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σ-compact Spaces in Nth-Topological Space

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Abstract. The principal aims of this paper include the introduction of the concept of σ-compact Spaces in n^{th}-topological spaces, the definition of compactness, discussion of various generalizations encompassing compactness in n^{th}-topological space, the properties of σ-compact Spaces. Furthermore, our study extends to the analysis of the notion of countable compactness in n^{th}-topological spaces. Our investigation extends to various generalizations of these spaces.

2020 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Tri-topological spaces, Compactness, Sigma compact spaces, Separation axioms

1. Introduction

The study of n^{th}-topological spaces arises as a natural extension of the concepts found in bi-topological and tri-topological spaces, building upon the foundation of single topological spaces. These extensions aim to provide a more comprehensive framework for understanding complex structures that involve multiple interrelated topologies. In this paper, we define an n^{th}-topological space as a non-empty set K equipped with n^{th} distinct topologies, denoted as η_1, η_2, ..., η_n, forming the structure $(K, \eta_1, \eta_2, ..., \eta_n)$.

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The idea of n^{th}-topological spaces began gaining attention in the early 21st century, specifically around the year 2000, as researchers sought to expand the theoretical understanding and practical applications of topological concepts to more intricate systems. Central to this study is the notion of n^{th}-compactness, which generalizes the classical concept of compactness by requiring the existence of a finite subcover under the n^{th}-topological framework. This generalization has proven to be a powerful tool in extending well-established results to broader contexts.

n^{th}-topological spaces have shown particular promise in their application to compact and metacompact spaces, serving as a bridge between classical topology and more specialized fields. The exploration of n^{th}-compactness has provided new insights into the behavior of finite subcovers and their interactions across multiple topologies. By expanding upon existing concepts such as n^{th}-metacompactness, researchers have developed a wealth of generalized theorems and illustrative examples.

This paper builds on these advancements, presenting several significant examples and discussing key results that extend the foundational theorems of topology to n^{th}-topological spaces. The works of [J. Oudetallah], [Jamal Oudetallah (2021)], [Jamal Oudetallah and M Al-Hawari (2018)], and [Jamal Oudetallah, Mohammad M. Rousa (2021)] are particularly noteworthy in this regard, as they provide a robust theoretical underpinning for the concepts discussed here. These contributions underline the versatility and applicability of n^{th}-topological spaces in modern topology, making them a crucial area of study for further exploration and development.

2. Literature review

The concept of a σ space in topological space (K, η) was presented by [7]. Recent research [2], [3], [6] has delved deeper into these areas. This paper explores the concept of n^{th}-σ presents associated conclusions. Then, we introduce the concept of topological space, n^{th}-topological space and some important concept in n^{th}-topological space like: open and closed sets, derived set, closure set, interior and exterior sets, separation axioms, etc... . Then we talk about the concept of σ space in n^{th}-topological spaces, discuss its features, and apply it to other spaces. We examine well-known definitions that will be applied in the sequel.

The terms η_u, η_{dis}, η_{cof} and η_{coc} represent the ordinary or usual topology, discrete topology, co-finite topology, and co-countable topology, respectively. The concept of tri-topological spaces can be represented as $K = (K, \eta_1, \eta_2, \eta_3)$ where η_1, η_2 and η_3 are topologies on K and the concept of n^{th}-topological space $(K, \eta_1, \eta_2, ..., \eta_n)$ where η_1, η_2 ... η_3 are topologies on K . This is connected to prior research on n^{th}-topological spaces, where each topology is a set of points that meet a set of axioms. [4] explained Hausdorff,regular and normal spaces in n^{th} topological spaces using a set of standard results known as Tietze extintion. The primary goal of this paper is to introduce and investigate a novel sort of n^{th}-σ space and n^{th}-topological spaces are sets containing n topologies.

3. Preliminaries

Definition 1.1 [3]: Let $K \neq \phi$, $\eta \subset$ P(K) = {A:A $\subseteq K$}, then η is called topology on K if the following conditions are satisfied :
i) ϕ , $K \in \eta$.
ii) Closed under intersection.
iii) The union of any collection of sets in η is also in η.

Definition 2.1[1] : Let K be a non empty set , $\eta_i \subset$ P(K) = {A:A $\subseteq K$}, where i=1,2,3,...,n .
we say that (K , η_1 , η_2, ... ,η_n) is n^{th}-topological space if η_i is topology on K , for all i=1,2,3,...,n .

Example consider K = {1,2,3}
$\eta_1 = \{\phi, K , \{1\}\} \subset$ P(K)
$\eta_2 = \{\phi , K, \{1\} , \{2\} , \{1,2\}\} \subset$ P(K)
$\eta_3 = \{\phi , K, \{2\} , \{3\} , \{2,3\}\} \subset$ P(K)
η_i's satisfies the condition of topological space , i=1,2,3 so, (K, η_1 , η_2 , η_3) is tri-topological space.
But, for example $\eta = \{ \phi, 1\}$ is not topological space because it is not contains K.

Definition 3.1 : Let (K , η_1 , η_2 , ..., η_n) be a n^{th}-topological space . E $\subset K$, then:
i) E is called n^{th}-Open set , If E $\in \eta_i$ for some i=1,2,3.
ii) E is called n^{th}-cloced set , If E$^c \in \eta_i$ for some i=1,2,3.
iii) E is called n^{th}-clopen set , If E and E^c are both in η_i for some i=1,2,3.

