac{x_1}{x_{2i}}\right)^{\theta}\right)} \tag{8}$$

By substituting $\hat{\lambda}_{\mathrm{ML}}(\theta)$ in the normal equation given by Eq.(6), the ML estimate of θ can be obtained by solving the following non-linear equation;

$$\begin{aligned} \frac{\partial \ell^*(\theta, \lambda|X)}{\partial \theta} &= \frac{D_1 + D_2}{\theta} + (D_1 + \hat{\lambda}_{\mathrm{ML}}(\theta) D_2) \log x_1 + (D_1 - n_1) \frac{\left(\frac{C_1}{x_1}\right)^{\theta} \log\left(\frac{C_1}{x_1}\right)}{1 + \left(\frac{C_1}{x_1}\right)^{\theta}} \\ &+ \hat{\lambda}_{\mathrm{ML}}(\theta)(D_2 - n_2) \frac{\left(\frac{C_2}{x_1}\right)^{\theta} \log\left(\frac{C_2}{x_1}\right)}{1 + \left(\frac{C_2}{x_1}\right)^{\theta}} - \sum_{i=1}^{D_1} \log x_{1i} - \hat{\lambda}_{\mathrm{ML}}(\theta) \sum_{i=1}^{D_2} \log x_{2i} \\ &- 2 \sum_{i=1}^{D_1} \frac{\left(\frac{x_1}{x_{1i}}\right)^{\theta} \log\left(\frac{x_1}{x_{1i}}\right)}{1 + \left(\frac{x_1}{x_{1i}}\right)^{\theta}} - (\hat{\lambda}_{\mathrm{ML}}(\theta) + 1) \sum_{i=1}^{D_2} \frac{\left(\frac{x_1}{x_{2i}}\right)^{\theta} \log\left(\frac{x_1}{x_{2i}}\right)}{1 + \left(\frac{x_1}{x_{2i}}\right)^{\theta}}, \end{aligned} \tag{9}$$

It is noteworthy to emphasis that the solution to Eq.(9) is analytically infeasible. Hence, the explicit derivation of the maximum likelihood estimator $\hat{\theta}_{\mathrm{ML}}$ poses a significant challenge. The required estimates can be obtained using numerical techniques, such as the Newton-Raphson method. It can be noted from Eq.(8) that the maximum likelihood estimator $\hat{\lambda}_{\mathrm{ML}}$ can be explicitly expressed as a function of the maximum likelihood estimator of the parameter θ.

2.2.2. EM algorithm

The EM algorithm in this section is employed as a viable alternative method to approximate the ML estimate of the unknown model parameters of the NPT distribution. This is done by taking into account the incomplete nature of the available sample, which is characterized as a partially constant-stress ALT under type-I GHCS. The algorithm in the discussion was initially put forth by Ref.[30] and subsequently received comprehensive analysis, along with its various adaptations, in the publication authored by Ref.[31]. The EM algorithm involves a cyclic process that consists of an expectation step (E-step) and a maximization step (M-step). The E-step involves the computation of an expectation function for the log-likelihood of the current estimation evaluation, utilizing the given parameters. On the other hand, the M-step entails the determination of the parameters that optimize the expected log-likelihood obtained from the E-step. For $j = 1, 2$, let $\mathbf{X}_j = \{x_{j1}, \ldots, x_{jD_j}\}$ be the observed data and $\mathbf{U}_j = \{u_{j1}, \ldots, u_{j(n_j - D_j)}\}$ be the censored data under the normal use and accelerated conditions, respectively. For a given $D_j, u_{j1}, \ldots, u_{j(n_j - D_j)}$ are not observable. We treated the censored observations as missing data. Thus, the combination of $(\mathbf{X}_j, \mathbf{U}_j)$ forms as $W_j = (\mathbf{X}_j, \mathbf{U}_j)$ represents the complete partially constant-stress ALT failure data set, for which the likelihood function

$L_c(w;\Phi)$can be expressed as;

$$L_c(w;\Phi) \propto \prod_{j=1}^{2}\left\{ \prod_{i=1}^{D_j} f_j(x_{ji;\Phi}) \prod_{i=1}^{n_j-D_j} f_j(u_{ji;\Phi}) \right\}. \tag{10}$$

After disregarding the constants in the above equation, the log-likelihood function denoted as $L_c(w;\Phi)$ can be represented as $\ell_c(w;\Phi)$;

$$\begin{aligned} \ell_c(w;\Phi) \propto\ & \lambda n_2 \log 2 + (n_1+n_2)\log\theta + n_2\log\lambda + \theta(n_1+\lambda n_2)\log\sigma \\ & + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\log(x_{ji}^{\theta}+\sigma^{\theta}) \\ & + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{n_j-D_j}\log u_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{n_j-D_j}\log(u_{ji}^{\theta}+\sigma^{\theta}). \end{aligned} \tag{11}$$

It is evident that the function $\ell_c(w;\Phi)$ exhibits a monotonic increase with respected to σ. Therefore, within the framework of the EM method, the maximum likelihood estimate (MLE) of σ is given by $\hat{\sigma}_{\text{EM}} = x_1$. By substituting the symbol σ into Eq.(11), we can derive the profile log-likelihood function for the parameters θ and λ;

$$\begin{aligned} \ell_c(w;\Phi) \propto\ & \lambda n_2 \log 2 + (n_1+n_2)\log\theta + n_2\log\lambda + \theta(n_1+\lambda n_2)\log x_1 \\ & + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\log(x_{ji}^{\theta}+x_1^{\theta}) \\ & + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{n_j-D_j}\log u_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{n_j-D_j}\log(u_{ji}^{\theta}+x_1^{\theta}). \end{aligned} \tag{12}$$

The EM algorithm consists of two primary steps: The initial phase, known as the expectation step (E-step), is succeeded by the maximization step (M-step), and this iterative process continues until meeting the specified convergence criteria. During each iteration, the missing data are imputed with expected values, resulting in the subsequent update of parameter estimations.

- **Expectation Step (E-step):**
  This phase entails calculating the conditional expectation of the log-likelihood, considering the incomplete data given the observed data. In order to meet the requirements of the E-stage, it is necessary to calculate the pseudo-log-likelihood function. This may be derived from the function $\ell_c(w;\Phi)$ by substituting any function of u_{ji}, denoted as $\mathrm{g}(u_{ji})$, with the corresponding conditional expectation "$\mathbf{E}\left[\mathrm{g}(u_{ji})|u_{ji} > C_j\right]$". consequently;

  $$\ell_s(\theta,\lambda) = \mathbf{E}\left[\ell_c(w;\theta,\lambda)|\mathbf{U}_j\right] = \lambda n_2\log 2 + (n_1+n_2)\log\theta + \theta(n_1+\lambda n_2)\log x_1$$

$$+ (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{n_j-D_j}\mathbf{E}\Big[\log u_{ji}\Big] - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{n_j-D_j}\mathbf{E}\Big[\log(u_{ji}^{\theta}+x_1^{\theta})\Big]$$

$$+ n_2\log\lambda + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\log(x_{ji}^{\theta}+x_1^{\theta}) \quad (13)$$

Following that, the expectation step (E-step) requires the calculation of the expected value of the logarithm of u_{ji}, given that u_{ji} is greater than C_j. The expression may be rewritten as $\mathbf{E}\Big[\log(u_{ji}^{\theta}+x_1^{\theta})\Big| u_{ji} > C_j\Big]$, where u_{ji} and x_1 are variables, θ is a parameter. The conditional probability function of the censored data, given the observed data, can be computed according to Ref.[32].

$$\begin{aligned} f_{U_j|X_j}(u_{ji}|x_{ji};\Phi) &= \frac{f(u_{ji},\Phi)}{1-F(x_{ji},\Phi)} \\ &= \frac{\theta S_j(\lambda)(C_j^{\theta}+\sigma^{\theta})^{S_j(\lambda)} u_{ji}^{\theta-1}}{(u_{ji}^{\theta}+\sigma^{\theta})^{S_j(\lambda)+1}}, \quad u_{ji} > C_j, i = 1,\ldots,D_j, j = 1,2. \end{aligned} \quad (14)$$

Hence, the required expected values of a NPT distribution from the left at C_j are, respectively, given by;

$$\begin{aligned} \mathbf{E}\Big[\log(u_{ji})|u_{ji} > C_j\Big] &= \theta S_j(\lambda)(C_j^{\theta}+x_1^{\theta})^{S_j(\lambda)}\int_{C_j}^{\infty}\frac{y^{\theta-1}\log(y)}{(y^{\theta}+x_1^{\theta})^{S_j(\lambda)+1}}dy \\ &= \log(C_j) + (C_j^{\theta}+x_1^{\theta})^{S_j(\lambda)}\int_{C_j}^{\infty}\frac{1}{y(y^{\theta}+x_1^{\theta})^{S_j(\lambda)}}dy \\ &= \Im_1(C_j,\theta,\lambda). \end{aligned} \quad (15)$$

and

$$\begin{aligned} \mathbf{E}\Big[\log(u_{ji}^{\theta}+x_1^{\theta})|u_{ji} > C_j\Big] &= \theta S_j(\lambda)(C_j^{\theta}+x_1^{\theta})^{S_j(\lambda)}\int_{C_j}^{\infty}\frac{y^{\theta-1}\log(y^{\theta}+x_1^{\theta})}{(y^{\theta}+x_1^{\theta})^{S_j(\lambda)+1}}dy \\ &= \log(C_j^{\theta}+x_1^{\theta}) + \frac{1}{S_j(\lambda)} \\ &= \Im_2(C_j,\theta,\lambda). \end{aligned} \quad (16)$$

- **Maximization Step (M-step)**:
  This step involves the maximization of the pseudo log-likelihood function Eq.(13) with respect to (θ,λ). In this regard suppose $(\theta_{(s)},\lambda_{(s)})$ denotes the s^{th} stage estimate of (θ,λ) then the next stage updated estimate $(\theta_{(s+1)},\lambda_{(s+1)})$ is obtained by maximizing the following function;

$$\mathrm{Q}(\theta,\lambda) = \lambda n_2\log 2 + (n_1+n_2)\log\theta + \theta(n_1+\lambda n_2)\log x_1 + (\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji}$$

$$- \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\log(x_{ji}^{\theta}+x_1^{\theta}) + (\theta-1)\sum_{j=1}^{2}(n_j-D_j)\Im_1(C_j,\theta_{(s)},\lambda_{(s)})$$

$$+ n_2\log\lambda - \sum_{j=1}^{2}(S_j(\lambda)+1)(n_j-D_j)\Im_2(C_j,\theta_{(s)},\lambda_{(s)}). \tag{17}$$

