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Advancements in Ostrowski type fractional Integral Inequalities via Applications of Jensen's and Young's Inequalities

Gauhar Rahman[1,*], Muhammad Samraiz[2], Çetin Yıldız[3,*], Maryam Ali Alghafli[4], Nabil Mlaiki[4]

[1] *Department of Mathematics and Statistics, Hazara University, Mansehra 21300, Pakistan*
[2] *Department of Mathematics, University of Sargodha P.O. Box 40100, Sargodha, Pakistan*
[3] *Deparment of Mathematics, K.K. Education Faculty, Atatürk University, 25240 Erzurum, Turkey*
[4] *Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia*

Abstract. Fractional integral operators and convexity have a close link due to their fascinating properties in the mathematical sciences. In this paper, we first establish an integral identity involving the generalized Hattaf-fractional integral operators. By using the Jensen integral inequality, Young's inequality, power-mean inequality, and Hölder inequality, we then apply this identity to provide some new generalizations of Ostrowski type inequality for the convexity of $|\aleph|$. Furthermore, we deduce several special cases from the main results. The results of this novel investigation should lead to new discoveries in the area of fractional calculus and inequalities.

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1. Introduction

Fractional integrals and derivatives have attracted a lot of attention from researchers nowadays. In many cases, fractional derivatives and fractional integrals provide more accurate representations of the frameworks than ordinary derivatives and integrals. The fractional calculus is currently widely employed in many scientific domains due to its numerous applications. The interest of researchers in fractional derivative and fractional

*Corresponding author.
*Corresponding author.
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Email addresses: gauhar55uom@gmail.com, drgauhar.rahman@hu.edu.pk (G. Rahman), muhammad.samraiz@uos.edu.pk;msamraizuos@gmail.com (M. Samraiz), cetin@atauni.edu.tr (Ç. Yıldız), mghafli@psu.edu.sa (M. A. Alghafli), nmlaiki@psu.edu.sa; nmlaiki2012@gmail.com (N. Mlaiki)

integration has grown recently due to their wide applications in diverse domain, for example (see, [1–5]). Dumitru and Arran [6] have given a new formula for fractional derivatives and integrals using the Mittag-Leffler kernel. While more theoretical ideas about fractional operators with Mittag-Leffler kernels (Atangana-Baleanu operators) and the higher-order case have been discussed in [7–9], the generalization to the generalized Mittag-Leffler kernels to gain a semigroup property has recently been developed in [10, 11].
In the beginning, many scientists working in different areas of theory of inequalities employed fractional calculus as an essential tool, for example, [12–16]. Shuang and Qi [17] proved a number of Hermite-Hadamard-type inequalities and examined specific methods for a class of s-convex functions. Mehrez and Agarwal [18] proved new integral inequalities and looked at specific cases of their discoveries with application to special means by employing the conventional Hermite-Hadamard inequalities. Park et al. [19] researched and used new generalized inequalities to stability analysis. By utilizing the local fractional approach, fractional integral inequalities were generated by Sarikaya et al. [20], expanding upon the findings found in the classical literature. In [21], Set et al. presented integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators. The various researchers examined a few noteworthy integral inequalities using various fractional methods. We refer the readers to the research conducted by [22–25].

2. Preliminaries

It is clear that the convex function is essential for the study of mathematical inequalities since it has several applications in the fields of pure and practical mathematics, mechanics, probability and statistics theory, economics, engineering, and optimization theory. Recently, there have been several mathematicians working on convexity's theories, variations, augmentations, generalizations, and refinements. For example, in a number of scientific and mathematical domains, it is a useful tool for presenting a variety of challenges and demonstrating awareness [26–29].
The convexity property can be used to generalize a number of well-known inequalities, such as the Opial type inequality, the Hermite–Hadamard inequality, the Ostrowski inequality, the Simpson inequality, the Bullen type inequality, and many more. The Ostrowski type inequality is one of the most extensively studied conclusions involving several kinds of convexities.

Definition 1. *[30] Let $\aleph : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function $\aleph \in L^1[r,s]$ with $r < s \in I$. If $|\aleph'(t)| \leq K$, for $t \in [r,s]$, then the Ostrowski type integral inequality is given by*

$$| \aleph(t) - \frac{1}{s-r}\int_r^s \aleph(t)dt | \leq K(s-r)\left[\frac{1}{4} + \frac{\left(t - \frac{r+s}{2}\right)}{(r+s)^2}\right],$$

where $\frac{1}{4}$ is the least possible value.

Mathematicians and scholars have been studying this inequality with significant attention and effort in recent years. This inequality was studied in 1997 by Dragomir and

Wang [31, 32] with relation to the lower and upper bounds of the first derivative. It was investigated by Barnett et al. and Cerone et al. [33, 34] that this inequality involving twice differentiable convex functions involved.

Definition 2. *[35] A function $\aleph : [r, s] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\aleph\left(\rho u + (1-\rho)v\right) \leq \rho\aleph(u) + (1-\rho)\aleph(v),$$

for all u, $v \in [r, s]$ and $\rho \in [0, 1]$.

Convex functions are a concept that is frequently utilized in inequality theory. The Hermite-Hadamard inequality, which derives upper and lower bounds from averages of the mean value of a convex function, is as follows:

Definition 3. *Given a convex mapping $\aleph : I \subseteq \mathbb{R} \to \mathbb{R}$, let $r < s$ be on the interval I of $\mathbb{R}$. Then the Hermite-Hadamard inequality is defined by*

$$\aleph\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \aleph(x)dx \leq \frac{\aleph(r)+\aleph(s)}{2}.$$

Definition 4. *[36] The ABC-fractional derivative is defined by*

$$^{ABC}\mathfrak{D}^{\kappa}_{r,\xi}\aleph(\xi) = \frac{M(\kappa)}{1-\kappa}\int_r^{\xi} \aleph'(\eta)\mathbb{E}_{\kappa}\left(\frac{-\kappa(\xi-\eta)^{\kappa}}{1-\kappa}\right)d\eta,$$

where $0 < \kappa < 1$, $\aleph' \in L^1(r, T]$ and $M(\kappa)$ is normalization function which satisfies the condition $M(0) = M(1) = 1$.