Example Let $K = \{x, y, z\}, \eta_1 = \{\phi, K, \{x\}\}, \eta_2 = \{\phi, K, \{y\}\}, \eta_3 = \{\phi, K, \{z\}\}$.
The sets : $\phi, K, \{x\}, \{y\}, \{z\}$ are n^{th}-open sets in K.
The sets : $\phi, K, \{y, z\}, \{x, z\}, \{x, y\}$ are n^{th}-closed sets in K.
The sets : ϕ ,K are n^{th}-clopen sets in K.

Definition 4.1[5] : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space , $K \neq \phi$ and A is subset of K ,then q $\in K$ is called n^{th}-Limit point of A If for all u_d n^{th}-open set such that $u_d \cap ($A $- \{d\}) \neq \phi$.
The set of all n^{th}-limit points is called n^{th}-derived set and it is denoted by A$'$ ={d : d is n^{th}-limit point of A }.
Properties of derived set: Let (K,η_1,η_2,..., η_n) be a n^{th}-topological space and let W,M $\subset K$, then:
i) the derived set of ϕ is ϕ.
ii) If W subset of M, then W' subset of M'.
iii) The derived set of union of W and M equal the union of the derived set of W and the derived set of M.
iv) The derived set of intersection of W and M equal the intersection of the derived set of

W and the derived set of M.

Example let K ={x, y, z, q} , η_1={$\phi, K, \{x\}, \{x, y\}\}, \eta_2 = \{\phi, K, \{x\}\}, \eta_3 = \{\phi, K, \{x\}, \{x, z\}\}$ $and\ A = \{x, z\}$.
Now y $\in K$ and {x,y} n^{th}-open set in η_1 and y $\in$ A we have
{x,y} $\cap$ (A $-$ $\{y\}$) ={x} $\neq \phi$
then y is a n^{th}-limit point of A. and the same argement we get also x and z are limit point, and A$'$= {x,y,z}.

Definitio 5.1 : Let $(K, \eta_1, \eta_2, ..., \eta_n)$ be a n^{th}-topological space , $K \neq \phi$ and W is subset of K , then the n^{th}-clousre set is denoted by $\overline{W}$ = W $\cup$ W'
Note that : $\overline{Z}$ is n^{th}-closure set.
Properties of closure set: let (K,η_1,η_2,...,η_n) be a n^{th}-topological space and W is subset of K. then :
i) $\overline{W}$ is n^{th}-closed set.
ii) $\overline{W}^C$ is n^{th}- open set.
iii) w $\in \overline{W}$ if and only if for all η_i-open set u_w and w $\in u_w$, we have $u_w \cap$ W $\neq \phi$.
iv) w $\notin \overline{W}$ if and only if for all η_i-open set u_w (i= 1,2,3,...,n) and w $\in u_w$, we have $u_w \cap$ W $= \phi$.

Definition 6.1 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space , $K \neq \phi$ and W $\subset$ D, then a piont d $\in$ W is said to be n^{th}-Interior piont of W if there exist at least one neighborhood of d (N(d,ε)) such that N(d,ε) $\subseteq$ W.
The set of all n^{th}-interior piont is called the n^{th}-Interior set and it is denoted by $A^\circ \equiv$ INT(W) =$\overline{A^C}^C$.
Note That : A° is n^{th}-open set.
Properties of interior set: Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space and let X,Y $\subset K$,then:
(i) $\phi^\circ = \phi$ and $K^\circ = K$.
(ii) $(X \cap Y)^\circ = X^\circ \cap Y^\circ$ and $X^\circ \cup Y^\circ \subset (X \cup Y)^\circ$.
(iii) X$^\circ$ is η_i-open set.
(iv) n $\in$ X$^\circ$ if and only if there exist η_i-open set u_n such that n $\in u_n \subset$ X.

Definition 7.1 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space , $K \neq \phi$ and W is subset of K, then the point d is said to be n^{th}-Exterior point of W, If there exist at least one neighborhood of d such that N(d,ε) $\cap$ W = ϕ
The set of all n^{th}-Exterior point is called n^{th}-Exterior set and it is denoted by
EX(W)=Int(W^c)= $\overline{W}^C$

Note That :Ex(W)is n^{th}-close set .
Properties of exterior set: Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space and let W,M $\subset K$,then:

(i) The exterior set of ϕ is K and the exterior set of K is ϕ.
(ii) if W $\subset$ M, then EX(M) $\subset$ EX(W).
(iii) EX(W) is η_i-open set, i=1,2,...,n.
(iv) m $\in$ EX(w) if and only if there exsit η_i-open set u_m such that m $\in u_K \subset W^c$.

Proof: (iii)
since EX(W) = INT(W^c), then EX(W) = $\overline{W^{cc}}^c$,but $W^{cc} = W$ thus, $EX(W) = \overline{W}^c$ and by definition of n^{th}-closure set we have $\overline{W}$ is η_i-closed set so,the complement of η_i-closed set is η_i-open set, therefor EX(W) is η_i-open set, i=1,2,...,n.

