EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5819
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A New Bivariate Transmuted Family of Distributions: Properties and Application

Amani Abdullah Alsalafi[1,2,*], Saman Hanif Shahbaz[1], Lutfiah Ismail Al-Turk[1]

[1] *Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Sana'a University, Sana'a, Yemen*

Abstract. The cubic transformation families of distributions are widely used to model complex univariate data. However, in many situations, the jointly modeling of two variables is necessary, making bivariate distributions essential. This paper introduces a novel family of bivariate probability distributions that extends the univariate cubic transformation family. The bivariate cubic transmuted (BCT) family of distributions is comprehensively discussed, with its statistical properties explored in detail. Within this family, the bivariate cubic transmuted Burr (BCTB) distribution is specifically analyzed. Its statistical properties are examined, and its parameters are estimated using the maximum likelihood estimation (MLE) method. To assess the performance of the estimation procedure, a Monte Carlo simulation study is conducted. Furthermore, the applicability of the proposed model is demonstrated by fitting it to real datasets, and its relevance is further discussed.

2020 Mathematics Subject Classifications: 60E05, 62E15, 62G30, 62F10

Key Words and Phrases: Bivariate cubic transmuted family, bivariate cubic transmuted Burr distribution, Burr distribution, maximum likelihood estimation

---

1. Introduction

In recent decades, significant advancements have been made in the development of continuous univariate distributions and families of univariate distributions. Despite these advancements, many datasets in reliability, science, and related fields deviate from traditional distributions. Consequently, there is a pressing need for modified, extended, and generalized distributions tailored to these specialized applications.
One such innovation is the transformed-transformer method, formerly referred to as the exponentiated $T-X$ family of distributions by [1]. This method has been pivotal in

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5819

Email addresses: amohammedalsalafi@stu.kau.edu.sa (A. A. Alsalafi),
shmohamad2@kau.edu.sa (S. H. Shahbaz),
lturk@kau.edu.sa (L. I. Al-Turk)

extending traditional distributions to address complex data behaviors.
Bivariate distributions also play a critical role, particularly in modeling extreme events. Return periods in bivariate distributions can be analyzed using separate single random variables or joint random variables. These return periods may involve one variable exceeding a certain magnitude, another variable meeting a threshold, or conditional scenarios where one variable's behavior depends on an other's magnitude. The mathematical exploration of new families of bivariate distributions is vital for enhancing their applicability to real-world problems.
Among the versatile techniques in distribution theory, the transmutation method introduced by [2] stands out. This technique employs a functional composition of a cumulative density function (CDF) and an inverse CDF (quantile function), leading to new distributions defined by the formula:

$$F(x) = (1+\lambda_1)G(x) - \lambda_1 G^2(x);$$

where $\lambda_1 \in [-1,1]$, $x \in R$ and $G(x)$ is the baseline distribution's CDF. This approach has resulted in numerous novel distributions, such as the Transmuted Weibull distribution by [3], the Transmuted Pareto distribution by [4], and others.
Further advancements include the cubic transmutation method. [5] studied the cubic transmuted Weibull distribution and its properties, while [6] obtained the CDF of cubic transmuted family of distrubution as:

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x);$$

where $\lambda_1, \lambda_2 \in [-1,1]$, $-2 \leqslant \lambda_1 + \lambda_1 \leqslant 1$. [7] and [8] expanded on this work by exploring parameter estimation and inference procedures. Notably, [9] introduced the cubic transmuted Rayleigh distribution with the following CDF:

$$F(x,\sigma,\lambda) = 1 - e^{-\frac{3x^2}{2\sigma^2}}[(1-\lambda)e^{-\frac{x^2}{\sigma^2}} + 3\lambda e^{-\frac{x^2}{2\sigma^2}} - 2\lambda],$$

where $\sigma \in R^+$ is the scale parameter and $\lambda \in [-1,1]$ is the shape parameter.
Bivariate distributions continue to evolve with innovations that integrate greater flexibility into modeling joint phenomena. [10] introduced generalized joint distributions, while [11] proposed the bivariate Gumbel-G family. These concepts were extended by incorporating transmuted families, as demonstrated by [12]. Moreover, [13] developed a bivariate transmuted family of distributions, which provided enhanced tools for analyzing interdependent phenomena in diverse fields.

The article is organized as follows: Section 2 introduced the bivariate transmuted family of distributions; Section 3 discussed the genesis of the BCT family, presenting its probability density function (PDF), cumulative distribution function (CDF), marginal distributions and conditional distributions; Section 4 examined key statistical properties of the BCT family; Section 5 explored the estimation of the family parameters; Section 6 proposed the BCTB distribution; Section 7 studied the statistical properties of the BCTB distribution; Section 8 details the parameter estimation for the proposed BCTB distribution; Section 9 provided simulations and applies the BCTB distribution to a real dataset; and finally, Section 10 concluded the study with key findings and implications.

2. The Bivariate Transmuted Family of Distributions

This research aims to expand the family of bivariate transformed distributions into the bivariate cubic state. Therefore, it is appropriate to discuss the family of bivariate T-X distributions.

[10] introduced a new technique to derive families of continuous distributions using two different distributions. A random variable X called a transformer is used to transform another random variable, T, This family of distributions is called the $\mathrm{T-X}$ family of distributions. The CDF of this family is:

$$F_{\mathrm{T-X}}(x)=\int_{\mathrm{a}}^{\mathrm{W}[G(x)]} r(u)du \tag{1}$$

where $G(x)$ is CDF of baseline distribution and $W(G(x))$ is some function of $G(x)$ such that

$$\left.\begin{array}{c} W[G(x)]\in[a,b] \\ W[G(x)] \text{ is differentiable and monotonically non-decreasing} \\ W[G(x)]\rightarrow \text{a as x} \rightarrow -\infty, \\ W[G(x)]\rightarrow \text{ b as x} \rightarrow +\infty \end{array}\right\} \tag{2}$$

where W [G(x)] satisfies the conditions Eq.(2). The pdf corresponding to the Eq.(1), is given by

$$f_{\mathrm{T-X}}(x)=\left\{\frac{d}{dx}W[G(x)]\right\} r\{W[G(x)]\}$$

The joint CDF of a simple bivariate T-X family of distributions is given by [14] as:

$$F_{X,Y}(x,\mathrm{y})=\int_{a_1}^{W_1[G_1(x)]}\int_{a_2}^{W_2[G_2(\mathrm{y})]} r\left(u_1,u_2\right)\mathrm{d}u_1\ \mathrm{d}u_2$$

where $W_1\left[G_1(x)\right]$ and $W_2\left[G_2(\mathrm{y})\right]$ have usual properties and $r\left(u_1,u_2\right)$ is any bivariate distribution with suitable support for random variable U_1 and U_2. If (u_1,u_2) is a bivariate distribution such that the support of U_1 and U_2 is $[0,1]\times[0,1]$ then a simpler version of bivariate $\mathrm{T-X}$ family is given as

$$F_{X,Y}(x,y)=\int_0^{G_1(x)}\int_0^{G_2(\mathrm{y})} r\left(u_1,u_2\right)\mathrm{d}u_1\ \mathrm{d}u_2. \tag{3}$$

[13] proposed a new family of bivariate transmuted distributions by using

$$r(u_1,u_2)=1+\lambda_1(1-2u_1)+\lambda_2(1-2u_2)+2\lambda_3(1-u_1-u_2);0\leqslant u_1,u_2\leq 1,$$

in Eq.(3) and the joint CDF of the family as:

$$F_{X,Y}(x,y) = G_1(x)G_2(y)[1 + (\lambda_1 + \lambda_3)(1 - G_1(x)) + (\lambda_2 + \lambda_3)(1 - G_2(y))], \quad (4)$$

Where $G_1(x)$ and $G_2(y)$ are any marginal CDF for all $(x, y) \in R^2$, $(\lambda_1, \lambda_2, \lambda_3)$ are the transmutation parameters such that $(\lambda_1, \lambda_2, \lambda_3) \in [-1, 1]$ under these conditions: $-1 \leqslant \lambda_1 - \lambda_2 \leqslant 1, -1 \leqslant \lambda_1 + \lambda_2 + 2\lambda_3 \leqslant 1, -1 \leqslant \lambda_1 + \lambda_3 \leqslant 1$ and $-1 \leqslant \lambda_2 + \lambda_3 \leqslant 1$.

the PDF corresponding to the CDF in Eq.(4) as:

$$f_{X,Y}(x,y) = g_1(x)g_2(y)[1 + (\lambda_1 + \lambda_3)(1 - 2G_1(x)) + (\lambda_2 + \lambda_3)(1 - 2G_2(y))].$$

Where $g_1(x)$ and $g_2(y)$ are the PDF of any base distribution corresponding to the CDF $G_1(x)$ and $G_2(y)$, respectively.

