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Sixteenth-Order Steffensen-Ostrowski Approach for Nonlinear Problems with Applications in Celestial, Predator-Prey and Neural Activation

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Abstract. The increasing demand for accurate and efficient solutions to nonlinear equations, driven by advancements across diverse research and engineering fields, highlights the critical need for innovative computational methods. This study addresses this need by introducing a novel derivative-free sixteenth-order iterative scheme derived from a weighted Steffensen-Ostrowski-type family. According to the Kung-Traub conjecture, this scheme is designed to achieve optimal convergence using only five function evaluations per iteration. A key innovation lies in employing a bivariate weight function in the third step and Lagrange interpolation in the fourth step, ensuring high accuracy and computational efficiency by avoiding derivative evaluation. The extensive convergence analysis shows that the proposed scheme is of sixteenth order and is validated through applications to real-world problems, including Kepler's celestial motion, an ideally mixed reactor, predator-prey models, neural activation dynamics, and periodic ecosystem growth. Numerical results demonstrate the superiority of the proposed scheme over existing four-point iterative schemes, particularly in terms of absolute error and computational convergence order. Furthermore, graphical analysis of complex polynomials illustrates the algorithm's attraction basins, offering a wide range of choosing from initial guesses to converge to the specified root more efficiently without divergence and hence showed more stable behavior. In this way we achieved significant computational improvements and higher convergence order. As the proposed scheme fall under the category of derivative free schemes, so it is more general as compared to existing schemes in literature and is considered to be a good alternative to the existing schemes especially where derivatives are unavailable.

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1. Introduction

The branch of computational mathematics that provides approximated solutions to a variety of mathematical problems is known as Numerical Analysis. These problems arise from practical applications in algebra, geometry, calculus, and span fields such as natural sciences, social sciences, engineering, medicine, and business. They often involve variables that change continuously. One of the key challenges in these fields is solving nonlinear equations, defined as:

$$g(k) = 0, \tag{1}$$

plays a significant role in computational and applied mathematics. Here, $g : D \subseteq \mathbb{C} \to \mathbb{C}$ is a sufficiently differentiable function on the interval D. Nonlinear equations in mathematics come in various forms, such as transcendental, integral, algebraic, and both ordinary and partial differential equations, often in combination. Analytic methods cannot solve these types of equations. Approximating the roots is a standard technique in numerical analysis. Numerical methods are key to tackling the complexity of nonlinear equations. A wealth of literature exists on solving these equations, including foundational works by Ostrowski [1], Kung and Traub [2], and Petković [3].

There are two widely used techniques for estimating the roots of a nonlinear algebraic equation: one-point and multipoint iteration. Newton's method was first presented by Thomas Simpson in 1740 as an iterative method of solving nonlinear equations. Newton's method, known for its one-point iterative approach, exhibits quadratic convergence. However, it has limitations, particularly its dependence on derivative evaluation and the necessity for a suitable initial guess to achieve root convergence. A recent suggestion to avoid introducing extra functions such as evaluating first or second derivatives, is to eliminate derivatives from the iteration process. As an example, this approach leads to the familiar Steffensen's method [2], where the forward difference approximation $\frac{g(k_j+g(k_j))-g(k_j)}{g(k_j)}$ substitutes the first-order derivative $g\prime(k_n)$ in Newton's method as follow:

$$k_{j+1} = k_j - \frac{g(k_j)}{g[k_j, v_j]},$$

where $v_j = k_j + g(k_j)$ and $g[k_j, v_j] = \frac{g(v_j)-g(k_j)}{v_j-k_j}$ is the divided difference of first order. Both the Steffensen approach and the Newton method acquire two function evaluations and converge quadratically. Kung and Traub's conjecture [2], posits that any without memory multipoint iterative method has convergence order cannot surpass the upper limit of 2^{n-1}, limited to n function evaluations. With advances in digital computers, arithmetic, and symbolic computing, higher-order multipoint methods have gained popularity. These methods offer more accurate and efficient root estimates in fewer iterations, with an efficiency index [4] superior to that of Newton's method. Techniques such as weight functions, Taylor expansions, and dynamical analysis have played a key role in developing both optimal and non-optimal multipoint methods with convergence orders up to eight [5–17]. Despite these advancements, achieving optimal four-point methods remains a significant challenge and an area of ongoing research. Some optimal sixteenth-order methods for solving nonlinear equations have been developed by several researchers (for example see, [18–25]). While most of these methods achieve high convergence orders, they often rely on derivative evaluations, which are computationally intensive and not always feasible for all problems.

These challanges inspired us to create a novel four-point optimal iterative approach based on the weighted family of Steffensen-Ostrowski type. It requires five functional evaluations and gives sixteenth-order convergence per iteration. The first two step of the suggested method is Ostrowski's method and third step is constructed by using the bivariate weight function. For the construction of the fourth step, Lagrange interpolation technique has been used. Lagrange interpolation is highly advantageous for its simplicity, derivative-free formulation, and flexibility in node selection, making it efficient for improving iterative methods for nonlinear equations.

This work aims to satisfy the Kung and Traub conjecture by achieving optimal sixteenth order convergence supported by an extensive convergence analysis. To evaluate its accuracy and effectiveness, the proposed approach is tested on real-world problems, including Kepler's equation of celestial motion, an ideally mixed reactor, predator-prey models, neural activation dynamics, and periodic ecosystem growth. Numerical results demonstrate the superiority of the proposed scheme over existing four-point iterative methods, particularly in terms of absolute error and convergence order. Additionally, graphical analysis of complex polynomials shows the algorithm's attraction basins. This analysis illustrating that it converges efficiently to the specified root from a wide range of initial guesses, with improved stability and minimum divergence. The proposed derivative-free scheme is more general than existing methods in the literature and offers a promising alternative in the cases where derivatives does not exist.

The rest of the paper is designed as: The formulation of a four-point iterative method along with the analysis of convergence is presented in Section 2. Some particular cases of weight functions are discussed in Section 3. Standard test functions and numerical experimentation along with the comparison of the developed methods with existing methods of equivalent order are depicted in Section 4. In Section 5, a detailed dynamical analysis of the presented approaches in a complex plane is demonstrated using a graphical tool basins of attraction. Finally, concluding remarks are given in Section 6.

2. A Formulation of Optimal sixteenth-order Scheme Using Lagrange Interpolation

In this section, an optimal iterative scheme has been presented which is based on Steffensen-Ostrowski's type weighted family.

2.1. Formulation of Scheme

For the construction of the scheme, take into account the three-point optimal eighth-order iterative method proposed by Kanwar et al. [26], based on Ostrowski's method:

$$
\begin{aligned}
y_j &= k_j - \frac{g(k_j)}{g[\zeta_j, k_j]},\ \zeta_j = k_j + \beta g(k_j)^3, \\
z_j &= y_j - \frac{g(y_j)}{2g[y_j, k_j] - g[\zeta_j, k_j]},
\end{aligned}
$$

$$k_{j+1} = z_j - \frac{g(z_j)}{g[y_j,z_j] + g[\zeta_j,y_j,z_j](z_j - y_j)} E(\lambda,\mu), \tag{2}$$

where, $\lambda = \frac{g(z_j)}{g(y_j)}$ and $\mu = \frac{g(y_j)}{g(k_j)}$. β is a parameter and $\beta \in \mathbb{R} \setminus \{0\}$.

With the help of Newton's technique at the fourth step of the scheme (2) we obtain:

$$\begin{aligned}
y_j &= k_j - \frac{g(k_j)}{g[\zeta_j,k_j]}, \ \zeta_j = k_j + \beta g(k_j)^4, \\
z_j &= y_j - \frac{g(y_j)}{2g[y_j,k_j] - g[\zeta_j,k_j]}, \\
t_j &= z_j - \frac{g(z_j)}{g[y_j,z_j] + g[\zeta_j,y_j,z_j](z_j - y_j)} E(\lambda,\mu), \\
k_{j+1} &= t_j - \frac{g(t_j)}{g'(t_j)},
\end{aligned} \tag{3}$$

where, $\beta \in \mathbb{R} \setminus \{0\}$, $\lambda = \frac{g(z_j)}{g(y_j)}$ and $\mu = \frac{g(y_j)}{g(k_j)}$.

The above scheme (3) does not meet the criteria for optimality as per the Kung-Traub conjecture [2], since per iteration it necessitates six function evaluations. Consequently, introducing a suitable approximation, we sought to reduce the number of function evaluations. The below given approximation is employed to replace the derivative $g'(t_j)$:

$$\begin{aligned}
g'(t_j) &= \frac{(t_j - z_j)(t_j - k_j)(t_j - y_j)g(\zeta_j)}{(\zeta_j - t_j)(\zeta_j - k_j)(\zeta_j - z_j)(\zeta_j - y_j)} \\
&+ \frac{(t_j - z_j)(t_j - \zeta_j)(t_j - k_j)g(y_j)}{(y_j - t_j)(y_j - \zeta_j)(y_j - z_j)(y_j - k_j)} \\
&+ \frac{(t_j - z_j)(t_j - \zeta_j)(t_j - y_j)g(k_j)}{(k_j - t_j)(k_j - \zeta_j)(k_j - y_j)(k_j - z_j)} \\
&+ \frac{(t_j - \zeta_j)(t_j - k_j)(t_j - y_j)g(z_j)}{(z_j - t_j)(z_j - \zeta_j)(z_j - k_j)(z_j - y_j)} \\
&+ \frac{g(t_j)}{(t_j - \zeta_j)} + \frac{g(t_j)}{(t_j - k_j)} + \frac{g(t_j)}{(t_j - y_j)} + \frac{g(t_j)}{(t_j - z_j)}.
\end{aligned} \tag{4}$$

where, $\zeta_j = k_j + \beta g(k_j)^4$ and $\beta \in \mathbb{R} \setminus \{0\}$. $g'(t_j)$ is the fourth-degree Interpolation by Lagrange [27] that interpolates t_j, z_j, y_j, k_j, and ζ_j and $\beta \in R \setminus \{0\}$.