By taking the derivatives of Eq.(17) with respect to θ and λ, respectively, and equating them to zero, as follows;

$$\frac{\partial \mathrm{Q}(\theta,\lambda)}{\partial\theta} = \frac{n_1+n_2}{\theta} + (n_1+\lambda n_2)\log x_{(1)} + \sum_{j=1}^{2}(n_j-D_j)\Im_1(C_j,\theta_{(s)},\lambda_{(s)})$$

$$+ \sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji} - \sum_{j=1}^{2}(S_j(\lambda)+1)\sum_{i=1}^{D_j}\frac{x_{ji}^{\theta}\log x_{ji}+x_1^{\theta}\log x_1}{x_{ji}^{\theta}+x_1^{\theta}} = 0, \tag{18}$$

and

$$\frac{\partial \mathrm{Q}(\theta,\lambda)}{\partial\lambda} = n_2\log 2 + \frac{n_2}{\lambda} + \theta\ n_2\log x_{(1)} - \sum_{i=1}^{D_2}\log(x_{2i}^{\theta}+x_1^{\theta}) - (n_2-D_2)\Im_2(C_2,\theta_{(s)},\lambda_{(s)}) = 0. \tag{19}$$

For this purpose, we first find $\theta_{(s+1)}$ by using the method of fixed point iteration as Ref. [33]. Where we have;

$$h(\theta) = \frac{n_1+n_2}{-(n_1+\hat{\lambda}(\theta)n_2)\log x_{(1)} - \sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji} - \sum_{j=1}^{2}(n_j-D_j)\Im_1(C_j,\theta_{(s)},\lambda_{(s)}) + 2A_1 + (\hat{\lambda}(\theta)+1)A_2}, \tag{20}$$

with

$$A_j = \left\{\sum_{i=1}^{D_j}\frac{x_{ji}^{\theta}\log(x_{ji})+x_1^{\theta}\log(x_1)}{x_{ji}^{\theta}+x_1^{\theta}}\right\}$$

and

$$\hat{\lambda}(\theta) = \frac{n_2}{-n_2\log(2) - \theta n_2\log(x_{(1)}) + \sum_{i=1}^{D_2}\log(x_{2i}^{\theta}+x_1^{\theta}) + (n_2-D_2)\Im_2(C_2,\theta_{(s)},\lambda_{(s)})}. \tag{21}$$

Here, $\theta_{(s)}$ and $\lambda_{(s)}$ are the estimate of θ and λ in the s^{th} step, respectively. Finally after finding $\theta^{(s+1)}$, the estimate $\lambda^{(s+1)}$ is derived as $\lambda^{(s+1)} = \hat{\lambda}(\theta^{(s+1)})$.

The MLEs of (θ,λ) can be obtained by repeating the E-step and M-step until convergence is achieved. Now, an iterative process can be employed to obtain the necessary maximum likelihood estimates of (θ,λ). This iterative procedure continues until.

$$\mid \theta^{(s+1)} - \theta^{(s)} \mid + \mid \lambda^{(s+1)} - \lambda^{(s)} \mid < \ \epsilon$$

for a predetermined small value of ϵ and some s. This approach converges to the local maximum likelihood as the log-likelihood increases with each iteration. In the expectation-maximization technique, we initialize the parameters using their maximum likelihood estimates based on the entire sample.

2.2.3. Bayes estimation

Unlike traditional statistics, Bayesian estimation incorporates prior information about life parameters. This approach combines the data provided with prior probabilities to infer the parameters of interest, making the inference process more objective and reasonable. A loss function is used to evaluate the difference between an estimated value and the true value of the parameter. In this section, we focus on objective Bayesian estimation of the unknown parameters θ, σ, and λ with respect to squared error (SE) and linear exponential (LINEX) loss functions. For the selection of prior distribution of unknown parameters is discussed and its simulation results are satisfactory in Ref.[8]. We adopt the same prior distribution as this Ref.[8]. The joint prior probability density function (PDF) of θ and σ is considered to be given by;

$$\pi_1^*(\theta,\sigma) \quad \propto \quad \theta^a \sigma^{\theta b-1} c^{-\theta}, \quad \theta > 0,\ 0 < \sigma < d,$$

for θ and σ, where a, b, c, d are positive constants and $d^b < c$. This joint prior has been used in [34, 35] for Bayesian inference of the two-parameter Pareto distribution.
Such a prior specifies $\pi_1^*(\theta)$ as a gamma distribution $Ga(a,\ \log c - b \log d)$ and $\pi_1^*(\sigma|\theta)$ as a power function distribution of the form;

$$\pi_1^*(\sigma|\theta)) \quad = \quad b\,\theta\,\sigma^{b\theta-1}\,d^{-b\theta}, \quad 0 < \sigma < d.$$

By letting $a = -1,\ b = 0,\ c = 1$ and $d \longrightarrow \infty$, this prior reduces to the non informative prior;

$$\pi_1^*(\sigma,\theta) \quad \propto \quad \frac{1}{\theta\sigma}\,, \quad \theta,\ \sigma > 0$$

On the other hand, the prior for the acceleration factor λ is assumed to be the non-informative prior, i.e.

$$\pi_2^*(\lambda) \quad \propto \quad \lambda^{-1}\,, \quad \lambda > 1.$$

Therefore, the joint prior PDF of θ, σ and λ can be written as,

$$\pi^*(\theta,\sigma,\lambda) \quad \propto \quad \theta^a \sigma^{\theta b-1} c^{-\theta} \lambda^{-1}, \quad \theta > 0,\ 0 < \sigma < d,\ \lambda > 1. \tag{22}$$

In combining the prior information from Eq.(22) with the likelihood function presented in Eq.(4), the joint posterior distribution can be represented as follows;

$$\begin{aligned}\pi(\theta,\sigma,\lambda|X) &= A^{-1}\ L(\theta,\sigma,\lambda|X)\ \pi^*(\theta,\sigma,\lambda)\\ &= A^{-1}\ 2^{\lambda n_2}\ \theta^{(D_1+D_2+a)}\ c^{-\theta}\ \lambda^{D_2-1}\ \sigma^{\theta(D_1+\lambda D_2+b)-1}\left[1+(\frac{C_1}{\sigma})^{\theta}\right]^{(D_1-n_1)}\\ &\times\left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda\ (D_2-n_2)}\prod_{i=1}^{D_1}\frac{x_{1i}^{-(\theta+1)}}{\left[1+(\frac{\sigma}{x_{1i}})^{\theta}\right]^2}\prod_{i=1}^{D_2}\frac{x_{2i}^{-(\lambda\theta+1)}}{\left[1+(\frac{\sigma}{x_{2i}})^{\theta}\right]^{\lambda+1}}\end{aligned}$$

$$= A^{-1}\ 2^{\lambda n_2}\ \theta^{(D_1+D_2+a+1)-1}\ c^{-\theta}\ \lambda^{D_2-1}\ \sigma^{\theta(D_1+\lambda D_2+b)-1}\ \left[1+(\frac{C_1}{\sigma})^{\theta}\right]^{(D_1-n_1)}$$
$$\times \left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda(D_2-n_2)}\ e^{-(\theta+1)\sum_{i=1}^{D_1}\log x_{1i}}\ e^{-2\sum_{i=1}^{D_1}\log(1+(\frac{\sigma}{x_{1i}})^{\theta})}$$
$$\times\ e^{-(\lambda\theta+1)\sum_{i=1}^{D_2}\log x_{2i}}\ e^{-(\lambda+1)\sum_{i=1}^{D_2}\log(1+(\frac{\sigma}{x_{2i}})^{\theta})}. \tag{23}$$

Where A is a normalizing constant, provided by;

$$A = \int_1^{\infty}\int_0^{d}\int_0^{\infty} L(\theta,\sigma,\lambda|X)\ \pi^*(\theta,\sigma,\lambda)\ d\theta d\sigma d\lambda.$$

It is clear that Eq.(23) is analytically tricky. Furthermore, the Bayesian estimation of a function with θ, σ and λ is also intractable because it is related to a ratio of three integrals. For solving the corresponding ratio of three integrals, some approximate approaches have been presented in the literature. Among them, the MH algorithm is a simulation method with wide applications in sampling from posterior density functions.
In this article, we use the MH algorithm to derive approximate explicit forms for the Bayesian estimates. In Bayesian statistics, the selection of the loss function is a fundamental step. There are many symmetric loss functions, among which is SE loss function is well known for its good mathematical properties. Let Υ^* be a Bayesian estimate of Υ. The form of SE loss function is;

$$\mathcal{L}_{SE}(\Upsilon^*, \Upsilon) = (\ \Upsilon - \Upsilon^*\ )^2$$

Now, we compute the Bayes estimate of $\Upsilon(\theta, \sigma, \lambda)$ under the SE loss function.

$$\begin{aligned}\Upsilon^*(\theta,\sigma,\lambda)_{SE} &= \mathbf{E}[\Upsilon(\theta,\sigma,\lambda)\ |X] \\ &= \int_1^{\infty}\int_0^{d}\int_0^{\infty} \Upsilon(\theta,\sigma,\lambda)\pi(\theta,\sigma,\lambda\ |X)\ d\theta d\sigma d\lambda. \end{aligned}\tag{24}$$

However, in many practical situations, overestimation and underestimation result in different losses, and the consequence is likely to be quite serious if one uses symmetric loss function indiscriminately. In the literature, many different asymmetric loss functions were used. Among them, LINEX is dominant, and this loss function can be expressed as;

$$\mathcal{L}_{LL}(\Upsilon^*, \Upsilon) = e^{h^*(\Upsilon^*-\Upsilon)} - h^*(\Upsilon^* - \Upsilon) - 1, \quad h^* \neq 0.$$

