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Fixed Point Results for Enriched Interpolative Type Multivalued Contractions via a Simulation Function

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Abstract. In this paper, using a simulation function in the sense of Khojasteh, we define multivalued enriched interpolative Kannan-type and Hardy-Rogers-type contractions from a convex metric space $\mathscr{U}$ to the collection of closed and bounded subsets of $\mathscr{U}$. We establish two fixed points for these types of multivalued contractions. Our results are illustrated by examples and followed by some corollaries. As a consequence of each main result, a theorem on data dependence of fixed point is proved.

2020 Mathematics Subject Classifications: 47H09, 47H10, 54H25

Key Words and Phrases: Convex metric space, simulation function, fixed point

1. Introduction

The Banach Contraction Principle [1] stands as a pivotal result in the field of fixed point theory, furnishing a robust framework for comprehending the existence and uniqueness of fixed points of mappings which are defined on a complete metric space. Due to various applications, this result was generalized and extended in numerous ways (see, for instance, [2–5] and references therein). Given the continuity of mappings satisfying the Banach contraction principle, a natural question arose regarding the existence of fixed

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points of discontinuous mappings that satisfy similar contractive criteria. Kannan [6] provided an affirmative response to this inquiry by establishing a contractive condition for a discontinuous map T, demonstrating the existence and uniqueness of fixed points within the context of complete metric spaces.

In 2018, Karapinar [7] utilized the interpolation technique to revisit Kannan type contractions. Prior to [7], interpolative techniques are used in interpolation theory, a field of functional analysis. Karapinar [7] formulated the interpolative Kannan-type contraction on a complete metric space $(\mathscr{U}, \rho)$ as follows:

$$\rho(Tx, Ty) \leq \lambda([\rho(x, Tx)]^{\alpha}.[\rho(y, Ty)]^{1-\alpha}),$$

for each $x, y \in \mathscr{U} \backslash Fix(T)$, where $Fix(T) = \{x \in \mathscr{U}, Tx = x\}$ and $\lambda \in [0, 1)$. Recent studies in the field of interpolative Ćirić-Reich-Rus type contractions [8–10] and Meir-Keeler type contractions [11, 12] can be also referred to [13, 14].

Recently, Berinde [15, 16] has extended the literature related to Banach contraction principle [1] in Banach spaces by introducing enriched contractions. Enriched contractions [17] refer to self-mappings T on the structure $\mathscr{U}$ of a normed linear space $(\mathscr{U}, \| \cdot \|)$. These mappings adhere to a symmetric contraction condition, expressed as $\| b(x-y) + Tx - Ty \| \leq \theta \| x - y \|$, where $b \in [0, \infty)$ and $\theta \in [0, b+1)$, for each $x, y \in \mathscr{U}$. Undoubtedly, the category of enriched contractions is more extensive, encompassing not only the conventional Banach contractions (where $b = 0$) but also incorporating Lipschitz-type and non-expansive mappings. The broader scope of the enriched contraction, which is an extension of Banach contractions, reinforces the assertion that within the Banach space context, a fixed point x^* is guaranteed to exist, and the Krasnoselskij iteration offers an approach to approximate the fixed point. This assertion has been substantiated by Berinde and Păcurar [17]. Additionally, it's worth noting that contractive mappings of Kannan type and of Chatterjea type, can similarly be enriched, as discussed in [18, 19]. In 2022, Rawat et al. [20] introduced the notion of an enriched ordered contraction to prove some novel fixed point theorems in a convex noncommutative Banach space. Recently, Gangwar et al. [21] defined λ-enriched multivalued nonexpansive mappings and (λ, θ)-enriched multivalued contractions on a double controlled metric type space and deduced some novel fixed point results along with an application to differential inclusions.

Nadler [22] presented a notable and widely acknowledged extension by introducing the notion of Hausdorff metric, which is defined over a collection of bounded and closed subsets on a complete metric space. He laid the groundwork for multivalued contraction mappings. To facilitate understanding, we revisit several standard notations and terms. Consider a metric space $(\mathscr{U}, \rho)$. The set $CB(\mathscr{U})$ (resp. $C(U)$) denotes the collection of those subsets of U which are nonempty bounded and closed (resp. compact) subsets. For $A, B \in CB(\mathscr{U})$, $H : CB(\mathscr{U}) \times CB(\mathscr{U}) \to [0, +\infty)$ defined as $H(A, B) = \max\{\mathfrak{D}^*(A, B), \mathfrak{D}^*(B, A)\}$, where $\mathfrak{D}^*(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(a, b)$, is known as the Hausdorff metric. Nadler formulated a theorem for fixed points applicable to set-valued mappings satisfying a symmetric contraction condition. Takahashi [23] defined a convex structure in a metric space and referred to it as a convex metric space. Takahashi also studied various characteristics of this metric space to conclude the that a fixed point exists for nonexpansive mappings in the framework of a convex metric space.

Very recently, Rawat et al. [24] enriched three types of existing interpolative contractions (Kannan, Hardy-Rogers and Matkowski) in the context of a convex metric space. In 2015, Khojasteh et al. [25] presented a novel approach to examining fixed points by introducing a simulation function. They introduced a new type of contraction mappings known as $\mathfrak{Z}$-contractions. Subsequently, other prominent researchers utilized the concept of $\mathfrak{Z}$-contractions to explore common fixed points and coincidence points in various metric space settings. De Hierro et al. [26] incorporated the notion of $\mathfrak{Z}$-contractions to establish results on coincidence points in metric spaces. Additionally, Argoubi et al. [27] demonstrated results within the framework of partially ordered metric spaces by employing some non-linear contractions based on simulation functions.