We formulate a four-step iterative method of order sixteenth utilizing a bivariate weight function and at each iterative step employing five evaluations of function. This formula achieved by integrating the provided approximation (4) in the fourth step of the iterative scheme (3), as shown below:

$$\begin{aligned}
y_j &= k_j - \frac{g(k_j)}{g[\zeta_j,k_j]}, \ \zeta_j = k_j + \beta g(k_j)^4, \\
z_j &= y_j - \frac{g(y_j)}{2g[y_j,k_j] - g[\zeta_j,k_j]},
\end{aligned}$$

$$\begin{aligned} t_j &= z_j - \frac{g(z_j)}{g[y_j,z_j]+g[\zeta_j,y_j,z_j](z_j-y_j)} E(\lambda,\mu), \\ k_{j+1} &= t_j - \frac{g(t_j)}{M_L}. \end{aligned} \tag{5}$$

where, $\beta \in \mathbb{R}\setminus\{0\}$, $\lambda = \frac{g(z_j)}{g(y_j)}$, $\mu = \frac{g(y_j)}{g(k_j)}$, and

$$\begin{aligned} M_L = & \frac{(t_j-z_j)(t_j-k_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-t_j)(\zeta_j-k_j)(\zeta_j-z_j)(\zeta_j-y_j)} \\ & +\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-k_j)g(y_j)}{(y_j-z_j)(y_j-t_j)(y_j-\zeta_j)(y_j-k_j)} \\ & +\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-y_j)g(k_j)}{(k_j-y_j)(k_j-t_j)(k_j-\zeta_j)(\zeta_j-z_j)} \\ & +\frac{(t_j-\zeta_j)(t_j-k_j)(t_j-y_j)g(z_j)}{(z_j-t_j)(z_j-\zeta_j)(z_j-k_j)(z_j-y_j)} \\ & +\frac{g(t_j)}{(t_j-\zeta_j)}+\frac{g(t_j)}{(t_j-k_j)}+\frac{g(t_j)}{(t_j-y_j)}+\frac{g(t_j)}{(t_j-z_j)}. \end{aligned}$$

Consider an analytic function $E : C^2 \to C$ in the vicinity of $(0,0)$, with $\lambda = \frac{g(z_j)}{g(y_j)} = O(e_j^2)$ and $\mu = \frac{g(y_j)}{g(k_j)} = O(e_j)$. The following theorem demonstrates that the order of convergence of the aforementioned scheme is optimal sixteenth-order.

Theorem 1. *Let g be a sufficiently differentiable function defined on an open interval D, with $\gamma \in D$ being a simple zero of g. Assume that the initial guess k^* is sufficiently close to $\gamma \in D$. Then, the four-point iterative scheme described in (5) achieves an optimal convergence order of sixteen, provided the following conditions for the weight function are satisfied:*

$$\begin{aligned} E_{00} &= 1 = E_{11}, \\ E_{01} &= 0 = E_{10} = E_{02}, \\ E_{03} &= -6. \end{aligned} \tag{6}$$

where $E_{i,j} = [\frac{\partial E(\lambda,\rho)}{\partial \lambda^i \rho^j}\frac{1}{i!j!}]_{(0,0)}$ and $i,\ j = 0,\ 1,\ 2,\ 3$. It possess the error relation as follows:

$$k_{j+1}-\gamma = (\frac{1}{6}c_3^3c_2^9 g - 6c_2^{12}c_4 - \ldots - \frac{1}{6}gc_3c_2^{13}b_2 + 3c_3^3c_2^9 a)e_j^{16} + O(e_j^{17}). \tag{7}$$

where $e_j = k_j - \gamma$ and $c_j = \frac{g^{(j)}(\gamma)}{j!g'(\gamma)}$, $j = 1, 2, 3\cdots$.

Proof. Expanding $g(k_j)$ around the simple zero γ, using Taylor's expansion, considering that $g(\gamma) = 0$:

$$g(k_j) = g'(\gamma)(e_j + c_2e_j^2 + c_3e_j^3 + c_4e_j^4 + c_5e_j^5 + \cdots + c_{16}e_j^{16}) + O(e_j^{17}). \tag{8}$$

where $c_j = \frac{g^{(j)}(\gamma)}{j!g'(\gamma)}$, $j = 1, 2, 3\cdots$. We define ζ_j as :

$$\begin{aligned} \zeta_j &= k_j + \beta g(k_j)^4, \\ \zeta_j - \gamma &= e_j + \beta e_j^4 + 4\beta c_2 e_j^5 + 2\beta(3c_2^2 + 2c_3)e_j^6 + \cdots. \end{aligned} \tag{9}$$

Now we expand $g(\zeta_j)$ using Taylor expansion about γ as:

$$g(\zeta_j) = g'(\gamma)(e_j + c_2e_j^2 + c_3e_j^3 + (c_4+\beta)e_j^4 + (c_5 + 6\beta c_2)e_j^5 + \cdots). \tag{10}$$

To compute divided difference $g[\zeta_j, k_j]$, we utilized equations (8) and (10) as follows:

$$\begin{aligned} g[\zeta_j, k_j] &= \frac{g(\zeta_j) - g(k_j)}{w_j - k_j}, \\ &= 1 + 2c_2e_j + 3c_3e_j^2 + 4c_4e_j^3 + (5c_5 + \beta c_2)e_j^4 + \cdots. \end{aligned} \tag{11}$$

Now, equations (8) and (11) substituting in the first step of method (5) and we get:

$$y_j - \gamma = c_2e_j^2 + (2c_3 - 2c_2^2)e_j^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_j^4 + \cdots. \tag{12}$$

Hence, equation (12) is achieved second order convergence of method (5) which is optimal. Taylor expansion of $g(y_n)$ about γ as well as considering equation (12):

$$g(y_j) = g'(\gamma)(c_2e_j^2 + (2c_3 - 2c_2^2)e_j^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_j^4 + \cdots). \tag{13}$$

To determine the divided difference $g[y_j, k_j]$, we used equations (8) and (13) subsequently, we arrive at the expression as follows:

$$\begin{aligned} g[y_j, k_j] &= \frac{g(k_j) - g(y_j)}{k_j - y_j}, \\ &= 1 + c_2e_j + (c_1 + c_2^2 + c_3)e_j^2 + (c_4 + 3c_3c_2 - 2c_2^3)e_j^3 + \cdots. \end{aligned} \tag{14}$$

In the second step of iterative method (5), we used equations (11), (13), and (14) to achieve fourth order convergence:

$$z_j - \gamma = (-c_2c_3 + c_2^3)e_j^4 + (-2c_2c_4 - 4c_2^4 + 8c_2^2c_3 - 2c_3^2)e_j^5 + \cdots. \tag{15}$$

Expanding $g(z_j)$ about γ by using of equation (15), we have:

$$g(z_j) = g'(\gamma)(c_2(-c_3 + c_2^2)e_j^4 + (-2c_2c_4 - 4c_2^4 + 8c_2^2c_3 - 2c_3^2)e_j^5 + \cdots). \tag{16}$$

Next, we derive the Taylor expansion of the divided differences used in the third step of the scheme (5). Employing equations (13) and (16) we find $g[y_j, z_j]$, as follows:

$$\begin{aligned} g[y_j, z_j] &= \frac{g(z_j) - g(y_j)}{z_j - y_j}, \\ &= 1 + c_2^2e_j^2 - 2c_2(-c_3 + c_2^2)e_j^3 + \cdots. \end{aligned} \tag{17}$$

To find $g[\zeta_j, y_j]$, we used equations (10) and (13) and get the following expression:

$$\begin{aligned} g[\zeta_j, y_j] &= \frac{g(y_j) - g(\zeta_j)}{y_j - \zeta_j}, \\ &= 1 + c_2 e_j + (c_3 + c_2^2)e_j^2 + (c_4 - 2c_2^3 + 3c_2 c_3)e_j^3 + \cdots. \end{aligned} \tag{18}$$

we calculated divided difference $g[\zeta_j, y_j, z_j]$, by using equations (17) and (18):

$$\begin{aligned} g[\zeta_j, y_j, z_j] &= \frac{g[y_j, z_j] - g[\zeta_j, y_j]}{z_j - \zeta_j}, \\ &= c_2 + c_3 e_j + (c_4 + c_2 c_3)e_j^2 + \cdots. \end{aligned} \tag{19}$$

Furthermore, we introduce two variables λ and μ, defined as follows:

$$\lambda = \frac{g(z_j)}{g(y_j)} = (c_1^2 + c_2^2 - c_3)e_j^2 + (-2c_4 + 4c_2 c_3 - 2c_2^3)e_j^3 + \cdots. \tag{20}$$

and

$$\mu = \frac{g(y_j)}{g(k_j)} = c_2 e_j + (2c_3 - 3c_2^2)e_j^2 + (3c_4 - 10c_2 c_3 + 8c_2^3)e_j^3 + \cdots. \tag{21}$$