Definition 8.1 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space , $K \neq \phi$ and W is subset of K , then the point d is said to be n^{th}-Boundary point of W, If every neighborhood of d satisfy that N(d,ε) $\cap$ A $\neq \phi$ and N(d,ε) $\cap$ $A^c \neq \phi$.
The set of all n^{th}-boundary point is called n^{th}-Boundary set , and it is denoted by Bd(W) = $\overline{W} - W^\circ = \overline{W} \cap \overline{W^c}$.
Note That : Bd (W) is n^{th}-Closed set.
Properties of boundary set: Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space and let X,Y $\subset K$,then:
(i) The boundary set of the empty set and K equal ϕ.
(ii) Bd(X) is η_i-closed set, i=1,2,...,n.
(iii) y $\in$ Bd(X) if and only if for all η_i-open set u_y such that y $\in u_y$ we have $u_y \cap$ X $\neq \phi$ and $u_y \cap X^c \neq \phi$.

Proof: (iii)
Let y $\in$ Bd(X) and u_y be a η_i-open set such that y u_y ,then y $\in$ ($\overline{\text{X}} \cap \overline{\text{X}^C}$) ,if and only if b $\in \overline{X}$ and y $\in \overline{X^c}$,if and only if y $\in$ (X $\cup$ X') and y $\in X^c \cup (X^c)'$,if and only if (y $\in$ X or y $\in$ X') and (y $\in X^c$ or y $\in (X^c)'$,if and only if y $\in$ X' and y $\in X^c$,if and only if $u_y \cap (X/\{y\}) \neq \phi$ and $y \cap X^c \neq \phi$,but $y \subset u_y$,so we have $u_y \cap$ X $\neq \phi$ and $u_y \cap X^c \neq \phi$.

Definition 9.1 : A n^{th}-topological space (K,η_1,η_2,...,η_n) is $T_\circ$-space, If for all c $\neq$ d in K , there exists η_i-open set u_c such that c $\in u_c$ and d $\notin u_c$ or there exists η_i-open set v_d such that c $\notin v_d$ and d $\in v_d$, for all i= 1,2,...,n.

Theorem 1 : Let (K,η_1,η_2,...,η_n) be a topological space, then the following are equivalent:
i) K is n^{th}-T_0-space .
ii) For all m $\neq$ n in K, we have m $\notin \overline{\{n\}}$ or n $\notin \overline{\{m\}}$.
iii) For all m $\neq$ n in K, we have $\overline{\{m\}} \neq \overline{\{n\}}$.
Proof: (**(i) implies (ii)**)
Let m $\neq$ n, then there exist η_i-open set such that m $\in um$ and n $\notin u - m$ or there exist η_j-open set v_n such that m $\in v_n$ and m $\notin v_n$ where i=1,2,...,n. so, we have m $\in u_m$ and $u_m \cap \{n\} = \phi$ or n in v_n and $v_n \cap \{m\} = \phi$. thus, m $\notin \overline{\{n\}}$ or n $\notin \overline{\{m\}}$.
(**(ii) implies (iii)**)

Let m $\neq$ n, then if m $\notin \overline{\{n\}}$ and m $\in \overline{\{m\}}$, then we have $\overline{\{m\}} \neq \overline{\{n\}}$. Additionally, if n $\notin \overline{\{n\}}$ and n $\in \overline{\{n\}}$, then we have $\overline{\{m\}} \neq \overline{\{n\}}$.

(**(iii) implies (i)**)

Let m $\neq$ n and by given $\overline{\{m\}} \neq \overline{\{n\}}$, but m $\in \overline{\{m\}}$ and n $\in \overline{\{n\}}$, then m $\notin K - \overline{\{m\}}$ $= v_n$ which is η_i open set in K since $\overline{\{m\}}$ is η_i-closed set in K and n $\in K - \overline{\{m\}} = v_n$ where i=1,2,...,n. thus, K is n^{th}-T_0-space.

Definition 10.1 : A n^{th}-topological space (K,η_1,η_2,...,η_n) is T_1-Space, If for all c $\neq$ d in K, there exists η_i-open set u_c such that c $\in u_c$ and d $\notin u_c$ and there exists η_i-open set v_d such that c $\notin v_d$ and d $\in v_d$, for all i= 1,2,...,n.

Definition 11.1 : A n^{th}-topological space (K,η_1,η_2,...,η_n) is T_2-Space, If for all c $\neq$ d in K, there exists η_i-open set u_c and there exists η_i-open set v_d such that c $\in u_c$ and d $\in$ v_d and $u_c \cap v_d \doteq \theta$, for all i= 1,2,...,n.

Definition 12.1 : A n^{th}-topological space (K,η_1,η_2,...,η_n) is $T_{2\frac{1}{2}}$-Space, If for all c $\neq$ d in K there exists η_i-closed set A_C and there exists η_i-closed set B_d in K, such that c $\in$ A_c, d $\in B_d$ and $A_c \cap B_d = \theta$, for all i= 1,2,...,n.

Definition 13.1 : A n^{th}-topological space (K,η_1,η_2,...,η_n) is n^{th}-regular space if for all c $\notin$ A and A n^{th}-closed set in K, there exist η_i-open set u_c and η_i-open set V_A in K such that c $\in u_c$, A $\subset V_A$ and $u_c \cap V_A \doteq \theta$

Theorem 2 : A space (K,η_1,η_2,...,η_n) is n^{th}-regular space if and only if for all m $\in$ u_m ,where u_m is η_i-open set ,there exist η_i-open set w_m such that m$\in w_m \subset \overline{w_m} \subset u_m$.