3. The Bivariate Cubic Transmuted Families of Distributions

In this section we proposed bivariate cubic transmuted (BCT) family using

$$\begin{aligned} r(u_1, u_2) = 1 &+ \lambda_1(1 - 2u_1) + \lambda_2 u_1(2 - 3u_1) \\ &+ \lambda_3(1 - 2u_2) + \lambda_4 u_2(2 - 3u_2) \\ &+ \lambda_5(1 - 3u_1^2)(1 - 3u_2^2);\ 0 < u_1, u_2 < 1. \end{aligned} \quad (5)$$

This function generalizes previous models and offers more flexible dependency structures between variables while also incorporating non-linear effects to capture more complex relationships. It ensures the validity of probabilistic properties, making it well-suited for constructing a new bivariate distribution. In addition, it can recover traditional cases where certain parameters are set to zero, which simplifies interpretation and application. Using (5)in (3), to get the cumulative distribution function for the BCT family of distributions (CDF) as:

$$\begin{aligned} F_{X,Y}(x,y) = G_1(x)G_2(y)[&1 + \lambda_1 + \lambda_3 + \lambda_5 + (\lambda_2 - \lambda_1)G_1(x) - (\lambda_2 + \lambda_5)G_1^2(x) \\ &+ (\lambda_4 - \lambda_3)G_2(y) - (\lambda_4 + \lambda_5)G_2^2(y) + \lambda_5 G_1^2(x)G_2^2(y)]. \end{aligned} \quad (6)$$

The joint PDF corresponding to Eq. (6) is easily obtained by differentiating Eq.(6) with respect to X and Y and is

$$\begin{aligned} f_{X,Y}(x,y) = g_1(x)g_2(y)[&1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) \\ &+ 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)] \end{aligned} \quad (7)$$

Where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in [-1, 1]$ under these conditions:$-2 \leq \lambda_1 + \lambda_3 + \lambda_5 \leq 0$,$-1 \leq \lambda_2 - \lambda_1 \leq 1$,$-1 \leq \lambda_2 + \lambda_5 \leq 1$, $-1 \leq \lambda_3 - \lambda_4 \leq 1$ and $-1 \leq \lambda_4 + \lambda_5 \leq 1$.

The marginal CDF of X and Y for the BCT family of distributions, given in (6), can be easily obtained as follows:

$$F_X(x) = G_1(x)[1 + \lambda_1(1 - G_1(x)) + \lambda_2 G_1(x)(1 - G_1(x))], \tag{8}$$

where $G_1(x)$ is the CDF of any baseline distribution and (λ_1, λ_2) are the transmutation parameters such that $(\lambda_1, \lambda_2) \in [-1, 1]$ and $-1 \leqslant \lambda_1 + \lambda_2 \leqslant 1$.

$$F_Y(y) = G_2(y)[1 + \lambda_3(1 - G_2(y)) + \lambda_4 G_2(y)(1 - G_2(y))]. \tag{9}$$

where $G_2(y)$ is the CDF of any baseline distribution and (λ_3, λ_4) are the transmutation parameters such that $(\lambda_3, \lambda_4) \in [-1, 1]$ and $-1 \leqslant \lambda_3 + \lambda_4 \leqslant 1$

Note that the distribution functions (8) and (9) are CDFs of the cubic transmuted family of distributions.

The marginal PDF of the random variables X and Y for the bivariate cubic transmuted family of distributions corresponding to (8) and (9), respectively, are given as

$$f_X(x) = g_1(x)[1 + \lambda_1(1 - 2G_1(x)) + \lambda_2 G_1(x)(2 - 3G_1(x))], \tag{10}$$

and

$$f_Y(y) = g_2(y)[1 + \lambda_3(1 - 2G_2(y)) + \lambda_4 G_2(y)(2 - 3G_2(y))], \tag{11}$$

where $g_1(x)$ and $g_2(y)$ are the PDF of any base distribution corresponding to the CDF $G_1(x)$ and $G_2(y)$, respectively. It is easy to see that (10) and (11) are density functions of the univariate cubic transmuted family of distributions which are CDFs of the Cubic Transmuted Family of Distributions proposed by [6].

The conditional distribution of X given Y = y is obtained as:

$$f_{X|Y}(x, y) = \frac{f_{X,Y}(x, y)}{f_Y(y)},$$

Using (7) and (11) in the above equation, the conditional distribution of X given Y = y for the bivariate cubic transmuted family of distributions is readily written as :

$$f_{X|y}(x, y) = \frac{g_1(x)\delta(x, y)}{[1 + \lambda_3(1 - 2G_2(y)) + \lambda_4 G_2(y)(2 - 3G_2(y))]}, \tag{12}$$

where

$$\begin{aligned}\delta(x, y) = [&1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) + 2(\lambda_4 - \lambda_3)G_2(y) \\ &- 3(\lambda_4 + \lambda_5)G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)].\end{aligned}$$

Again, the conditional PDF of Y given X = x is obtained by using

$$f_{Y|x}(y,x) = \frac{f_{X,Y}(x,y)}{f_X(x)},$$

Using the joint density function of X and Y and given in (7) the marginal density function of X given in (10), the conditional density function of Y given X = x is:

$$f_{Y|x}(y,x) = \frac{g_2(y)\delta(x,y)}{[1+\lambda_1(1-2G_1(x))+\lambda_2 G_1(x)(2-3G_1(x))]}. \tag{13}$$

Conditional distributions can be studied for any baseline distribution.

In the next section, some properties of the proposed bivariate cubic transmuted family of distributions will be discussed.

4. Statistical Properties

Some useful properties will be included in the following subsection.

4.1. The Product and Ratio Moments

The $(r,s)^{th}$ product moment of the random variables X and Y, following bivariate cubic transmuted families of distributions are obtained in the next theorem.

Theorem 1. *Let X and Y have joint bivariate cubic transmuted families of distributions then the $(r,s)^{th}$ product moment of X and Y is*

$$\mu_{x,y}^{r,s} = E(X^r Y^s) = \eta\mu_x^r\mu_y^s + \delta_2\mu_{x(2:2)}^r\mu_y^s - \delta_6\mu_{x(3:3)}^r\mu_y^s + \delta_4\mu_x^r\mu_{y(2:2)}^s - \delta_7\mu_x^r\mu_{y(3:3)}^s + \lambda_5\mu_{x(3:3)}^r\mu_{y(3:3)}^s.$$

Where $\eta = (1+\lambda_1+\lambda_3+\lambda_5)$,$\delta_2 = (\lambda_2-\lambda_1)$, $\delta_4 = (\lambda_4-\lambda_3)$, $\delta_6 = (\lambda_2+\lambda_5)$, $\delta_7 = (\lambda_4+\lambda_5)$, $\mu_{x,y}^{r,s}$ represents the joint raw moment of the random variables X and Y, $\mu_{y(2:2)}^s, \mu_{y(3:3)}^s$ are the s^{th} moment of larger observation in sample of size 2 and 3 from $G_2(y)$, $\mu_{x(2:2)}^r, \mu_{x(3:3)}^r$ are the r^{th} moment of larger observation in sample of size 2 and 3 from $G_2(x)$.

Proof. *The product moments are defined as:*

$$\mu_{x,y}^{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f_{X,Y}(x,y)dxdy.$$

Now using the bivariate cubic transmuted family PDF from (7), The product moments for the bivariate cubic transmuted family of distributions are written as

$$E(X^r Y^s) = (1+\lambda_1+\lambda_3+\lambda_5)\mu_x^r\mu_y^s + (\lambda_2-\lambda_1)\mu_{x(2:2)}^r\mu_y^s - (\lambda_2+\lambda_5)\mu_{x(3:3)}^r\mu_y^s + (\lambda_4-\lambda_3)\mu_x^r\mu_{y(2:2)}^s - (\lambda_4+\lambda_5)\mu_x^r\mu_{y(3:3)}^s + \lambda_5\mu_{x(3:3)}^r\mu_{y(3:3)}^s.$$

which completes the proof.

The moment of the $(r,s)^{th}$ ratio of the random variables X and Y is obtained in the following.

Result. Let X and Y follow the bivariate cubic transmuted family of distribution The $(r,s)^{th}$ ratio moments of the random variables X and Y is

$$\mu_{x,y}^{r,-s} = E(\frac{X^r}{Y^s}) = \eta\mu_x^r\mu_y^{-s} + (\lambda_2-\lambda_1)\mu_{x(2:2)}^r\mu_y^{-s} - (\lambda_2+\lambda_5)\mu_{x(3:3)}^r\mu_y^{-s} + (\lambda_4-\lambda_3)\mu_x^r\mu_{y(2:2)}^{-s} - (\lambda_4+\lambda_5)\mu_x^r\mu_{y(3:3)}^{-s} + \lambda_5\mu_{x(3:3)}^r\mu_{y(3:3)}^{-s},$$

where μ_y^{-s} is s^{th} negative moment of Y and $\mu_{y(2:2)}^{-s}, \mu_{y(3:3)}^{-s}$ are s^{th} negative moment of larger observation in a sample of size 2 and 3 from $G_2(y)$.

4.2. The Conditional Moments

In the following, the conditional moments are derived for the bivariate cubic transmuted family of distributions. The conditional moment r^{th} of X given Y = y is obtained in the following theorem.

Theorem 2. *Let X and Y follow the bivariate cubic transmuted family of distributions.*

$$E(X^r/y) = \frac{1}{\Phi(y)}[(\eta + 2(\lambda_4-\lambda_3)G_2(y) - 3(\lambda_4+\lambda_5)G_2^2(y))\mu_x^r + (\lambda_2-\lambda_1)\mu_{x(2:2)}^r - (\lambda_2+\lambda_5-3\lambda_5G_2^2(y))\mu_{x(3:3)}^r], \tag{14}$$

where $\Phi(y) = (1+\lambda_3) + 2(\lambda_4-\lambda_3)G_2(y) - 3\lambda_4G_2^2(y)$, $\eta = 1+\lambda_1+\lambda_3+\lambda_5$, μ_x^r *is* r^{th} *raw moment of X and* $\mu_{x(2:2)}^r$, $\mu_{x(3:3)}^r$ *is* r^{th} *raw moment of larger observation in a sample of size 2 and 3 respectively from* $G_1(x)$.