Inspired by the results mentioned earlier, this study introduces enriched interpolative Kannan type contractions (EIK-contractions) and enriched interpolative Hardy-Rogers type contractions (EIHR-contractions) for multivalued mappings via a simulation function. The research also establishes several fixed-point theorems utilizing multivalued mappings by employing these contractions. To support our findings, illustrative examples are also provided. As a consequence of each main result, a theorem on data dependence of fixed point is proved.

2. Preliminaries

Khojasteh et. al. [25] introduced a new approach in fixed point theory by using the concept of a simulation function and thus generalized many known results, starting with Banach contraction principle. A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following three conditions:

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(x, y) < y - x, \ \forall \ x, y > 0$;
(ζ_3) If sequences $\{x_n\}$ and $\{y_n\}$ in the interval $[0, \infty)$ satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$, then

$$\lim_{n\to\infty} \sup \zeta(x_n, y_n) < 0.$$

The collection of all simulation functions will be denoted by $\mathfrak{Z}$.

Definition 1. *Consider a self-mapping T on a metric space $(\mathscr{U}, \rho)$. If for each $x, y \in \mathscr{U}$*

$$\zeta(\rho(Tx, Ty), \rho(x, y)) \geq 0,$$

then T is known as a $\mathfrak{Z}$-contraction with respect to ζ .

The concept of a simulation function was broadened by Roldán et al. [28] by just replacing the property (ζ_3) with (ζ_3'):

(ζ_3') If $\{x_n\}$ and $\{y_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$ and $x_n < y_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \sup \zeta(x_n, y_n) < 0.$$

A C-class function [29] $\mathcal{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ fulfills the following for each $x, y \in [0,\infty)$:

(i) $\mathcal{G}(x,y) \leq x$;

(ii) $\mathcal{G}(x,y) = x$ implies either $y = 0$ or $x = 0$.

Definition 2. *[30] Consider a mapping* $\mathcal{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$. *It is said to fulfill property* $C_{\mathcal{G}}$ *if for each* $x, y \in [0,\infty)$, *there is some constant* $C_{\mathcal{G}} \geq 0$ *for which:*

(i) $\mathcal{G}(x,y) > C_{\mathcal{G}}$ *implies* $x > y$;

(ii) $\mathcal{G}(y,y) \leq C_{\mathcal{G}}$, *for each* y.

Definition 3. *[30] A* $\mathfrak{Z}_{\mathcal{G}}$ *simulation function is any mapping* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *which fulfills the following:*

(i) $\zeta(0,0) = 0$;

(ii) $\zeta(x,y) < \mathcal{G}(y,x)$, $\forall\ x, y > 0$, *where* $\mathcal{G}$ *is a* C*-class function;*

(iii) If sequences $\{x_n\}$ *and* $\{y_n\}$ *in the interval* $[0,\infty)$ *satisfy* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$, *then*

$$\lim_{n\to\infty} \sup \zeta(x_n, y_n) < C_{\mathcal{G}}.$$

Lemma 1. *[31] Consider a metric space* $(\mathscr{U}, \rho)$ *and* $A, B \subseteq \mathscr{U}$. *Then, for every* $a \in A$, *there is some* $b \in B$ *so that for* $q > 1$, *we obtain*

$$\rho(a,b) \leq q\ H(A,B).$$

Rawat et al. [24] defined an enriched interpolative Kannan type contraction (EIK-contraction) for single valued mappings as follows.

Definition 4. *Let* $(\mathscr{U}, d, W)$ *be a convex metric space. A self-mapping* $T : \mathscr{U} \to \mathscr{U}$ *is an EIK-contraction if there exist* $\lambda \in [0,1), c \in [0,1)$ *and* $\alpha \in (0,1)$, *such that*

$$d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq c[d(x, W(x,Tx;\lambda))]^{\alpha}.[d(y, W(y,Ty;\lambda))]^{1-\alpha},$$

for all $x, y \in X \backslash Fix(T)$.

They further demonstrated the next result.

Theorem 1. *[24] Let* $(\mathscr{U}, \rho)$ *be a convex complete metric space and* $T : \mathscr{U} \to \mathscr{U}$ *be an EIK-contraction mapping. Then* T *admits a fixed point.*

Karapinar et al. [32] defined a multivalued interpolative Hardy-Rogers type contraction (IHR-contraction) as follows.

Definition 5. *Let* $(\mathscr{U}, d)$ *be a metric space. We say that* $T : \mathscr{U} \to CB(\mathscr{U})$ *is a multivalued interpolative HR-contraction via a simulation function* $\mathcal{Z}_G$ *, if there exist* $k \in [0,1)$ *and* $\alpha, \beta, \gamma \geq \geq 0$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$\zeta(H(Tx,Ty), R(x,y)) \geq C_G,$$

where

$$R(x,y) = k[d(x,y)]^{\alpha}[D(x,Tx)]^{\beta}[D(y,Ty)]^{\gamma}[12(D(x,Ty) + D(x,Tx))]^{1-\alpha-\beta-\gamma},$$

for all $x, y \in \mathscr{U} \backslash Fix(T)$.