The constant h^* represented the weight of error on different decisions. Under the above loss function, the Bayesian estimate of the function $\Upsilon(\theta, \sigma, \lambda)$ can be calculated by;

$$\begin{aligned}\Upsilon^*(\theta,\sigma,\lambda)_{LL} &= -\frac{1}{h^*}\mathbf{ln}[\mathbf{E}(e^{-h^*\Upsilon(\theta,\sigma,\lambda)}\ |X)] \\ &= -\frac{1}{h^*}\mathbf{ln}\Big[\int_1^{\infty}\int_0^{D^*}\int_0^{\infty} e^{-h^*\Upsilon(\theta,\sigma,\lambda)}\ \pi(\theta,\sigma,\lambda\ |X)\ d\theta d\sigma d\lambda\Big]. \end{aligned}\tag{25}$$

For θ, σ and λ , the full conditionals can be expressed as;

$$\pi_1(\theta|\sigma,\lambda) \propto \theta^{(D_1+D_2+a)} c^{-\theta} \sigma^{\theta(D_1+\lambda D_2+b)} \left[1+(\frac{C_1}{\sigma})^{\theta}\right]^{(D_1-n_1)} \left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda(D_2-n_2)}$$
$$\times e^{-\theta \sum_{i=1}^{D_1} \log x_{1i} - 2\sum_{i=1}^{d_1} \log(1+(\frac{\sigma}{x_{1i}})^{\theta}) - \lambda\theta \sum_{i=1}^{D_2} \log x_{2i} - (\lambda+1)\sum_{i=1}^{D_2} \log(1+(\frac{\sigma}{x_{2i}})^{\theta})} \quad (26)$$

$$\pi_2(\sigma|\theta,\lambda) \propto \sigma^{\theta(D_1+\lambda D_2+b-1)} \left[1+(\frac{C_1}{\sigma})^{\theta}\right]^{(D_1-n_1)} \left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda(D_2-n_2)}$$
$$\times e^{-2\sum_{i=1}^{D_1} \log(1+(\frac{\sigma}{x_{1i}})^{\theta}) - (\lambda+1)\sum_{i=1}^{D_2} \log(1+(\frac{\sigma}{x_{2i}})^{\theta})} \quad (27)$$

$$\pi_3(\lambda|\theta,\sigma) \propto \lambda^{D_2-1}\, \sigma^{\theta\lambda D_2} \left[1+(\frac{C_2}{\sigma})^{\theta}\right]^{\lambda(D_2-n_2)} e^{\lambda n_2 \log 2 - \lambda\theta \sum_{i=1}^{D_2} \log x_{2i} - \lambda \sum_{i=1}^{D_2} \log(1+(\frac{\sigma}{x_{2i}})^{\theta})} \quad (28)$$

The conditional posterior functions of the parameters θ, σ and λ in Eqs.(26–28) do not present standard form. Therefore, we generate random samples from these distributions using the Metropolis–Hastings (M–H) algorithm with proposal distribution.
Since Gibbs sampling is not a clear-cut alternative and the conditional posteriors of θ, σ and λ in Eqs.(26–28) do not give conventional forms, the application of the M–H sampler is necessary for the implementation of the Markov chain Monte Carlo (MCMC) approach. A detailed discussion about MCMC and M-H algorithm can be found in Refs.[36],[37], and [38]. The procedures outlined in Algo.(1) are utilized to derive Bayesian estimation for parameters θ, σ and λ .

Algorithm 1. MH Sampling for Bayesian estimation.

Step 1: Set $l = 1$ and the initial value of the parameters as $(\theta^{(0)}, \sigma^{(0)}, \lambda^{(0)}) = (\hat{\theta}_{\text{ML}}, \hat{\sigma}_{\text{ML}}, \hat{\lambda}_{\text{ML}})$.

Step 2: Generate $\theta^{(l)}, \sigma^{(l)}$ and $\lambda^{(l)}$ such that $\theta^{(l)} \sim N(\theta^{(l-1)}, \mathbf{Var}(\hat{\theta}_{\text{ML}}))$, $\sigma^{(l)} \sim N(\sigma^{(l-1)}, \mathbf{Var}(\hat{\sigma}_{\text{ML}}))$, and $\lambda^{(l)} \sim N(\lambda^{(l-1)}, \mathbf{Var}(\hat{\lambda}_{\text{ML}}))$.

Step 3: Obtain;

$$\begin{cases} P_\theta = \min\left(\dfrac{\pi_1(\theta^l|\sigma^{l-1}, \lambda^{l-1})}{\pi_1(\theta^{l-1}|\sigma^{l-1}, \lambda^{l-1})},\ 1\right), \\ P_\sigma = \min\left(\dfrac{\pi_2(\sigma^l|\theta^l, \lambda^{l-1})}{\pi_2(\sigma^{l-1}|\theta^l, \lambda^{l-1})},\ 1\right), \\ P_\lambda = \min\left(\dfrac{\pi_3(\lambda^l|\theta^l, \sigma^l)}{\pi_3(\lambda^{l-1}|\theta^l, \sigma^l)},\ 1\right). \end{cases}$$

Step 4: From a Uniform $U(0,1)$ distribution, we produce u_1, u_2 and u_3.

Step 5: If $u_1 \leq P_\theta$, set $\theta^{(l)} = \theta^{(l)}$, else $\theta^{(l)} = \theta^{(l-1)}$. Similarly, If $u_2 \leq P_\sigma$, set $\sigma^{(l)} = \sigma^{(l)}$, else $\sigma^{(l)} = \sigma^{(l-1)}$ and If $u_3 \leq P_\lambda$, set $\lambda^{(l)} = \lambda^{(l)}$, else $\lambda^{(l)} = \lambda^{(l-1)}$.

Step 5: Set $l = l + 1$. Then repeat steps $2 - 5$ for B times to generate $(\theta^{(l)}, \sigma^{(l)}, \lambda^{(l)})$, for $l = 1, \ldots, B$.

Step 6: After discarding the first M number of burn-in samples, the remaining $B-M$ samples are used to obtain the Bayesian estimates of θ, σ, and λ, where the Bayes estimate of $\Phi = \Phi(\theta, \sigma, \lambda)$ under the SE and LINEX loss functions can now be computed as;

$$\hat{\Phi}_{SE} = \frac{1}{B-M} \sum_{i=M+1}^{B} \Phi(\theta^{(i)}, \sigma^{(i)}, \lambda^{(i)}) \quad , \quad \hat{\Phi}_{LL} = -\frac{1}{h^*} \log\left[\frac{1}{B-M} \sum_{i=M+1}^{B} e^{-h^* \Phi(\theta^{(i)}, \sigma^{(i)}, \lambda^{(i)})}\right].$$

2.3. Interval estimation

In this section, the asymptotic confidence interval (CI), normal bootstrap confidence interval (N-boot) and Bayesian CI (credibility interval) have been studied for parameters θ, σ and λ.

2.3.1. Asymptotic Confidence Interval

Rather of acquiring specific estimates for unknown parameters, there may be a desire to obtain a range of values that could potentially encompass these parameters with a designated level of probability. The aforementioned ranges are sometimes referred to as interval estimations.

In this context, we employ the asymptotic properties of the maximum likelihood estimators (MLEs) to construct the asymptotic confidence intervals (ACIs) for the unknown parameters $\Phi = (\theta, \sigma, \lambda)^T$. Using the large sample theory reveals that the MLEs, $\hat{\Phi}_{\mathbf{ML}} = (\hat{\theta}_{\mathbf{ML}}, \hat{\sigma}_{\mathbf{ML}}, \hat{\lambda}_{\mathbf{ML}})$ follow an asymptotic distribution that may be approximated by a normal distribution. This normal distribution has a mean of Φ and a variance-covariance matrix of $I^{-1}(\Phi)$. In this study, we employ the asymptotic variance-covariance matrix (AVCM) denoted as $I^{-1}(\hat{\Phi})$ to estimate $I^{-1}(\Phi)$. The estimation is achieved by inverting the observed Fisher information matrix. In this particular instance, the AVCM assumes the structure.

$$I^{-1}(\hat{\Phi}) = -\begin{pmatrix} \frac{\partial^2 \ell(\Phi)}{\partial\theta^2} & \frac{\partial^2 \ell(\Phi)}{\partial\theta\partial\sigma} & \frac{\partial^2 \ell(\Phi)}{\partial\theta\partial\lambda} \\ & \frac{\partial^2 \ell(\Phi)}{\partial\sigma^2} & \frac{\partial^2 \ell(\Phi)}{\partial\sigma\partial\lambda} \\ & & \frac{\partial^2 \ell(\Phi)}{\partial\lambda^2} \end{pmatrix}_{(\hat{\Phi})} = \begin{pmatrix} \hat{\varpi}_{11}^2 & \hat{\varpi}_{12} & \hat{\varpi}_{13} \\ & \hat{\varpi}_{22}^2 & \hat{\varpi}_{23} \\ & & \hat{\varpi}_{33}^2 \end{pmatrix}, \tag{29}$$