Definition 5. *[37, 38] The AB-fractional operator for $\aleph \in L^1(r, T]$ and $0 < \kappa < 1$ is defined by*

$$^{AB}\mathcal{I}^{\kappa}_{r,\xi}\aleph(\xi) = \frac{1-\kappa}{M(\kappa)}\aleph(\xi) + \frac{\kappa}{M(\kappa)\Gamma(\kappa)}\int_r^{\xi}(\xi-\eta)^{\kappa-1}\aleph(\eta)d\eta. \tag{1}$$

Definition 6. *[39] The Hattaf fractional derivative is defined by*

$$\mathfrak{D}^{\kappa,\sigma,\delta}_{r,\xi,\omega}\aleph(\xi) = \frac{M(\kappa)}{1-\kappa}\frac{1}{\omega(\xi)}\int_r^{\xi}\mathbb{E}_{\sigma}\left(\frac{-\kappa(\xi-\eta)^{\delta}}{1-\kappa}\right)\frac{d}{dt}(\omega\aleph)(\eta)d\eta,$$

where $0 < \kappa < 1$, $\aleph' \in L^1(r, T]$ $\omega \in C^1(a, b)$, $\omega, \omega' > 0$ on $[a, b]$.

Definition 7. *[39] The left sided Hattaf fractional operator for $\aleph \in L^1(r, T]$ and $0 < \kappa < 1$ is defined by*

$$\mathcal{I}^{\kappa,\sigma,\omega}_{r,\xi}\aleph(\xi) = \frac{1-\kappa}{M(\kappa)}\aleph(\xi) + \frac{\kappa}{M(\kappa)\Gamma(\sigma)\omega(\xi)}\int_r^{\xi}(\xi-\eta)^{\sigma-1}\omega(\eta)\aleph(\eta)d\eta. \tag{2}$$

Definition 8. *The right sided Hattaf fractional operator for* $\aleph \in L^1(r, s)$ *and* $0 < \kappa < 1$ *is defined by*

$$\mathcal{I}_{s,\xi}^{\kappa,\sigma,\omega}\aleph(\xi) = \frac{1-\kappa}{M(\kappa)}\aleph(\xi) + \frac{\kappa}{M(\kappa)\Gamma(\sigma)\omega(\xi)}\int_{\xi}^{s}(\eta-\xi)^{\sigma-1}\omega(\eta)\aleph(\eta)d\eta. \tag{3}$$

Remark 1. *i. If we consider* $\sigma = \kappa$ *in* (2) *and* (3)*, then we get AB-operator defined in* (1)*.*
ii. If we consider $\sigma = \kappa$ *and* $\omega = 1$ *in* (2) *and* (3)*, then we get AB-operator defined in* (1)*.*

Definition 9. *The left sided Hattaf fractional operator for* $\omega = 1$*,* $\aleph \in L^1(r, T]$ *and* $0 < \kappa < 1$ *is defined by*

$$\mathcal{I}_{r,\xi}^{\kappa,\sigma}\aleph(\xi) = \frac{1-\kappa}{M(\kappa)}\aleph(\xi) + \frac{\kappa}{M(\kappa)\Gamma(\sigma}\int_{r}^{\xi}(\xi-\eta)^{\sigma-1}\aleph(\eta)d\eta. \tag{4}$$

Definition 10. *The right sided Hattaf fractional operator for* $\omega = 1$*,* $\aleph \in L^1(r, s)$ *and* $0 < \kappa < 1$ *is defined by*

$$\mathcal{I}_{s,\xi}^{\kappa,\sigma}\aleph(\xi) = \frac{1-\kappa}{M(\kappa)}\aleph(\xi) + \frac{\kappa}{M(\kappa)\Gamma(\sigma)}\int_{\xi}^{s}(\eta-\xi)^{\sigma-1}\aleph(\eta)d\eta. \tag{5}$$

This paper aims to derive an integral identity by incorporating the Hattaf fractional integral operators (4) and (5) and uses them to prove the refinement of Ostrowski type integral inequalities for differentiable convex functions.

3. Main Result

In this section, we first prove the following fractional integral identity which will be used in our main findings.

Lemma 1. *Assume that* $\aleph : [r, s] \to \mathbb{R}$ *represents a differentiable function on* (r, s)*, where* $\aleph' \in L_1[r, s]$ *and* $r < s$*. Next, for modified Hattaf-fractional integral operators, we have the identity given below:*

$$\begin{aligned}
&\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r) + \mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right] \\
=&\frac{(t-r)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma}\aleph'(\rho t+(1-\rho)r)d\rho - \frac{(s-t)^{\sigma+1}}{s-r}\int_0^1 \rho^{\varrho}\aleph'(\rho t+(1-\rho)s)d\rho,
\end{aligned}$$

where $\kappa \in (0, 1]$*,* $t \in [r, s]$*.*

Proof. For simplicity, let us consider

$$\begin{aligned} I =& \frac{(t-r)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma}\aleph'(\rho t+(1-\rho)r)d\rho - \frac{(s-t)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma}\aleph'(\rho t+(1-\rho)s)d\rho \\ =& \frac{(t-r)^{\sigma+1}}{s-r}I_1 - \frac{(s-t)^{\sigma+1}}{s-r}I_2, \end{aligned} \tag{6}$$

where

$$\begin{aligned} I_1 =& \int_0^1 \rho^{\sigma}\aleph'(\rho t+(1-\rho)r)d\rho \\ =& \frac{\rho^{\sigma}\aleph(\rho t+(1-\rho)r)}{t-r}\ |_0^1 + \frac{\sigma}{t-r}\int_0^1 \rho^{\sigma-1}\aleph(\rho t+(1-\rho)r)d\rho \end{aligned}$$