Proof. *The* r^{th} *conditional moment of X given* $Y = y$ *is obtained by*

$$E(X^r|y) = \int_{-\infty}^{\infty} x^r f(x|y)dx.$$

Using f(x—y) from (12), the above integral becomes

$$E(X^r/y) = \int_{-\infty}^{\infty} \frac{x^r}{1+\lambda_3-2\lambda_3G_2(y)+2\lambda_4G_2(y)-3\lambda_4G_2^2(y)}[1+\lambda_1+\lambda_3+\lambda_5 + 2(\lambda_2-\lambda_1)G_1(x) - 3(\lambda_2+\lambda_5)G_1^2(x) + 2(\lambda_4-\lambda_3)G_2(y) - 3(\lambda_4+\lambda_5)G_2^2(y) + 9\lambda_5G_1^2(x)G_2^2(y)]g_1(x)dx.$$

After simplifying, we can easily get (14).

Again, the conditional moment s^{th} of Y given X = x is given in the following theorem.

Theorem 3. *Let X and Y be two random variables having joint bivariate cubic transmuted family of distributions then the s^{th} conditional moment of Y given $X = x$ is*

$$E(Y^s|x) = \frac{1}{\Phi(x)}\big[(\eta + 2(\lambda_2 - \lambda_4)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x))\mu_y^s + (\lambda_4 - \lambda_3)\mu_{y(2:2)}^s - (\lambda_4 + \lambda_5 + 3\lambda_5 G_2^2(x))\mu_{y(3:3)}^s\big], \tag{15}$$

where $\Phi(x) = (1 + \lambda_1) + 2(\lambda_2 - \lambda_1)G_1(x) - 3\lambda_2 G_1^2(x)$, $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$, μ_y^s *is* r^{th} *raw moment of y and* $\mu_{y(2:2)}^s$, $\mu_{y(3:3)}^s$ *is* r^{th} *raw moment of larger observation in a sample of size 2 and 3 respectively from* $G_2(y)$.

Proof. *The s^{th} conditional moment of Y given $X = x$ is defined as*

$$E(Y^s|x) = \int_{-\infty}^{\infty} y^s f(y|x)dy,$$

where f(y—x) is the conditional distribution of Y given $X = x$, given in (13). Using the conditional distribution, the above integral becomes

$$E(Y^s|x) = \int_{-\infty}^{\infty} \frac{y^s}{[1 + \lambda_1 - 2\lambda_1 G_1(x) + 2\lambda_2 G_1(x) - 3\lambda_2 G_1^2(x)]}\big[1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) + 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)\big]g_2(y)dy.$$

This, in simplification, becomes (15) and hence the proof is complete.

4.3. The Bivariate Reliability and Hazard Rate Functions

The reliability function of the BCT family is given by this theorem.

Theorem 4. *Let X and Y be two random variables having a bivariate cubic transmuted family of distributions then the bivariate reliability function of X and Y is*

$$R(x, y) = 1 + G_1(x)(\eta G_2(y) - \delta_1) + \delta_2 G_1^2(x)(1 + G_2(y)) + G_1^3(x)(\lambda_2 - \delta_6 G_2(y)) - \delta_3 G_2(y) + \delta_4 G_2^2(y)(1 - G_1(x)) + G_2^3(y)(\lambda_4 - \delta_7 G_1(x) + \lambda_5 G_1^3(x)), \tag{16}$$

where $\delta_1 = (1+\lambda_1)$,$\delta_2 = (\lambda_2-\lambda_1)$, $\delta_3 = (1+\lambda_3)$, $\delta_4 = (\lambda_3-\lambda_4)$, $\eta = (1+\lambda_1+\lambda_3+\lambda_5)$, $\delta_6 = (\lambda_2 + \lambda_5)$, $\delta_7 = (\lambda_4 + \lambda_5)$.

Proof. *The bivariate reliability function of X and Y is defined as*

$$R(x, y) = 1 - [F_X(x) + F_Y(y) - F_{X,Y}(x, y)].$$

Using (8), (9) and (6) in the above equation, the bivariate reliability function for the bivariate cubic transmuted family of distribution completes the proof.
The bivariate reliability function can be obtained for different choices of the baseline CDF $G_1(x)$ and $G_2(y)$

The hazard rate is an important function in reliability analysis since it shows changes in the probability of failure over the lifetime of a component and is also known as the failure or failure rate function. The bivariate hazard rate function of X and Y is defined as:

$$h(x,y) = \frac{f_{X,Y}(x,y)}{R(x,y)}, \tag{17}$$

by using the bivariate density function (7) and the bivariate reliability function (16) in (17), we can derive the bivariate hazard rate function for the bivariate cubic transmuted family of distributions.

4.4. Random Number Generation

The random sample from the bivariate cubic transmuted family of distributions can be generated by using the conditional distribution method. In this method, the random sample from a bivariate distribution is generated by using two steps which are illustrated below:

Step 1: Obtain a random sample using the marginal distribution of X. This is done by equating the distribution function of x, given in (8), to p_1, where p_1 is a uniform random variable, and solving it for x, that is obtain x by solving

$$\begin{aligned}(1+\lambda_1)G_1(x) + (\lambda_2 - \lambda_1)G_1^2(x) - \lambda_2 G_1^3(x) =& p_1 \\ \delta_1 G_1(x) + \delta_2 G_1^2(x) - \lambda_2 G_1^3(x) =& p_1,\end{aligned}$$

or

$$\lambda_2 G_1^3(x) - \delta_2 G_1^2(x) - \delta_1 G_1(x) + p_1 = 0.$$

Writing G (x) = w above equation can be written as

$$\lambda_2 w^3 - \delta_2 w^2 - \delta_1 w + p_1 = 0,$$

or

$$t_1 w^3 + t_2 w^2 + t_3 w + p_1 = 0,$$

where $t_1 = \lambda_2$, $t_2 = -\delta_2$, $t_3 = -\delta_1$.
In simplification, this can be written as

$$w = G(x) = -\frac{t_2}{3t_1} - \frac{2^{\frac{1}{3}}\zeta_1}{3t_1\zeta_3^{\frac{1}{3}}} + \frac{\zeta_3^{\frac{1}{3}}}{3 \times 2^{\frac{1}{,}3} t_1}$$

where
$\zeta_1 = \left|-t_2^2 + 3t_1t_3\right|$,
$\zeta_2 = -t_2^3 + 9t_1t_2t_3 - 27t_1^2p_1$,
and

$\zeta_3 = \zeta_2 + \sqrt{4\zeta_1^3 + \zeta_2^2}$.

The quantile function of the marginal distribution of X in the bivariate cubic transmuted family of distributions is:

$$x = G^{-1}[-\frac{t_2}{3t_1} - \frac{2^{\frac{1}{3}}\zeta_1}{3t_1\zeta_3^{\frac{1}{3}}} + \frac{\zeta_3^{\frac{1}{3}}}{3\times 2^{\frac{1}{3}}t_1}]. \tag{18}$$

and can be computed for any baseline distribution. The random sample from the bivariate cubic transmuted family of distributions can be obtained by using uniform random numbers in ζ_2 above and then using the quantile function.

Step 2: For the application of the second step the conditional CDF of Y given X = x is needed and is obtained as:

$$\begin{aligned} F_{Y|X}(Y|X) &= \int_{-\infty}^{y} f_{Y|x}(u|x)du \\ &= \int_{-\infty}^{y} \frac{g_2(u)}{\sigma}[\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) \\ &+ 2(\lambda_4 - \lambda_3)G_2(u) - 3(\lambda_4 + \lambda_5)G_2^2(u) + 9\lambda_5 G_1^2(x)G_2^2(u)]du, \end{aligned}$$

Where
$\eta = (1 + \lambda_1 + \lambda_3 + \lambda_5)$ and $\sigma = [1 + \lambda_1 - 2\lambda_1 G_1(x) + 2\lambda_2 G_1(x) - 3\lambda_2 G_1^2(x)]$
Making the transformation $w = G_2(u)$, $w^2 = G_2^2(u)$ then $dw = g_2(u)du$

$$\begin{aligned} F_{Y|X}(Y|X) &= \frac{1}{\sigma}\int_{0}^{G_2(y)} [\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) \\ &+ 2(\lambda_4 - \lambda_3)w - 3(\lambda_4 + \lambda_5)w^2 + 9\lambda_5 G_1^2(x)w^2]dw \\ &= \frac{G_2(y)}{\sigma}[\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) \\ &+ (\lambda_4 - \lambda_3)G_2(y) - (\lambda_4 + \lambda_5)G_2^2(y) + 3\lambda_5 G_1^2(x)G_2^2(y)]. \end{aligned} \tag{19}$$

Now using random observation obtained from (18) in (19), and then setting $F_{Y|X}(Y|X)$ is equal to u, where u is a uniform random number, and the random observation y is obtained by solving

$$\begin{aligned} u = \frac{G_2(y)}{\sigma}&[\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) \\ &+ (\lambda_4 - \lambda_3)G_2(y) - (\lambda_4 + \lambda_5)G_2^2(y) + 3\lambda_5 G_1^2(x)G_2^2(y)], \end{aligned}$$

for y.The solution of the above equation in terms of $G_2(y)$ is

$$\begin{aligned} u\sigma = G_2(y)&[\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x)] \\ &+ G_2^2(y)(\lambda_4 - \lambda_3) - G_2^3(y)[(\lambda_4 + \lambda_5) + 3\lambda_5 G_1^2(x)], \end{aligned}$$

Writing $G_2(y) = w_1$ above equation can be written as

$$r_1 w_1^3 + r_2 w_1^2 + r_3 w_1 + p_2 = 0,$$

where $r_1 = [(\lambda_4 + \lambda_5) + 3\lambda_5 G_1^2(x)]$, $r_2 = -(\lambda_4 - \lambda_3)$,$r_3 = -[\eta + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x)]$, $p_2 = u\sigma$.
In simplification, this can be written as

$$w_1 = G_2(y) = -\frac{r_2}{3r_1} - \frac{2^{\frac{1}{3}}\varpi_1}{3r_1\varpi_3^{\frac{1}{3}}} + \frac{\varpi_3^{\frac{1}{3}}}{3 \times 2^{\frac{1}{3}} r_1},$$

where
$\varpi_1 = \left|-r_2^2 + 3r_1 r_3\right|$,
$\varpi_2 = -r_2^3 + 9r_1 r_2 r_3 - 27r_1^2 p_2$,
and
$\varpi_3 = \varpi_2 + \sqrt{4\varpi_1^3 + \varpi_2^2}$,
So

$$y = G_2^{-1}[-\frac{r_2}{3r_1} - \frac{2^{\frac{1}{3}}\varpi_1}{3r_1\varpi_3^{\frac{1}{3}}} + \frac{\varpi_3^{\frac{1}{3}}}{3 \times 2^{\frac{1}{3}} r_1}]. \tag{20}$$

Thus, the random observation y can be obtained from (20) for any baseline distribution $G_2(y)$ and different parameter choices.