Proof:

($\rightarrow$)

Let m $\in u_m$, then m $\notin {u_m}^c$,but ${u_m}^c$ is η_i-closed set, then we can say that ${u_m}^c$=M. so, by definition of n^{th}-regular space, there exist η_i-open set w_m and v_M such that a $\in w_m$,M $\subset$ v_M and $w_m \cap v_M = \phi$,but clearly $w_m \subset \overline{w_m}$.

It is enough to show $\overline{w_m} \subset u_m$, now $w_m \cap v_M = \phi$, then we can say that $w_m \subset {v_M}^c$, then $\overline{w_m} \subset \overline{{v_M}^c} = {v_M}^c$. so, we have $\overline{w_m} \subset {v_M}^c$,but ${u_m}^c$= M $\subset v_M$,then ${u_m}^c \subset v_M$,then ${v_M}^c$ $\subset u_m$,thus $\overline{w_m} \subset u_m$.

($\leftarrow$)

Let m $\notin$ M and M is η_i-open set, then m $\in M^c$ and M^c is η_i-open set, then by given there exist η_i-open set w_m such that m $\in w_m \subset \overline{w_m} \subset M^c$. now we have two givens, m $\in w_m$ and M $\subset \overline{w_m}^c$,where w_m and $\overline{w_m}^c$ is η_i-open sets, i=1,2,...,n. ..... (*).

it is enough to show that $w_m \cap \overline{w_m}^c = \phi$,suppose not ,then there exist y such that y $\in$ $(w_m \cap \overline{w_m}^c)$,that is implies y $\in w$ and y $\in \overline{w_m}^c$, then y $\in w_m$ and y $\notin w$ and y $\in {w_m}'$,then we have y $\in$ ($w_m \cap {w_m}^c$) that is contradiction.

so $,w_m \cap \overline{w_m}^c = \phi$ ..... (**)
By (*) and (**) we have, a space $(K,\eta_1,\eta_2,...,\eta_n)$ is n^{th}-regular space.

Definition 14.1 : A n^{th}-topological space $(K,\eta_1,\eta_2,...,\eta_n)$ is T_3-Space if K is n^{th}-regular space and T_1-Space.

Definition 15.1 : A n^{th}-topological space $(K,\eta_1,\eta_2,...,\eta_n)$ is n^{th}-normal apace if for all M, W be a n^{th}- closed set in K and M $\cap$ W = θ, there exists η_i-open set u_M and η_i-open set V_W in M. such that M $\subset u_M$, W $\subset V_B$ and $u_A \cap V_B \doteq \theta$.

Definition 16.1 : A n^{th}- topological space $(K,\eta_1,\eta_2,...,\eta_n)$ is T_4-Space if K is n^{th} normal space and T_1-Space.

Definition 17.1 : A n^{th}-topological space $(K,\eta_1,\eta_2,...,\eta_n)$ and let A subset of K. the complement of n^{th}-α open set is called n^{th}-α closed set.

4. Nth-compact space and some type of it

This section includes several important concepts in n^{th}-topological spaces.

Definition 2.1 [1] : Let (K, η) be a topological space and Q = $\{W_\alpha : \alpha \in \lambda, W_\alpha \subset K\}$ is called :
i) cover of K if and only if $\bigcup_{\alpha\in\lambda} W\alpha = K$.
ii) open cover of K if and only if Q is cover and W_α is open set ,where $\alpha \in \lambda$.
iii) closed cover of K if and only if Q is cover and W_α is closed set ,where $\alpha \in \lambda$.
iv) $C = \{R_\gamma : \gamma \in \Gamma\}$ is a subcover of Q if and only if :
(i) C $\subset$ Q (ii) $\bigcup_{\gamma\in\Gamma} R_\gamma = K$
A space (K, η) is called compact space, if every open cover of K has a finite subcover.

Example [7]
($\mathbb{R}$, η) is not compact.

Proof: by contradiction , assume that $(\mathbb{R}, \eta_u)$ is compact ,so every open cover of $\mathbb{R}$ has a finite subcover, but Q = $\{(-n, n) : n = 1, 2, 3, ...\}$ is open cover of K because $\bigcup_{n=1}^{\infty} (-n, n)$=$\mathbb{R}$ and (-n,n) is open set,so E has a finite subcover say C=$\{(-n_1, n_1), (-n_2, n_2).(-n_3, n_3), \ldots, (-n_m$,then $\bigcup_{i=1}^{m} (-n_i, n_i) = \mathbb{R}$,then $(a, b) = \mathbb{R}$ where $a = \min_{i=1,...,m}\{-n_i\}$ *and* $b = \max_{i=1,...,m}\{n_i\}$,then $\mathbb{R}$=(a,b) $\subset [a, b] \Rightarrow \mathbb{R} \subset$ [a,b] $\equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted.
$\therefore$ $(\mathbb{R}, \eta_u)$ is not compact space.

Definition 2.2 : Let $(K,\eta_1,\eta_2,...,\eta_n)$ be an n^{th}-topological space and let W subset of K is called an n^{th}-α^*-open set in k, if W subset of η_1 $int^*(\eta_2 CL(\eta_3 int^* W)$.