From equations (20) and (21) it was observed that λ exhibits second-order convergence, while μ demonstrates first-order convergence. Now, the bivariate weight function $E(\lambda, \mu)$ was expanded using a Taylor series expansion up to the fifth term, as illustrated:

$$\begin{aligned} E(\lambda, \mu) &= E_{00} + (\lambda E_{10} + \mu E_{01}) + \frac{1}{2!}(\lambda^2 E_{20} + 2\lambda\mu E_{11} + \mu^2 E_{02}) \\ &\quad + \frac{1}{3!}(\lambda^3 E_{30} + 3\lambda^2\mu E_{21} + 3\lambda\mu^2 E_{12} + \mu^3 E_{03}) + \frac{1}{4!}(\lambda^4 E_{40} \\ &\quad + 4\lambda^3\mu E_{31} + 6\lambda^2\mu^2 E_{22} + 4\lambda\mu^3 E_{13} + \lambda^4 E_{04}) + \frac{1}{5!}(\lambda^5 E_{50} \\ &\quad + 5\lambda^4\mu E_{41} + 10\lambda^3\mu^2 E_{32} + 10\lambda^2\mu^3 E_{23} + 5\lambda\mu^4 E_{14} \\ &\quad + \mu^5 E_{05}). \end{aligned} \tag{22}$$

In third step of scheme (5), we substituted equations (16), (17), (19), and (22) as:

$$\begin{aligned} t_j - \gamma &= \left(-c_2 c_3 + c_2^3 + c_2 E_{00} c_3 - c_2^3 E_{00}\right) e_j^4 + (-4c_2^4 + 8c_2^2 c_3 - 2c_2 c_4 \\ &\quad - 2c_3^2 - c_2^2(c_2 - c_3)E_{01}(-4c_2^4 + 8c_2^2 c_3 - 2c_2 c_4 - 2c_3^2)E_{00})e_j^5 \\ &\quad + \Big(10c_2^5 - 30c_2^3 c_3 - \beta c_2^2 + 12c_2^2 c_4 + 18c_2 c_3^2 - 3c_2 c_5 - 7c_3 c_4 \Big) \\ &\quad - \frac{1}{2}E_{02} c_2^5 + \frac{1}{2}E_{02} c_2^3 c_3 - E_{10} c_2^5 + 2E_{10} c_2^3 c_3 - E_{10} c_2 c_3^2 + 7E_{01} c_2^5 \\ &\quad - 13E_{01} c_2^3 c_3 + 4E_{01} c_2 c_3^2 - 10E_{00} c_2^5 + 30E_{00} c_2^3 c_3 + E_{00}\beta c_2^2 \\ &\quad - 12E_{00} c_2^2 c_4 - 18E_{00} c_2 c_3^2 - 3E_{00} c_2 c_5 + 7E_{00} c_3 c_4 \Big) e_j^6 + \cdots. \end{aligned} \tag{23}$$

Equation (23) demonstrates fourth-order convergence. To achieve eighth-order convergence, we employed the following values:

$$\begin{aligned} E_{00} &= 1 = E_{11}, \\ E_{01} &= 0 = E_{10} = E_{02}, \\ E_{03} &= -6. \end{aligned} \tag{24}$$

By substituting the values (24) in equation (23) we get the error term as:

$$t_j - \gamma = (-c_2^2 c_4 c_3 - \frac{3}{2}(b_2 c_3^2 c_2^3) - \frac{1}{2}(a c_3^2 c_2^3) + \cdots - \frac{1}{2}(b_2 c_2^7))e_j^8 + \cdots. \tag{25}$$

By utilizing equation (25), we successfully attained optimal eighth-order convergence. Employing Taylor series expansion, we computed $g(t_j)$, yielding the following expression:

$$g(t_j) = g'(\gamma)(-\frac{1}{24}\{c_2(24c_3c_4c_2 + 12c_2^6 b_2 - 48c_3^2c_2^2 + c_2^6 g + 12c_2^6 a - \cdots)\}e_j^8 + \cdots). \tag{26}$$

Firstly, we calculate $\frac{(t_j-k_j)(t_j-z_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-t_j)(\zeta_j-k_j)(\zeta_j-z_j)(\zeta_j-y_j)}$ term by utilizing equations (9), (10), (12), (15), and (25) as:

$$\frac{(t_j-k_j)(t_j-z_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-t_j)(\zeta_j-k_j)(\zeta_j-z_j)(\zeta_j-y_j)} = \frac{-c_2^2(-c_3+c_2^2)}{\beta}e_j + \cdots. \tag{27}$$

Now, to determine the term $\frac{(t_j-\zeta_j)(t_j-z_j)(t_j-y_j)g(k_j)}{(k_j-t_j)(k_j-y_j)(k_j-\zeta_j)(\zeta_j-z_j)}$ we used (8), (9), (12), (15), and (25) given as:

$$\begin{aligned} \frac{(t_j-k_j)(t_j-z_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-z_j)(\zeta_j-k_j)(\zeta_j-t_j)(\zeta_j-y_j)} &= \frac{-c_2^2(-c_3+c_2^2)}{\beta}e_j \\ &+ \frac{2c_2(-7c_2^2c_3+4c_2^4+2c_3^2+c_2c_4)}{\beta}e_j^2 + \cdots. \end{aligned} \tag{28}$$

Also, to find $\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-k_j)g(y_j)}{(y_j-z_j)(y_j-t_j)(y_j-\zeta_j)(y_j-k_j)}$ we used (9), (12), (13), (15), and (25) as:

$$\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-k_j)g(y_j)}{(y_j-z_j)(y_j-t_j)(y_j-\zeta_j)(y_j-k_j)} = (c_3-c_2^2)e_j^2 + (2c_4-2c_2c_3)e_j^3 + \cdots. \tag{29}$$

We used (9), (12), (15), (16), and (15) to find $\frac{(t_j-\zeta_j)(t_j-k_j)(t_j-y_j)g(z_j)}{(z_j-t_j)(z_j-\zeta_j)(z_j-k_j)(z_j-y_j)}$ term given as:

$$\frac{(t_j-\zeta_j)(t_j-y_j)(t_j-k_j)g(z_j)}{(z_j-t_j)(z_j-k_j)(z_j-y_j)(z_j-\zeta_j)} = 1 + (-c_3+c_2^2)e_j^2 + (-2c_4+2c_2c_3)e_j^3 + \cdots. \tag{30}$$

We computed $\frac{g(t_j)}{(t_j-w_j)}$ by using equations (10), (25), and (26) as:

$$\frac{g(t_j)}{(t_j-\zeta_j)} = \frac{1}{24}c_2(24c_2c_3c_4 - \cdots - 36b_2c_3c_2^4)e_j^7 + \cdots. \tag{31}$$

Using equations (25) and (26) we evaluated $\frac{g(t_j)}{(t_j-k_j)}$ as:

$$\frac{g(t_j)}{(t_j-k_j)} = \frac{1}{24}c_2(24c_2c_3c_4 - \cdots - 36b_2c_3c_2^4)e_j^7 + \cdots . \tag{32}$$

$\frac{g(t_j)}{(t_j-y_j)}$ is calculated by using equations (12), (25), and (26):

$$\frac{g(t_j)}{(t_j-y_j)} = (c_2c_3c_4 - 2c_2^2c_3^2 - \cdots - \frac{3}{2}b_2c_3c_2^4)e_j^6 + \cdots . \tag{33}$$

Last term $\frac{g(t_j)}{(t_j-z_j)}$ is calculated by using equations (15), (25), and (26):

$$\frac{g(t_j)}{(t_j-z_j)} = (\frac{1}{24}c_2^4g + \frac{1}{2}c_2^4b_2 + \cdots + \frac{1}{2}b_2c_3^2)e_j^4 + \cdots . \tag{34}$$

Substituting equations (27)-(34) in equation in M_L, we obtained:

$$M_L = 1 - \frac{1}{12}c_2^2(-12b_2c_3^3 + 36b_2c_2^2c_3^2 - \cdots - 24ac_3c_2^4)e_j^8 + \cdots . \tag{35}$$

Now, utilizing equations (26) and (35) in fourth step of iterative scheme (5) we obtained error term as:

$$e_{j+1} = (\frac{1}{6}(c_3^3gc_2^9) - 6c_2^{12}c_4 - \ldots - \frac{1}{6}(b_2c_2^{13}gc_3) + 3c_3^3ac_2^9)e_j^{16} + O(e_j^{17}). \tag{36}$$

Above error relation (36) demonstrates that the sixteenth-order convergence approach (5) is optimal.

3. Some Particular Cases

We introduce two specific weight functions: Case 1 and Case 2.