4.5. The Dependence Measures

In this section, three important dependence measures are obtained for the bivariate cubic transmuted family of distributions are obtained. These dependence measures include Kendall's τ, Spearman's ρ, and local dependence measures. The results for the measures of dependence are derived in the following subsections.

4.5.1. Kendall's Tau Coefficient

Kendall's τ for the BCT family proposal in the following theorem:

Theorem 5. *Let X and Y be two random variables that have a bivariate cubic transmuted family of distributions. The Kendall's τ for the variables X and Y are then given as:*

$$\tau = \frac{5}{90}\left[(2\lambda_1 + \lambda_2)(2\lambda_3 + \lambda_4) - 3\lambda_5(15 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)\right]. \tag{21}$$

Proof. *The Kendall's The Kendall's τ coefficient for two continuous random variables is computed by using:*

$$\tau = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{X,Y}(x, y) f_{X,Y}(x, y)dxdy - 1,$$

Using (6) and (7), the above integral becomes

$$\begin{aligned}\tau = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&[G_1(x)G_2(y)[1+\lambda_1-\lambda_1G_1(x)+\lambda_2G_1(x)-\lambda_2G_1^2(x)\\&+\lambda_3-\lambda_3G_2(y)+\lambda_4G_2(y)-\lambda_4G_2^2(y)\\&+\lambda_5-\lambda_5G_1^2(x)-\lambda_5G_2^2(y)+\lambda_5G_1^2(x)G_2^2(y)]\\&\times g_1(x)g_2(y)[1+\lambda_1-2\lambda_1G_1(x)+2\lambda_2G_1(x)-3\lambda_2G_1^2(x)\\&+\lambda_3-2\lambda_3G_2(y)+2\lambda_4G_2(y)-3\lambda_4G_2^2(y)+\lambda_5\\&-3\lambda_5G_1^2(x)-3\lambda_5G_2^2(y)+9\lambda_5G_1^2(x)G_2^2(y)]]dxdy-1,\end{aligned}$$

Making the transformation $u = G_1(x)$ and $v = G_2(y)$, we then interpret as:

$$\begin{aligned}\tau = 4\int_0^1\int_0^1&\Big[\big[uv(1+\lambda_1+\lambda_3+\lambda_5)+(\lambda_2-\lambda_1)u^2v-(\lambda_2+\lambda_5)u^3v\\&-(\lambda_3-\lambda_4)uv^2-(\lambda_4+\lambda_5)uv^3+\lambda_5u^3v^3\big]\\&\times\Big[(1+\lambda_1+\lambda_3+\lambda_5)+2(\lambda_2-\lambda_1)u-3(\lambda_2+\lambda_5)u^2\\&-2(\lambda_3-\lambda_4)v-3(\lambda_4+\lambda_5)v^2+9\lambda_5u^2v^2\Big]\Big]dudv-1.\end{aligned}$$

In simplification, the above integral reduces to (21), and hence the proof is complete.

4.5.2. Spearman's ρ

Spearman's ρ is another measure of dependence between two variables. The Spearman coefficient ρ for the bivariate cubic transmuted family of distributions in the following theorem

Theorem 6. *Let X and Y be two random variables having a bivariate cubic transmuted family of distributions then the Spearman's ρ for X and Y*

$$\rho = -\frac{1}{300}\big[25(2\lambda_1+\lambda_2)(2\lambda_3+\lambda_4)-\lambda_5(15-\lambda_1+\lambda_2)(15-\lambda_3+\lambda_4)\big], \quad (22)$$

Proof. *The Spearman's ρ for two continuous random variables is obtained as:*

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_{X,Y}(x,y)-F_X(x)F_Y(y)]f_X(x)f_Y(y)dxdy.$$

Now, using (6), (8), (9), (10), (11) and making the transformation $u = G_1(x)$ and $v = G_2(y)$ the above equation becomes

$$\begin{aligned}\rho = 12\int_0^1\int_0^1&[[(uv+\lambda_1uv-\lambda_1u^2v+\lambda_2u^2v-\lambda_2u^3v\\&+\lambda_3uv-\lambda_3uv^2+\lambda_4uv^2-\lambda_4uv^3+\lambda_5uv-\lambda_5u^3v-\lambda_5uv^3+\lambda_5u^3v^3)\\&-(uv+\lambda_1uv-\lambda_1u^2v+\lambda_2u^2v-\lambda_2u^3v)(1+\lambda_3-\lambda_3v+\lambda_4v-\lambda_4v^2)]\\&\times(1+\lambda_1-2\lambda_1u+2\lambda_2u-3\lambda_2u^2)(1+\lambda_3-2\lambda_3v+2\lambda_4v-3\lambda_4u^2)]dudv.\end{aligned}$$

which, on simplification, becomes (22) and hence the proof is complete.

It can be easily seen that Kendall's τ is always larger than Spearman's ρ for the bivariate cubic transmuted family of distributions.

4.6. The Local Dependence Measure

A local dependence measure defines the strength of the local association between two random variables X and Y. The bivariate local dependence function for the bivariate cubic transmuted family of distribution is the following theorem.

Theorem 7. *The local dependence function for the bivariate cubic transmuted family of distribution is*

$$\gamma(x,y) = \frac{(5184)\delta_2\delta_6\delta_4\delta_7\lambda_5 g_1(x)g_2(y)G_1(x)G_2(y)}{[\eta - +2\delta_2 G_1(x) - 3\delta_6 G_1^2(x) - 2\delta_4 G_2(y) - 3\delta_7 G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)]^2}, \quad (23)$$

where $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$, $\delta_2 = \lambda_2 - \lambda_1$, $\delta_6 = \lambda_2 + \lambda_5$, $\delta_4 = \lambda_3 - \lambda_4$, $\delta_7 = \lambda_4 + \lambda_5$
Proof. *The local bivariate dependence function for two continuous random variables has been defined by [15] as*

$$\gamma(x,y) = \frac{\partial^2}{\partial x \partial y} log f(x,y),$$

Using the bivariate density function, given in (7), the above equation becomes

$$\begin{aligned}\gamma(x,y) = \frac{\partial^2}{\partial x \partial y} log\big[g_1(x)g_2(y)[1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) \\ - 3(\lambda_2 + \lambda_5)G_1^2(x) - 2(\lambda_3 - \lambda_4)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y) \\ + 9\lambda_5 G_1^2(x)G_2^2(y)]\big].\end{aligned}$$

which, in simplification, becomes (23) and hence the proof is complete.

The local dependence function can be computed for given values of the parameters and different choices of baseline distributions.

5. Estimation of Parameters

In this section, the MLE of the parameters is discussed for the bivariate cubic transmuted family of distributions under the assumption that all the parameters of the baseline distributions $G_1(x)$ and $G_2(y)$ are known. Let $X_1, X_2, , X_n$ be a random sample of size n taken from the bivariate density in (7), then the corresponding log-likelihood function can be written as:

$$\begin{aligned}\ell = \sum_{i=1}^{n} log(g_1(x_i)) + \sum_{i=1}^{n} log(g_2(y_i)) + \sum_{i=1}^{n} log\big[1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x_i) \\ - 3(\lambda_2 + \lambda_5)G_1^2(x_i) - 2(\lambda_3 - \lambda_4)G_2(y_i) - 3(\lambda_4 + \lambda_5)G_2^2(y_i) + 9\lambda_5 G_1^2(x_i)G_2^2(y_i)\big].\end{aligned} \quad (24)$$

The MLEs of λ_1, λ_2, λ_3, λ_4 and λ_5 which maximize the log-likelihood function in (24) are obtained by simultaneous solution of the likelihood equations. Now, the derivatives of (24) writ the unknown parameters are

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2G_1(x_i)}{\Phi(x_i, y_i)}, \tag{25}$$

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2G_1(x_i) - 3G_1^2(x_i)}{\Phi(x_i, y_i))}, \tag{26}$$

$$\frac{\partial \ell}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{1 - 2G_2(y_i)}{\Phi(x_i, y_i))}, \tag{27}$$

$$\frac{\partial \ell}{\partial \lambda_4} = \sum_{i=1}^{n} \frac{2G_2(y_i) - 3G_2^2(y_i)}{\Phi(x_i, y_i)}, \tag{28}$$

and

$$\frac{\partial \ell}{\partial \lambda_5} = \sum_{i=1}^{n} \frac{1 - 3G_2^2(y_i) - 3G_1^2(x_i) + 9G_1^2(x_i)G_2^2(y_i)}{\Phi(x_i, y_i)}. \tag{29}$$

where $\Phi(x_i, y_i) = \big[1+\lambda_1+\lambda_3+\lambda_5+2(\lambda_2-\lambda_1)G_1(x_i)-3(\lambda_2+\lambda_5)G_1^2(x_i)-2(\lambda_3-\lambda_4)G_2(y_i)-3(\lambda_4+\lambda_5)G_2^2(y_i)+9\lambda_5 G_1^2(x_i)G_2^2(y_i)\big]$.
The MLE of λ_1,λ_2, λ_3, λ_4 and λ_5 is obtained by equating (25), (26),(27), (28) and (29) to zero by solving the resulting equations numerically.