Example let $(K,\eta_1,\eta_2,...,\eta_n)$ be an n^{th}-topological space where $k = \{1,2,3\}$,
$\eta_1=\{\phi,K,\{1\}\}$,
$\eta_2=\{\phi,K,\{1\},\{1,2\}\}$,
$\eta_3= \{\phi,K,\{1\},\{1,3\}\}$
then the n^{th}-α^*-open set are $\{\phi,\{1\},\{1,2\},\{1,3\},K\}$.

Theorem 1 : If W is n^{th} open set, then W is n^{th} α^*-open set.

Proof:
W is n^{th}-open set $\Rightarrow$ W $\subset \eta_1$ $int(\eta_2 int(\eta_3 int A)$
$\Rightarrow$ W $\subset \eta_1$ $int(\eta_2 CL(\eta_3 int A)$
$\Rightarrow$ W $\subset \eta_1$ $int^*(\eta_2 CL(\eta_3 int^* A)$.
$\Rightarrow$ W is n^{th} α^*-open set.
Note: The converse is not true.

Example: Let $(K,\eta_1,\eta_2,...,\eta_n)$ be an n^{th}-topological space where $K = \{1,2,3\}$,
$\eta_1 = \{\phi, K, \{1\}\}$,
$\eta_2 = \{\phi, K, \{1\}, \{1, 2\}\}$,
$\eta_3 = \{\phi, K, \{1\}, \{1, 3\}\}$, then:
n^{th} α^*-open set are $\{\phi,\{1\},\{1,2\},\{1,3\},K\}$.
n^{th} α^*-closed set are $\{\phi,\{2\},\{3\},\{2,3\},K\}$.
W=$\{1,2\}$ is n^{th} α^*-open set. But, W=$\{1,2\}$ is not n^{th} open set.

Definition 3.2 : Let $(K,\eta_1,\eta_2,...,\eta_n)$ be an n^{th}-topological space and let W subset of K is called an n^{th}-α^*-closed set in K , if W $\supset \eta_1$ $int^*(\eta_2 CL(\eta_3 int^* A)$.

Theorem 2: Every n^{th}-closed set is n^{th} α^*-closed set.

Proof: W is n^{th}-closed set implies w subset of η_1 cl η_2 cl η_3 cl W implies $W^c \subset \eta_1$ int η_2 int η_3 int W^c implies $W^c \subset \eta_1$ $int^* \eta_2$ int η_3 int^* W^c implies $W^c \subset \eta_1$ int^* η_2 CLη_3 int^* W^c implies W^c is n^{th} α^*-open implies W is n^{th} α^*-closed set.

Definition 4.2 : Let $(K,\eta_1,\eta_2,...,\eta_n)$ be an n^{th}-topological space and U = $\{W_\alpha : \alpha \in \lambda, W_\alpha \subset K\}$ is called :
i) n^{th}-cover of K if and only if $\bigcup_{\alpha\in\lambda} W\alpha = K$.
ii) n^{th}-open cover of K if and only if U is n^{th}-cover and W_α is η_i-open set ,where $\alpha \in \lambda$, i=1,2,...,n.
iii) n^{th}-closed cover of K if and only if U is n^{th}-cover and W_α is η_i-closed set, where $\alpha \in \lambda$, i=1,2,...,n.

iv) $E = \{M_\gamma : \gamma \in \Gamma\}$ is a n^{th}partite subcover of U if and only if :
(a) E $\subset$ U (b) $\bigcup_{\gamma\in\Gamma} M_\gamma = K$

A space $(K, \eta_1, \eta_2, ..., \eta_n)$ is called n^{th}-compact space, if every n^{th}-open cover of K has a finite n^{th} subcover.

Example : The n^{th}-topological space ($\mathbb{R}$, η_{u_1}, η_{u_2},..., η_{u_n}) is not n^{th}-compact space.

Proof: by contradiction , assume that $(\mathbb{R}, \eta_1, \eta_2, ..., \eta_n)$ is n^{th}-compact, so every n^{th}-open cover of $\mathbb{R}$ has a finite n^{th} subcover ,but E = $\{(-n, n) : n = 1, 2, 3, ...\}$ is n^{th}-open cover of K because $\bigcup_{n=1}^{\infty} (-n, n)$=$\mathbb{R}$ and (-n,n) is η_{ui}-open set ,i=1,2,...,n ,so E has a finite n^{th}-subcover say C=$\{(-n_1, n_1), (-n_2, n_2).(-n_3, n_3), ..., (-n_m, n_m)\}$,then $\bigcup_{i=1}^{m} (-n_i, n_i) = \mathbb{R}$,then
$(x, y) = \mathbb{R}$ where $x = \min_{i=1,...,m}\{-n_i\}$ and $y = \max_{i=1,...,k}\{n_i\}$,then $\mathbb{R}$=(x,y) $\subset [x, y]$
$\Rightarrow \mathbb{R} \subset$ [x,y] $\equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted.$\therefore$ $(\mathbb{R}, \eta_{u_1}, \eta_{u_2}, ..., \eta_{u_n})$ is not n^{th}-compact space.

Theorem 3 : Let (K, η) be a topological space and U $\subset$ k,then U is compact space if and only if U is closed set and bounded set.

Theorem 4 : Let (K,η_1,η_2,...,η_n) be a n^{th} topological space and U $\subset$ K,then U is n^{th}-compact space if and only if U is η_i-closed set and n^{th}-bounded set.