They further demonstrated the next result.

Theorem 2. *[32] Let* $(\mathscr{U}, \rho)$ *be a complete metric space and* T *be a multivalued IHR-contraction via a simulation function* $\mathfrak{Z}_G$. *Then* $Fix(T) \neq \phi$.

Now, we define some basic preliminaries related to convex metric spaces.

Definition 6. *[23] Let* $\mathscr{U}$ *be a metric space. A continuous function* $\mathscr{W} : \mathscr{U} \times \mathscr{U} \times [0,1] \to \mathscr{U}$ *is known as a convex structure on* $\mathscr{U}$*, if for every* $\lambda \in [0,1]$ *and* $x, y \in \mathscr{U}$*, the next inequality holds:*

$$\rho(u, W(x,y;\lambda)) \leq \lambda\rho(u,x) + (1-\lambda)\rho(u,y), \text{ for each } x \in \mathscr{U}. \tag{1}$$

A metric space $\mathscr{U}$ *with a convex structure* $\mathscr{W}$ *on* $\mathscr{U}$ *is called a Takahashi convex metric structure, or simply with a convex metric structure and will be denoted as* $(\mathscr{U}, \rho, \mathscr{W})$.

The lemmas below outline some fundamental properties of a convex metric space.

Lemma 2. *[23] Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex metric space. For any* $x, y \in \mathscr{U}$ *and any* $\lambda \in [0,1]$*, the following holds:*

$$\rho(x,y) = \rho(x, \mathscr{W}(x,y;\lambda)) + \rho(\mathscr{W}(x,y;\lambda), y).$$

Lemma 3. *[33] Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex metric space. For any* $x, y \in \mathscr{U}$ *and any* $\lambda, \lambda_1, \lambda_2 \in [0,1]$*, the following holds:*

(i) $\mathscr{W}(x,x;\lambda) = x; \mathscr{W}(x,y;0) = y$ *and* $\mathscr{W}(x,y;1) = x$.

(ii) $|\lambda_1 - \lambda_2|\rho(x,y) \leq \rho(\mathscr{W}(x,y;\lambda_1), \mathscr{W}(x,y;\lambda_2))$.

Lemma 4. *[23] Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex metric space. For any* $x, y \in \mathscr{U}$ *and any* $\lambda \in [0,1]$*, the following holds:*

$$\rho(x, \mathscr{W}(x,y;\lambda)) = (1-\lambda)\rho(x,y) \text{ and } \rho(\mathscr{W}(x,y;\lambda), y) = \lambda\rho(x,y).$$

Lemma 5. *[33] Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex metric space and* $T : \mathscr{U} \to \mathscr{U}$ *be a mapping. For each* $\lambda \in [0,1),$*, define the mapping* $T_\lambda : \mathscr{U} \to \mathscr{U}$ *as follows:*

$$T_\lambda x = \mathscr{W}(x, Tx; \lambda), \ x \in \mathscr{U}. \tag{2}$$

Then, $Fix(T) = Fix(T_\lambda)$.

3. Main results

Definition 7. *Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex metric space. A multivalued mapping* $T : \mathscr{U} \to CB(\mathscr{U})$ *is an EIK-contraction via a simulation function* $\mathfrak{Z}_{\mathcal{G}}$*, if there are* $\alpha \in [0, 1)$ *and* $\lambda, p \in (0, 1)$ *so that*

$$\zeta(H(\mathscr{W}(x, Tx; \lambda), \mathscr{W}(y, Ty; \lambda)), Q(x, y)) \geq C_{\mathcal{G}}. \tag{3}$$

Here,

$$Q(x, y) = p[\mathfrak{D}^*(x, \mathscr{W}(x, Tx; \lambda)]^{\alpha}[\mathfrak{D}^*(y, \mathscr{W}(y, Ty; \lambda)]^{1-\alpha}$$

for each $x, y \in \mathscr{U}/Fix(T)$.

Theorem 3. *Let* $(\mathscr{U}, \rho, \mathscr{W})$ *be a convex complete metric space and* $T : \mathscr{U} \to CB(\mathscr{U})$ *be a multivalued EIK-contraction via a simulation function* $\mathfrak{Z}_{\mathcal{G}}$*. Then* $Fix(T) \neq \phi$.

Proof. Using the multivalued EIK-contraction condition (3), the mapping $T_\lambda : \mathscr{U} \to CB(\mathscr{U})$ given by (2) satisfies

$$\zeta(H(T_\lambda x, T_\lambda y), Q(x, y)) \geq C_{\mathcal{G}}, \tag{4}$$

where

$$Q(x, y) = p[\mathfrak{D}^*(x, T_\lambda x)]^{\alpha}[\mathfrak{D}^*(y, T_\lambda y)]^{1-\alpha}$$

for each $x, y \in \mathscr{U}/Fix(T)$, that is, T_λ is an interpolative Kannan type contraction. Let $y_0 \in \mathscr{U}$ and define a sequence $y_n \in T_\lambda y_{n-1}$, for each $n \geq 1$. If we have $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then y_{n_0} is fixed point of T_λ and thus a fixed point of T. So there is nothing to prove.
Let $y_n \neq y_{n+1}$ for each $n \geq 0$. Since $0 < p < 1$ and $y_n \in T_\lambda y_{n-1}$ for each $n \geq 1$, we can choose $q > 1$ so that $qp < 1$, then from Lemma 1 there is $y_{n+1} \in T_\lambda y_n$, for each $n \geq 1$ so that

$$\rho(y_n, y_{n+1}) \leq qH(T_\lambda y_{n-1}, T_\lambda y_n). \tag{5}$$

Taking $x = y_n$ and $y = y_{n-1}$, from equation (3), we obtain

$$\zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), Q(y_n, y_{n-1}) \geq C_{\mathcal{G}}. \tag{6}$$