6. The Bivariate Cubic Transmuted Burr Distribution

In this section, the bivariate cubic transmuted Burr distribution (BCTB) is proposed by using the distribution function of Burr-**XII** distribution as the baseline distribution in the bivariate cubic transmuted family of distributions, given in (6)
Suppose that the two random variables X and Y have the Burr distributions with the following CDF:

$$G_1(x) = 1 - (1 + x^{a_1})^{-b_1}; x, a_1, b_1 \geq 0, \tag{30}$$

and

$$G_2(y) = 1 - (1 + y^{a_2})^{-b_2}; y, a_2, b_2 \geq 0. \tag{31}$$

Using the above CDF in (6), the distribution function of the BCTB distribution proposed is:

$$\begin{aligned} F_{BCTB}(x, y) = TxTy\Big[&1 + \lambda_1 + \lambda_3 + \lambda_5 + (\lambda_2 - \lambda_1)Tx - (\lambda_2 + \lambda_5)Tx^2 - (\lambda_3 - \lambda_4)Ty \\ &- (\lambda_4 + \lambda_5)Ty^2 + \lambda_5 Tx^2 Ty^2\Big], \end{aligned} \tag{32}$$

where $Tx = 1 - (1 + x^{a_1})^{-b_1}$ and $Ty = 1 - (1 + y^{a_2})^{-b_2}$.

In Figure (1), the CDF of the BCTB distribution increases monotonically, with higher values appearing in the upper right corners of each graph. The surface curvature varies across the plots, reflecting the influence of the parameters on the dependence structure and shape of the distribution.

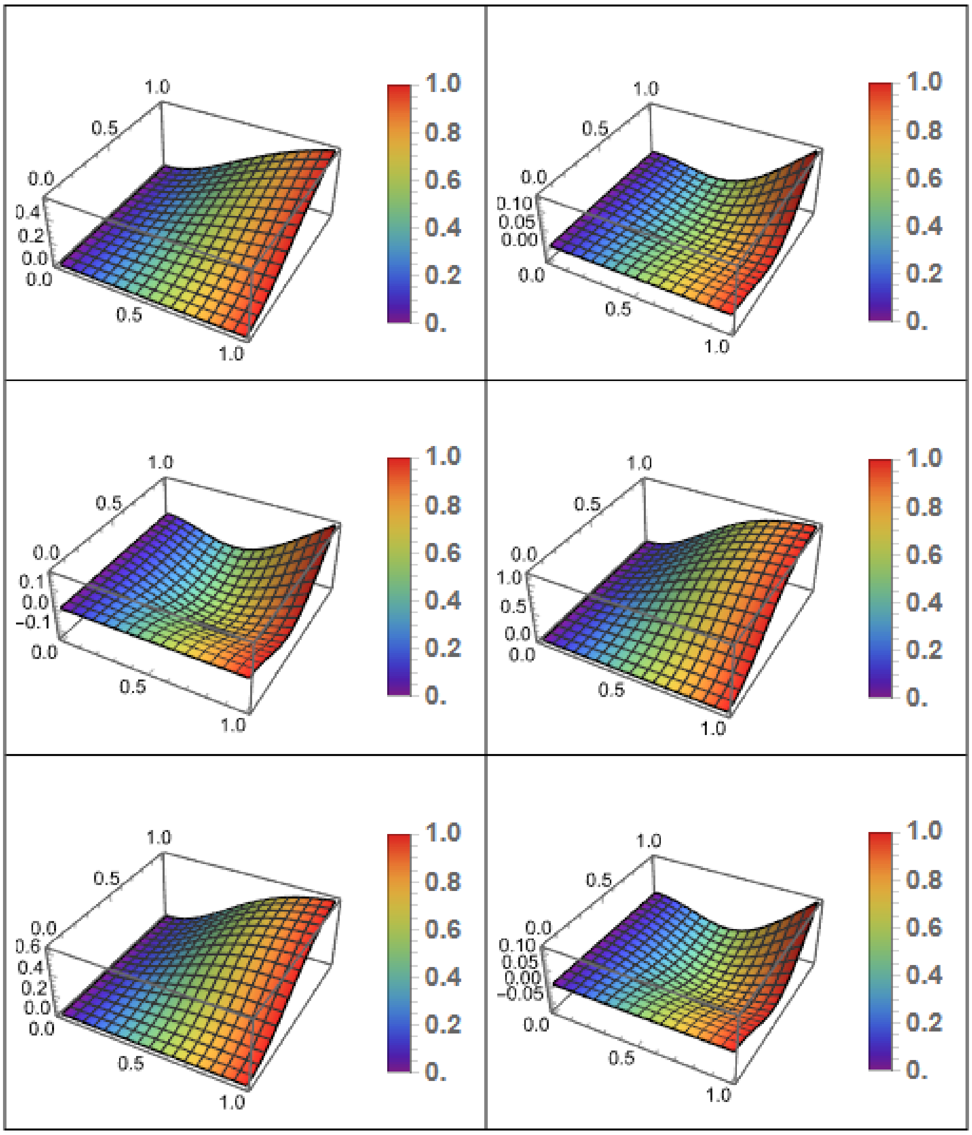

Figure 1: The CDF of BCTB distribution

for x, y, a_1, a_2, b_1, $b_2 > 0$. The density function of the BCTB distribution corresponding to (32) is obtained by using the following PDF of Burr distribution

$$g_1(x) = a_1 b_1 x^{a_1-1}(1+x^{a_1})^{-(b_1+1)}; x, a_1, b_1 \geq 0, \tag{33}$$

and

$$g_2(y) = a_2 b_2 y^{a_2-1}(1+y^{a_2})^{-(b_2+1)}; y, a_2, b_2 \geq 0. \tag{34}$$

in (7), the joint density function of the proposed BCTB distribution is given as

$$\begin{aligned} f_{BCTB}(x,y) = a_1 a_2 b_1 b_2 x^{a_1-1} y^{a_2-1}(1+x^{a_1})^{-(b_1+1)}(1+y^{a_2})^{-(b_2+1)}\Big[1+\lambda_1+\lambda_3+\lambda_5 \\ + 2(\lambda_2-\lambda_1)Tx - 3(\lambda_2+\lambda_5)Tx^2 - 2(\lambda_3-\lambda_4)Ty - 3(\lambda_4+\lambda_5)Ty^2 \\ + 9\lambda_5 Tx^2 Ty^2\Big], \end{aligned} \tag{35}$$

where $(b_1, b_2) > 0$, are the scale parameters, $(a_1, a_2) > 0$ are the shape parameters, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ are the transmutation parameters such that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in [-1, 1]$ under these conditions:$-1 \leq 1+\lambda_1+\lambda_3+\lambda_5 \leq 1$,$-1 \leq \lambda_2-\lambda_1 \leq 1$,$-1 \leq \lambda_2+\lambda_5 \leq 1$, $-1 \leq \lambda_3-\lambda_4 \leq 1$ and $-1 \leq \lambda_4+\lambda_5 \leq 1$.

In Figure (2), PDF plot of the BCTB distribution is shown. Among the six plots in Figure (2), some show centrally located peaks, indicating symmetric or balanced distributions. Other plots exhibit off-centered peaks or more pronounced slopes, reflecting skewed distributions or stronger interactions between the two variables.