By substituting $\rho = \frac{u-r}{t-r}$ in the integral part, we get

$$I_1 = \frac{\aleph(t)}{t-r} + \frac{\sigma}{(t-r)^{\sigma+1}}\int_r^t (u-r)^{\sigma-1}\aleph(u)\,du. \tag{7}$$

Similarly, one can get

$$I_2 = -\frac{\aleph(t)}{s-t} + \frac{\sigma}{(s-t)^{\sigma+1}}\int_t^s (s-u)^{\sigma-1}\aleph(u)\,du. \tag{8}$$

By substituting (7) and (8) in (6) and then after a simple computation, we get the desired Lemma 1.

Remark 2. *Applying Lemma 1 for $\sigma=\kappa$, we get the Lemma 1 given in [40].*

Theorem 1. *Assume that $\aleph:[r,s]\to\mathbb{R}$ be a differentiable function on (r,s), with $\aleph'\in L_1[r,s]$ and $r<s$. Then, the following inequality holds for Hattaf-fractional integral operators* (4) *and* (5) *if $|\aleph'|$ is a convex function*

$$\begin{aligned} &\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \\ &-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big| \\ \le& \frac{(t-r)^{\sigma+1}}{s-r}\left\{\frac{\aleph'(t)}{\sigma+2}+\frac{\aleph'(r)}{(\sigma+1)(\sigma+2)}\right\} + \frac{(s-t)^{\sigma+1}}{s-r}\left\{\frac{\aleph'(t)}{\sigma+2}+\frac{\aleph'(s)}{(\sigma+1)(\sigma+2)}\right\}, \end{aligned}$$

where $t\in[r,s]$, $\kappa\in(0,1]$.

Proof. By using Lemma 1, we have

$$\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$

$$
\begin{aligned}
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
&\leq\frac{(t-r)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}|\aleph'(\rho t+(1-\rho)r)|d\rho-\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}|\aleph'(\rho t+(1-\rho)s)|d\rho.
\end{aligned}
$$

By applying the convexity of $|\aleph'|$, we get

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
&\leq\frac{(t-r)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}[\rho\left|\aleph'(t)\right|+(1-\rho)\left|\aleph'(r)\right|]d\rho\\
&+\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}[\rho\left|\aleph'(s)\right|+(1-\rho)\left|\aleph'(t)\right|]d\rho.
\end{aligned}
$$

After solving the above integrals, we get

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
&\leq\frac{(t-r)^{\sigma+1}}{s-r}\left\{\frac{1}{(\sigma+2)}|\aleph'(t)|+\frac{1}{(\sigma+1)(\sigma+2)}|\aleph'(r)|\right\}\\
&+\frac{(s-t)^{\sigma+1}}{s-r}\left\{\frac{1}{(\sigma+2)}|\aleph'(t)|+\frac{1}{(\sigma+1)(\sigma+2)}|\aleph'(s)|\right\},
\end{aligned}
$$

which complete the desired proof.

Corollary 1. *Applying Theorem 1 for $|\aleph'|\leq K$ where $K>0$, we get the following inequality*

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
&\leq\frac{K}{s-r}\left(\frac{1}{\sigma+2}+\frac{1}{(\sigma+1)(\sigma+2)}\right)\left\{(t-r)^{\sigma+1}+(s-t)^{\sigma+1}\right\}.
\end{aligned}
$$

Corollary 2. *Applying Corollary 1 for $t=\frac{r+s}{2}$, we get the following inequality*

$$
\Big|\frac{(s-r)^{\sigma-1}}{2^{\sigma-1}}\aleph\left(\frac{r+s}{2}\right)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]
$$

$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,\frac{r+s}{2}}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,\frac{r+s}{2}}\aleph(s)\right]\Bigg|$$
$$\leq K\left(\frac{1}{\sigma+1}\right)\frac{(s-r)^{\sigma}}{2^{\sigma}}.$$

Remark 3. *Applying Theorem 1 for $\sigma=\kappa$, we get Theorem 1 proved by Ahmad et al. [40].*

Remark 4. *Applying Corollary 2 for $\sigma=\kappa$, we get Corollary 2 proved earlier by Ahmad et al. [40].*

Theorem 2. *Let $\aleph:[r,s]\to\mathbb{R}$ be a differentiable function on (r,s) with $r<s$ and $\aleph'\in L_1[r,s]$. For Hattaf-fractional integral operators, we have the following inequality if $|\aleph'|^q$ is a convex function.*

$$\Bigg|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Bigg|$$
$$\leq\frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\left[\frac{|\aleph'(t)|^q+|\aleph'(r)|^q}{2}\right]^{\frac{1}{q}}$$
$$+\frac{(s-t)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\left[\frac{|\aleph'(s)|^q+|\aleph'(t)|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $t\in[r,s]$, $\kappa\in(0,1]$ and $M(\kappa)>0$.