By Definition 3, we obtain

$$\mathcal{C}_{\mathcal{G}} \leq \zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), Q(y_n, y_{n-1})) < \mathcal{G}(Q(y_n, y_{n-1}), H(T_\lambda y_n, T_\lambda y_{n-1}).$$

From Definition 2, we obtain

$$\begin{aligned} H(T_\lambda y_n, T_\lambda y_{n-1}) &< Q(y_n, y_{n-1}) \\ &= p[\mathfrak{D}^*(y_n, T_\lambda y_n]^{\alpha}.[\mathfrak{D}^*(y_{n-1}, T_\lambda y_{n-1}]^{1-\alpha}. \end{aligned}$$

Using equation (5) and substituting $pq = \theta < 1$, we obtain

$$\begin{aligned}\rho(y_n, y_{n+1}) &< pq\ [\mathfrak{D}^*(y_n, T_\lambda y_n]^\alpha.[\mathfrak{D}^*(y_{n-1}, T_\lambda y_{n-1}]^{1-\alpha} \\ &= \theta[\mathfrak{D}^*(y_n, T_\lambda y_n]^\alpha.[\mathfrak{D}^*(y_{n-1}, T_\lambda y_{n-1}]^{1-\alpha}.\end{aligned}$$

Since we know $y_n \in T_\lambda y_{n-1}$, one gets $\mathfrak{D}^*(y_{n-1}, Ty_{n-1}) \leq \rho(y_{n-1}, y_n),\ \forall\ n \geq 1$. Therefore, we obtain

$$\rho(y_n, y_{n+1}) < \theta[\rho(y_n, y_{n+1})]^\alpha[\rho(y_{n-1}, y_n]^{1-\alpha}. \tag{7}$$

Suppose if possible $\rho(y_{n-1}, y_n) < \rho(y_n, y_{n+1})$ for some $n \geq 1$, then

$$\begin{aligned}\rho(y_n, y_{n+1}) &< \theta[\rho(y_n, y_{n+1})]^\alpha[\rho(y_n, y_{n+1})]^{1-\alpha} \\ &= \theta\rho(y_n, y_{n+1}).\end{aligned}$$

This leads to a contradiction as $\theta < 1$. Therefore,

$$\rho(y_n, y_{n+1}) \leq \rho(y_{n-1}, y_n).$$

From equation (7), we obtain

$$\rho(y_n, y_{n+1}) \leq \theta\rho(y_{n-1}, y_n), \tag{8}$$

which further implies

$$\begin{aligned}\rho(y_n, y_{n+1}) &< \theta\rho(y_{n-1}, y_n) \\ &< \theta^2\rho(y_{n-2}, y_{n-1}) \\ &< \dots \\ &< \theta^n\rho(y_0, y_1).\end{aligned}$$

Taking $n \to \infty$, we get $\rho(y_n, y_{n+1}) \to 0$. Let $m, n \in \mathbb{N},\ m > n$, then

$$\begin{aligned}\rho(y_n, y_m) &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq \frac{\theta^n}{1-\theta}\rho(y_0, y_1).\end{aligned}$$

As $n \to \infty$, we obtain $\rho(y_n, y_m) \to 0$. This implies $\{y_n\}$ is a Cauchy sequence and using the completeness of $\mathscr{U}$, there is $y^* \in \mathscr{U}$ so that $\lim_{n\to\infty} y_n = y^*$. Suppose $y \notin Ty$, then $y \notin T_\lambda y$. Since $T_\lambda y_n$ is closed for each $n \geq 0$, therefore $y_n \notin T_\lambda y_n$. From equation (4), we get

$$C_\mathcal{G} \leq \zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), Q(y_n, y_{n-1}) < C_\mathcal{G},$$

which leads to a contradiction. Hence, $y \in Ty$, which implies $Fix(T) \neq \phi$.

Now, we present an example in support of our first theorem.

Example 1. *Let $\mathscr{U} = \mathbb{R}$ be equipped with the Euclidean metric $\rho(x, y) = |x - y|$, $\forall\ x, y \in \mathscr{U}$, and $T : \mathscr{U} \to CB(\mathscr{U})$ be given as*

$$T(x) = \{-x, 1 - x\}.$$

Also, let $\zeta(t, r) = \frac{1}{2}r - t$, $\mathcal{G}(r, t) = r - t$ for each $r, t \in [0, \infty)$ and $C_{\mathcal{G}} = 0$.
Taking $\lambda = \frac{1}{2}$, we obtain

$$T_{\frac{1}{2}}(x) = \left\{0, \frac{1}{2}\right\}.$$

Since $Fix(T) = \{0, \frac{1}{2}\}$, we get for each $x, y \in \mathscr{U}/\{0, \frac{1}{2}\}$, $H(T_\lambda x, T_\lambda y) = 0$ and for $\alpha \in (0, 1)$, $Q(x, y) = c[\mathfrak{D}^*(x, \{0, \frac{1}{2}\})]^\alpha [\mathfrak{D}^*(y, \{0, \frac{1}{2}\})]^{1-\alpha} > 0$, which implies

$$\begin{aligned}\zeta(H(T_\lambda x, T_\lambda y), Q(x, y) &= \frac{1}{2}Q(x, y)\\ &\geq C_{\mathcal{G}}.\end{aligned}$$