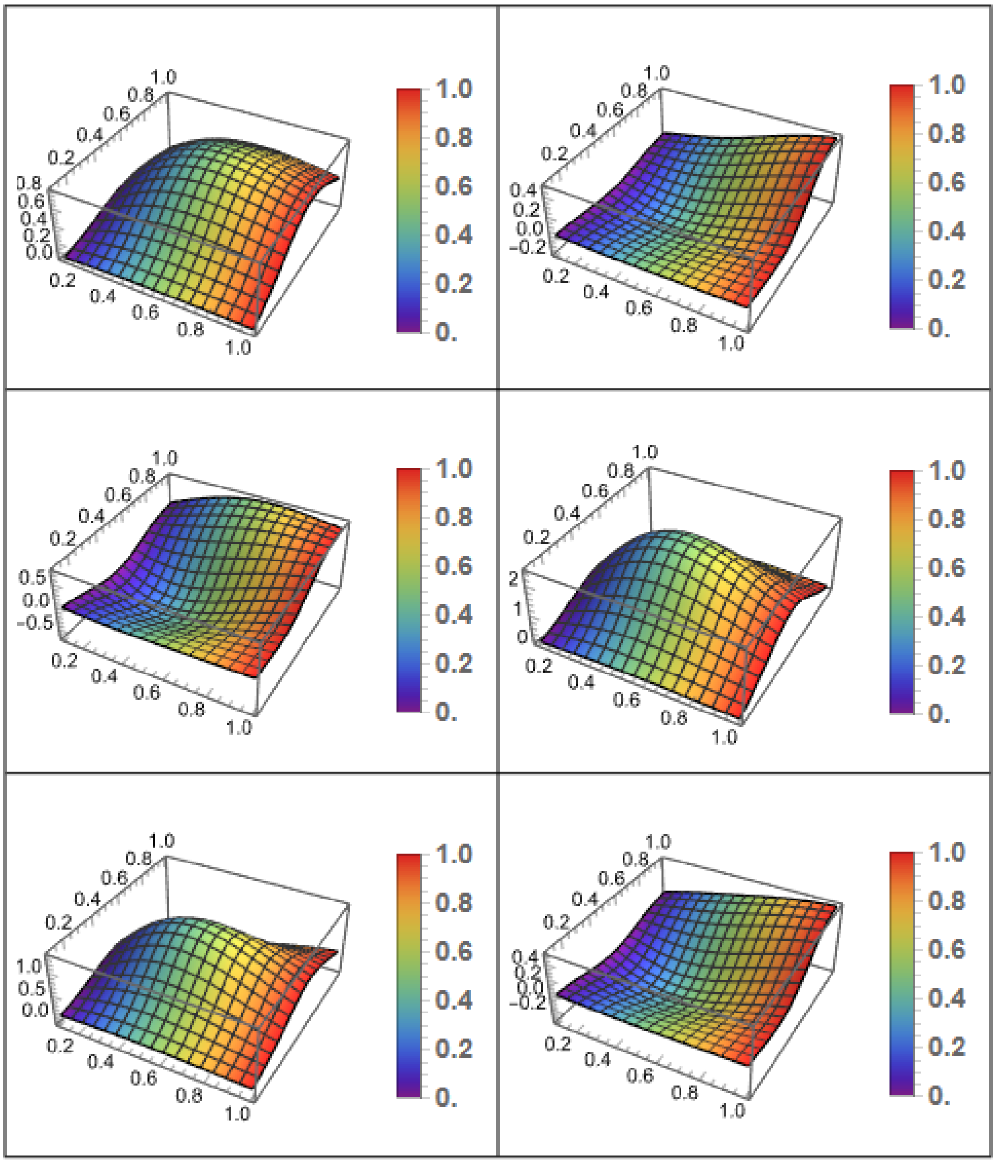

Figure 2: The PDF of BCTB distribution

The following special cases can be immediately obtained from the distribution and density functions given in (32) and (35):

1. we can obtain the bivariate cubic transmuted Lomax distribution (BCTL for short) by setting $a_1 = a_2 = 1$ in (32) and (35).
2. we can obtain the bivariate cubic transmuted log-logistic distribution (BCTLog-L for short) by setting $b_1 = b_2 = 1$ in (32) and (35).

The marginal CDF of X for BCTB distribution is obtained by using (30) in (8), as:

$$F_{MCT}(x) = Tx\Big[1 - \lambda_2(1 + x^{a_1})^{-2b_1} + (\lambda_1 + \lambda_2)(1 + x^{a_1})^{-b_1}\Big].$$

Now using (31) in (9). The marginal CDFs of the random variable Y for the BCTB distribution is

$$F_{MCT}(y) = Ty\Big[1 - \lambda_4(1 + y^{a_2})^{-2b_2} + (\lambda_3 + \lambda_4)(1 + y^{a_2})^{-b_2}\Big].$$

Using the PDF of Burr distribution (33) and (30) in (10), the marginal density function of X for the BCTB distribution is

$$f_X(x) = \frac{a_1 b_1 x^{a_1-1}}{(1 + x^{a_1})^{(b_1+1)}}[1 + \lambda_1 + 2Tx(\lambda_2 - \lambda_1) - 3Tx^2\lambda_2, \tag{36}$$

Again, the marginal PDF of Y for the BCTB distribution are obtained by using (34) and (31) in (11), and is given as

$$f_Y(y) = \frac{a_2 b_2 y^{a_2-1}}{(1 + y^{a_2})^{(b_2+1)}}[1 + \lambda_3 - 2Ty(\lambda_3 - \lambda_4) - 3Ty^2\lambda_4]. \tag{37}$$

The conditional PDF of the random variable X given $Y = y$ is obtained using (35) and (37)in (12). The conditional distribution of X given Y = y for the BCTB distribution is

$$\begin{aligned} f_{BCTB}(x|y) = \frac{a_1 b_1 x^{a_1-1}}{(1 + x^{a_1})^{(b_1+1)}\Delta_{BCTB}(y)}&[1 + \lambda_1 + \lambda_3 + \lambda_5 \\ &+ 2Tx(\lambda_2 - \lambda_1) - 3Tx^2(\lambda_2 + \lambda_5) - 2Ty(\lambda_3 - \lambda_4) - 3Ty^2(\lambda_4 + \lambda_5) \\ &+ 9\lambda_5 Tx^2 Ty^2], \end{aligned}$$

where

$$\Delta_{BCTB}(y) = [1 + \lambda_3 - 2Ty(\lambda_3 - \lambda_4) - 3Ty^2\lambda_4],$$

and The conditional distribution of Y given X = x for BCTB distribution is obtained by using (35) and (36) in (13) and is

$$\begin{aligned} f_{BCTB}(y|x) = \frac{a_2 b_2 y^{a_2-1}}{(1 + y^{a_2})^{(b_2+1)}\Delta_{BCTB}(x)}&\Big[1 + \lambda_1 + \lambda_3 + \lambda_5 \\ &+ 2Tx(\lambda_2 - \lambda_1) - 3Tx^2(\lambda_2 + \lambda_5) \\ &- 2Ty(\lambda_3 - \lambda_4) - 3Ty^2(\lambda_4 + \lambda_5) + 9\lambda_5 Ty^2 Tx^2\Big], \end{aligned}$$

where

$$\Delta_{BCTB}(x) = [1 + \lambda_1 - 2(\lambda_1 - \lambda_2)Tx - 3\lambda_2 Tx^2].$$

7. Statistical Properties

This section discusses some important distributional properties of the BCTB distribution and the special distributions obtained from the proposed BCTB distribution.

7.1. The Conditional Moments

In the following, conditional moment r^{th} for the BCTB distribution is obtained using the raw momen r^{th} and the r^{th} moment of the second and third order statistics in a sample of sizes 2 and 3 for the BurrXII distribution is given in the following result.

Result: For a fixed positive integer r r^{th} raw moment of a Burr random variable, Z, with two positive shape parameters, namely a , b and r^{th} raw moment of its second-order statistics in a sample of size 2 and 3 are, respectively, given as

$$\mu_Z^r = E(Z^r) = bB(1 + \frac{r}{a}, b - \frac{r}{a}), \tag{38}$$

$$\mu_{Z(2:2)}^r = E(Z_{(2:2)}^r) = 2bB(1 + \frac{r}{a}, b - \frac{r}{a}) - 2bB(1 + \frac{r}{a}, 2b - \frac{r}{a}) \tag{39}$$

, and

$$\mu_{Z(3:3)}^r = E(Z_{(3:3)}^r) = 6bB(1 + \frac{r}{a}, b - \frac{r}{a}) - 12bB(1 + \frac{r}{a}, 2b - \frac{r}{a}) + 6bB(1 + \frac{r}{a}, 3b - \frac{r}{a}), \tag{40}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the complete beta function, μ_Z^r represents the r^{th} raw moment (also known as the non-central moment) of the random variable Z, $\mu_{Z(2:2)}^r$ and $\mu_{Z(3:3)}^r$ the r^{th} raw moment of the largest order statistic in a sample of size 2 and 3 from the distribution of Z.
The conditional moment r^{th} for the BCTB distribution is derived in the following theorem.

Theorem 8. *If random variables X and Y have joint BCTB distribution then the r^th conditional moment of X given $Y = y$ is*

$$\begin{aligned} \mu_{X|y}^r = \frac{b_1}{\Phi_{BCTB}(y)} \Bigg[& A_1 \cdot B\left(1 + \frac{r}{a_1}, b_1 - \frac{r}{a_1}\right) + A_2 \cdot B\left(1 + \frac{r}{a_1}, 2b_1 - \frac{r}{a_1}\right) \\ & + A_3 \cdot B\left(1 + \frac{r}{a_1}, 3b_1 - \frac{r}{a_1}\right) \Bigg], \end{aligned} \tag{41}$$

where:

$$A_1 = \eta + 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y) + 2(\lambda_2 - \lambda_1) - 6(\lambda_2 + \lambda_5 - 3\lambda_5 G_2^2(y)),$$

$$A_2 = -2(\lambda_2 - \lambda_1) + 12(\lambda_2 + \lambda_5 - 3\lambda_5 G_2^2(y)),$$

$$A_3 = -6(\lambda_2 + \lambda_5 - 3\lambda_5 G_2^2(y)).$$

Here, $B(\cdot,\cdot)$ is the Beta function and $\Phi_{BCTB}(y) = (1 - \lambda_3 - \lambda_4 + 2(2\lambda_4 + \lambda_3)(1 + y^{a_2})^{-b_2} - 3\lambda_4(1 + y^{a_2})^{-2b_2}$ is the normalization constant.

Proof. The r^{th} conditional moment of X given $Y = y$ when X and Y have the bivariate cubic transmuted family of distributions. are given in (14) as

$$\mu^r_{X|y} = E(X^r/y) = \frac{1}{\Phi(y)}\Big[\big(\eta + 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y)\big)\mu^r_x + (\lambda_2 - \lambda_1)\mu^r_{x(2:2)} - \big(\lambda_2 + \lambda_5 - 3\lambda_5 G_2^2(y)\big)\mu^r_{x(3:3)}\Big],$$

where $\Phi(y) = (1 + \lambda_3) + 2(\lambda_4 - \lambda_3)G_2(y) - 3\lambda_4 G_2^2(y)$, $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$. Now, for the BCTB distribution, the random variable X is Burr(a_1, b_1) and the random variable Y is Burr(a_2, b_2) hence, using (38) and (39), $\mu^r_{x(2:2)}, \mu^r_{x(3:3)}$the r^{th} raw moment of X and r^{th} raw moment of larger observation in a sample of size 2 and 3 for X are, respectively, given as

$$\mu^r_x = E(X^r) = b_1 B(1 + \frac{r}{a_1}, b_1 - \frac{r}{a1}) \tag{42}$$

,

$$\mu^r_{x(2:2)} = E(X^r_{(2:2)}) = 2b_1 B(1 + \frac{r}{a1}, b_1 - \frac{r}{a1}) - 2b_1 B(1 + \frac{r}{a1}, 2b_1 - \frac{r}{a1}) \tag{43}$$

and

$$\mu^r_{x(3:3)} = E(X^r_{(3:3)}) = 6b_1 B(1 + \frac{r}{a1}, b_1 - \frac{r}{a1}) - 12b_1 B(1 + \frac{r}{a1}, 2b_1 - \frac{r}{a1}) + 6b_1 B(1 + \frac{r}{a1}, 3b_1 - \frac{r}{a1}). \tag{44}$$

Furthermore, $G_2(y)$ is given in (31) and hence $\Phi_{BCTB}(y) = (1 - \lambda_3 - \lambda_4 + 2(2\lambda_4 + \lambda_3)(1 + y^{a_2})^{-b_2} - 3\lambda_4(1 + y^{a_2})^{-2b_2}$.
Using (31), (42),(43) and (44) in (14), the conditional moment r^{th} of the random variable X given $Y = y$ for the BCTB distribution and for simplification becomes (41) and hence the proof is complete.