Proof: Suppose $M \subset \mathbb{R}$ is n^{th}-compact. For each $m \in M$, consider the open interval $(m - 1, m + 1) = W_m$. Each W_m is η_i-open in $\mathbb{R}$, so $\{W_m : m \in M\}$ forms an n^{th}-open cover of M. Since M is n^{th}-compact, there exist finitely many points $m_1, m_2, \ldots, m_n \in M$ such that $M \subseteq \bigcup_{i=1}^{n} W_{m_i}$.
Let $Q = \max(m_1, m_2, \ldots, m_n)$ and $e = \min(m_1, m_2, \ldots, m_n)$. Then $M \subseteq \bigcup_{i=1}^{n} W_{m_i} \subseteq [q - 1, q + 1]$, so Y is n^{th}-bounded. Since M is n^{th}-compact in the Euclidean space $\mathbb{R}$ (denoted as T_2 space), Y is η_i-closed set.
Conversely, suppose Y is η_i-closed and n^{th}-bounded in $\mathbb{R}$. If Y is n^{th}-bounded, then $Y \subset [x, y]$ for some $x < y$ in $\mathbb{R}$. Since Y is η_i-closed in the n^{th}-compact subset $[x, y]$, then Y is n^{th}-compact set.
Therefore, Y is η_i-closed and n^{th}-bounded if and only if M is n^{th}-compact.

Definition 5.2 [6] : Let (K, η) be a topological space, and let L be a family of subsets of K. We say that L has a finite intersection property (f.i.p) if and only if the intersection of any finite number of members of L is non-empty.

Definition 6.2 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space, and let L be a family of subsets of K. We say that L has a finite intersection property (f.i.p) if and only if the intersection of any finite number of members of L is non-empty.

Theorem 5 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space. Then K is n^{th}-compact if and only if every family of n^{th}-closed subsets of K with the finite intersection property (f.i.p) has a non-empty intersection.

Proof:

Suppose K is n^{th}-compact space. If there exists a family of n^{th}-closed subsets of K, say $\{W_\alpha : \alpha \in \lambda\}$, with f.i.p such that $\bigcap_{\alpha\in\lambda} W_\alpha = \emptyset$, then $\bigcup_{\alpha\in\lambda}(K \setminus W_\alpha) = K \setminus \bigcap_{\alpha\in\lambda} W_\alpha = K \setminus \emptyset = K$. Since W_α is η_i-closed set in K for all $\alpha \in \lambda$, $K \setminus W_\alpha$ is η_i-open set in K for all $\alpha \in \lambda$. Therefore, $N = \{K \setminus W_\alpha : \alpha \in \lambda\}$ is an n^{th}-open cover of K. By the compactness of K, N has a finite subcover of K, say $\{K \setminus W_{\alpha_i} : i = 1, 2, \dots, n\}$. Thus, $K = \bigcup_{i=1}^{n}(K \setminus W_{\alpha_i}) = K \setminus \bigcap_{i=1}^{n} W_{\alpha_i}$. This contradicts $\bigcap_{\alpha\in\lambda} W_\alpha = \emptyset$, proving that every family of n^{th}-closed subsets of K with f.i.p has a non-empty intersection.
Conversely, suppose every family of n^{th}-closed subsets of K with f.i.p has a non-empty intersection. If K is not n^{th}-compact, then there exists an n^{th}-open cover of K, say $\{w_\alpha : \alpha \in \lambda\}$. Since w_α is n^{th}-open for all $\alpha \in \lambda$, $\{K \setminus w_\alpha : \alpha \in \lambda\}$ is a family of n^{th}-closed subsets of K.
Claim: $\{K \setminus w_\alpha : \alpha \in \lambda\}$ has f.i.p. If not, there exist $w_1, w_2, \dots, w_n$ such that $\bigcap_{i=1}^{n}(K \setminus w_i) = \emptyset$, hence $\bigcup_{i=1}^{n} w_i = K$. This implies $\{w_i : i = 1, 2, \dots, n\}$ is a finite subcover of K, which is a contradiction. Therefore, $\{K \setminus w_\alpha : \alpha \in \lambda\}$ has f.i.p.
By assumption, $\bigcap_{\alpha\in\lambda} w_\alpha \neq \emptyset$. So, $\emptyset \neq K \setminus \bigcap_{\alpha\in\lambda} w_\alpha = \bigcup_{\alpha\in\lambda}(K \setminus (K \setminus w_\alpha)) = \bigcup_{\alpha\in\lambda} w_\alpha$, which is a contradiction. Hence, K must be compact.

Theorem 6 : Every n^{th}-closed subset of a n^{th}-compact space is n^{th}-compact.

Proof:

Suppose that T is a n^{th}-closed subset in a compact space K. Let $E = \{t_\alpha : \alpha \in \lambda\}$ be an n^{th}-open cover of T. Then K = T ∪ (K - T) = $\bigcup_{\alpha\in\lambda} t_\alpha$∪ (K - T) is an n^{th}-open cover of K. Since K is n^{th}-compact, $T \cup (K - T)$ can be reduced to a n^{th} finite subcover. Say n^{th}-finite subcover

Theorem 7 [4] : Let W be a compact subset in a n^{th}-T_2-space K. Then for all $n \notin W$ there exists an open set U_n containing n such that W ∩ $U_n = \emptyset$

Theorem 8 : Let W be a n^{th}-compact subset in a n^{th}-T_2-space K. Then for all $n \notin W$ there exists a η_i-open set U_n containing n such that W ∩ $U_n = \emptyset$, i=1,2,...,n.