The notation $\hat{\Phi}$ indicates that the derivatives are being evaluated at the estimated values of θ, σ, and λ. After a little calculation, one has;

$$\begin{aligned} \frac{\partial^2 \ell(\Phi)}{\partial\theta^2} &= -\frac{D_1 + D_2}{\theta^2} + (D_1 - n_1)\frac{\left(\frac{C_1}{\sigma}\right)^\theta (\log(\frac{C_1}{\sigma}))^2(1 + (\frac{C_1}{\sigma})^\theta) - \left((\frac{C_1}{\sigma})^\theta \log(\frac{C_1}{\sigma})\right)^2}{(1 + (\frac{C_1}{\sigma})^\theta)^2} \\ &+ \lambda(D_2 - n_2)\frac{\left(\frac{C_2}{\sigma}\right)^\theta (\log(\frac{C_2}{\sigma}))^2(1 + (\frac{C_2}{\sigma})^\theta) - \left((\frac{C_2}{\sigma})^\theta \log(\frac{C_2}{\sigma})\right)^2}{(1 + (\frac{C_2}{\sigma})^\theta)^2} \\ &- \sum_{j=1}^{2}(S_j(\lambda) + 1)\sum_{i=1}^{D_j} \frac{\left(\frac{\sigma}{x_{ji}}\right)^\theta (\log(\frac{\sigma}{x_{ji}}))^2(1 + (\frac{\sigma}{x_{ji}})^\theta) - \left((\frac{\sigma}{x_{ji}})^\theta \log(\frac{\sigma}{x_{ji}})\right)^2}{(1 + (\frac{\sigma}{x_{ji}})^\theta)^2}. \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial^2 \ell(\Phi)}{\partial\sigma^2} &= (D_1 - n_1)\frac{\frac{\theta}{\sigma^2}(\frac{C_1}{\sigma})^\theta(\theta(\frac{C_1}{\sigma})^\theta + (\frac{C_1}{\sigma})^\theta + 1)}{(1 + (\frac{C_1}{\sigma})^\theta)^2} + \lambda(D_2 - n_2)\frac{\frac{\theta}{\sigma^2}(\frac{C_2}{\sigma})^\theta(\theta(\frac{C_2}{\sigma})^\theta + (\frac{C_2}{\sigma})^\theta + 1)}{(1 + (\frac{C_2}{\sigma})^\theta)^2} \\ &- \frac{\theta(D_1 + \lambda D_2)}{\sigma^2} - \sum_{j=1}^{2}(S_j(\lambda) + 1)\sum_{i=1}^{D_j} \frac{\frac{\theta}{x_{ji}^2}(\frac{\sigma}{x_{ji}})^{(\theta-2)}(\theta - (\frac{\sigma}{x_{ji}})^\theta - 1)}{(1 + (\frac{\sigma}{x_{ji}})^\theta)^2}, \end{aligned} \tag{31}$$

$$\frac{\partial^2 \ell(\Phi)}{\partial\lambda^2} = -\frac{D_2}{\lambda^2}, \tag{32}$$

$$\frac{\partial^2 \ell(\Phi)}{\partial\theta\partial\sigma} = \frac{\partial^2 \ell(\Phi)}{\partial\sigma\partial\theta} = \frac{n_1 + \lambda n_2}{\sigma} + \frac{(D_1 - n_1)\sigma^{\theta-1}(1 + \theta\log\sigma)}{\sigma^\theta + C_1^\theta} + \frac{(D_2 - n_2)\sigma^{\theta-1}\lambda(1 + \theta\log\sigma)}{\sigma^\theta + C_2^\theta}$$

$$+\frac{(n_1-D_1)\theta\sigma^{\theta-1}(\sigma^{\theta}\log\sigma+C_1^{\theta}\log C_1)}{(\sigma^{\theta}+C_1^{\theta})^2}+\frac{(n_2-D_2)\theta\sigma^{\theta-1}\lambda(\sigma^{\theta}\log\sigma+C_2^{\theta}\log C_2)}{(\sigma^{\theta}+C_2^{\theta})^2}$$
$$-(1+\lambda)\sum_{i=1}^{D_2}\left(\frac{\sigma^{\theta-1}+\theta\sigma^{\theta-1}\log\sigma}{\sigma^{\theta}+x_i^{\theta}}-\frac{\theta\sigma^{\theta-1}(\sigma^{\theta}\log\sigma+x_i^{\theta}\log x_i)}{(\sigma^{\theta}+x_i^{\theta})^2}\right)$$
$$-2\sum_{i=1}^{D_1}\left(\frac{\sigma^{\theta-1}+\theta\sigma^{\theta-1}\log\sigma}{\sigma^{\theta}+x_i^{\theta}}-\frac{\theta\sigma^{\theta-1}(\sigma^{\theta}\log\theta+x_i^{\theta}\log x_i)}{(\sigma^{\theta}+x_i^{\theta})^2}\right), \tag{33}$$

$$\frac{\partial^2\ell(\Phi)}{\partial\theta\partial\lambda}=\frac{\partial^2\ell(\Phi)}{\partial\lambda\partial\theta}=\frac{(D_2-n_2)(\sigma^{\theta}\log\sigma+C_2^{\theta}\log D_2)}{\sigma^{\theta}+C_2^{\theta}}-\sum_{i=1}^{D_2}\frac{\sigma^{\theta}\log\sigma+x_i^{\alpha}\log x_i}{\sigma^{\theta}+x_i^{\theta}}, \tag{34}$$

and

$$\frac{\partial^2\ell(\Phi)}{\partial\sigma\partial\lambda}=\frac{\partial^2\ell(\Phi)}{\partial\lambda\partial\sigma}=\frac{(D_2-n_2)\theta\sigma^{\theta-1}}{\sigma^{\theta}+C_2^{\theta}}-\sum_{i=1}^{D_2}\frac{\theta\sigma^{\theta-1}}{\sigma^{\theta}+x_i^{\theta}}. \tag{35}$$

Using the asymptotic distribution of the maximum likelihood estimators (MLE), specifically the multivariate normal distribution with a mean vector of zero and a variance−covariance matrix as described in Eq.(29), it can be observed that the expression $\sqrt{n}(\hat{\Phi}-\Phi)$ follows this distribution. Hence, for any arbitrary value of $0<\upsilon<1$, the $100(1-\upsilon)\%$ Asymptotic Confidence Intervals (ACIs) for the unknown parameters can be expressed in the following manner;

$$\begin{cases}\hat{\theta}\pm Z_{\upsilon/2}\hat{\varpi}_{11},\\ \hat{\sigma}\pm Z_{\upsilon/2}\hat{\varpi}_{22},\\ \hat{\lambda}\pm Z_{\upsilon/2}\hat{\varpi}_{33}.\end{cases} \tag{36}$$

The conventional tabular normal values $Z_{\upsilon/2}$ with tail $(\upsilon/2)$ are denoted correspondingly.

2.3.2. Bootstrap Methods

The bootstrap method is a widely recognized resampling technique used in statistics to estimate the sampling distribution of a statistic. It achieves this by repeatedly sampling with replacement from the observed data. The main objective of this method is to offer an empirical approximation of the sampling distribution of a statistic, which proves useful for making inferences and constructing confidence intervals. The bootstrap method, originally introduced by Ref.[39], has been rigorously examined and extensively debated in the academic literature. In this subsection , we use the parametric bootstrap method to construct normal bootstrap (N-boot) confidence interval (CI) for the unknown model parameter (θ,σ,λ) , see Ref.[40]. The procedures outlined in Algo.(2) are utilized to derive $100(1-\upsilon)\%$ Normal bootstrap confidence intervals for these parameters.

Algorithm 2. Normal bootstrap CI for $(\theta, \sigma, \lambda)$.

Step 1: Based on the original type-I GHCS $X_j = \{X_{j1}, \ldots, X_{jdj}\}, j = 1, 2,$ and $X = \{X_1, X_2\}$, obtain the estimates of $(\theta, \sigma, \lambda)$, say $(\hat{\theta}_{\mathbf{ML}}, \hat{\sigma}_{\mathbf{ML}}, \hat{\lambda}_{\mathbf{ML}})$.

Step 2: Employ the censoring plan n_j, l_j, m_j, T and $(\hat{\theta}_{\mathbf{ML}}, \hat{\sigma}_{\mathbf{ML}}, \hat{\lambda}_{\mathbf{ML}})$ to generate a type-I GHCS bootstrap sample $X_j^* = \{X_{j1}^*, \ldots, X_{jdj}^*\}, j = 1, 2.$

Step 3: From the ordered observations obtained in Step 2, the bootstrap sample estimates of $(\theta, \sigma, \lambda)$ are computed, namely $(\hat{\theta}^*, \hat{\sigma}^*, \hat{\lambda}^*)$.

Step 4: To get B^* bootstrap samples, repeat Steps 2-3 B several times.

Step 5: Arrange all $(\hat{\theta}^*, \hat{\sigma}^*, \hat{\lambda}^*)$ in ascending order and denote,

$$\left(\hat{\theta}^{*[1]}, \hat{\theta}^{*[2]}, \ldots\ \hat{\theta}^{*[B^*]}\right), \quad \left(\hat{\sigma}^{*[1]}, \hat{\sigma}^{*[2]}, \ldots\ \hat{\sigma}^{*[B^*]}\right), \quad \textbf{and} \quad \left(\hat{\lambda}^{*[1]}, \hat{\lambda}^{*[2]}, \ldots\ \hat{\lambda}^{*[B^*]}\right)$$

Step 6: To rigorously construct the Normal bootstrap (N-boot) confidence intervals for $\kappa_j,\ j = 1, 2, 3$, where $\kappa_1 = \theta, \kappa_2 = \sigma$ and $\kappa_3 = \lambda$ by the following, formulae:

$$\left(2\hat{\kappa}_{j\mathbf{ML}}^* - \overline{\kappa_j^*} - Z_{1-\frac{\upsilon}{2}}\sqrt{S(\hat{\kappa_j^*})}, \quad 2\hat{\kappa}_{j\mathbf{ML}}^* + \overline{\kappa_j^*} - Z_{1-\frac{\upsilon}{2}}\sqrt{S(\hat{\kappa_j^*})}\right)$$

where Z_P is P^{th} the quantile of the standard normal distribution,

$$\overline{\kappa_j^*} = (B^*)^{-1}\sum_{i=1}^{B^*}\hat{\kappa}_j^{*[i]}, \quad \textbf{and} \quad S(\hat{\kappa_j^*}) = (B^* - 1)^{-1}\sum_{i=1}^{B^*}\left(\hat{\kappa}_j^{*[i]} - \overline{\kappa_j^*}\right)^2$$

2.3.3. Highest Posterior Density Credible Interval

In subs. (2.2.3), for Bayesian point estimation, $B - M$ samples of $(\theta, \sigma, \lambda)$ were;

$$\left(\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \ \ldots, \ \hat{\theta}_{(B-M)}\right), \ \left(\hat{\sigma}_{(1)}, \hat{\sigma}_{(2)}, \ \ldots, \ \hat{\sigma}_{(B-M)}\right), \ \textbf{and} \ \left(\hat{\lambda}_{(1)}, \hat{\lambda}_{(2)}, \ \ldots, \ \hat{\lambda}_{(B-M)}\right).$$

For arbitrary $0 < \upsilon < 1$, a $100(1 - \upsilon)\%$ confidence interval for parameter $(\theta, \sigma, \lambda)$ can be composed in the following form,

$$\left(\hat{\theta}_{[(B-M)\frac{\upsilon}{2}]}, \hat{\theta}_{[(B-M)(\frac{1-\upsilon}{2})]}\right), \ \left(\hat{\sigma}_{[(B-M)\frac{\upsilon}{2}]}, \hat{\sigma}_{[(B-M)(\frac{1-\upsilon}{2})]}\right), \textbf{and} \left(\hat{\lambda}_{[(B-M)\frac{\upsilon}{2}]}, \hat{\lambda}_{[(B-M)(\frac{1-\upsilon}{2})]}\right)$$

Where $[(B - M)\frac{\upsilon}{2}]$ represents the maximum integer less than $(B - M)\frac{\upsilon}{2}$; similarly, $[(B - M)(\frac{1-\upsilon}{2})]$ represents the maximum integer less than $[(B - M)(\frac{1-\upsilon}{2})]$. Repeat the above steps M^* times to obtain M^* interval estimates with the form above, from which the interval with the smallest length is selected as the highest posterior density credible interval.