Case 1: To obtain sixteenth-order convergence we utilize the bivariate polynomial of the following form:

$$H(\lambda,\mu) = 1 + \lambda\mu + 5\lambda^2 - \mu^3 - 8\mu^2\lambda.$$

In the scheme's third step (5), **SFHM-1** is indicated and explained as follows:

$$\begin{aligned}
y_j &= k_j - \frac{g(k_j)}{g[\zeta_j,k_j]},\ \zeta_j = k_j + \beta g(k_j)^4,\\
z_j &= y_j - \frac{g(y_j)}{2g[y_j,k_j] - g[\zeta_j,k_j]},\\
t_j &= z_j - \frac{g(z_j)}{g[y_j,z_j] + g[\zeta_j,y_j,z_j](z_j-y_j)}(1 + \lambda\mu + 5\lambda^2 - \mu^3 - 8\mu^2\lambda + \lambda^3),\\
k_{j+1} &= t_j - \frac{g(t_j)}{M_L}.
\end{aligned} \tag{37}$$

where $\beta \in \mathbb{R}\setminus\{0\}$, $\lambda = \frac{g(z_j)}{g(y_j)}$, $\mu = \frac{g(y_j)}{g(k_j)}$ and

$$\begin{aligned} M_L \;=\;& \frac{(t_j-z_j)(t_j-k_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-k_j)(\zeta_j-t_j)(\zeta_j-z_j)(\zeta_j-y_j)} \\ &+\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-y_j)g(k_j)}{(k_j-t_j)(k_j-\zeta_j)(k_j-y_j)(\zeta_j-z_j)} \\ &+\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-k_j)g(y_j)}{(y_j-z_j)(y_j-t_j)(y_j-\zeta_j)(y_j-k_j)} \\ &+\frac{(t_j-k_j)(t_j-\zeta_j)(t_j-y_j)g(z_j)}{(z_j-t_j)(z_j-\zeta_j)(z_j-k_j)(z_j-y_j)} \\ &+\frac{g(t_j)}{(t_j-\zeta_j)}+\frac{g(t_j)}{(t_j-k_j)}+\frac{g(t_j)}{(t_j-y_j)}+\frac{g(t_j)}{(t_j-z_j)}. \end{aligned}$$

Case 2: Using weight function of the form

$$H(\lambda,\mu) = 1+\lambda\mu+(\lambda\mu)^3-\mu^3-8\mu^2\lambda.$$

The new scheme, termed **SFHM-2**, takes on the following form in the third step of (5).

$$\begin{aligned} y_j &= k_j - \frac{g(k_j)}{g[\zeta_j,k_j]}, \zeta_j = k_j+\beta g(k_j)^4, \\ z_j &= y_j - \frac{g(y_j)}{2g[y_j,k_j]-g[\zeta_j,k_j]}, \\ t_j &= z_j - \frac{g(z_j)}{g[\zeta_j,y_j,z_j](z_j-y_j)+g[y_j,z_j]}(1+\lambda\mu+(\lambda\mu)^3-\mu^3-8\mu^2\lambda), \\ k_{j+1} &= t_j - \frac{g(t_j)}{M_L}, \end{aligned} \tag{38}$$

where $\beta \in \mathbb{R}\setminus\{0\}$, $\lambda = \frac{g(z_j)}{g(y_j)}$, $\mu = \frac{g(y_j)}{g(k_j)}$ and

$$\begin{aligned} M_L \;=\;& \frac{(t_j-z_j)(t_j-k_j)(t_j-y_j)g(\zeta_j)}{(\zeta_j-t_j)(\zeta_j-z_j)(\zeta_j-k_j)(\zeta_j-y_j)} \\ &+\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-y_j)g(k_j)}{(k_j-y_j)(k_j-t_j)(k_j-\zeta_j)(\zeta_j-z_j)} \\ &+\frac{(t_j-z_j)(t_j-\zeta_j)(t_j-k_j)g(y_j)}{(y_j-z_j)(y_j-t_j)(y_j-\zeta_j)(y_j-k_j)} \\ &+\frac{(t_j-k_j)(t_j-\zeta_j)(t_j-y_j)g(z_j)}{(z_j-t_j)(z_j-\zeta_j)(z_j-k_j)(z_j-y_j)} \\ &+\frac{g(t_j)}{(t_j-\zeta_j)}+\frac{g(t_j)}{(t_j-k_j)}+\frac{g(t_j)}{(t_j-y_j)}+\frac{g(t_j)}{(t_j-z_j)}. \end{aligned}$$

4. Computational Analysis

In this section, we delve into examining the convergence behavior, effectiveness, robustness, and validity of the schemes proposed in section 2. To assess accuracy, we employ the absolute error between consecutive iterations $|k_{j+1}-k_j|$ for the initial three iterations. The theoretical order of convergence is confirmed through the calculation of the computational order of convergence (COC).

$$p=\frac{\ln|(k_j-k_{j-1})/(k_{j-1}-k_{j-2})|}{\ln|(k_{j-1}-k_{j-2})/(k_{j-2}-k_{j-3})|}.$$

The Maple 16 programming package uses multi-precision arithmetic with 2000 significant decimal digits to ensure high accuracy and prevent the loss of significant digits. Different standard test functions and some real-world problems have been used to check the efficiency of our proposed schemes. The sixteenth-order convergence techniques that were employed for comparison are listed below.

For the sake of comparison, we adopt the iterative scheme introduced by Sharma et al. [21], referred to as **CM-1**. This method was constructed by using Rational Interpolation, outlined as:

$$\begin{aligned}
z_j &= k_j-\frac{g(k_j)}{g'(k_j)},\\
w_j &= z_j-\frac{g'(k_j)g(z_j)}{(g[z_j,k_j])^2},\\
u_j &= w_j-\frac{1}{h_1-h_2+h_3}\frac{g'(k_j)g(w_j)}{(g[w_j,k_j])^2},\\
k_{j+1} &= u_j-\frac{1}{H_1-H_2+H_3-H_4}\frac{g'(k_j)g(u_j)}{(g[u_j,k_j])^2},
\end{aligned}\tag{39}$$

where

$$\begin{aligned}
h_1 &= \frac{z_j-w_j}{z_j-k_j},\ h_2=\frac{(w_j-k_j)^2g'(k_j)}{(z_j-w_j)(z_j-k_j)g[z_j,k_j]},\\
h_3 &= \frac{(w_j-k_j)g'(k_j)}{(z_j-w_j)g[w_j,k_j]},\ H_1=\frac{(z_j-u_j)(w_j-u_j)}{(z_j-k_j)(w_j-k_j)},\\
H_2 &= \frac{(u_j-k_j)^2(u_j-w_j)g'(k_j)}{(z_j-k_j)(z_j-u_j)(z_j-w_j)g[z_j,k_j]},\\
H_3 &= \frac{(u_j-k_j)^2(z_j-u_j)g'(k_j)}{(w_j-u_j)(w_j-z_j)(w_j-k_j)g[w_j,k_j]},\\
H_4 &= \frac{(u_j-k_j)(2u_j-z_j-w_j)g'(k_j)}{(w_j-u_j)(z_j-u_j)g[u_j,k_j]}.
\end{aligned}$$

We juxtapose the numerical results of our newly developed methods with an iterative four-point scheme devised by Sharifi et al. [20]. Specifically, We take into account their scheme's special case, denoted as **CM-2**, which is formulated as follows:

$$\begin{aligned} y_j &= k_j - \frac{g(k_j)}{g'(k_j)}, \\ z_j &= y_j - (-6t_j^3 + 5t_j^2 + 2t_j + 1)\frac{g(y_j)}{g'(k_j)}, \\ w_j &= z_j - (1 + 2t_j + 4u_j + 6t_j^2 + k_j)\frac{g(z_j)}{g'(k_j)}, \\ k_{j+1} &= w_j - (I(t_j) + J(k_j) + K(u_j) + L(t_j, u_j) \\ &\quad + M(p_j, q_j, r_j) + N(t_j, k_j, u_j, r_j)).\frac{g(w_j)}{g'(k_j)}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} I(t_j) &= 6t_j^2 + 2t_j,\ J(k_j) = -k_j^3 + k_j + 1, \\ K(u_j) &= 4u_j - 4u_j^2, \\ L(t_j, u_j) &= t_j u_j + 6t_j^2 u_j + 2t_j^3 u_j - 10 t_j u_j^2, \\ M(p_j, q_j, r_j) &= r_j + 2q_j + 8p_j, \\ N(t_j, k_j, u_j, r_j)) &= 2t_j r_j + 2k_j u_j + 6t_j^2 r_j - 4k_j^2 u_j + 24 t_j^4 u_j, \\ t_j &= \frac{g(y_j)}{g(k_j)},\ u_j = \frac{g(z_j)}{g(k_j)},\ p_j = \frac{g(w_j)}{g(k_j)},\ k_j = \frac{g(z_j)}{g(y_j)}, \\ q_j &= \frac{g(w_j)}{g(y_j)},\ r_j = \frac{g(w_j)}{g(z_j)}. \end{aligned}$$

Sivakumar et al. [28] introduced the following four-point iterative method utilizing the divided difference technique, known as **CM-3**:

$$\begin{aligned} y_j &= k_j - u(k_j)\ , z_j = k_j + g(k_j)^4, \\ z_j &= k_j - u(k_j)[\frac{g(k_j) - g(y_j)}{g(k_j) - 2g(y_j)}], \\ w_j &= z_j - \frac{g(z_j)}{q'(z_j)}, \\ k_{j+1} &= w_j - \frac{g(w_j)}{r'(w_j)}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} u(k_j) &= \frac{g(k_j)}{g'(k_j)}, \\ q'(z_j) &= a1 + 2a_2(z_j - k_j) + 3a_3(z_j - k_j)^2, \\ r'(w_j) &= b_1 + 2b_2(w_j - k_j) + 3b_3(w_j - k_j)^2 + 4b_4(w_j - k_j)^3, \\ a1 &= g'(k_j) = b_1, \end{aligned}$$

$$
\begin{aligned}
a_2 &= \frac{g[y_j,k_j,k_j](z_j-k_j)-g[z_j,k_j,k_j](y_j-k_j)}{z_j-y_j},\\
a_3 &= \frac{g[z_j,k_j,k_j]-g[y_j,k_j,k_j]}{z_j-y_j},\\
b_4 &= \frac{(g[y_j,k_j,k_j](S_3-S_2)+g[z_j,k_j,k_j](S_3+S_1)+g[w_j,k_j,k_j](S_2-S_1))}{-S_1^2S_2+S_1S_2^2+S_1^2S_3-S_2^2S_3-S_1S_3^2+S_2S_3^2},\\
b_3 &= \frac{(g[y_j,k_j,k_j](S_2^2-S_3^2)+g[z_j,k_j,k_j](S_3^2-S_1^2)+g[w_j,k_j,k_j](S_1^2-S_2^2))}{-S_2S_1^2+S_2^2S_1-S_3S_2^2-S_3^2S_1+S_3S_1^2+S_3^2S_2},\\
b_2 &= \frac{\begin{array}{c}(g[y_j,k_j,k_j](-S_2^2S_3+S_2S_3^2)+g[z_j,k_j,k_j](-S_1^2S_3+S_1S_3^2)\\ +g[w_j,k_j,k_j](-S_1^2S_2+S_1S_2^2))\end{array}}{-S_1^2S_2+S_1S_2^2+S_1^2S_3-S_2^2S_3-S_1S_3^2+S_2S_3^2},\\
S_1 &= y_j-k_j,\ S_2=z_j-k_j,\ S_3=w_j-k_j.
\end{aligned}
$$