7.2. Random Number Generation

The random sample from the BCTB distribution is generated using the following two steps.

Step 1: Using the inverse of distribution function of Burr distribution (30) in (18), the random observation for x is obtained as

$$x = [(1 - u_1)^{-\frac{1}{b_1}} - 1]^{\frac{1}{a_1}}, \tag{45}$$

where

$$u_1 = -\frac{t_2}{3t_1} - \frac{2^{\frac{1}{3}}\zeta_1}{3t_1\zeta_3^{\frac{1}{3}}} + \frac{\zeta_3^{\frac{1}{3}}}{3\times 2^{\frac{1}{3}}t_1},$$

where $t_1 = \lambda_2$, $t_2 = -\delta_2$, $t_3 = -\delta_1$ and ζ_1, ζ_2, ζ_3, δ_1, δ_2 are defined earlier. Random sample X of the BCTB distribution can be generated using (45) for various parameters choices a_1, b_1, λ_1, and λ_2.

Step 2: Using the inverse distribution function of Burr distributions (31) in (20), the random observation of Y from the bivariate transmuted Burr distribution is generated as

$$y = [(1-u_2)^{-\frac{1}{b_2}} - 1]^{\frac{1}{a_2}}, \tag{46}$$

where

$$u_2 = -\frac{r_2}{3r_1} - \frac{2^{\frac{1}{3}}\varpi_1}{3r_1\varpi_3^{\frac{1}{3}}} + \frac{\varpi_3^{\frac{1}{3}}}{3\times 2^{\frac{1}{3}}r_1},$$

where r_1, and r_2, ϖ_1, ϖ_3 are defined earlier. The random sample for Y from the BCTB distribution can be obtained using (46).

8. Estimation of Parameters

let $X_1, X_2, , X_n$ be a random sample of size n be taken from the joint density function of the BCTB distribution in (35), then the corresponding log-likelihood function for BCTB distribution can be written as:

$$\begin{aligned} \ell &= n\cdot log(a_1) + n\cdot log(a_2) + n\cdot log(b_1) + n\cdot log(b_2) + (a_1-1)\sum_{i=1}^{n} log(x_i) \\ &+ (a_2-1)\sum_{i=1}^{n} log(y_i) - (b_1+1)\sum_{i=1}^{n} log(1+x_i^{a_1}) - (b_2+1)\sum_{i=1}^{n} log(1+y_i^{a_2}) \\ &+ \sum_{i=1}^{n} log\Big[1+\lambda_1+\lambda_3+\lambda_5+2(\lambda_2-\lambda_1)Tx_i - 3(\lambda_2+\lambda_5)Tx_i^2 - 2(\lambda_3-\lambda_4)Ty_i \\ &- 3(\lambda_4+\lambda_5)Ty_i^2 + 9\lambda_5 Tx_i^2 Ty_i^2\Big]. \end{aligned} \tag{47}$$

The MLE of $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$and λ_5 are obtained by maximizing the log likelihood function (47). The derivatives of (47) with respect to unknown parameters are

$$\begin{aligned} \frac{\partial \ell}{\partial a_1} &= \frac{n}{a_1} + \sum_{i=1}^{n} log(x_i) - (b_1+1)\sum_{i=1}^{n}\frac{x_i^{a_1} log(x_i)}{1+x_i^{a_1}} \\ &+ \sum_{i=1}^{n}\frac{2b_1 x_i^{a1}(1+x_i^{a1})^{-1-b_1} log(x_i)\left[\delta_2 - 3\delta_6 Tx_i + 9\lambda_5 Tx_i Ty_i^2\right]}{\delta(x_i, y_i)}, \end{aligned} \tag{48}$$

$$\frac{\partial \ell}{\partial a_2} = \frac{n}{a_2} + \sum_{i=1}^{n} log(y_i) - (b_2+1)\sum_{i=1}^{n} \frac{y_i^{a_2} log(y_i)}{1+y_i^{a_2}} + \sum_{i=1}^{n} \frac{-2b_2 y_i^{a2}(1+y_i^{a2})^{-1-b_2} log(y_i)\left[\delta_4 + 3\delta_7 Ty_i - 9\lambda_5 Tx_i^2 Ty_i\right]}{\delta(x_i,y_i)}, \tag{49}$$

$$\frac{\partial \ell}{\partial b_1} = \frac{n}{b_1} - \sum_{i=1}^{n} log(1+x_i^{a_1}) + \sum_{i=1}^{n} \frac{2(1+x_i^{a_1})^{-b_1} log(1+x_i^{a_1})\left[\delta_2 - 3\delta_6 Tx_i + 9\lambda_5 Tx_i Ty_i^2\right]}{\delta(x_i,y_i)}, \tag{50}$$

$$\frac{\partial \ell}{\partial b_2} = \frac{n}{b_2} - \sum_{i=1}^{n} log(1+y_i^{a_2}) + \sum_{i=1}^{n} \frac{2(1+y_i^{a_2})^{-b_2} log(1+y_i^{a_2})\left[\delta_4 + 3\delta_7 Ty_i - 9\lambda_5 Tx_i^2 Ty_i\right]}{\delta(x_i,y_i)}, \tag{51}$$

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1-2Tx_i}{\delta(x_i,y_i)}, \tag{52}$$

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2Tx_i - 3Tx_i^2}{\delta(x_i,y_i)}, \tag{53}$$

$$\frac{\partial \ell}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{1-2Ty_i}{\delta(x_i,y_i)}, \tag{54}$$

$$\frac{\partial \ell}{\partial \lambda_4} = \sum_{i=1}^{n} \frac{2Ty_i - 3Ty_i^2}{\delta(x_i,y_i)}, \tag{55}$$

$$\frac{\partial \ell}{\partial \lambda_5} = \sum_{i=1}^{n} \frac{1-3Tx_i^2 - 3Ty_i^2 + 9Tx_i^2 Ty_i^2}{\delta(x_i,y_i)}, \tag{56}$$

where $\delta(x_i,y_i) = \left[1+\lambda_1+\lambda_3+\lambda_5+2Tx_i(\lambda_2-\lambda_1) - 3(\lambda_2+\lambda_5)Tx_i^2 - 2(\lambda_3-\lambda_4)Ty_i - 3(\lambda_4+\lambda_5)Ty_i^2 + 9\lambda_5 Tx_i^2 Ty_i^2\right]$.
The corresponding MLE of the parameter vector $\Theta = (a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is obtained by equating the above derivatives to zero and solving the resulting equations numerically.

9. Simulation and Numerical Studies

In this section, a simulation study is carried out to observe the performance of the MLE procedure. In addition, the BCTB distribution has been applied to real-life data sets to investigate its applicability.

9.1. Simulation Study

This subsection, the method of the MLE for the parameters of the BCTB distribution parameters is obtained through a Monte Carlo simulation study using the R-package "max-Lik" and "stats4". The sample size n=20, 50, 100 and 200 is considered, and m the number of samples is set to 10000. The ML estimates of the parameters $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are obtained numerically as the following ste:

1. The ML estimates of parameters$a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are computed by solving the system of non-linear equations from (48) to (56), simultaneously.

2. The bias and MSE of the estimates are calculated using the following:

$$Bias(\hat{\Theta}) = \bar{\hat{\Theta}} - \Theta,$$

$$MSE(\hat{\Theta}) = Var(\hat{\Theta}) + (Bias(\hat{\Theta}))^2,$$

   where $\Theta = (a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, $\hat{\Theta} = (\hat{a_1}, \hat{a_2}, \hat{b_1}, \hat{b_2}, \hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_3}, \hat{\lambda_4}, \hat{\lambda_5})$ and $\bar{\hat{\Theta}}$ is the mean of Θ.

   It is clear from the summarized results of the simulation study for different parameters of Table (1) that when the sample size increases, the estimates of all parameters improve. That is, the bias and the MSE width decrease as the sample sizes increase.