3. Numerical Application

3.1. Monte Carlo Simulations

To investigate the effectiveness of frequentest and Bayesian estimations produced in the preceding sections, Monte Carlo simulations were carried out. We compare the efficacy of different estimators and confidence intervals of the NPT parameters (θ, σ) and the acceleration factor λ, based on the true parameter values of $(\theta, \sigma, \lambda)$, namely Set 1: $(0.5, 0.5, 1.3)$ and Set 2: $(1.0, 1.0, 2.0)$. We generate 1000 type-I GHCS based on various choices of $(n_j, (l_j, m_j), T)$, $j = 1, 2$. The sample size $(n_j, (l_j, m_j), T)$ is fixed from the data under type-I GHCS and is set to $(n_1, n_2) = (30, 40), (50, 60), (70, 80)$ with three sets of fixed numbers $(l_1, m_1) = (15, 25), (35, 45),$
$(55, 65), (l_2, m_2) = (20, 35), (40, 55), (60, 75)$ and $T = 5.0, 7.0$ presented respectively for each size. The process of generating the type-I GHCS data is shown in Algo.(3).

Algorithm 3. Steps for generating type-I GHCS datasets.

Step 1: Create n_j independent variables $\epsilon_j = (\epsilon_{j1}, \epsilon_{j2}, \dots, \epsilon_{jn_j})$ from Uniform$(0, 1)$ distribution, $j = 1, 2$.

Step 2: For given l_j, m_j, T, set $D_j,\ j = 1, 2$ as;

$$D_j = \begin{cases} l_j & \textit{if} \quad T < l_j \\ d_j^* & \textit{if} \quad l_j \le T < m_j \\ m_j & \textit{if} \quad T \ge m_j \end{cases}$$

Step 3: Record the type-I GHCS data as $X_j = (X_{j1}, X_{j2}, \dots, X_{jD_j}),\ j = 1, 2$, are derived by the inverse function method;

$$\begin{cases} X_1 = \sigma \left(\frac{1+\epsilon_1}{1-\epsilon_1}\right)^{1/\theta} \\ X_2 = \sigma \left(\frac{2}{(1-\epsilon_2)^{1/\lambda}} - 1\right)^{1/\theta} \end{cases}$$

Once the samples have been generated, using **R** (4.1.3) software with the "maxLik" package introduced by Ref.[41], we obtain the MLEs of parameters $(\theta, \sigma, \lambda)$ based on both NR and EM algorithm methods. We set the real values of the parameters $(\theta, \sigma, \lambda)$ as the initial guesses of EM algorithm and the convergence is assumed when the absolute differences between the successive estimates are less than 10^{-5}. We computed the $100(1 - \upsilon)\%$ confidence intervals (CIs) for $(\theta, \sigma, \lambda)$ based on the asymptotic normal distribution of the MLEs. We have also computed normal bootstrap (N-boot) confidence intervals as well. We have reported the ALs and the CPs in each case. What's more, for the numbers of bootstrap re-sampling, we set it as $B^* = 1000$. For comparing the performance of the

Bayesian estimates, the associated values of the hyper-parameters $(a,\ b,\ c,\ d)$ were taken as $(0.02, 1.0, 4.0, 2.0)$ and, $(0.05, 2.0, 6.0, 3.0)$ for the given Sets 1 and 2, respectively. The M-H within the Gibbs algorithm given in Subs. (2.2.3) is used to obtain Bayesian estimates. For this algorithm, we took a normal proposal density to generate θ and λ from Eqs.(26) and (28), a power function distribution to generate σ from Eq.(27), using the 'coda' package proposed by Ref.[42], and we take into account the MLEs as initial guess values. We generate $B = 11,000$ MCMC samples and discard the first $M = 1000$ values as burn-in period as described in Subs. (2.2.3). The last $10,000$ Markov chains are used to establish the empirical distribution to estimate the posterior distribution of $(\theta, \sigma, \lambda)$. Consequently, the Bayes estimates under the specified squared error (SE) and linear exponential (LINEX) loss functions must be rigorously derived from the empirical posterior distribution of $(\theta, \sigma, \lambda)$. Furthermore, the $100(1 - \upsilon)\%$ credible intervals for $(\theta, \sigma, \lambda)$ must be precisely computed by extracting the two symmetric quantiles from the same empirical posterior distribution of $(\theta, \sigma, \lambda)$, respectively. For evaluating the Bayes estimators under LINEX loss function we take $h^* = (-0.8, 0.8)$. Also, ALs and the CPs of Bayesian credible intervals are calculated. The interval estimates are computed using the nominal $\upsilon = 0.05$ significance level in each case and the performances of different estimation methods are compared based on 1000 replications. A comparison of several point estimations of $\Phi_j, j = 1, 2, 3$, where $\Phi_1 = \theta, \Phi_2 = \sigma$, and $\Phi_3 = \lambda$, is then made based on two different criteria, namely average values (Avg) and mean-square errors (MSEs), by the following formulae;

$$\mathbf{Avg} = (N^*)^{-1} \sum_{i=1}^{N^*} \hat{\Phi}_j^{(i)} \quad \textbf{and} \quad \mathbf{MSE} = (N^*)^{-1} \sum_{i=1}^{N^*} \left(\hat{\Phi}_j^{(i)} - \Phi_j\right)^2.$$

Respectively, where N^* is the number of generated sequence data, and $\hat{\Phi}_j^{(i)}$ denotes the calculated estimate at the $i^{\mathbf{th}}$ sample of $\Phi_j,\ j = 1, 2, 3$. Additionally, the evaluation of various interval estimates of $\Phi_j,\ j = 1, 2, 3$ is determined by two other standards, namely the average confidence lengths (ACLs) and coverage percentages (CPs), by the following formulae;

$$\mathbf{ACL}_{(1-\upsilon)\%}(\Phi_p) = (N^*)^{-1} \textstyle\sum_{i=1}^{N^*} \left(U^*_{\hat{\Phi}_p^{(i)}} - L^*_{\hat{\Phi}_p^{(i)}}\right) \textbf{ and } \mathbf{CP}_{(1-\upsilon)\%}(\Phi_p) = (N^*)^{-1} \textstyle\sum_{i=1}^{N^*} \mathbf{1}_{\left(L^*_{\hat{\Phi}_p^{(i)}}\ ;\ U^*_{\hat{\Phi}_p^{(i)}}\right)}(\Phi_p).$$

Respectively, where $\mathbf{1}_{(\cdot)}$ is the indicator function, and $(L^*_{(\cdot)}\ ,\ U^*_{(\cdot)})$ are the lower and upper bounds of asymptotic (or credible) interval estimates. The results of the Avg, MSEs, ALs and CPs for the Bayes estimates for $(\theta, \sigma, \lambda)$ are strictly presented in Tabs.(1)-(4).
On the basis of the results reported in Tabs.(1)-(4), some points can be drawn which are stated as follows;

- The proposed point (or interval) estimates of θ, σ and λ have shown good performance based on given true parameter set.
- In every instance, as expected, the estimation results are satisfactory based on the Avg. The MSEs of all estimates decrease as the sample size increases, confirming the consistency of each estimation method.

- The MLEs of θ, σ and λ using the EM algorithm have smaller MSEs than the MLEs using the NR algorithm. Hence, the MLEs via the EM algorithm perform better than those obtained by the NR method.
- In terms of Avg and MSEs, Bayes estimation using MCMC performs better than the other approaches (ML, N-boot).
- Due to having the smallest MSE and narrowest width, MCMC CRIs are, overall, the most satisfactory.
- The improvement of CPs and CIs is signifcant with increased total and efective sample sizes.
- The estimates produced by the ML, bootstrap, and Bayesian approaches are highly similar and have high CPs (around 0.95).
- In most simulations, the Bayes estimates outperform the MLEs for the estimation of θ, σ, and λ. So, in general, we would recommend using the Bayes estimate of the unknown parameters of NPT distribution based on type-I GHCS under partially constant-stress ALT.

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Table 1: Avg and MSEs of for the parameters with $(\theta, \sigma, \lambda) = (0.5, 0.5, 1.3)$.