Proof. By applying Lemma 1, we have

$$\Bigg|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Bigg|$$
$$\leq\frac{(t-r)^{\sigma+1}}{s-t}\int_0^1\rho^{\sigma}\,|\,\aleph'(\rho t+(1-\rho)r)\,|\,d\rho$$
$$+\frac{(s-t)^{\sigma+1}}{s-t}\int_0^1\rho^{\sigma}\,|\,\aleph'(\rho t+(1-\rho)s)\,|\,d\rho.$$

By employing Hölder inequality, we obtain

$$\Bigg|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \tag{9}$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Bigg|$$

$$\leq \frac{(t-r)^{\sigma+1}}{s-t}\left[\left(\int_0^1 \rho^{\varrho p} d\rho\right)^{\frac{1}{p}}\left(\int_0^1 \mid \aleph'(\rho t+(1-\rho)r)\mid^q d\rho\right)^{\frac{1}{q}}\right]$$
$$+\frac{(s-t)^{\varrho+1}}{s-t}\left[\left(\int_0^1 \rho^{\sigma p} d\rho\right)^{\frac{1}{p}}\left(\int_0^1 \mid \aleph'(\rho t+(1-\rho)s)\mid^q d\rho\right)^{\frac{1}{q}}\right]. \tag{10}$$

By employing the convexity of $\mid \aleph' \mid^q$, we have

$$\int_0^1 \mid \aleph'(\rho t+(1-\rho)r)\mid^q d\rho \leq \int_0^1 [\rho \mid \aleph'(t)\mid^q +(1-\rho)\mid \aleph'(r)\mid^q] d\rho$$
$$=\frac{\mid \aleph'(t)\mid^q + \mid \aleph'(r)\mid^q}{2} \tag{11}$$

and

$$\int_0^1 \mid \aleph'(\rho t+(1-\rho)s)\mid^q d\rho \leq \int_0^1 [\rho \mid \aleph'(t)\mid^q +(1-\rho)\mid \aleph'(s)\mid^q] d\rho$$
$$=\frac{\mid \aleph'(t)\mid^q + \mid \aleph'(s)\mid^q}{2}. \tag{12}$$

Substituting (11) and (12) in (9) and then solving the integrals, we get the required inequality.

Corollary 3. *Applying Theorem 2 for $|\aleph'| \leq K$ where $K > 0$, we get the following inequality*

$$\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big|$$
$$\leq \frac{K}{s-r}\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\left\{(t-r)^{\sigma+1}+(s-t)^{\sigma+1}\right\}.$$

Corollary 4. *Applying Corollary 3 for $t=\frac{r+s}{2}$, we get the following inequality*

$$\Big|\frac{(s-r)^{\sigma-1}}{2^{\sigma-1}}\aleph\left(\frac{r+s}{2}\right)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,\frac{r+s}{2}}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,\frac{r+s}{2}}\aleph(s)\right]\Big|$$
$$\leq K\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\frac{(s-r)^{\sigma}}{2^{\sigma}}.$$

Remark 5. *Applying Theorem 2 for $\sigma=\kappa$, we get Theorem 2 proved by Ahmad et al. [40].*

Remark 6. *Applying Corollary 4 for $\sigma = \kappa$, we get Corollary 4 proved earlier by Ahmad et al. [40].*

Theorem 3. *Let $\aleph : [r,s] \to \mathbb{R}$ be a differentiable function on (r,s), where $\aleph' \in L_1[r,s]$ and $r < s$. The following inequality holds for Hattaf-fractional integral operators (4) and (5) if $|\aleph'|^q$ is a convex function.*

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma} + (s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r) + \aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r) + \mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big| \\
&\leq \frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{p(\sigma p+1)} + \frac{|\aleph'(t)|^q + |\aleph'(r)|^q}{2q}\right) \\
&+\frac{(s-t)^{\sigma+1}}{s-t}\left(\frac{1}{p(\sigma p+1)} + \frac{|\aleph'(s)|^q + |\aleph'(t)|^q}{2q}\right),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $t \in [r,s]$, $\kappa \in (0,1]$ and $M(\kappa) > 0$.

Proof. By utilizing Lemma 1, we have

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma} + (s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r) + \aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r) + \mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big| \\
&\leq \frac{(t-r)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma}\, |\aleph'(\rho t + (1-\rho)r)|\, d\rho \\
&+\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma}\, |\aleph'(\rho t + (1-\rho)s)|\, d\rho.
\end{aligned}$$

By employing Young inequality as $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$ in above, we have

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma} + (s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r) + \aleph(s)\right] \qquad (13)\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r) + \mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big| \\
&\leq \frac{(t-r)^{\sigma+1}}{s-r}\left[\frac{1}{p}\int_0^1 \rho^{\sigma p} d\rho + \frac{1}{q}\int_0^1 |\aleph'(\rho t + (1-\rho)r)|^q\, d\rho\right] \\
&+\frac{(s-t)^{\sigma+1}}{s-r}\left[\frac{1}{p}\int_0^1 \rho^{\sigma p} d\rho + \frac{1}{q}\int_0^1 |\aleph'(\rho t + (1-\rho)s)|^q\, d\rho\right]. \qquad (14)
\end{aligned}$$

Since $|\aleph'|^q$ is convex, so therefore we have

$$\int_0^1 |\aleph'(\rho t + (1-\rho)r)|^q\, d\rho \leq \int_0^1 [\rho\, |\aleph'(t)|^q + (1-\rho)\, |\aleph'(r)|^q] d\rho$$

$$= \frac{|\aleph'(t)|^q + |\aleph'(r)|^q}{2} \tag{15}$$

and

$$\int_0^1 |\aleph'(\rho t + (1-\rho)s)|^q \, d\rho \leq \int_0^1 [\rho |\aleph'(t)|^q + (1-\rho)|\aleph'(s)|^q] d\rho$$
$$= \frac{|\aleph'(t)|^q + |\aleph'(s)|^q}{2}. \tag{16}$$

Substituting (15) and (16) in (13) and then by solving the integrals, we get the desired proof.