We have taken another method developed by Nusrat et al. [29] for comparison, which constructs optimal sixteenth-order derivative-free root-finding methods based on Hermite interpolation, referred to as **CM-4**.

$$
\begin{aligned}
y_j &= k_j-\frac{g(k_j)}{g[w_j,k_j]},\\
w_j &= k_j+g(k_j)^4, \quad j\geq 0,\\
z_j &= y_j-\frac{g(y_j)}{2g[y_j,k_j]-g[w_j,k_j]},\\
t_j &= z_j-\frac{g(z_j)}{K_3'(z_j)},\\
k_{j+1} &= t_j-\frac{g(t_j)}{K_4'(t_j)},
\end{aligned}
$$

where

$$
\begin{aligned}
K_3'(z) &= g[z,k]\left(2+\frac{z-k}{z-y}\right)-\frac{(z-k)^2}{(y-k)(z-y)}g[y,k]+g[w,k]\frac{z-y}{y-k},\\
K_4'(t) &= g[t,z]+(t-z)g[t,z,y]+(t-z)(t-y)g[t,z,y,k]\\
&\quad +(t-z)(t-y)(t-k)g[t,z,y,k,2],\\
g[t,z,y,k,2] &= \frac{1}{(t-k)^2(t-y)}\Big(g[t,z]-g[z,y]\Big)-\frac{1}{(t-k)^2(z-k)}\Big(g[z,y]-g[y,k]\Big)\\
&\quad -\frac{1}{(t-k)(z-k)^2}\Big(g[y,k]-g[w,k]\Big)+\frac{1}{(t-k)(z-k)(y-k)}\Big(g[y,k]-g[w,k]\Big).
\end{aligned}
$$

We adopt the approach proposed by Soleymani et al. [30], which constructs an optimal 16th-order iterative class for approximating simple zeros of nonlinear equations, referred to as **CM-5** is given as:

$$
y_j = k_j-\frac{g(k_j)}{g'(k_j)},
$$

$$
\begin{aligned}
z_j &= y_j - \frac{g(y_j)}{g(k_j)-2g(y_j)}\frac{g(k_j)}{g'(k_j)},, \\
w_j &= z_j - \left[1+\frac{g(z_j)}{g(k_j)}\right]\frac{g[k_j,y_j]g(z_j)}{g[k_j,z_j]g[y_j,z_j]}, \\
k_{j+1} &= w_j - \frac{(1+b_5(w_j-k_j))^2 g(w_j)}{g'(k_j)+2b_3(w_j-k_j)+(3b_4+b_3b_5)(w_j-k_j)^2+2b_4b_5(w_j-k_j)^3}, \quad (42)
\end{aligned}
$$

where

$$
\begin{aligned}
b_5 &= \frac{g[z,k]WY(w-y)+Z(g[w,k](y-z)Y+g[y,k](z-w)W)-(w-y)(w-z)(y-z)g'(k)}{(YZ(z-y)g(w)+(w-z)((w-y)(y-z)g(k)+WZg(y))+WY(y-w)g(z))}, \\
b_2 &= g'(k)+g(k)b_5, \\
b_4 &= \frac{(z-k)g[y,k]+(k-y)g[z,k]+(y-z)b_2+((z-k)g(y)+(k-y)g(z))b_5}{(k-z)(k-y)(y-z)}, \\
b_3 &= g[w,z,k]+g[w,k]b_5-(w-k)b_4, \quad (43)
\end{aligned}
$$

wherein $y-k=Y$, $z-k=Z$, and $w-k=W$.

Now, our focus shifts to assessing the effectiveness of the proposed schemes through their application to various test functions and real-world problems provided below:

Example 1. ***Perfectly Mixed Reactor***

Consider a problem involving a Perfectly Mixed Reactor [31]. In this reactor type, the concentration within the tank equals the output concentration of the reactor. The objective is to determine the concentration of a chemical within a completely mixed reactor. The objective here is to ascertain the concentration of a chemical within a completely mixed reactor. This is governed by a differential equation:

$$\frac{dc}{dk} = (c_{in}-c)\frac{Q_f}{V_R}. \quad (44)$$

Here the inflow chemical concentration is represented by c_{in}, while Q_f denotes the flow rate of fluid entering the reactor with a volume V_R. The solution to equation (44) is given by:

$$c(k) = (1-e^{-\frac{k}{\tau}})c_{in}, \quad (45)$$

where $\tau = \frac{V_R}{Q_f}$, represents the mean residence time. Now, let's consider the equation representing the concentration of chemical within completely mixed reactor:

$$g_1(k) = (1-e^{-0.04k})c_{in} + c_o e^{-0.04k}. \quad (46)$$

Here, $c_{in} = 10$ represents the inflow concentration, and $c_o = 4$ is the initial concentration of the chemical.

*The exact root of $g_1(k)$ is $\gamma = -12.770059497670\cdots$. We choose the initial approximation for $g_1(k)$ is $k^* = -13.51$. Table 1 explains the numerical results for test function $g_1(k)$. Table 1 shows that the recently developed approaches **SFHM-1** and **SFHM-2** outperform than the previously published methods when compared in terms of COC and successive iterations. These methods*

Table 1: Convergence behavior for $g_1(k)$

$g_1(k) = 10 - 6e^{-0.04k}, \quad k^* = -13.51$				
Methods	$\lvert k_2 - k_1 \rvert$	$\lvert k_3 - k_2 \rvert$	$\lvert k_4 - k_3 \rvert$	**COC**
SFHM-1	7.39×10^{-1}	1.63×10^{-28}	3.53×10^{-471}	16.00
SFHM-2	7.39×10^{-1}	9.23×10^{-29}	1.97×10^{-475}	16.00
CM-1	7.39×10^{-1}	7.34×10^{-28}	7.27×10^{-460}	15.99
CM-2	7.39×10^{-1}	5.43×10^{-25}	1.41×10^{-410}	15.07
CM-3	7.39×10^{-1}	1.00×10^{-29}	D	D
CM-4	7.08×10^{-15}	7.08×10^{-15}	7.08×10^{-15}	0
CM-5	7.08×10^{-15}	7.08×10^{-15}	7.08×10^{-15}	0

exhibit a computational order of convergence of 16*, indicating highly accurate results. On the other hand, the methods **CM-1** and **CM-2** show slightly lower performance, with computational orders of convergence of* 15.99 *and* 15.07*, respectively. The scheme **CM-3** fails to converge, as indicated by the divergence observed in its results. Meanwhile, **CM-4** and **CM-5** both fail to progress using the suggested initial values, resulting in a computational order of convergence of zero.*

Example 2. ***Kepler's Equation of Celestial Motion***

In the field of Celestial mechanics, Kepler's equation [32] serves as a fundamental cornerstone, establishing a vital link between the temporal dynamics and spatial configurations of celestial bodies, notably planets, concerning a designated starting point. This equation, known for its significance, intricately relates the mean anomaly, eccentric anomaly, and the eccentricity of the orbital path formulated as:

$$g_2(k) = k - esin(k) - M. \tag{47}$$

where k symbolizes the eccentric anomaly, M embodies the mean anomaly and e represents the eccentricity of the orbit. In practical astronomical scenarios, consider a satellite orbiting along an elliptical trajectory characterized by an eccentricity of $e = 0.9995$ *and a mean anomaly* $M = 0.01$*.Compute the eccentric anomaly while maintaining the criteria where* $0 \le e \le 1$ *and* $0 \le M \le \pi$.

The exact root of function $g_2(k)$ *is* $0.3899777749463\cdots$*. Table 2 provides a thorough comparison between the established counterparts and the suggested four-point iterative schemes **SFHM-1** and **SFHM-2** in order to provide more understanding of the computational approaches used. In particular, we investigate how the initial approximation* $k^* = 0.381$ *influence the convergence behavior and computational efficiency of these iterative techniques. Among the previously published methods, **CM-1** and **CM-2** also exhibit a strong performance with a COC of* 16.0*. However, **CM-3** shows a slightly reduced COC of* 15.8*, indicating a minor drop in accuracy. On the other hand, **CM-4** and **CM-5** display significantly lower COC values of* 5.53 *and* 3.94*, respectively, highlighting their comparatively weaker performance for this specific example. These results confirm the superior efficiency and robustness of **SFHM-1** and **SFHM-2** in handling this nonlinear equation, further emphasizing the importance of advanced iterative methods in this domain.*

Example 3. ***Population Dynamics in Predator-Prey Ecosystems***

Table 2: Convergence behavior for $g_2(k)$

$g_2(k) = k - 0.9995\sin(k) - 0.01, \quad k^* = 0.381$				
Methods	$\|k_2 - k_1\|$	$\|k_3 - k_2\|$	$\|k_4 - k_3\|$	**COC**
SFHM-1	8.98×10^{-3}	2.25×10^{-28}	6.02×10^{-438}	16.0
SFHM-2	0.8×10^{-2}	2.25×10^{-28}	5.95×10^{-438}	16.0
CM-1	0.8×10^{-2}	2.99×10^{-26}	5.07×10^{-402}	16.0
CM-2	0.8×10^{-2}	3.86×10^{-23}	4.30×10^{-350}	16.0
CM-3	0.8×10^{-2}	4.57×10^{-28}	7.61×10^{-433}	15.8
CM-4	5.59×10^{-1}	5.84×10^{-2}	6.21×10^{-14}	5.53
CM-5	7.09×10^{-1}	1.62×10^{-2}	6.21×10^{-14}	3.94

Consider a predator-prey ecosystem [33] where the prey population k, experiences both natural growth and nonlinear mortality due to overpopulation and competition. This ecosystem could apply to various animal populations where the death rate increases nonlinearly due to overcrowding effects, resource limitations, or intensified predation as population density grows.