(n_1, n_2)	(l_1, m_1) (l_2, m_2)	T	**Par**	**Criteria**	**MLE** NR	**MLE** EM	**Bayesian** SELF	**Bayesian** LLF $h^* = -0.8$	**Bayesian** LLF $h^* = 0.8$
$(30, 40)$	$(15, 25)$ $(20, 35)$	5.0	$\hat{\theta}$	**Avg**	**0.392**	**0.432**	**0.476**	**0.482**	**0.472**
				MSE	**0.015**	**0.006**	**0.008**	**0.009**	**0.008**
			$\hat{\sigma}$	**Avg**	**0.521**	**0.521**	**0.514**	**0.517**	**0.512**
				MSE	**0.001**	**0.001**	**0.003**	**0.004**	**0.003**
			$\hat{\lambda}$	**Avg**	**1.135**	**1.394**	**1.432**	**1.567**	**1.342**
				MSE	**0.077**	**0.015**	**0.012**	**0.021**	**0.076**
		10.0	$\hat{\theta}$	**Avg**	**0.405**	**0.445**	**0.502**	**0.506**	**0.497**
				MSE	**0.013**	**0.027**	**0.011**	**0.012**	**0.010**
			$\hat{\sigma}$	**Avg**	**0.528**	**0.528**	**0.595**	**0.598**	**0.592**
				MSE	**0.001**	**0.001**	**0.013**	**0.024**	**0.014**
			$\hat{\lambda}$	**Avg**	**1.173**	**1.406**	**1.506**	**1.632**	**1.418**
				MSE	**0.077**	**0.074**	**0.137**	**0.237**	**0.117**
$(50, 60)$	$(35, 45)$ $(40, 55)$	5.0	$\hat{\theta}$	**Avg**	**0.423**	**0.521**	**0.518**	**0.520**	**0.516**
				MSE	**0.008**	**0.006**	**0.007**	**0.008**	**0.007**
			$\hat{\sigma}$	**Avg**	**0.515**	**0.515**	**0.547**	**0.548**	**0.546**
				MSE	**0.001**	**0.001**	**0.006**	**0.006**	**0.005**
			$\hat{\lambda}$	**Avg**	**1.132**	**1.232**	**1.392**	**1.439**	**1.349**
				MSE	**0.064**	**0.034**	**0.090**	**0.113**	**0.075**
		10.0	$\hat{\theta}$	**Avg**	**0.421**	**0.521**	**0.519**	**0.521**	**0.517**
				MSE	**0.008**	**0.004**	**0.006**	**0.007**	**0.006**
			$\hat{\sigma}$	**Avg**	**0.516**	**0.516**	**0.569**	**0.571**	**0.568**
				MSE	**0.001**	**0.001**	**0.007**	**0.008**	**0.007**
			$\hat{\lambda}$	**Avg**	**1.141**	**1.234**	**1.392**	**1.439**	**1.351**
				MSE	**0.066**	**0.045**	**0.093**	**0.115**	**0.078**
$(70, 80)$	$(55, 65)$ $(60, 75)$	5.0	$\hat{\theta}$	**Avg**	**0.420**	**0.532**	**0.499**	**0.500**	**0.497**
				MSE	**0.008**	**0.007**	**0.003**	**0.004**	**0.004**
			$\hat{\sigma}$	**Avg**	**0.511**	**0.511**	**0.520**	**0.529**	**0.521**
				MSE	**0.002**	**0.002**	**0.003**	**0.004**	**0.003**
			$\hat{\lambda}$	**Avg**	**1.181**	**1.380**	**1.405**	**1.436**	**1.377**
				MSE	**0.051**	**0.045**	**0.094**	**0.109**	**0.082**
		10.0	$\hat{\theta}$	**Avg**	**0.431**	**0.532**	**0.514**	**0.516**	**0.513**
				MSE	**0.006**	**0.007**	**0.005**	**0.004**	**0.004**
			$\hat{\sigma}$	**Avg**	**0.509**	**0.509**	**0.520**	**0.521**	**0.519**
				MSE	**0.001**	**0.001**	**0.002**	**0.003**	**0.002**
			$\hat{\lambda}$	**Avg**	**1.148**	**1.345**	**1.351**	**1.378**	**1.325**
				MSE	**0.047**	**0.048**	**0.048**	**0.055**	**0.043**

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Table 2: Avg and MSEs of for the parameters with $(\theta, \sigma, \lambda) = (1.0, 1.0, 2.0)$.

$(n_1, n_2)^{(l_1,m_1)}_{(l_2,m_2)}$	T	Par	Criteria	MLE NR	MLE EM	Bayesian SELF	Bayesian LLF $h^* = -0.8$	Bayesian LLF $h^* = 0.8$
$(30, 40)^{(15,25)}_{(20,35)}$	7.0	$\hat{\theta}$	**Avg**	**1.098**	**1.072**	**0.934**	**0.936**	**0.928**
			MSE	**0.086**	**0.079**	**0.066**	**0.062**	**0.073**
		$\hat{\sigma}$	**Avg**	**1.319**	**1.319**	**1.291**	**1.423**	**1.397**
			MSE	**0.097**	**0.097**	**0.076**	**0.078**	**0.066**
		$\hat{\lambda}$	**Avg**	**1.519**	**1.572**	**1.373**	**1.468**	**1.386**
			MSE	**0.244**	**0.278**	**0.052**	**0.078**	**0.053**
	10.0	$\hat{\theta}$	**Avg**	**1.154**	**1.131**	**0.879**	**0.913**	**0.869**
			MSE	**0.083**	**0.080**	**0.044**	**0.052**	**0.043**
		$\hat{\sigma}$	**Avg**	**1.134**	**1.134**	**1.096**	**1.125**	**0.983**
			MSE	**0.017**	**0.017**	**0.050**	**0.066**	**0.050**
		$\hat{\lambda}$	**Avg**	**1.370**	**1.412**	**1.531**	**1.612**	**1.435**
			MSE	**0.171**	**0.188**	**0.103**	**0.126**	**0.105**
$(50, 60)^{(35,45)}_{(40,55)}$	7.0	$\hat{\theta}$	**Avg**	**1.014**	**0.983**	**0.869**	**0.923**	**0.895**
			MSE	**0.033**	**0.034**	**0.030**	**0.036**	**0.029**
		$\hat{\sigma}$	**Avg**	**1.082**	**1.082**	**0.884**	**0.896**	**0.865**
			MSE	**0.085**	**0.085**	**0.074**	**0.082**	**0.061**
		$\hat{\lambda}$	**Avg**	**1.714**	**1.787**	**1.731**	**1.752**	**1.693**
			MSE	**0.203**	**0.269**	**0.090**	**0.099**	**0.089**
	10.0	$\hat{\theta}$	**Avg**	**1.023**	**1.005**	**0.867**	**0.902**	**0.849**
			MSE	**0.030**	**0.031**	**0.023**	**0.032**	**0.019**
		$\hat{\sigma}$	**Avg**	**1.072**	**1.072**	**0.857**	**0.879**	**0.847**
			MSE	**0.009**	**0.009**	**0.004**	**0.025**	**0.001**
		$\hat{\lambda}$	**Avg**	**1.543**	**1.581**	**1.725**	**1.865**	**1.658**
			MSE	**0.125**	**0.141**	**0.122**	**0.156**	**0.069**
$(70, 80)^{(55,65)}_{(60,75)}$	7.0	$\hat{\theta}$	**Avg**	**1.021**	**0.991**	**0.962**	**0.976**	**0.936**
			MSE	**0.017**	**0.018**	**0.015**	**0.026**	**0.011**
		$\hat{\sigma}$	**Avg**	**1.082**	**1.082**	**0.913**	**0.963**	**0.896**
			MSE	**0.012**	**0.012**	**0.003**	**0.019**	**0.001**
		$\hat{\lambda}$	**Avg**	**1.485**	**1.537**	**1.822**	**1.836**	**1.756**
			MSE	**0.037**	**0.042**	**0.022**	**0.056**	**0.015**
	10.0	$\hat{\theta}$	**Avg**	**0.944**	**0.928**	**0.918**	**0.968**	**0.897**
			MSE	**0.020**	**0.023**	**0.018**	**0.032**	**0.009**
		$\hat{\sigma}$	**Avg**	**1.096**	**1.096**	**0.919**	**0.936**	**0.895**
			MSE	**0.010**	**0.010**	**0.009**	**0.023**	**0.002**
		$\hat{\lambda}$	**Avg**	**1.597**	**1.634**	**1.716**	**1.768**	**1.659**
			MSE	**0.142**	**0.169**	**0.139**	**0.189**	**0.106**

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Table 3: ALs and CPs for the parameters with $(\theta, \sigma, \lambda) = (0.5, 0.5, 1.3)$.

(n_1, n_2)	(l_1, m_1) (l_2, m_2)	T	Par	Criteria	MLE	N-boot	Bayesian
$(30, 40)$	$(15, 25)$ $(20, 35)$	5.0	$\hat{\theta}$	**AL**	**0.295**	**0.529**	**0.324**
				95% **CP**	**0.852**	**0.872**	**0.868**
			$\hat{\sigma}$	**AL**	**0.406**	**0.599**	**0.288**
				95% **CP**	**0.897**	**0.886**	**0.854**
			$\hat{\lambda}$	**AL**	**1.933**	**1.789**	**1.324**
				95% **CP**	**0.912**	**93.8**	**0.867**
		10.0	$\hat{\theta}$	**AL**	**0.201**	**0.517**	**0.311**
				95% **CP**	**0.789**	**0.897**	**0.967**
			$\hat{\sigma}$	**AL**	**0.451**	**0.599**	**0.315**
				95% **CP**	**0.852**	**0.916**	**0.921**
			$\hat{\lambda}$	**AL**	**1.463**	**1.765**	**1.311**
				95% **CP**	**0.890**	**0.923**	**0.873**
$(50, 60)$	$(35, 45)$ $(40, 55)$	5.0	$\hat{\theta}$	**AL**	**0.221**	**0.567**	**0.293**
				95% **CP**	**0.893**	**0.936**	**0.827**
			$\hat{\sigma}$	**AL**	**0.348**	**0.577**	**0.173**
				95% **CP**	**0.917**	**0.944**	**0.878**
			$\hat{\lambda}$	**AL**	**1.597**	**1.679**	**1.275**
				95% **CP**	**0.908**	**0.962**	**0.925**
		10.0	$\hat{\theta}$	**AL**	**0.167**	**0.568**	**0.292**
				95% **CP**	**0.908**	**0.962**	**0.925**
			$\hat{\sigma}$	**AL**	**0.359**	**0.577**	**0.186**
				95% **CP**	**0.923**	**0.938**	**0.932**
			$\hat{\lambda}$	**AL**	**1.271**	**1.672**	**1.266**
				95% **CP**	**0.946**	**0.950**	**0.889**
$(70, 80)$	$(55, 65)$ $(60, 75)$	5.0	$\hat{\theta}$	**AL**	**0.150**	**0.608**	**0.221**
				95% **CP**	**0.902**	**0.917**	**0.871**
			$\hat{\sigma}$	**AL**	**0.311**	**0.567**	**0.112**
				95% **CP**	**0.918**	**0.915**	**0.885**
			$\hat{\lambda}$	**AL**	**1.245**	**1.563**	**1.035**
				95% **CP**	**0.902**	**0.956**	**0.938**
		10.0	$\hat{\theta}$	**AL**	**0.140**	**0.612**	**0.229**
				95% **CP**	**0.919**	**0.923**	**0.909**
			$\hat{\sigma}$	**AL**	**0.310**	**0.566**	**0.118**
				95% **CP**	**0.927**	**0.936**	**0.910**
			$\hat{\lambda}$	**AL**	**1.112**	**1.556**	**0.996**
				95% **CP**	**0.884**	**0.922**	**0.918**

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Table 4: ALs and CPs for the parameters with $(\theta, \sigma, \lambda) = (1.0, 1.0, 2.0)$.