Corollary 5. *Applying Theorem 3 for $|\aleph'| \leq K$ where $K > 0$, we get the following inequality*

$$\Big| \left[\frac{(t-r)^\sigma + (s-t)^\sigma}{s-r} \right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)} \Gamma(\sigma) [\aleph(r) + \aleph(s)]$$
$$- \frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)} \left[\mathcal{I}_{r,t}^{\kappa,\sigma} \aleph(r) + \mathcal{I}_{s,t}^{\kappa,\sigma} \aleph(s) \right] \Big|$$
$$\leq \left(\frac{1}{p(\sigma p + 1)} + \frac{K^q}{q} \right) \left\{ \frac{(t-r)^{\sigma+1}}{s-r} + \frac{(s-t)^{\sigma+1}}{s-r} \right\}.$$

Corollary 6. *Applying Corollary 5 for $t = \frac{r+s}{2}$, we get the following inequality*

$$\Big| \frac{(s-r)^{\sigma-1}}{2^{\sigma-1}} \aleph\left(\frac{r+s}{2}\right) + \frac{\sigma(1-\kappa)}{\kappa(s-r)} \Gamma(\sigma) [\aleph(r) + \aleph(s)]$$
$$- \frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)} \left[\mathcal{I}_{r,\frac{r+s}{2}}^{\kappa,\sigma} \aleph(r) + \mathcal{I}_{s,\frac{r+s}{2}}^{\kappa,\sigma} \aleph(s) \right] \Big|$$
$$\leq \left(\frac{1}{p(\sigma p + 1)} + \frac{K^q}{q} \right) \left\{ \frac{(s-r)^\sigma}{2^\sigma} \right\}.$$

Remark 7. *Applying Theorem 3 for $\sigma = \kappa$, we get Theorem 3 proved by Ahmad et al. [40].*

Remark 8. *Applying Corollary 6 for $\sigma = \kappa$, we get Corollary 6 proved earlier by Ahmad et al. [40].*

Theorem 4. *Let $\aleph : [r,s] \to \mathbb{R}$ be a differentiable function on (r,s), where $\aleph' \in L_1[r,s]$ and $r < s$. The following inequality holds for Hattaf-fractional integral operators if $|\aleph'|^q$ is a convex function*

$$\Big| \left[\frac{(t-r)^\sigma + (s-t)^\sigma}{s-r} \right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)} \Gamma(\sigma) [\aleph(r) + \aleph(s)]$$

$$
\begin{aligned}
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
\leq&\frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)^{1-\frac{1}{q}}\left[\frac{|\aleph'(t)|^q}{(\sigma+2)}+\frac{|\aleph'(r)|^q}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}\\
+&\frac{(s-t)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)^{1-\frac{1}{q}}\left[\frac{|\aleph'(s)|^q}{(\sigma+1)(\sigma+2)}+\frac{|\aleph'(t)|^q}{(\sigma+2)}\right]^{\frac{1}{q}},
\end{aligned}
$$

where $q\geq 1$, $t\in[r,s]$, $\kappa\in(0,1]$ *and* $M(\kappa)>0$.

Proof. By utilizing Lemma 1, we have

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
\leq&\frac{(t-r)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}\,|\aleph'(\rho t+(1-\rho)r)|\,d\rho\\
+&\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}\,|\aleph'(\rho t+(1-\rho)s)|\,d\rho.
\end{aligned}
$$

By employing power mean inequality, we obtain

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
\leq&\frac{(t-r)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma}d\rho\right)^{1-\frac{1}{q}}\left(\int_0^1\rho^{\sigma}\,|\aleph'(\rho t+(1-\rho)r)|^q\,d\rho\right)^{\frac{1}{q}}\\
+&\frac{(s-t)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma}d\rho\right)^{1-\frac{1}{q}}\left(\int_0^1\rho^{\sigma}\,|\aleph'(\rho t+(1-\rho)s)|^q\,d\rho\right)^{\frac{1}{q}}.
\end{aligned}
$$

Now, by utilizing the convexity of $|\aleph'|^q$, we get

$$
\begin{aligned}
&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|\\
\leq&\frac{(t-r)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma}d\rho\right)^{1-\frac{1}{q}}\left(\int_0^1\rho^{\sigma}[\rho\,|\aleph'(t)|^q+(1-\rho)\,|\aleph'(r)|^q]d\rho\right)^{\frac{1}{q}}\\
+&\frac{(s-t)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma}d\rho\right)^{1-\frac{1}{q}}\left(\int_0^1\rho^{\sigma}[\rho\,|\aleph'(t)|^q+(1-\rho)\,|\aleph'(s)|^q]d\rho\right)^{\frac{1}{q}}
\end{aligned}
$$

$$\leq \frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)^{1-\frac{1}{q}}\left[\frac{|\aleph'(t)|^q}{(\sigma+2)}+\frac{|\aleph'(r)|^q}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}$$
$$+\frac{(s-t)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)^{1-\frac{1}{q}}\left[\frac{|\aleph'(t)|^q}{(\sigma+2)}+\frac{|\aleph'(s)|^q}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}},$$

which complete the proof.