Also,

$$\mathcal{G}(Q(x, y), H(T_\lambda x, T_\lambda y)) = Q(x, y),$$

and

$$C_{\mathcal{G}} \leq \zeta(H(T_\lambda x, T_\lambda y), Q(x, y) < \mathcal{G}(Q(x, y), H(T_\lambda x, T_\lambda y)).$$

Thus, T is a multivalued EIK-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$ and all requirements outlined in Theorem 3.1 are met. Here, $Fix(T) = \{0, \frac{1}{2}\}$.

Corollary 4. *Let $(\mathscr{U}, \rho)$ be a complete metric space which. If T is a multivalued interpolative Kannan type contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$, then $Fix(T) \neq \phi$.*

Proof. Taking $\lambda = 0$ and using the same method of proof as in Theorem 3.1, we achieve the intended result.

Corollary 5. *Let $(\mathscr{U}, \rho)$ be a convex complete metric space. If a self mapping T is an EIK-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$, then T possesses a fixed point.*

Definition 8. *Let $(\mathscr{U}, \rho, \mathscr{W})$ be a convex metric space. Then $T : \mathscr{U} \to CB(\mathscr{U})$ is a multivalued EIHR-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$, if there is some $p \in (0, 1)$ and $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ so that*

$$\zeta(H(\mathscr{W}(x, Tx; \lambda), \mathscr{W}(y, Ty; \lambda)), Q(x, y) \geq C_{\mathcal{G}}. \tag{9}$$

Here, $C_{\mathcal{G}} \geq 0$ and

$$\begin{aligned}Q(x, y) = p.\rho(x, y)^\alpha.[\mathfrak{D}^*(x, \mathscr{W}(x, Tx; \lambda))]^\beta.[\mathfrak{D}^*(y, \mathscr{W}(y, Ty; \lambda)))]^\gamma.\\ \left[\frac{1}{2}(\mathfrak{D}^*(x, \mathscr{W}(y, Ty; \lambda)) + \mathfrak{D}^*(y, \mathscr{W}(x, Tx; \lambda)))\right]^{1-\alpha-\beta-\gamma}\end{aligned}$$

for each $x, y \in \mathscr{U}/Fix(T)$.

Theorem 6. *Let $(\mathscr{U}, \rho, \mathscr{W})$ be a convex complete metric space. If $T : \mathscr{U} \to CB(\mathscr{U})$ is a multivalued EIHR-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$, then $Fix(T) \neq \phi$.*

Proof. Using the multivalued EIHR-contraction condition (9), the mapping $T_\lambda : \mathscr{U} \to CB(\mathscr{U})$ given by (2) satisfies

$$\zeta(H(T_\lambda x, T_\lambda y), Q(x, y)) \geq C_{\mathcal{G}}, \tag{10}$$

where $C_{\mathcal{G}} \geq 0$ and

$$Q(x, y) = p.\rho(x, y)^\alpha.[\mathfrak{D}^*(x, T_\lambda x)]^\beta.[\mathfrak{D}^*(y,T_\lambda y)]^\gamma. \left[\frac{1}{2}(\mathfrak{D}^*(x, T_\lambda y) + \mathfrak{D}^*(y, T_\lambda x)\right]^{1-\alpha-\beta-\gamma}$$

for each $x, y \in \mathscr{U}/Fix(T)$, that is, T_λ is an IHR-contraction. Let $y_0 \in \mathscr{U}$ and define a sequence $y_n \in T_\lambda y_{n-1}$, for each $n \geq 1$. If we have $y_{n_0} = y_{n_0+1}$ for a particular $n_0 \in \mathbb{N}$, then y_{n_0} is a fixed point of T_λ and thus a fixed point of T. So there is nothing to prove. Let $y_n \neq y_{n+1}$ for each $n \geq 0$. Since $0 < p < 1$ and $y_n \in T_\lambda y_{n-1}$ for each $n \geq 1$, we can choose $q > 1$ so that $qp < 1$, then from Lemma 1 there is some $y_{n+1} \in T_\lambda y_n$, for each $n \geq 1$ so that

$$\rho(y_n, y_{n+1}) \leq qH(T_\lambda y_{n-1}, T_\lambda y_n). \tag{11}$$

Taking $x = y_n$ and $y = y_{n-1}$, from equation (3), we obtain

$$\zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), Q(y_n, y_{n-1}) \geq C_{\mathcal{G}}. \tag{12}$$

By Definition 3, we obtain

$$C_{\mathcal{G}} \leq \zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), Q(y_n, y_{n-1})) < \mathcal{G}(Q(y_n, y_{n-1}), H(T_\lambda y_n, T_\lambda y_{n-1}).$$