In such ecosystems, the growth rate of the prey population can be modeled as:

$$g_3(k) = -20k^5 - \frac{1}{2}k + \frac{1}{2}. \tag{48}$$

From the five roots of the function $g_3(k)$, we choose $\gamma = 0.427677296931003\cdots$. For the function $g_3(k)$, the initial approximation is $k^* = 0.5$. Table 3 illustrates the numerical results.

Table 3: Convergence analysis for $g_3(k)$

$g_3(k) = -20k^5 - \frac{1}{2}k + \frac{1}{2}, \quad k^* = 0.5$				
Methods	$\|k_2 - k_1\|$	$\|k_3 - k_2\|$	$\|k_4 - k_3\|$	**COC**
SFHM-1	7.23×10^{-2}	4.57×10^{-11}	2.03×10^{-157}	15.90
SFHM-2	7.23×10^{-2}	1.45×10^{-12}	3.48×10^{-182}	15.85
CM-1	7.23×10^{-2}	2.42×10^{-10}	7.89×10^{-145}	15.86
CM-2	7.23×10^{-2}	2.57×10^{-8}	2.78×10^{-109}	15.65
CM-3	7.23×10^{-2}	1.42×10^{-11}	2.32×10^{-166}	15.94
CM-4	1.23×10^{-15}	1.21×10^{-230}	0	*D*
CM-5	5.16×10^{-14}	5.41×10^{-202}	0	*D*

*Table 3 presents the results for $g_3(k)$ with $k^* = 0.5$. The proposed methods, **SFHM-1** and **SFHM-2** demonstrating strong and reliable performance. Among the previously proposed methods, **CM-1** and **CM-3** also deliver comparable results with COC values But slower in terms of error reduction. However, **CM-2** exhibits a slightly lower COC of 15.09, indicating reduced efficiency in this case. The methods **CM-4** and **CM-5** fail to converge, as shown by their divergence behavior. This highlights their inability to handle the nonlinearity of this example effectively. The results reaffirm the superior performance and reliability of the proposed methods, **SFHM-1** and **SFHM-2**, for this nonlinear equation while also pointing out limitations in some of the compared schemes.*

Example 4. *We now consider a different nonlinear test function to be:*

$$g_4(k) = (k+2)e^k - 1. \tag{49}$$

The nonlinear function $g_4(k)$ exhibits an exact root of $\gamma = -0.4428544010023\cdots$. and $k^* = -0.41$ is our first approximation. Table 4 displays the comparison findings of $g_4(k)$. In terms

Table 4: Convergence analysis for $g_4(k)$

$g_4(k) = (k+2)e^k - 1, \quad k^* = -0.41$				
Methods	$\lvert k_2 - k_1\rvert$	$\lvert k_3 - k_2\rvert$	$\lvert k_4 - k_3\rvert$	**COC**
SFHM-1	3.28×10^{-2}	2.73×10^{-27}	1.69×10^{-428}	15.99
SFHM-2	3.28×10^{-2}	7.54×10^{-28}	5.94×10^{-438}	15.99
CM-1	3.28×10^{-2}	2.56×10^{-26}	5.73×10^{-412}	15.99
CM-2	3.28×10^{-2}	4.87×10^{-24}	2.70×10^{-372}	15.95
CM-3	3.28×10^{-2}	4.05×10^{-28}	D	D
CM-4	3.02×10^{-26}	4.26×10^{-411}	2.00×10^{-2000}	4.12
CM-5	9.15×10^{-25}	1.62×10^{-385}	1.00×10^{-2000}	4.47

*of computational order of convergence and the error between two consecutive iterations, Table 4 demonstrates that the suggested schemes **SFHM-1** and **SFHM-2** exhibit superior performance compared to the other approaches presented for comparison. In contrast, the methods **CM-1**, **CM-2**, and **CM-3** show relatively weaker outcomes, with **CM-3** even displaying divergence. Furthermore, while **CM-4** and **CM-5** achieve convergence, their computational orders of convergence remain lower than those of the proposed schemes.*

Example 5. ***Neural Activation in Response to Complex Stimuli***

The nonlinear equation:

$$g_5(k) = \sin(2\cos(k)) - 1 - k^2 + e^{\sin k^3}. \tag{50}$$

represents the activation response of a neural circuit to varying intensities or complexities of external stimuli, where k represents the stimulus intensity or frequency. In neuroscience, neurons or neural networks often respond to external stimuli in a complex, nonlinear fashion. Sensory processing in the brain, such as auditory or visual inputs, may elicit responses influenced by multiple factors, including the frequency, amplitude, and complexity of the stimulus [34].

This model can be applied to studying sensory processing, especially in response to oscillatory or periodic stimuli, where nonlinearities play a critical role in signal processing and perception.

*We choose $\gamma = -0.7848959\cdots$ as the root for the function $g_5(k)$, and the appropriate starting approximation is $k^* = -0.7$. Table 5 lists the comparative results for test function $g_5(k)$. Table 5 demonstrates that the suggested techniques **SFHM-1** and **SFHM-2** are not only convergent but also more efficient computationally than the existing approaches **CM-21**-**CM-3** in terms of error of successive iteration for the test function $g_5(k)$. Also, the methods **CM-21** and **CM-3** exhibit zero computational convergence order.*

Table 5: Convergence analysis for $g_5(k)$

$g_5(k) = \sin(2\cos(k)) - 1 - k^2 + e^{\sin(k^3)}, \quad k^* = -0.7$				
Methods	$\lvert k_2 - k_1\rvert$	$\lvert k_3 - k_2\rvert$	$\lvert k_4 - k_3\rvert$	**COC**
SFHM-1	8.48×10^{-2}	3.04×10^{-19}	5.96×10^{-297}	15.91
SFHM-2	8.48×10^{-2}	1.80×10^{-16}	5.11×10^{-252}	16.05
CM-1	8.48×10^{-2}	6.51×10^{-17}	1.54×10^{-259}	16.00
CM-2	8.48×10^{-2}	2.09×10^{-13}	6.21×10^{-205}	16.01
CM-3	8.48×10^{-2}	5.13×10^{-19}	6.98×10^{-295}	16.02
CM-4	1.63×10^{-15}	1.62×10^{-51}	1.62×10^{-51}	0
CM-5	1.63×10^{-15}	1.62×10^{-51}	1.62×10^{-51}	0

Example 6. ***Population Growth in Ecosystems with Periodic Resources***

We consider the nonlinear test function to represent the growth of a biological population where k represents a variable such as time, resource availability, or environmental conditions.

$$g_6(k) = \ln(1+k^2) + e^k \sin(k). \tag{51}$$

In ecology, population dynamics can be influenced by both intrinsic factors (like growth rates) and extrinsic factors (such as resource availability, seasonal changes, or environmental conditions) [35]. This model can be applied to studying ecosystem populations where resource availability and environmental cycles (like seasonal changes) significantly impact growth.

The exact root of the above equation is $\gamma = 0.0$ *Numerical results are shown in Table 6 with the starting approximation* $k^* = 0.01$ *of function* $g_6(k)$*. Table 6 illustrates the convergence per-*

Table 6: Convergence analysis for $g_6(k)$

$g_6(k) = \ln(1+k^2) + e^k \sin(k), \quad k^* = 0.01$				
Methods	$\lvert k_2 - k_1\rvert$	$\lvert k_3 - k_2\rvert$	$\lvert k_4 - k_3\rvert$	**COC**
SFHM-1	9.99×10^{-3}	1.59×10^{-29}	4.48×10^{-459}	16.02
SFHM-2	9.99×10^{-3}	3.21×10^{-27}	6.19×10^{-419}	15.99
CM-1	9.99×10^{-3}	7.45×10^{-27}	9.80×10^{-413}	15.99
CM-2	1.23×10^{-2}	2.05×10^{-26}	6.89×10^{-405}	15.97
CM-3	9.99×10^{-3}	1.51×10^{-28}	D	D
CM-4	1.26×10^{-28}	7.83×10^{-443}	0	D
CM-5	4.68×10^{-26}	4.09×10^{-399}	4.94×10^{-3991}	9.63

*formance of newly proposed methods (**SFHM-1** and **SFHM-2**) compared to established methods (**CM-1**, **CM-2**, **CM-3**, **CM-4**, and **CM-5**) for the nonlinear equation* $g_6(k)$*. The proposed methods, **SFHM-1** and **SFHM-2**, demonstrate remarkable performance with significantly smaller error levels and higher COC values of* 16.02 *and* 15.99*, respectively, outperforming all other methods. Among the established methods, **CM-1** and **CM-2** also achieve required COC values but their error levels are slightly higher than those of the proposed techniques. The methods **CM-3** and **CM-4** encounter divergence and **CM-5** exhibits the weakest performance with a low COC of* 9.63 *and larger error levels.*