Corollary 7. *Applying Theorem 4 for* $|\aleph'| \leq K$ *where* $K > 0$ *, we get the following inequality*

$$\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|$$
$$\leq\frac{K}{s-r}\left(\frac{1}{\sigma+1}\right)\left\{(t-r)^{\sigma+1}+(s-t)^{\sigma+1}\right\}.$$

Corollary 8. *Applying Corollary 7 for* $t=\frac{r+s}{2}$*, we get the following inequality*

$$\Big|\left[\frac{(s-r)^{\sigma-1}}{2^{\sigma-2}}\right]\aleph\left(\frac{r+s}{2}\right)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,\frac{r+s}{2}}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,\frac{r+s}{2}}^{\kappa,\sigma}\aleph(s)\right]\Big|$$
$$\leq\frac{K}{s-r}\left(\frac{1}{\sigma+1}\right)\frac{(s-r)^{\sigma}}{2^{\sigma}}.$$

Remark 9. *Applying Theorem 4 for* $\sigma=\kappa$*, we get Theorem 4 proved by Ahmad et al. [40].*

Remark 10. *Applying Corollary 8 for* $\sigma=\kappa$*, we get Corollary 8 proved earlier by Ahmad et al. [40].*

Theorem 5. *Let* $\aleph:[r,s]\to\mathbb{R}$ *be a differentiable function on* (r,s)*, where* $\aleph'\in L_1[r,s]$ *and* $r<s$*. For Hattaf-fractional integral operators (4) and (5), we have the following inequality if* $|\aleph'|$ *is a concave function*

$$\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]$$
$$-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big|$$
$$\leq\frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)\Big|\aleph'\left(\frac{(\sigma+1)t+r}{\sigma+2}\right)\Big|+\frac{(s-t)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma+1}\right)\Big|\aleph'\left(\frac{(\sigma+1)t+s}{\sigma+2}\right)\Big|,$$

where $\kappa\in(0,1]$ *and* $M(\kappa)>0$*.*

Proof. By utilizing Lemma 1, we have

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big| \\
\leq & \frac{(t-r)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma} \mid \aleph'(\rho t+(1-\rho)r) \mid d\rho \\
&+\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1 \rho^{\sigma} \mid \aleph'(\rho t+(1-\rho)s) \mid d\rho .
\end{aligned}$$

By employing the Jensen integral inequality, we get

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big| \\
\leq & \frac{(t-r)^{\sigma+1}}{s-r}\left(\int_0^1 \rho^{\sigma} d\rho\right) \mid \aleph'\left(\frac{\int_0^1 \rho^{\sigma}(\rho t+(1-\rho)r)d\rho}{\int_0^1 \rho^{\sigma}d\rho}\right) \mid \\
&+\frac{(s-t)^{\sigma+1}}{s-r}\left(\int_0^1 \rho^{\sigma} d\rho\right) \mid \aleph'\left(\frac{\int_0^1 \rho^{\sigma}(\rho t+(1-\rho)s)d\rho}{\int_0^1 \rho^{\sigma}d\rho}\right) \mid .
\end{aligned}$$

After simple calculation of above integrals, we get the required inequality.

Corollary 9. *Applying Theorem 5 for $|\aleph'| \leq K$ where $K > 0$, we get the following inequality*

$$\begin{aligned}
&\Big| \left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right] \aleph(t) + \frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right] \\
&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}_{r,t}^{\kappa,\sigma}\aleph(r)+\mathcal{I}_{s,t}^{\kappa,\sigma}\aleph(s)\right]\Big| \\
\leq & \frac{K}{s-r}\left(\frac{1}{\sigma+1}\right)\left\{(t-r)^{\sigma+1}+(s-t)^{\sigma+1}\right\}.
\end{aligned}$$

Remark 11. *Applying Theorem 5 for $\kappa = \sigma$, we get Theorem 5 proved by Ahmad et al. [40].*

Remark 12. *Applying Corollary 9 for $\kappa = \sigma$, we get Corollary 9 proved earlier by Ahmad et al. [40].*

Theorem 6. *Assume that the function $\Psi : [r, s] \to \mathbb{R}$ has a continuous derivative on $[r, s]$ and is strictly increasing and positive. Let $\aleph : [r, s] \to \mathbb{R}$ be a differentiable function on*

(r,s), where $\aleph' \in L_1[r,s]$ and $r<s$. For Hattaf-fractional integral operators, we have the following inequality if $|\aleph'|^q$ is a concave function

$$\begin{aligned}&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big|\\&\leq\frac{(t-r)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\mid\aleph'\left(\frac{t+r}{2}\right)\mid+\frac{(s-t)^{\sigma+1}}{s-r}\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\mid\aleph'\left(\frac{s+t}{2}\right)\mid,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, $t\in[r,s]$, $\kappa\in(0,1]$ and $M(\kappa)>0$.

Proof. By utilizing Lemma 1, we have

$$\begin{aligned}&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big|\\&\leq\frac{(t-r)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}\mid\aleph'(\rho t+(1-\rho)r)\mid d\rho\\&+\frac{(s-t)^{\sigma+1}}{s-r}\int_0^1\rho^{\sigma}\mid\aleph'(\rho t+(1-\rho)s)\mid d\rho.\end{aligned}$$

By employing the Hölder integral inequality, we obtain

$$\begin{aligned}&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big|\\&\leq\frac{(t-r)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma p}d\rho\right)^{\frac{1}{p}}\left(\int_0^1\mid\aleph'(\rho t+(1-\rho)r)\mid^q d\rho\right)^{\frac{1}{q}}\\&+\frac{(s-t)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma p}d\rho\right)^{\frac{1}{p}}\left(\int_0^1\mid\aleph'(\rho t+(1-\rho)s)\mid^q d\rho\right)^{\frac{1}{q}}.\end{aligned}$$

Now, by employing the convexity of $\mid\aleph'\mid^q$ and Jensen integral inequality, we obtain

$$\begin{aligned}&\Big|\left[\frac{(t-r)^{\sigma}+(s-t)^{\sigma}}{s-r}\right]\aleph(t)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\&-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,t}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,t}\aleph(s)\right]\Big|\\&\leq\frac{(t-r)^{\sigma+1}}{s-r}\left(\int_0^1\rho^{\sigma p}d\rho\right)^{\frac{1}{p}}\left(\int_0^1\mid\aleph'(\rho t+(1-\rho)r)\mid^q d\rho\right)^{\frac{1}{q}}\end{aligned}\tag{17}$$

$$+\frac{(s-t)^{\sigma+1}}{s-r}\left(\int_0^1 \rho^{\sigma p}d\rho\right)^{\frac{1}{p}}\left(\int_0^1 \mid \aleph'(\rho t+(1-\rho)s)\mid^q d\rho\right)^{\frac{1}{q}}. \quad (18)$$