From Definition 2, we obtain

$$H(T_\lambda y_n, T_\lambda y_{n-1}) < Q(y_n, y_{n-1}), \tag{13}$$

where

$$\begin{aligned} Q(y_n, y_{n-1}) &= p[\rho(y_{n-1}, y_n)^\alpha[\mathfrak{D}^*(y_{n-1}, T_\lambda y_{n-1})]^\beta[\mathfrak{D}^*(y_n, T_\lambda y_n))]^\gamma \\ &\times [\frac{1}{2}(\mathfrak{D}^*(y_{n-1}, T_\lambda y_n) + \mathfrak{D}^*(y_n, T_\lambda y_{n-1}))]^{1-\alpha-\beta-\gamma} \\ &\leq p[\rho(y_{n-1}, y_n)^\alpha[\rho(y_{n-1}, y_n)]^\beta[\rho(y_n, y_{n+1}))]^\gamma \\ &\times [\frac{1}{2}(\rho(y_{n-1}, y_{n+1}) + \rho(y_n, y_n))]^{1-\alpha-\beta-\gamma} \\ &\leq p[\rho(y_{n-1}, y_n)^\alpha[\rho(y_{n-1}, y_n)]^\beta[\rho(y_n, y_{n+1}))]^\gamma \\ &\times [\frac{1}{2}(\rho(y_{n-1}, y_n) + \rho(y_n, y_{n+1}) + d(y_n, y_n))]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

Suppose if possible $\rho(y_{n-1}, y_n) < \rho(y_n, y_{n+1})$ for some $n \geq 1$, then

$$\frac{1}{2}(\rho(y_{n-1}, y_n) + \rho(y_n, y_{n+1}) + \rho(y_n, y_n)) \leq d(y_n, y_{n+1}),$$

which further implies

$$\begin{aligned} p[\rho(y_{n-1}, y_n)^{\alpha}[\rho(y_{n-1}, y_n)]^{\beta}[\rho(y_n, y_{n+1}))]^{\gamma}.[\frac{1}{2}(\rho(y_{n-1}, y_n) + \rho(y_n, y_{n+1}) + \rho(y_n, y_n))]^{1-\alpha-\beta-\gamma} \\ < p[\rho(y_n, y_{n+1})]^{\alpha+\beta+\gamma}.[\rho(y_n, y_{n+1})]^{1-\alpha-\beta-\gamma} \\ = pd(y_n, y_{n+1}) \end{aligned}$$

i.e., $R(y_n, y_{n-1}) < pd(y_n, y_{n+1})$. From equation (9) and (11), we get

$$\begin{aligned} \rho(y_n, y_{n+1}) &\leq pq\rho(y_n, y_{n+1}) \\ &= \theta\rho(y_n, y_{n+1}). \end{aligned}$$

This leads to a contradiction as $\theta < 1$. Therefore, we obtain

$$\rho(y_n, y_{n+1}) \leq \rho(y_{n-1}, y_n).$$

From equations (11) and (13), one writes

$$\rho(y_n, y_{n+1}) \leq qH(T_\lambda y_{n-1}, T_\lambda y_n) < pq\rho(y_{n-1}, y_n). \tag{14}$$

This implies that

$$\begin{aligned} \rho(y_n, y_{n+1}) &< \theta\rho(y_{n-1}, y_n) \\ &< \theta^2\rho(y_{n-2}, y_{n-1}) \\ &< \dots \\ &< \theta^n\rho(y_0, y_1). \end{aligned}$$

Taking $n \to \infty$, we obtain $\rho(y_n, y_{n+1}) \to 0$. Let $m, n \in \mathbb{N}$ and $m > n$, then

$$\begin{aligned} \rho(y_n, y_m) &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq \frac{\theta^n}{1-\theta}\rho(y_0, y_1). \end{aligned}$$

As $n \to \infty$, we get $\rho(y_n, y_m) \to 0$. This implies $\{y_n\}$ is a Cauchy sequence and using the completeness of $\mathscr{U}$ there is some $v^* \in U$ so that $\lim_{n\to\infty} y_n = y^*$. Suppose $y \notin Ty$, then $y \notin T_\lambda y$. Since $T_\lambda y_n$ is closed for each $n \geq 0$, $y_n \notin T_\lambda y_n$. From equation (10), we obtain

$$C_{\mathcal{G}} \leq \zeta(H(T_\lambda y_n, T_\lambda y_{n-1}), R(y_n, y_{n-1}) < C_{\mathcal{G}},$$

which leads to a contradiction. Hence, $y \in Ty$, which implies $Fix(T) \neq \phi$.

The next example justifies the previous theorem.

Example 2. *Let $\mathscr{U} = [0,1]$ be equipped with the Euclidean metric $\rho(x,v) = |x-v|$, for each $x, y \in \mathscr{U}$, and $T : \mathscr{U} \to CB(\mathscr{U})$ be given as*

$$T(x) = \left\{\frac{1-x}{2}, \frac{-x}{2}\right\}.$$

Also, let $\zeta(t,r) = \frac{1}{2}r - t$, $\mathcal{G}(r,t) = r - t$ for each $r, t \in [0,\infty)$ and $C_{\mathcal{G}} = 0$. Taking $\lambda = \frac{1}{3}$, we obtain

$$T_{\frac{1}{3}}(x) = \left\{0, \frac{1}{3}\right\}.$$

Since $Fix(T) = \{0, \frac{1}{2}\}$, we get for each $x, y \in \mathscr{U}/\{0, \frac{1}{2}\}$, $H(T_\lambda x, T_\lambda y) = 0$ and for $\alpha \in (0,1)$, $Q(x,y) = c[\mathfrak{D}^*(x, \{0, \frac{1}{2}\})]^\alpha [\mathfrak{D}^*(y, \{0, \frac{1}{2}\})]^{1-\alpha} > 0$, which implies

$$\begin{aligned}\zeta(H(T_\lambda x, T_\lambda y), Q(x,y) &= \frac{1}{2}Q(x,y) \\ &\geq C_{\mathcal{G}}.\end{aligned}$$