(n_1, n_2)	(l_1, m_1) (l_2, m_2)	T	**Par**	**Criteria**	**MLE**	**N-boot**	**Bayesian**
$(30, 40)$	$(15, 25)$ $(20, 35)$	5.0	$\hat{\theta}$	**AL**	**1.164**	**0.534**	**0.775**
				95% **CP**	**0.975**	**0.923**	**0.965**
			$\hat{\sigma}$	**AL**	**1.927**	**1.324**	**1.275**
				95% **CP**	**0.960**	**0.882**	**97.2**
			$\hat{\lambda}$	**AL**	**1.444**	**1.653**	**1.022**
				95% **CP**	**0.935**	**0.893**	**0.965**
		10.0	$\hat{\theta}$	**AL**	**0.978**	**0.563**	**0.797**
				95% **CP**	**0.938**	**0.913**	**0.923**
			$\hat{\sigma}$	**AL**	**1.083**	**0.698**	**0.828**
				95% **CP**	**0.966**	**0.946**	**0.977**
			$\hat{\lambda}$	**AL**	**1.710**	**1.653**	**1.030**
				95% **CP**	**0.975**	**0.943**	**96.3**
$(50, 65)$	$(35, 45)$ $(40, 55)$	5.0	$\hat{\theta}$	**AL**	**0.942**	**0.698**	**0.707**
				95% **CP**	**0.965**	**0.930**	**0.973**
			$\hat{\sigma}$	**AL**	**1.363**	**0.963**	**0.673**
				95% **CP**	**0.956**	**0.933**	**0.968**
			$\hat{\lambda}$	**AL**	**1.457**	**1.036**	**1.008**
				95% **CP**	**0.954**	**0.936**	**0.976**
		10.0	$\hat{\theta}$	**AL**	**0.721**	**0.563**	**0.633**
				95% **CP**	**0.965**	**0.963**	**0.955**
			$\hat{\sigma}$	**AL**	**0.947**	**0.678**	**0.585**
				95% **CP**	**0.955**	**0.906**	**0.965**
			$\hat{\lambda}$	**AL**	**1.671**	**1.324**	**1.029**
				95% **CP**	**0.923**	**0.903**	**0.935**
$(70, 80)$	$(55, 65)$ $(60, 75)$	5.0	$\hat{\theta}$	**AL**	**0.701**	**0.456**	**0.653**
				95% **CP**	**0.975**	**0.943**	**0.985**
			$\hat{\sigma}$	**AL**	**0.895**	**0.678**	**0.490**
				95% **CP**	**0.925**	**0.930**	**0.965**
			$\hat{\lambda}$	**AL**	**1.440**	**0.896**	**0.984**
				95% **CP**	**0.955**	**0.939**	**0.965**
		10.0	$\hat{\theta}$	**AL**	**0.610**	**0.568**	**0.537**
				95% **CP**	**0.985**	**0.936**	**0.966**
			$\hat{\sigma}$	**AL**	**0.891**	**0.783**	**0.483**
				95% **CP**	**0.968**	**0.936**	**0.973**
			$\hat{\lambda}$	**AL**	**1.576**	**0.963**	**0.973**
				95% **CP**	**0.966**	**0.936**	**0.956**

3.2. Illustrative Example and real data analysis

In this section, we introduce a numerical investigation of the estimation methods discussed in previous sections for the NPT distribution using simulated data and a real data set.

3.2.1. Type-I GHCS data

Here, the estimation methods described in the previous sections is applied to the set of simulated type-I GHCS data under the constant-stress PALT. A data set of system lifetime is generated from the NPT model with $\theta = 0.5, \sigma = 1.0$ and $\lambda = 1.5$, respectively. Based on $(n_1 = n_2 = 60, l_1 = l_2 = 40, m_1 = m_2 = 50, T = 10)$, using the algorithm described in Algo.(3). We simulate two samples of size $D_1 = 40$ and $D_2 = 42$ from the NPT(θ, σ) and NPT$(\theta, \sigma, \lambda)$. The simulated data are given in Tab.(5).

Table 5: Simulated type-I GHC samples with constant-stress PALT.

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Normal condition	**1.147 1.149 1.206 1.289 1.310 1.485 2.021 2.130 2.149 2.233 2.267**
	2.324 2.394 2.632 2.662 3.166 3.436 4.072 4.104 4.918 5.267 5.522
	5.778 6.450 7.219 8.413 8.527 9.288 9.600 10.32 10.34 11.48 12.97
	18.72 20.14 23.34 26.70 33.34 37.09 38.81
Accelerated condition	**1.039 1.103 1.235 1.260 1.280 1.317 1.966 2.039 2.082 2.146 2.333**
	2.410 2.443 2.565 2.667 2.759 2.837 3.028 3.067 3.090 3.549 3.755
	3.807 3.945 4.234 4.369 4.616 4.860 4.945 5.131 5.519 5.956 6.403
	6.608 6.862 7.030 7.086 7.148 7.225 7.579 8.327 8.404

Based on the observed censored data in Tab.(5) and the methods adopted in Subs. (2.2.3), we calculated the MLEs of θ, σ and λ. The values are shown in Tab.(6). In order to obtain the Bayes estimates of θ, σ and λ, 10,000 samples were generated based on the Metropolis–Hastings algorithm. These Bayes estimates were obtained after discarding the initial 1000 burn-in iterative values. Bayes estimates were calculated with the informative priors, and the hyper parameters as $(a = 0.002, b = 2.0, c = 5.0, d = 2.0)$. For LLF, $h^* = -0.8$ and $h^* = 0.8$ were used as two options for the constant h^*. These choices give more weight to underestimation and exaggeration, respectively. The MLEs of the unknown parameters were considered as their initial values, while the diagonal elements of the reciprocal of the observed Fisher information matrix were considered as the variance of the MLEs. The estimates are displayed in Tab.(7). The 95% N-boot CIs, together with the HPD credible intervals, are presented in Tab.(7). The numerical illustration demonstrates that the means of the Bayes estimates and maximum likelihood estimates (MLEs) obtained by

Table 6: Estimated values of θ, σ, and λ.

Method⟶		MLEs		Bayesian		
Parameter↓		**NR**	**EM**	**SELF**	**LLF**	
					$h^* = -0.8$	$h^* = 0.8$
$\theta = 0.5$	**Estimate**	**0.4544**	**0.4643**	**0.4706**	**0.4722**	**0.4691**
$\sigma = 1.0$	**Estimate**	**1.0390**	**1.0648**	**1.1499**	**1.1528**	**1.1469**
$\lambda = 1.5$	**Estimate**	**1.5732**	**1.4408**	**1.7870**	**1.8489**	**1.7324**

Table 7: 95% confdence interval (CI) estimates of θ, σ and λ.

Method⟶		MLEs	N-boot	Bayesian
$\theta = 0.5$	95% **CI**	**(0.4079, 0.5010)**	**(0.3704, 1.3103)**	**(0.3567, 0.5448)**
$\sigma = 1.0$	95% **CI**	**(0.6944, 1.3836)**	**(0.9629, 2.0927)**	**(0.9812, 1.2213)**
$\lambda = 1.5$	95% **CI**	**(1.3264, 2.0545)**	**(1.4541, 2.6006)**	**(1.1472, 2.2459)**

the Newton-Raphson (NR) or expectation-maximization (EM) algorithm for the unknown parameters θ, σ, and λ exhibit a high degree of proximity to the true values.

3.2.2. Insulating Fluid

This application provides an analysis of the time-to-breakdown (measured in seconds) of an insulating fluid during a voltage endurance test conducted under varying stress levels. The longevity of various industrial components is contingent upon the durability of their electrical insulation. Enhancing the efficiency of the life test can be achieved by implementing accelerating circumstances, such as increasing the voltage. Ref.[43] conducted a study on accelerated tests, wherein varying magnitudes of voltage were applied to distinct experimental groups. According to Ref.[43], there are two stress levels that have been identified; **30** Kilovolt, which represents typical use, and **32** Kilovolt, which represents rapid stress. The data is shown in Tab.(8).

Table 8: oil breakdown times of insulating fluid.

Constant Stress Condition
194.90, 175.88, 144.12, 139.07, 47.30, 43.40, 22.66, 21.02, 20.46, 17.05, 7.74
Accelerated Stress Condition
215.10, 100.58, 89.29, 82.85, 53.24, 27.80, 15.93, 13.95, 9.88, 3.91, 2.75, 0.79, 0.69, 0.40, 0.27

Before performing parameter estimation, the adequacy of the dataset's fit to the NPT distribution was assessed. The Kolmogorov-Smirnov test (KS test) was employed as a means to quantify the deviation of the data from the underlying distribution. The KS distance and p-value of the data obtained from the constant stress condition were determined to be **0.2146** and **0.6178**, respectively. Similarly, the KS distance and p-value of the data obtained from the accelerated stress condition were found to be **0.2179** and **0.4149**, respectively. Thus, we ensured that the NPT distribution is considered an appropriate distribution for this data. For further illustration, the empirical cumulative distribution plot with fitted theoretical NPT distribution CDF where the model parameters is estimated from the complete real life data based on maximum likelihood method. In this goodness-of-fit approach, probability-probability (P−P) and Quantile-Quantile (Q−Q) plots are also provided in Figs.(2) and (3) which indicates that the NPT distribution provides a good fit to the data as well.

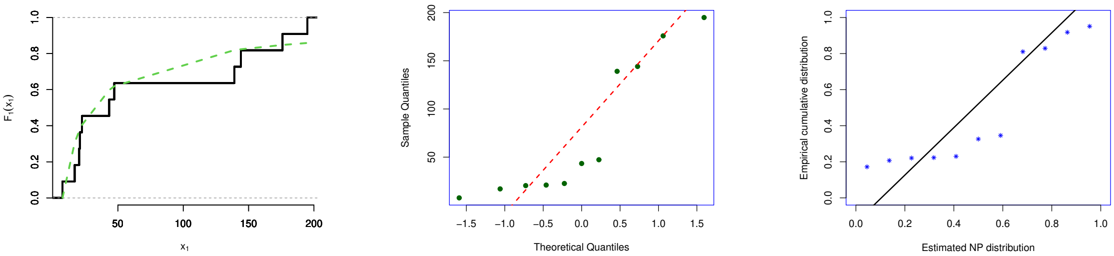

Figure 2: ECD, P-P, and Q-Q plots for normal conditions.