Definition 7.2 : Let (K,η_1,η_2,...,η_n) be a n^{th}-topological space ,then a set G in (K,η_1,η_2,...,η_n) is called n^{th}-Dense set If $\overline{G} = K$.
On another hand, if G is dense in (K,η_1,η_2,...,η_n),then for all η_i-open set u we have $u \cap S$ $\neq \phi$.

Definition 8.2 : Let $(K,\eta_1,\eta_2,...,\eta_n)$ and $(Q,\Lambda_1,\Lambda_2,...,\Lambda_n)$ are n^{th}-topological space ,then the function
h:$(K,\eta_1,\eta_2,...,\eta_n) \to (Q,\Lambda_1,\Lambda_2,...,\Lambda_n)$ is called n^{th}-open function if $h(u) = v$,where u is η_i-open set and v is σ_i-open set.

5. σ-compact space in Nth-topological space

Now, we will show some concept of σ-compact space in topological space, σ-compact space in n^{th}-topological space and some theorems and their properties.

Definition 1.3 [2] : Let (K,η) be a topological space ,then it is called σ-compact space if every open cover of K has a countable subcover of K.On the same time A topological space K is called **σ-compact** if it can be expressed as a countable union of compact subspaces.

Definition 2.3 : Let $(K,\eta_1,\eta_2,...,\eta_n)$ be a n^{th}-topological space ,then it is called n^{th} σ topological space if every η_i-open cover of K has a n^{th} countable subcover of k.

Theorem 1 : Any closed subspace of a n^{th} σ-compact space is also n^{th} σ-compact.

Proof:
Let $(K,\eta_1,\eta_2,...,\eta_n)$ be a n^{th} σ-compact space, meaning $K = \bigcup_{n_i=1}^{\infty} M_(n_i)$, where each $M_(n_i)$ is compact in K. For any closed subspace Q $\subseteq K$, each $M_n \cap$ Q is compact (since compactness is preserved under closed subspaces). Thus, Q $= \bigcup_{n_i=1}^{\infty} M_(n_i) \cap F$, which is a countable union of compact sets in Q, proving that Q is n^{th} σ-compact, i=1,2,...,n.

Theorem 2 : The continuous image of a n^{th} σ-compact space is also n^{th} σ-compact space.

Proof:
Let h : $(K,\eta_1,\eta_2,...,\eta_n) \to (J,\Lambda_1,\Lambda_2,\Lambda_3)$ be a continuous map and $D = \bigcup_{n_i=1}^{\infty} M_(n_i)$, where each $M_(n_i)$ is compact. Since h($M_(n_i)$) is compact (compactness preserved under continuous maps), we have h$(K) = \bigcup_{n_i=1}^{\infty} f(M_(n_i))$, which is a countable union of compact sets in J. Hence, h(K) is n^{th} σ-compact, for all i=1,2,...,n.

Theorem 3 : The product of a n^{th} σ-compact space with a n^{th} compact space is n^{th} σ-compact.

Proof:
Let $K = \bigcup_{n_i=1}^{\infty} M_(n_i)$, where each $M_(n_i) \subset K$ is n^{th} compact, and let Y be n^{th} compact. Then $k \times$ Y $= \bigcup_{n_i=1}^{\infty} (M_(n_i) \times Y)$, where each $M_(n_i) \times Y$ is n^{th} compact in K $\times Y$ (since

the product of n^{th} compact spaces is n^{th} compact). Therefore, $k \times$ Y is n^{th} σ-compact.

Theorem 5 : The countable union of n^{th} σ-compact subspaces is n^{th} σ-compact.

Proof:
Let $K = \bigcup_{i=1}^{\infty} K_i$, where each D_i is n^{th} σ-compact. Then, for each i, we can write $K_i = \bigcup_{j=1}^{\infty} S_{ij}$, where S_{ij} are n^{th} compact. Thus,$K = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} S_{ij}$, which is a n^{th} countable union of n^{th} compact sets, proving K is n^{th} σ-compact.

Theorem 6 : Every second-countable compact space is n^{th} σ-compact space.

proof:
Since $(K, \eta_1, \eta_2, ..., \eta_n)$ is n^{th} topological space and K is second countable, there exists a countable base $\{V_i\}_{i=1}^{\infty}$. Each compact subset $M_i = \overline{V_i}$ is compact in a second-countable space, and since $K = \bigcup_{i=1}^{\infty} M_i$, K is n^{th} σ-compact space.

Theorem 7 : Every compact subspace of a n^{th} σ-compact space is n^{th} σ-compact space.

proof:
Let $(K, \eta_1, \eta_2, ..., \eta_n)$ be n^{th} σ-compact and $M \subset K$ compact. Since $K = \bigcup_{n=1}^{\infty} M_(n_i)$ where each $M_(n_i)$ is n^{th} compact, M = $\bigcup_{n=1}^{\infty} (M \cap M_(n_i))$, which is a countable union of compact sets. Thus, M is n^{th} σ-compact space, i=1,2,...,n.