Also,

$$\mathcal{G}(R(x,y), H(T_\lambda x, T_\lambda y)) = Q(x,y).$$

One writes

$$C_{\mathcal{G}} \leq \zeta(H(T_\lambda x, T_\lambda y), Q(x,y) < \mathcal{G}(Q(x,y), H(T_\lambda x, T_\lambda y)).$$

Thus, T is a multivalued EIHR-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$ and all requirements outlined in Theorem 3.5 are met. Here, $Fix(T) = \{0, \frac{1}{3}\}$.

Corollary 7. *Let $(\mathscr{U}, \rho)$ be a complete metric space. If T is a multivalued IHR-contraction via simulation function $\mathfrak{Z}_{\mathcal{G}}$, then $Fix(T) \neq \phi$.*

Proof. Taking $\lambda = 0$, and using the same method of proof as in Theorem 3.5, we achieve the intended result.

Corollary 8. *Let $(\mathscr{U}, \rho)$ be a complete metric space. If a self mapping $T : \mathscr{U} \to \mathscr{U}$ is an IHR-contraction via a simulation function $\mathfrak{Z}_{\mathcal{G}}$, then T possesses a fixed point.*

4. Data Dependence Results

We propose data dependence results for multivalued EIK-contractions and multivalued EIHR-contractions via a simulation function $\mathfrak{Z}_{\mathcal{G}}$. Here $p_1, p_2 \in (0,1)$ are the constants for T_1 and T_2, respectively as given in Definition 7 and Definition 8.

Theorem 9. *Let $(\mathscr{U},\rho)$ be a convex metric space and $T_i : \mathscr{U} \to CB(\mathscr{U})$ be two multivalued EIK-contraction operators via a simulation function $\mathfrak{Z}_{\mathcal{G}}$. Suppose that there is some $\alpha > 0$ so that $H(T_1y, T_2y) \leq \alpha$ for each $y \in \mathscr{U}$. Then*
(i)$Fix(T_i)$ is a closed subset of $\mathscr{U}$ for $i \in \{1,2\}$.
(ii) $H(Fix(T_1,), Fix(T_2)) \leq \frac{\alpha}{1-max\{p_1,p_2\}}$.

Proof. From Theorem 3.1, $Fix(T_i) \neq \phi$ for $i \in \{1,2\}$. Let $\{y_n\}$ be a sequence in $Fix(T_i) = Fix(T_{i_\lambda})$ for $i \in \{1,2\}$ so that $y_n \to y^*$ as $n \to \infty$, then for $i \in \{1,2\}$

$$\zeta(H(T_{i_\lambda}y_n, T_{i_\lambda}y_{n-1}), Q(y_n, y_{n-1}) \geq C_{\mathcal{G}}.$$

By Definition 3, we obtain

$$C_{\mathcal{G}} \leq \zeta(H(T_{i_\lambda}y_n, T_{i_\lambda}y_{n-1}), Q(y_n, y_{n-1}) < \mathcal{G}(Q(y_n, y_{n-1}), H(T_{i_\lambda}y_n, T_{i_\lambda}y_{n-1})).$$

From Definition 2, we obtain

$$H(T_{i_\lambda}y_n, T_{i_\lambda}y_{n-1}) < Q(y_n, y_{n-1}),$$

which implies

$$\begin{aligned}\mathfrak{D}^*(y_n, T_{i_\lambda}y_{n-1}) < Q(y_n, y_{n-1}) &= p_i[\mathfrak{D}^*(y_n, T_{i_\lambda}y_n]^{\alpha}.[\mathfrak{D}^*(y_{n-1}, T_{i_\lambda}y_{n-1}]^{1-\alpha} \\ &= 0.\end{aligned}$$

As $n \to \infty$, we get that $\mathfrak{D}^*(x, T_{i_\lambda}x) = 0$. Since $T_{i_\lambda}x \in CB(\mathscr{U})$, we have $x \in T_{i_\lambda}x$. Hence, $x \in Fix(T_{i_\lambda}) = Fix(T_i)$ for $i \in \{1,2\}$.
Let $y_0 \in Fix(T_1) = Fix(T_{1_\lambda})$ be arbitrary. Then for $q > 1$, there is some $y_1 \in T_{2_\lambda}y_0$ so that

$$\rho(y_0, y_1) \leq qH(T_{1_\lambda}y_0, T_{2_\lambda}y_0).$$

Next, for $y_1 \in T_{2_\lambda}y_0$ there is some $y_2 \in T_{2_\lambda}y_1$ so that

$$\rho(y_1, y_2) \leq qH(T_{2_\lambda}y_0, T_{2_\lambda}y_1).$$

Similarly, we derive the sequence of successive approximations for T_{2_λ} beginning from y_0, such that $y_{n+1} \in T_{2_\lambda}y_n$ for each $n \geq 1$ and

$$\rho(y_n, y_{n+1}) \leq qH(T_{2_\lambda}y_{n-1}, T_{2_\lambda}y_n).$$

From equation (8) (taking p_2 in place of θ), we obtain

$$\rho(y_n, y_{n+1}) \leq qp_2\rho(y_{n-1}, y_n) \text{ for each } n \geq 1.$$

Hence, for $m \geq 1$ and $n \in \mathbb{N}$, we obtain

$$\rho(y_{n+m}, y_n) \leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \cdots + \rho(v_{n+m-1}, y_{n+m})$$

$$\leq (qp_2)^n \rho(y_0, y_1) + (qp_2)^{n+1} \rho(y_0, y_1) + \cdots + (qp_2)^{n+m-1} \rho(y_0, y_1)$$
$$\leq \frac{(qp_2)^n}{1 - qp_2} \rho(y_0, y_1).$$