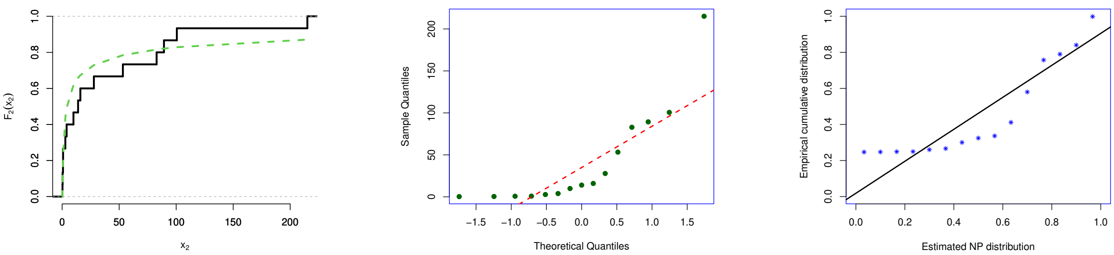

Figure 3: ECD, P-P, and Q-Q plots for accelerated conditions.

The maximum likelihood estimates (MLEs) are presented in Tab.(9).In the absence of prior knowledge about the unknown population parameters, adopting Bayes estimates is justified. In this scenario, it is postulated that the prior distributions of θ and σ are improper, specifically with parameters $a = -1$, $b = 0$, $c = 1$, and $d = \infty$, as there is a lack of prior knowledge available. As previously mentioned, the Gibbs method utilized Metropolis within it to produce a total of $11,000$ Markov Chain Monte Carlo (MCMC) samples. These samples were generated using the maximum likelihood estimators (MLEs) of the parameters θ, σ, and λ as the initial values at the onset of the procedure. Fig.(5) displays trace plots for the initial 1000 Markov Chain Monte Carlo (MCMC) iterations of the parameters θ, σ, and λ. The MCMC technique shows strong convergence. Furthermore, Fig.(4) displays the histogram plots of the generated samples for the parameters θ, σ, and λ. The histograms of the obtained samples exhibit a strong resemblance to the theoretical posterior density functions. For Bayesian estimations under asymmetric LLF, it is known in the literature that, $h^* < 0$ implies that underestimation results in more penalty than overestimation and the reverse is true for $h^* > 0$. When h^* close to zero, the LLF becomes symmetric and behaves roughly like the SELF. The MLEs relative to both NR, EM techniques and Bayesian MCMC method with chosen $h^* = -3$ and 3 of unknown parameters are computed and listed in Tab.(9). Moreover, the results of 95% approximate CI, N-boot and HPD intervals of unknown parameters are given in Tab.(10).

Table 9: Estimated values of θ, σ and λ from insulating fluid.

Method⟶		**MLEs**		**Bayesian**		
Parameter↓		**NR**	**EM**	**SELF**	**LLF**	
					$h^* = -0.3$	$h^* = 0.3$
θ	**Estimate**	**0.2279**	**0.2111**	**0.2241**	**0.2233**	**0.2249**
σ	**Estimate**	**0.2700**	**0.2734**	**0.3336**	**0.3316**	**0.3362**
λ	**Estimate**	**1.4243**	**1.6554**	**1.7704**	**1.6526**	**1.9369**

Table 10: 95% confdence interval (CI) estimates of θ, σ and λ from insulating fluid.

Method		**MLEs**	**N-boot**	**Bayesian**
θ	95% **CI**	**(0.1544, 0.3015)**	**(0.1676, 0.5900)**	**(0.1008, 0.3144)**
σ	95% **CI**	**(0.0252, 0.5652)**	**(0.0037, 0.7070)**	**(0.1014, 0.5016)**
λ	95% **CI**	**(0.0777, 2.7709)**	**(0.1113, 3.6775)**	**(0.5968, 2.9076)**

In conclusion, it can be inferred that the estimated distribution of the non parametric NPT model has a strong alignment with the provided data.

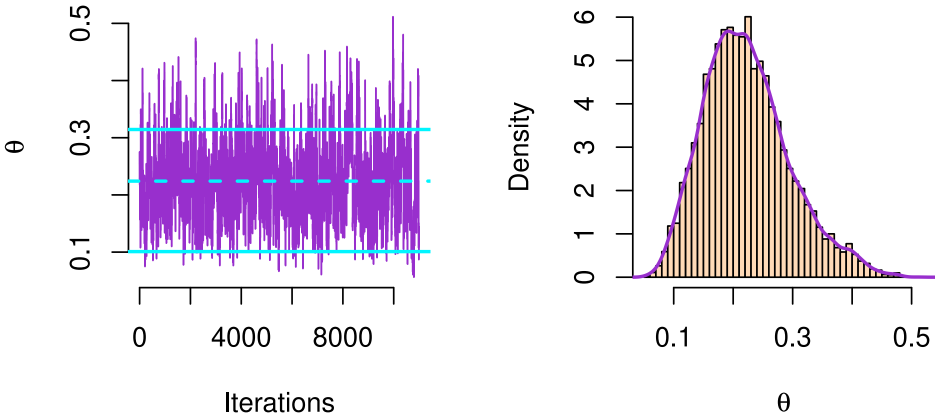

Figure 4: Density (right) and Trace (left) plots of θ from insulating fluid data.

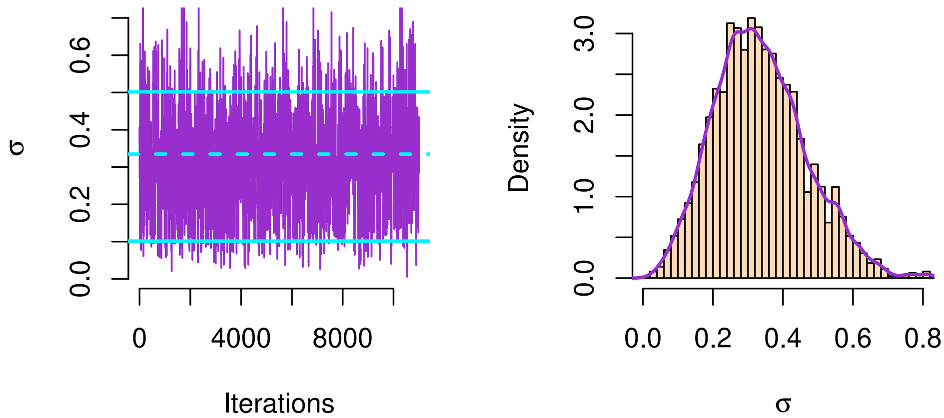

Figure 5: Density (right) and Trace (left) plots of σ from insulating fluid data.

4. Conclusions

This study discusses multiple parameter estimation approaches to estimate the two parameters of the NPT distribution. The estimation methods are applied to a constant stress partially accelerated life test, utilising type-I GHCS data. The maximum likelihood estimators (MLEs) and Bayes estimates of the parameters were obtained, along with the calculation of the acceleration factor and the related confidence intervals. Additionally, the EM technique has been utilised to derive the maximum likelihood estimators (MLEs) for the undetermined parameters. The associated Hessian matrix is also presented in

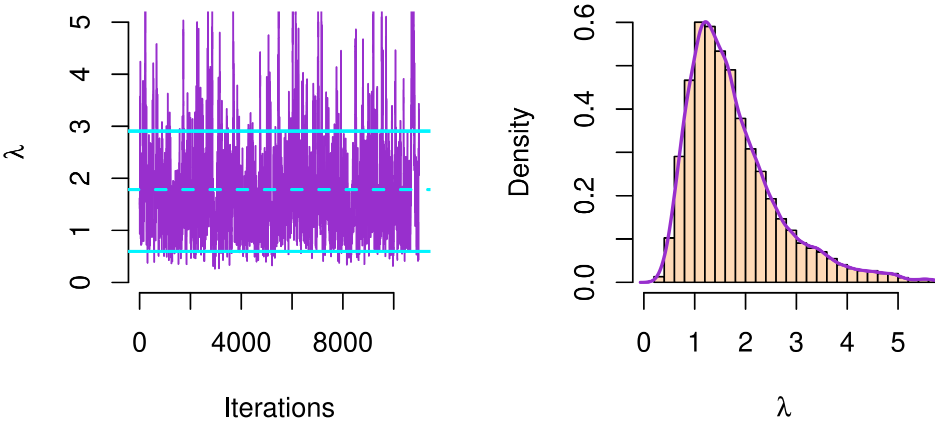

Figure 6: Density (right) and Trace (left) plots of λ from insulating fluid data.

the paper. In the absence of explicit formulations for the maximum likelihood estimators (MLEs) of certain parameters, we rely on the Bayesian approach for assistance. The Bayes estimates are derived under the assumption of dependent priors, using the two loss functions(squared error (SE) and linear exponential (LINEX)). The posterior distributions of the unknown parameters suggest that certain parameters do not conform to a widely recognised distribution. As a result, we employed Metropolis-Hastings sampling in the Gibbs sampling procedure to compute the Bayes estimates together with their corresponding credible intervals. Moreover, the empirical findings indicate that the utilization of informative priors in the Bayes approach yielded superior outcomes compared to the maximum likelihood (ML) technique, regardless of whether the Newton-Raphson (NR) or expectation-maximization (EM) algorithm was employed. Furthermore, it is well acknowledged that the estimates obtained using the Expectation-Maximization (EM) approach tend to exhibit superior performance when compared to those acquired using the Newton-Raphson (NR) method. This is seen in the significantly lower values of mean squared errors (MSEs) and average widths of the interval estimators. In this study, our primary focus has been on type-I GHCS and NPT distribution. However, it is worth noting that the methodology discussed can also be applied to other distribution and censoring schemes. There are several additional tasks that can be pursued in this area. These topics present potential avenues for future investigation. In conclusion, we propose the utilization of Markov Chain Monte Carlo (MCMC) and Expectation-Maximization (EM) methodologies in conjunction with partially accelerated life testing techniques, specifically employing type-I GHCS data, for the purpose of life testing and reliability modeling.

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