Theorem 8 : If K is n^{th} σ-compact space, then any quotient space $\mathbb{R} = K \sim$ is n^{th} σ-compact space.

proof:
Let $K = \bigcup_{n=1}^{\infty} M_(n_i)$ with each $M_(n_i)$ n^{th} σ-compact space. Since the quotient map Q : $K \to \mathbb{R}$ is continuous, $Q(M_(n_i))$ is n^{th} σ-compact in $\mathbb{R}$. Thus, $\mathbb{R} = \bigcup_{n=1}^{\infty} Q(M_(n_i))$, proving that $\mathbb{R}$ is n^{th} σ-compact space, i=1,2,...,n.

Theorem 9 : Every n^{th} σ-compact space is separable if it is a metric space.

proof:
Let K be an n^{th} σ-compact metric space. Since K can be expressed as a countable union of compact sets $M_(n_i)$, each $M_(n_i)$ is separable (n^{th} compact subsets of metric spaces are separable). Therefore, we can find a countable dense subset $T_(n_i)$ for each $M_(n_i)$. The union T = $\bigcup_{n=1}^{\infty} T_(n_i)$ is dense in k, showing that K is separable, for all i=1,2,...,n.

Example: The real numbers $\mathbb{R}$ with the standard topology are n^{th} σ-compact and separable (since the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$).

Theorem 10 : Any locally compact in n^{th} σ-compact space is n^{th} σ-compact space.

proof:
Let $(K, \eta_1, \eta_2, ..., \eta_n)$ be n^{th} topological space and K is n^{th} σ-compact space and let K be a locally compact n^{th} σ-compact space, and let $K = \bigcup_{n=1}^{\infty} M_(n_i)$, where each $M_(n_i)$ is n^{th} compact. For each point $d_i \in K$, there exists a neighborhood $V_(d_i)$ that is compact. Thus, we can cover K by a countable union of compact neighborhoods, confirming that K is n^{th} σ-compact space.

Example: The space $\mathbb{R}^n$ is locally compact and n^{th} σ-compact space, as it can be covered by compact sets (closed balls) in a countable manner.

Theorem 11 : The space C(K) of continuous functions on aN n^{th} σ-compact space, K is n^{th} σ-compact under the compact open topology.

proof:
Let $K = \bigcup_{n=1}^{\infty} M_(n_i)$, where each $M_(n_i)$ is n^{th} compact. The compact open topology is generated by sets of the form $\{h : M \to \mathbb{R} \mid h(M) \subseteq U\}$. Each n^{th} compact set $M_(n_i)$ gives rise to a countable family of compact sets C$(M_(n_i))$ in C(K). Thus, C(K) can be expressed as a countable union of compact sets, proving it is n^{th} σ-compact.

Theorem 12 : If K is n^{th} σ-compact space and Q is a closed subset of K, then Q is n^{th} σ-compact space.

proof:
Since $K = \bigcup_{n=1}^{\infty} M_(n_i)$ is n^{th} σ-compact Space, the intersection $M_(n_i) \cap$ Q is compact for each n_i. Therefore, Q $= \bigcup_{n=1}^{\infty} (M_(n_i) \cap Q)$, which is a countable union of compact sets. Thus, Q is n^{th} σ-compact Space, for all i=1,2,...,n.

Theorem 13 : Every n^{th} σ-compact space can be expressed as a countable union of locally finite open covers.

proof:
Let $K = \bigcup_{n=1}^{\infty} M_(n_i)$ be n^{th} σ-compact space. Each compact set $M_(n_i)$ can be covered by a locally finite collection of open sets. The union of these open covers from all $M_(n_i)$ remains locally finite. Thus, K can be expressed as a countable union of locally finite open covers, for all i=1,2,...,n.

Example: The space $\mathbb{R}$ can be covered by intervals $(n, n+1)$ for each n $\in \mathbb{Z}$.

Theorem 15 : A n^{th} σ-compact Hausdorff space is second-countable.

proof:
Suppose K is a n^{th} σ-compact Hausdorff space that may be written as a countable union of compact sets. Each compact subset is second-countable, resulting in a countable base for the topology. The union of these countable bases produces a countable base for K, indicating that it is second-countable.

Example: The space $\mathbb{R}^n$ is Hausdorff, n^{th} σ-compact, and second-countable, meaning it may be covered by balls in a countable way.

6. Conclusion

In this paper, we obtained some results related to σ-compact spaces, and applied this concept in n^{th}-topological spaces. Several characteristics of these spaces and their interactions with other topologies are presented. Also, our study of σ-compact spaces solved some important mathematical problems in n^{th}-topological spaces.

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References

[1] Dugundji ;J. ,(1966). Topology, Allyn and Bacon, Boston .
[2] J. Oudetallah , On feebly pairwiese expandable space, J. Math. Comput. Sci. 11 (2021), No. 5, 6216-6225
[3] J. Oudetallah, Nearly Expandability in bitopological spaces, Advances in Mathematics: Scientific Journal 10 (2021), 705-712.
[4] J. Kelley, General topology, Van Nostrand Company, 1955.. kyungpook Math.J.,32, No. 2(1992), 273-284
[5] Kim,Y. W. (1968). Pairwise Compactness. Publ. Math. Debrecen.15, 87-90.
[6] Levine; N. , (1963). Semi-Open Sets and Semi-Continuity in Topological Spaces , Amer. Math. Monthly, 70, 36-41.
[7] Willard ; S., (1970).General Topology , Addison- Wesley Publishing Company, Inc.