Example 7. *We use the following nonlinear test function:*

$$g_7(k) = (k-1)^3 - 1. \tag{52}$$

*With three approximate roots in the preceded equation, we choose $\gamma = 2$ We chose $k^* = 2.1$ as our first approximation. The numerical findings are displayed in Table 7. When comparing concerning computational order of convergence and absolute error, the suggested schemes **SFHM-1** and **SFHM-2** outperform **CM-1**-**CM-5**, as demonstrated by Table 7 for the function $g_7(k)$.*

Table 7: Convergence analysis for $g_7(k)$

$g_7(k) = (k-1)^3 - 1, \quad k^* = 2.1$				
Methods	$\lvert k_2 - k_1 \rvert$	$\lvert k_3 - k_2 \rvert$	$\lvert k_4 - k_3 \rvert$	**COC**
SFHM-1	9.99×10^{-2}	7.46×10^{-18}	1.27×10^{-275}	15.98
SFHM-2	9.99×10^{-2}	3.75×10^{-17}	1.92×10^{-263}	15.96
CM-1	9.99×10^{-2}	3.13×10^{-16}	1.03×10^{-247}	15.96
CM-2	9.99×10^{-2}	1.13×10^{-16}	1.34×10^{-252}	15.78
CM-3	9.99×10^{-2}	7.85×10^{-18}	4.16×10^{-275}	15.97
CM-4	6.38×10^{-14}	1.83×10^{-210}	0	*D*
CM-5	2.94×10^{-15}	5.66×10^{-231}	0	*D*

5. Dynamical Investigation of Proposed Family

Basins of attraction are used to study the dynamical behavior of a rational function connected to an iterative approach, which offers essential perspectives on the convergence and stability of the method. It was initially proposed by Vrscay and Gilbert [36] to analyze the complicated dynamics of iteration systems. Later, several researchers used this strategy in their writings [37, 38]. A graphic that illustrates the behavior of such an algorithm as a function of the different beginning positions is called the basin of attraction.

Let a rational mapping represented by $\chi : \mathbb{C} \to \mathbb{C}$ on the complex plane, where $\mathbb{C}$ represents the Riemann sphere. The orbit of the point $\rho_o \in \mathbb{C}$ is represented by the set that follows as,

$$\left\{\rho_o, \chi(\rho_o), \chi^2(\rho_o), \ldots, \chi^n(\rho_o), \ldots\right\}.$$

If a point $\rho_0 \in \mathbb{C}$ fulfills $\chi^n(\rho_0) = \rho_0$, where n is the smallest value that satisfies this condition, then the point is termed a periodic point with a minimal period n. A periodic point with a minimum period of one is called a fixed point of χ. Fixed points are categorized based on the multiplier $|\chi'(\rho_0)|$. Taking into consideration the associated multiplier, a fixed point ρ_0 can be classified as follows:

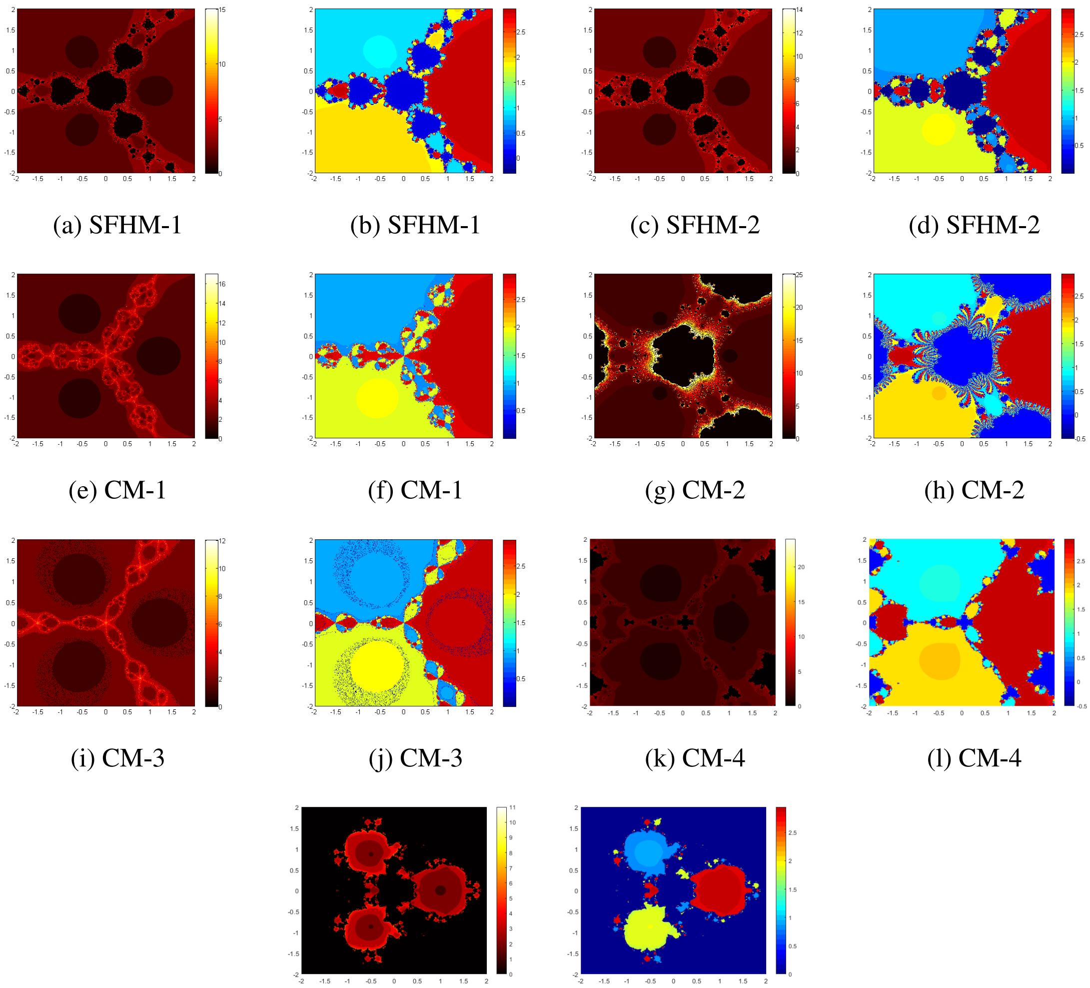
- If $|\chi'(\rho_0)| < 1$, then the point is an attractor.
- If $|\chi'(\rho_0)| > 1$, then the point is repelling.
- If $|\chi'(\rho_0)| = 1$, then the point is neutral.

- If $|\chi'(\rho_0)| = 0$, then the point is a superattractor.

The collection of preimages of any order is referred to as $C(\alpha)$ if α is a rational function that attracts fixed points.

$$R(\alpha) = \{\rho_0 \in \mathbb{C} \mid \chi^n(\rho_0) \to \alpha, n \to \infty\}.$$

A set of points known as the Fatou set is where orbits converge while attempting to attract a fixed point. The Julia set, the closure of a collection of repelling fixed points is its counterpart and establishes the limits separating the attraction basins.



(a) SFHM-1 (b) SFHM-1 (c) SFHM-2 (d) SFHM-2

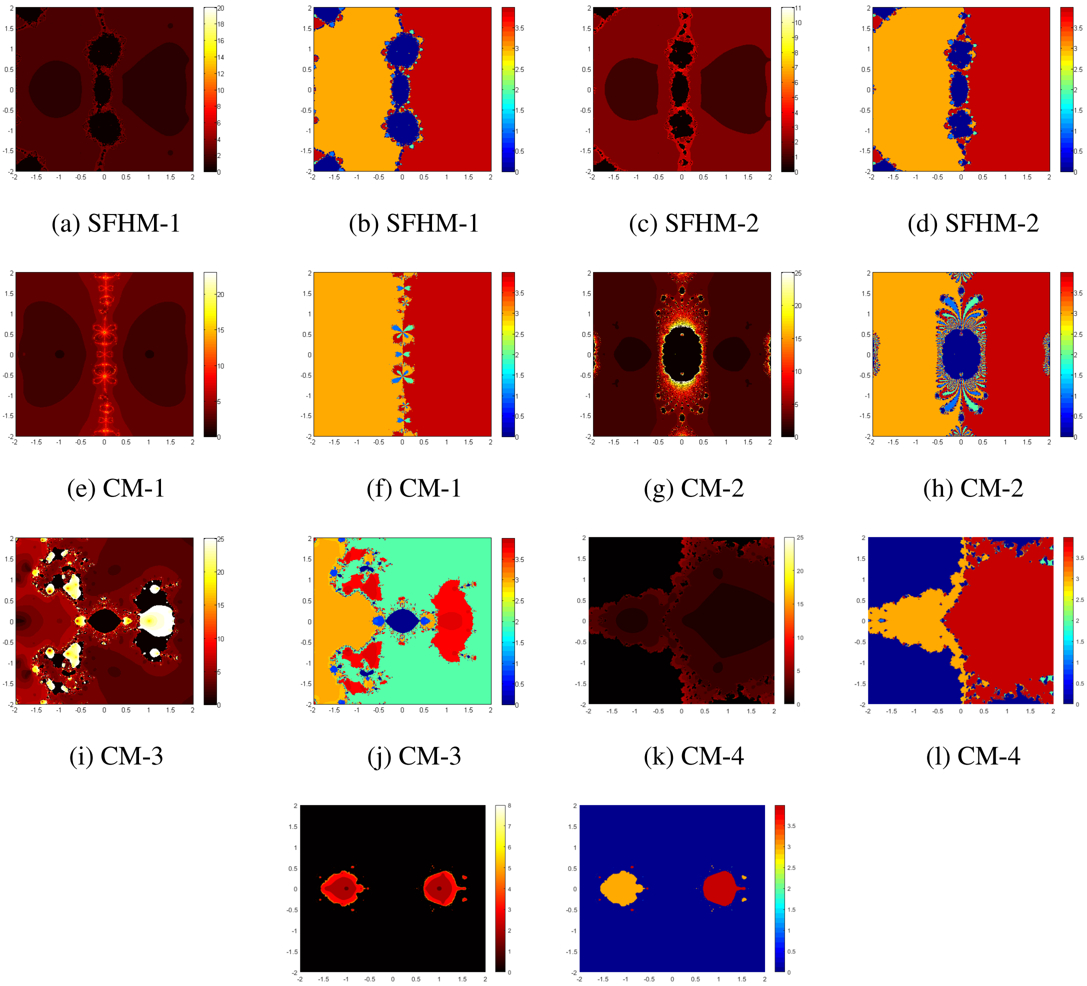
(e) CM-1 (f) CM-1 (g) CM-2 (h) CM-2

(i) CM-3 (j) CM-3 (k) CM-4 (l) CM-4

(m) CM-5 (n) CM-5

Figure 1: Dynamical behavior of different methods for $z_3(\rho)$

To establish a basin of attraction, we take two distinct approaches. The first approach utilizes a rectangular box defined by $[-2,2] \times [-2,2] \in \mathbb{C}$. Iterations are performed up to a maximum