Now, since

$$\begin{aligned}\int_0^1 \mid \aleph'(\rho t+(1-\rho)r)\mid^q d\rho &\leq \int_0^1 \rho^0 \mid \aleph'(\rho t+(1-\rho)r)\mid^q d\rho\\ &\leq \left(\int_0^1 \rho^0 d\rho\right)\mid \aleph'\left(\frac{1}{\int_0^1 \rho^0 d\rho}\int_0^1(\rho t+(1-\rho)r)d\rho\right)\mid^q\\ &\leq \mid \aleph'\left(\frac{t+r}{2}\right)\mid^q. \quad (19)\end{aligned}$$

Similarly

$$\int_0^1 \mid \aleph'(\rho t+(1-\rho)s)\mid^q d\rho \leq \mid \aleph'\left(\frac{t+s}{2}\right)\mid^q. \quad (20)$$

By substituting (20) and (19) in (17) and then solving the integrals, we get the desire assertion.

Corollary 10. *Applying Theorem 6 for $t=\frac{r+s}{2}$, we get the following inequality*

$$\begin{aligned}&\Big|\frac{(s-r)^{\sigma-1}}{2^{\sigma-1}}\aleph\left(\frac{r+s}{2}\right)+\frac{\sigma(1-\kappa)}{\kappa(s-r)}\Gamma(\sigma)\left[\aleph(r)+\aleph(s)\right]\\ &-\frac{\sigma M(\kappa)\Gamma(\sigma)}{\kappa(s-r)}\left[\mathcal{I}^{\kappa,\sigma}_{r,\frac{r+s}{2}}\aleph(r)+\mathcal{I}^{\kappa,\sigma}_{s,\frac{r+s}{2}}\aleph(s)\right]\Big|\\ &\leq\left(\frac{1}{\sigma p+1}\right)^{\frac{1}{p}}\frac{1}{s-r}\Big\{(s-t)^{\sigma+1}\mid \aleph'\left(\frac{3r+s}{4}\right)\mid+(s-t)^{\sigma+1}\mid \aleph'\left(\frac{r+3s}{4}\right)\mid\Big\}.\end{aligned}$$

Remark 13. *Applying Theorem 6 for $\sigma=\kappa$, we get Theorem 6 proved by Ahmad et al. [40].*

Remark 14. *Applying Corollary 10 for $\sigma=\kappa$, we get Corollary 10 proved earlier by Ahmad et al. [40].*

4. Concluding Remarks

In this paper, we developed Ostrowski-type inequalities for convex functions containing the Hattaf fractional integral operators. The results presented in this paper are original to the best of our knowledge. Given the widespread applications of convex functions in numerous scientific fields, our unique advancements are believed to be applicable to several special functions, including convexity, interval analysis, quantum calculus, fractional calculus, and coordinates. The inequalities in term of AB operators will be restored if we put $\kappa=\sigma$ and classical inequalities if we put $\kappa=\sigma=1$. One can obtain Grüss type inequalities, Chebyshev type inequalities, Reverse Minkowski's type inequalities and certain other type inequalities by using Hattaf fractional integral operators.

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Declaration

Competing Interests

All authors declare no conflict of interest.

Author's Contributions

The authors have worked equally when writing this paper. All authors read and approved the final manuscript.

References

[1] M. Samraiz, Z. Perveen, T. Abdeljawad, S. Iqbal, and S. Naheed. On certain fractional calculus operators and their applications in mathematical physics. *Physica Scripta*, 95(11):115210, 2020.

[2] M. Samraiz, Z. Perveen, G. Rahman, K. S. Nisar, and Devendra Kumar. On (k, s)-Hilfer Prabhakar fractional derivative with applications in mathematical physics. *Frontiers in Physics*, 8:1–9, 2020.

[3] A. Nazir, G. Rahman, A. Ali, S. Naheed, K. S. Nisar, W. Albalawi, and H. Y. Zahran. On generalized fractional integral with multivariate Mittag-Leffler function and its applications. *Alexandria Engineering Journal*, 61:9187–9201, 2022.

[4] M. Samraiz, A. Mehmood, S. Naheed, G. Rahman, A. Kashuri, and K. Nonlaopon. On novel fractional operators involving the multivariate Mittag–Leffler function. *Mathematics*, 10(21):3991, 2022.

[5] M. Samraiz, M. Umer, T. Abdeljawad, S. Naheed, G. Rahman, and K. Shah. On Riemann-type weighted fractional operators and solutions to Cauchy problems. *Computer Modeling in Engineering and Sciences*, 36(1):901–918, 2023.

[6] D. Baleanu and A. Fernandez. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Communications in Nonlinear Science and Numerical Simulation*, 59:444–462, 2018.

[7] T. Abdeljawad and D. Baleanu. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *Journal of Nonlinear Sciences and Applications*, 10(3):1098–1107, 2017.

[8] T. Abdeljawad. A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel. *Journal of Inequalities and Applications*, 2017:130, 2017.

[9] W. H. Huang, M. Samraiz, A. Mehmood, D. Baleanu, G. Rahman, and S. Naheed. Modified Atangana-Baleanu fractional operators involving generalized Mittag-Leffler function. *Alexandria Engineering Journal*, 75:639–648, 2023.