Table 1: Simulation Results for the BCTB Distribution

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
$a_1 = 2.5$	2.5001	0.0183	0.0130	2.5002	0.0366	0.0136	2.4998	-0.0366	0.0134	2.5001	0.0183	0.0119
$a_2 = 1.25$	1.2501	0.0183	0.0131	1.2498	-0.0366	0.0159	1.2500	0.0000	0.0063	1.2500	0.0000	0.0137
$b_1 = 3.0$	2.9997	-0.0549	0.0141	3.0001	0.0183	0.0119	3.0000	0.0000	0.0066	3.0000	0.0000	0.0099
$b_2 = 2.0$	2.0000	0.0000	0.0083	1.9999	-0.0183	0.0091	2.0002	0.0366	0.0131	1.9999	-0.0183	0.0090
$\lambda_1 = 0.1$	0.1000	0.0000	0.0057	0.1000	0.0000	0.0058	0.1000	0.0000	0.0058	0.1000	0.0000	0.0058
$\lambda_2 = 0.2$	0.2000	0.0000	0.0116	0.2001	0.0183	0.0115	0.2002	0.0366	0.0115	0.2000	0.0000	0.0116
$\lambda_3 = 0.25$	0.2498	-0.0366	0.0145	0.2499	-0.0183	0.0144	0.2501	0.0183	0.0145	0.2499	-0.0183	0.0143
$\lambda_4 = -0.15$	-0.1499	0.0183	0.0086	-0.1500	0.0000	0.0086	-0.1501	-0.0183	0.0087	-0.1500	0.0000	0.0086
$\lambda_5 = 0.15$	0.1500	0.0000	0.0087	0.1501	0.0183	0.0087	0.1501	0.0366	0.008	0.1501	0.0183	0.0086

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
$a_1 = 3.0$	3.0001	0.0183	0.0119	3.0000	0.0000	0.0144	3.0001	0.0183	0.0072	3.0001	0.0183	0.0072
$a_2 = 2.0$	2.0000	0.000	0.0099	2.0000	0.0000	0.0088	1.9999	-0.0183	0.0128	2.0002	0.0366	0.0107
$b_1 = 1.0$	0.9999	-0.0183	0.0074	1.0001	0.0183	0.0115	1.0000	0.0000	0.0124	1.0001	0.0183	0.0087
$b_2 = 1.5$	1.5000	0.0000	0.0076	1.5000	0.0000	0.0106	1.4999	-0.0183	0.0137	1.4999	-0.0183	0.0113
$\lambda_1 = 0.01$	0.0100	0.0000	0.0006	0.0100	0.0000	0.0006	0.0100	0.0000	0.0006	0.0100	0.0000	0.0006
$\lambda_2 = 0.04$	0.0400	0.0000	0.0023	0.0400	0.0000	0.0023	0.0400	0.0000	0.0023	0.0400	0.0000	0.0023
$\lambda_3 = 0.05$	0.0500	0.0000	0.0029	0.0500	0.0000	0.0029	0.0500	0.0000	0.0029	0.0500	0.0000	0.0029
$\lambda_4 = -0.25$	-0.2499	0.018	0.0144	-0.2501	-0.0183	0.0144	-0.2499	0.0183	0.0144	-0.2503	-0.0549	0.0145
$\lambda_5 = 0.03$	0.0300	0.0000	0.0017	0.0300	0.0000	0.0017	0.0300	0.0000	0.0017	0.0300	0.0000	0.0017

9.2. Real Data Applications

In the following, the BCTB distribution is applied to model a real data set. The data set is analyzed using the proposed BCTB distribution along with several other distributions. The “bbmle” R package is used for most numerical computations. To perform a comparative analysis among the suggested models and to evaluate the goodness of fit, we compute AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion). Recall that smaller values of these selection criteria indicate better goodness-of-fit for the distribution. We have applied the BCTB distribution to model a real data set. The data were modeled using the BCTB distribution proposed for the other distributions. The other distributions that we have used in the study are the bivariate Burr (BB) distribution by [16], the Gumbel bivariate Burr (GBB) and the bivariate transmuted Burr (BTB) distribution by [13]. To check the applicability, real-life application have been conducted for the proposed BCTB distribution, which are described by the following data set. A data set with GNI for all countries from 2016 (X) and 2017 (Y) is useful for analyzing economic growth trends, understanding disparities, and conducting cross-country comparisons.
Table (2) gives some descriptive statistics of the data.

Table 2: Summary Statistics for Dataset

	Min.	Q_1	Median	Mean	Q_3	Max.
X	0.0644	0.3839	1.1118	1.773906	2.43105	11.8088
Y	0.0663	0.39695	1.11	1.798834	2.51975	11.6818

The results of MLE and SE are listed in Table (3) for the BCTB distribution and the BTB distribution. From Table (4), we can see that the proposed BCTB distribution has the smallest AIC and BIC values, and therefore is considered the best fit for the data based on goodness-of-fit criteria.

Table 3: MLEs and SEs for selected Distribution

Distribution	Parameter	Estimate	SE
BCTB	a_1	1.862	0.04113
	a_2	1.837	0.04218
	b_1	1.811	0.03470
	b_2	2.066	0.05780
	λ_1	-0.2454	0.06176
	λ_2	-0.2483	0.06157
	λ_3	0.002906	0.1127
	λ_4	0.009309	0.09424
	λ_5	-0.3706	0.00001056
BTB	k	9.8129	8.3886
	c	0.8827	8.3886
	λ_1	-0.1280	5.6453
	λ_2	-0.1280	14.5339
	λ_3	-0.8720	8.3886
BB	P	0.04161	0.01231
	a_1	0.009436	0.01974
	a_2	0.0002489	0.0005095
	B_1	9.860	2.029
	B_2	12.88	2.649
GBB	K_1	0.00000016335	0.0007430
	K_2	0.0000001892	0.0008853
	c_1	0.07567	0.05968
	c_2	0.06122	0.06796
	γ	1.000	0.2391

Table 4: Selection Criteria for Selected Distributions

Distribution	LogLik	AIC	BIC
BCTB	5272.72	-10527.44	-10498.17
BTB	78.8980	-143.7970	-175.0395
BB	-473.7103	957.4205	968.6631
GBB	48.5202	-87.0404	-118.2829

10. Conclusions

In this paper, we construct a novel family of distributions, named the Bivariate Cubic Transmuted (BCT) family of distributions. This family introduced a new approach to modeling and analyzing multivariate data with greater flexibility and precision. Explicit mathematical expressions for the probability density function (PDF), the cumulative distribution function (CDF), and key statistical properties of the proposed family are rigorously derived. General parameter estimation techniques for the BCT family are provided using the maximum likelihood estimation (MLE) method, ensuring accurate parameter inference.

As a specific case, we introduced and thoroughly investigated the Bivariate Cubic Transmuted Burr (BCTB) distribution, a member of this family. We established and explored its theoretical properties, including marginal distributions, dependence structure, and moments. Furthermore, the practical utility of the BCTB distribution is demonstrated through its application to a real dataset. Comparative analyzes reveal that the BCTB distribution offers superior fit and performance compared to existing models, making it a robust tool for modeling bivariate data in various fields.

References

[1] Ahmad Alzaghal, Felix Famoye, and Carl Lee. Exponentiated T-X family of distributions with some applications. *International Journal of Statistics and probability*, 2(3):31, 2013.

[2] William T Shaw and Ian RC Buckley. The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv preprint arXiv:0901.0434*, 2009.

[3] Gokarna Aryal and Chris Tsokos. Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of pure and applied mathematics*, 4(2):89–102, 2011.

[4] Faton Merovci and Lukan Puka. Transmuted Pareto distribution. In *ProbStat Forum*, volume 7, pages 1–11, 2014.

[5] Kareema Abed AL-Kadim and Maysaa Mohammed. The cubic transmuted Weibull distribution. *Journal of University of Babylon*, 3:862–876, 2017.

[6] Md Mahabubur Rahman, Bander Al-Zahrani, and Muhammad Qaiser Shahbaz. A general transmuted family of distributions. *Pakistan Journal of Statistics and Operation Research*, pages 451–469, 2018.

[7] Md Mahabubur Rahman, Bander Al-Zahrani, and Muhammad Qaiser Shahbaz. Cubic transmuted Weibull distribution: properties and applications. *Annals of Data Science*, 6(1):83–102, 2019.

[8] Md Mahabubur Rahman, Jumanah Ahmed Darwish, Saman Hanif Shahbaz, GG Hamedani, and Muhammad Qaiser Shahbaz. A new cubic transmuted log-logistic

distribution: Properties, applications, and characterizations. *Advances and Applications in Statistics*, 91(3):335–361, 2024.

[9] Md Mahabubur Rahman. Cubic transmuted Rayleigh distribution:\\theory and application. *Austrian Journal of Statistics*, 51(3):164–177, 2022.

[10] Ayman Alzaatreh, Carl Lee, and Felix Famoye. A new method for generating families of continuous distributions. *Metron*, 71(1):63–79, 2013.

[11] MS Eliwa and M El-Morshedy. Bivariate gumbel-g family of distributions: statistical properties, bayesian and non-bayesian estimation with application. *Annals of data science*, 6(1):39–60, 2019.

[12] Md Mahabubur Rahman, Bander Al-Zahrani, Saman Shahbaz, and Muhammad Qaiser Shahbaz. Transmuted probability distributions: A review. *Pakistan Journal of Statistics and Operation Research*, pages 83–94, 2020.

[13] Jumanah Darwish, Lutfiah Al Turk, and Muhammad Qaiser Shahbaz. The bivariate transmuted family of distributions: Theory and applications. *Computer Systems Science and Engineering*, 36(1):131–144, 2021.

[14] M. Ganji, H. Bevrani, and N.H Golzar. A new method for generating continuous bivariate distribution families. *Journal of The Iranian Statistical Society*, 17(1):109–129, 2018.

[15] Paul Holland and Yuchung Wang. Dependence function for continuous bivariate densities. *Communications in Statistics-Theory and Methods*, 16(3):863–876, 1987.

[16] Frederick Durling. The bivariate burr distribution. In *A Modern Course on Statistical Distributions in Scientific Work: Volume 1—Models and Structures Proceedings of the NATO Advanced Study Institute held at the University of Calgagry, Calgary, Alberta, Canada July 29–August 10, 1974*, pages 329–335. Springer, 1975.