(a) SFHM-1 (b) SFHM-1 (c) SFHM-2 (d) SFHM-2

(e) CM-1 (f) CM-1 (g) CM-2 (h) CM-2

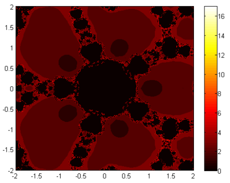
(i) CM-3 (j) CM-3 (k) CM-4 (l) CM-4

(m) CM-5 (n) CM-5

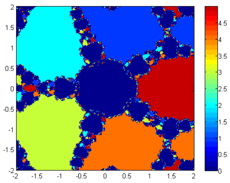
Figure 2: Dynamical behavior of different methods for $z_4(\rho)$

of 25, and the root of the sequence generated using the iterative technique converges with the stopping condition $|g(u_j)| \leq 10^{-5}$. In this approach, the points are darkened based on the number of iterations to visualize the convergence speed. Depending on the root of convergence of the iterative algorithm, we assign a color to each initial guess ρ_0. Dark colors are assigned to indicate divergence.

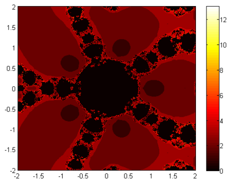
In the second approach, each root is given a distinct shade, and the points are also colored based on the distance to the closest root. In other words, after identifying the nearest root, we assign its color to the starting point. Consequently, this approach illustrates which of the basic iterative methods converges. We use an error estimate of less than 10^{-5} and a maximum of 25 iterations. With this method, each starting estimate is assigned a unique shade, and the convergence of finding a root of a specific nonlinear polynomial determines the number of iterations.



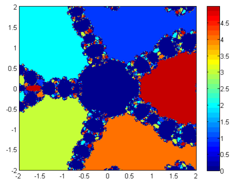
(a) SFHM-1



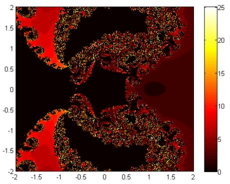
(b) SFHM-1



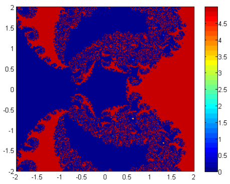
(c) SFHM-2



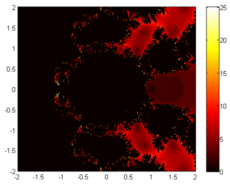
(d) SFHM-2



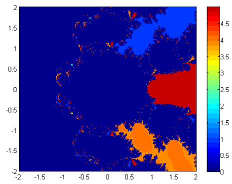
(e) CM-1



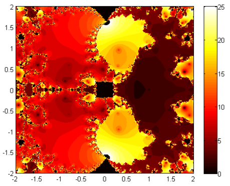
(f) CM-1



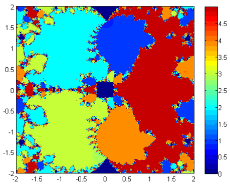
(g) CM-2



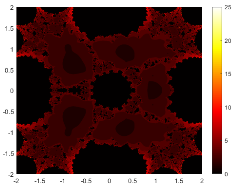
(h) CM-2



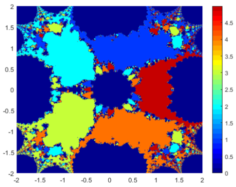
(i) CM-3



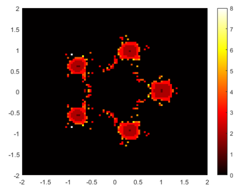
(j) CM-3



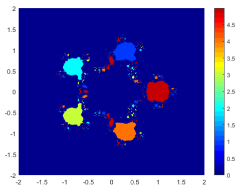
(k) CM-4



(l) CM-4



(m) CM-5



(n) CM-5

Figure 3: Dynamical behavior of different methods for $z_5(\rho)$

Two distinct strategies were utilized to create polynomiographs using MATLAB R2014a. To create dynamic planes, we consider three complex functions for dynamic testing. These functions are provided by:

$$
\begin{aligned}
z_3(\rho) &= \rho^3 - 1,\ \rho = 1.0,\ -0.5000 + 0.86605I,\ -0.5000 - 0.86605I, \\
z_4(\rho) &= \rho^4 - 10\rho^2 + 9,\ \rho = 1.0,\ -3.0,\ 3.0,\ -1.0, \\
z_5(\rho) &= \rho^5 - 1,\ \rho = 1.0,\ 0.3090 - 0.951I,\ 0.309 + 0.951I, \\
&\quad -0.809 - 0.587I,\ 0.809 + 0.587I
\end{aligned}
$$

The complex planes of the suggested techniques (**SFHM-1**) and (**SFHM-2**) as well as the previously constructed algorithms by Sharifi et al. [20] (**CM-1**), Sharma et al. [21] (**CM-2**), Sivakumar et al. [28] (**CM-3**), Nusrat et al. [29] (**CM-4**), and Soleymani et al. [30] (**CM-5**) are shown in Figures 1-3. There are two different kinds of basins of attraction illustrated in all these figures:

Based on the larger and darker regions of convergence observed in Figures 1-3 when compared to other techniques, it is evident that our proposed methods, **SFHM-1** and **SFHM-2**, exhibit a significantly higher order of convergence. These methods also encompass a broader range of initial conditions that successfully lead to convergence. Furthermore, the convergence regions of **SFHM-1** and **SFHM-2** are not only wider but also display substantially reduced chaotic behavior compared to the competing methods **CM-1**, **CM-2**, and **CM-3**. The methods **CM-4** and **CM-5** exhibit extensive divergence regions across all tested complex functions, indicating their limited reliability and effectiveness in these scenarios.

For the complex polynomial $z_3(\rho)$, **SFHM-1** and **SFHM-2** exhibit larger and darker convergence regions with significantly reduced chaotic behavior compared to **CM-2**, **CM-4**, and particularly **CM-5**, which demonstrates an extensive divergence region. Although **CM-1** and **CM-3** display relatively smaller divergence regions, they exhibit higher chaotic behavior, making them less stable in this case. A similar pattern is observed for the complex polynomial $z_4(\rho)$, where **SFHM-1** and **SFHM-2** consistently demonstrate superior stability and broader convergence regions. In the case of $z_5(\rho)$, these methods continue to perform effectively, maintaining a large convergence region with minimal chaotic behavior and reduced divergence. In contrast, the other methods exhibit significantly larger divergence regions and pronounced chaotic behavior, leading to slower convergence and instability in initial conditions.

Consequently, the results indicate that **SFHM-1** and **SFHM-2** maintain their stability even as the degree of the complex polynomial increases, confirming their effectiveness for high-degree polynomials. Their ability to provide larger, more stable convergence regions while ensuring accuracy and efficiency establishes them as superior alternatives to the existing techniques **CM-1**-**CM-5**.

6. Conclusion

In this research, we have presented novel four-point iterative methods, referred to as **SFHM-1** and **SFHM-2**, which are developed within the framework of a weighted Steffensen-Ostrowski family. These methods utilize the Lagrange interpolation technique, involving five function evaluations per iteration, achieving a validated sixteenth-order convergence and supporting the Kung-Traub conjecture. Additionally, we have demonstrated the practical advantages of these methods by applying them to complex, real-world scenarios, including Kepler's celestial motion, ideally mixed reactors, and models of predator-prey dynamics, neural activation, and periodic ecosystem growth. Compared to existing iterative techniques such as **CM-1**, **CM-2**, **CM-3**, **CM-4**,and **CM-5**, our methods exhibit superior convergence speed and efficiency. A detailed basin of attraction analysis further illustrates the robustness and adaptability of **SFHM-1** and **SFHM-2** across a diverse range of scenarios. The consistency between our theoretical predictions and numerical results underscores the practical relevance and reliability of the proposed methods. Our findings suggest that the **SFHM-1** and **SFHM-2** schemes provide a competitive alternative for solving nonlinear equations, with potential applications across various complex problem domains, encouraging further research and practical exploration.

Future directions in this field involve extending higher-order iterative methods to complex, multi-dimensional problems and systems of equations. Exploring hybrid approaches that integrate machine learning for parameter estimation holds significant potential for boosting convergence rates and enhancing robustness. Additionally, further theoretical investigations into stability, error dynamics, and adaptation to diverse nonlinear problems will contribute to the broader applicability of these methods. This progress could lead to more efficient and reliable numerical solvers capable of addressing the growing complexity of real-world applications across various scientific and engineering disciplines.

Acknowledgements

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