[10] T. Abdeljawad and D. Baleanu. On fractional derivatives with generalized Mittag-Leffler kernels. *Advances in Difference Equations*, 2018:468, 2018.

[11] T. Abdeljawad. Fractional operators with generalized Mittag-Leffler kernels and their iterated differintegrals. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 29:2, 2019.

[12] S. S. Dragomir. Ostrowski type inequalities for Riemann–Liouville fractional integrals of absolutely continuous functions in terms of norms. *RGMIA Research Report Collection*, 20:49, 2017.

[13] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak. Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57:2403–2407, 2013.

[14] M. Z. Sarikaya and H. Yildirim. On Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Mathematical Notes*, 17:1049–1059, 2017.

[15] H. M. Srivastava, Z. H. Zhang, and Y. D. Wu. Some further refinements and extensions of the Hermite–Hadamard and Jensen inequalities in several variables. *Mathematical and Computer Modelling*, 54:2709–2717, 2011.

[16] A. Rafiq, N. A. Mir, and F. Ahmad. Weighted Chebyshev–Ostrowski type inequalities. *Applied Mathematics and Mechanics*, 28:901–906, 2007.

[17] Y. Shuang and F. Qi. Integral inequalities of Hermite–Hadamard type for extended s-convex functions and applications. *Mathematics*, 6(11):223, 2018.

[18] K. Mehrez and P. Agarwal. New Hermite–Hadamard type integral inequalities for convex functions and their applications. *Journal of Computational and Applied Mathematics*, 350:274–285, 2019.

[19] M. J. Park, O. M. Kwon, and J. H. Ryu. Generalized integral inequality: application to time-delay systems. *Applied Mathematics Letters*, 77:6–12, 2018.

[20] M. Z. Sarikaya, T. Tunc, and H. Budak. On generalized some integral inequalities for local fractional integrals. *Applied Mathematics and Computation*, 276:316–323, 2016.

[21] E. Set, S. I. Butt, A. O. Akdemir, A. Karaoğlan, and T. Abdeljawad. New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators. *Chaos, Solitons and Fractals*, 143:110554, 2021.

[22] G. Rahman, K. S. Nisar, and F. Qi. Some new inequalities of the Grüss type for conformable fractional integrals. *AIMS Mathematics*, 3(4):575–583, 2018.

[23] G. Rahman, K. S. Nisar, A. Ghaffar, and F. Qi. Some inequalities of the Grüss type for conformable k-fractional integral operators. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 114:9, 2020.

[24] G. Rahman, Z. Ullah, A. Khan, E. Set, and K. S. Nisar. Certain Chebyshev type inequalities involving fractional conformable integral operators. *Mathematics*, 7:364, 2019.

[25] G. Rahman, T. Abdeljawad, F. Jarad, and K. S. Nisar. Bounds of generalized proportional fractional integrals in general form via convex functions and their applications. *Mathematics*, 8:113, 2020.

[26] T. Toplu, M. Kadkal, and İ. İşcan. On n-polynomial convexity and some related

inequalities. *AIMS Mathematics*, 5:1304–1318, 2020.

[27] M. Kadkal and İ. İşcan. Exponential type convexity and some related inequalities. *Journal of Inequalities and Applications*, 2009:82, 2020.

[28] S. I. Butt, M. Tariq, A. Aslam, H. Ahmad, and T. A. Nofel. Hermite–Hadamard type inequalities via generalized harmonic exponential convexity. *Journal of Function Spaces*, 2021:5533491, 2021.

[29] M. Tariq. New Hermite–Hadamard type inequalities via p-harmonic exponential type convexity and applications. *Universal Journal of Mathematics and Applications*, 4:59–69, 2021.

[30] D. S. Mitrinović, J. Pečarić, and A. M. Fink. *Inequalities Involving Functions and Their Integrals and Derivatives*, volume 53. Springer Science and Business Media, Dordrecht, The Netherlands, 2012.

[31] S. S. Dragomir and S. Wang. A new inequality of Ostrowski type in L_1 norm and applications to some special means and to some numerical quadrature rules. *Tamkang Journal of Mathematics*, 28:239–244, 1997.

[32] S. S. Dragomir and S. Wang. Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. *Applied Mathematics Letters*, 11:105–109, 1998.

[33] N. S. Barnett and S. S. Dragomir. An Ostrowski type inequality for double integrals and applications for cubature formulae. *Soochow Journal of Mathematics*, 27:109–114, 2001.

[34] P. Cerone, S. S. Dragomir, and J. Roumeliotis. An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications. *East Asian Mathematical Journal*, 15:1–9, 1999.

[35] Z. Retkes. An extension of the Hermite–Hadamard inequality. *Acta Scientiarum Mathematicarum (Szeged)*, 74(1):95–106, 2008.

[36] K. M. Owolabi. Modelling and simulation of a dynamical system with the Atangana–Baleanu fractional derivative. *European Physical Journal Plus*, 133(1):15, 2018.

[37] D. Kumar, J. Singh, and D. Baleanu. Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel. *Physica A: Statistical Mechanics and its Applications*, 492:155–167, 2018.

[38] Z. Jianke, W. Gaofeng, Z. Xiaobin, and Z. Chang. Generalized Euler–Lagrange equations for fuzzy fractional variational problems under gH-Atangana–Baleanu differentiability. *Journal of Function Spaces*, 2018:2740678, 2018.

[39] K. Hattaf. A new generalized definition of fractional derivative with non-singular kernel. *Computation*, 8:49, 2020.

[40] H. Ahmad, M. Tariq, S. K. Sahoo, S. Askar, A. E. Abouelregal, and K. M. Khedher. Refinements of Ostrowski type integral inequalities involving Atangana–Baleanu fractional integral operator. *Symmetry*, 13:2059, 2021.