Taking $1 < q < min\{\frac{1}{p_1}, \frac{1}{p_2}\}$ and as $n \to \infty$, we conclude that the $\{y_n\}$ is a Cauchy sequence in $(\mathscr{U}, \rho)$. Therefore, there is some $y^* \in \mathscr{U}$ so that $y_n \to y^*$ as $n \to \infty$.
Claim: y^* is a fixed point of T_2.
Suppose if possible $y^* \notin T_2 y^*$, which implies $y^* \notin T_{2_\lambda} y^*$, then $y_{n_k} \notin T_{2_\lambda} y_{n_k}$. From Definition 3 and the contraction condition, we obtain

$$C_{\mathcal{G}} \leq \lim_{n\to\infty} \sup \zeta(H(T_{2_\lambda} y^*, T_{2_\lambda} y_{n_k}), Q(y^*, y_{n_k})) < C_{\mathcal{G}}.$$

This leads to a contradiction. Hence, y^* is a fixed point of T_2.
Taking $m \to \infty$, then for each $n \in \mathbb{N}$ we obtain

$$\rho(y^*, y_n) \leq \frac{(qp_2)^n}{1 - qp_2} \rho(y_0, y_1),$$

which implies

$$\rho(y_0, y^*) \leq \frac{1}{1 - qp_2} d(y_0, y_1)$$
$$\leq \frac{qk'}{1 - qp_2}.$$

Similarly, for every $x_0 \in Fix(T_2)$, there is some $x^* \in Fix(T_1)$, for which

$$\rho(x_0, x^*) \leq \frac{1}{1 - qp_2} d(x_0, x_1)$$
$$\leq \frac{qk'}{1 - qp_2}.$$

Hence,

$$H(Fix(T_1), Fix(T_2)) \leq \frac{qk'}{1 - \max\{qp_1, qp_2\}}$$

Letting $q \to 1$, we achieve the intended result.

Theorem 10. *Let $(\mathscr{U}, \rho)$ be a metric space and $T_i : \mathscr{U} \to CB(\mathscr{U})$ be two multivalued EIHR-contractions via a simulation function $\mathfrak{Z}_{\mathcal{G}}$. Suppose that there is some $\alpha > 0$ so that $H(T_1 y, T_2 y) \leq \alpha$ for each $y \in \mathscr{U}$. Then*
(i)$Fix(T_i)$ is a closed subset of $\mathscr{U}$ for $i \in \{1, 2\}$.
(ii) $H(Fix(T_1,), Fix(T_2)) \leq \frac{\alpha}{1 - max\{p_1, p_2\}}$.

Proof. From Theorem 3.5, $Fix(T_i) \neq \phi$ for $i \in \{1, 2\}$. Let $\{y_n\}$ be a sequence in $Fix(T_i) = Fix(T_{i_\lambda})$ for $i \in \{1, 2\}$ so that $y_n \to y^*$ as $n \to \infty$, then for $i \in \{1, 2\}$

$$\zeta(H(T_{i_\lambda} y_n, T_{i_\lambda} y_{n-1}), Q(y_n, y_{n-1})) \geq C_{\mathcal{G}}$$

By Definition 3, we obtain

$$C_{\mathcal{G}} \leq \zeta(H(T_{i_\lambda} y_n, T_{i_\lambda} y_{n-1}), Q(y_n, y_{n-1})) < \mathcal{G}(Q(y_n, y_{n-1}), H(T_{i_\lambda} y_n, T_{i_\lambda} y_{n-1})).$$

Using Definition 2, one writes

$$H(T_{i_\lambda} y_n, T_{i_\lambda} y_{n-1}) < Q(y_n, y_{n-1})$$

which implies

$$\begin{aligned}\mathfrak{D}^*(y_n, T_{i_\lambda} y_{n-1}) < Q(y_n, y_{n-1}) = p_i[\mathfrak{D}^*(y_{n-1}, y_n)^{\alpha}[\mathfrak{D}^*(y_{n-1}, T_{i_\lambda} y_{n-1})]^{\beta}[\mathfrak{D}^*(y_n, T_{i_\lambda} y_n))]^{\gamma} \\ \times [\frac{1}{2}(\mathfrak{D}^*(y_{n-1}, T_{i_\lambda} y_n) + \mathfrak{D}^*(y_n, T_{i_\lambda} y_{n-1}))]^{1-\alpha-\beta-\gamma}.\end{aligned}$$

Now, using the same method of proof as in Theorem 4.1, we achieve the intended result.

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Authors' contributions

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Competing interests

The authors declare that they have no conflicts of interest.

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