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Dual Approach to the Generalization of Extended Bessel Function

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Abstract. In this paper, we will discuss the geometrical interpretation of generalized Bessel function, which is defined as:

$$ {}_kH_{\xi,b}(z) = z \ . \ {}_kh_{\xi,b}(z) \ = z + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^{r+1}}{r! \ 4^r \ k^r \ (\xi)_{r,k}} $$

where $\xi = v + k \in (0, +\infty)$, $k \in \mathbb{R}^+$, $v > -k$, $b \in \mathbb{R}$. The generalization of Pochammer's symbol in the form of inequality:

$$ (q)_{r,k} > q(q+\beta)^{r-1} $$

for $q > 0, k \in \mathbb{R}^+, 0 \leq \beta \leq \beta_0 = \sqrt{2} \simeq 1.4142..., r \in \mathbb{N} \backslash \{1, 2\}$, which is proved by using the generalization of Lemma [1]. This has been proved by many authors by using different methods. Using this inequality to analyse the order of starlikeness and convexity. We will prove this Lemma by the same technique used by Zayed and Bulboaca (partial derivative and two-variable extremum technique). We will give the geometrical interpretation of Generalized Bessel k-function for different values of k. Providing some examples for better understanding of the reader regarding our approach.

2020 Mathematics Subject Classifications: 30C45, 33C10, 33B15

Key Words and Phrases: Univalent, Starlikeness, Convexity, Pochammer, Gamma Function, Generalized Bessel function

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1. Introduction

Mathematical Functions that we call Special Functions are very important in different fields like Applied Mathematics, Physics, Economics, Engineering, Statistics etc. Its a fact that Special Functions have an imperative space in the solution of many problems. Due to the unique properties of special functions, these functions play crucial role in solving differential equations used as mathematical models in different fields. It is linked strongly with analytic functions series expansion for real and complex variables. Lots of work has been done in this field of mathematics such as Gamma Functions, Pochammer, Bessel functions, Hypergeometric Functions, Zeta Function etc.

Here we restrict to the study of Bessel functions which also have important role in mathematical modeling. Friedrich Wilhelm Bessel was the first who used Bessel functions to study three body motion. Bessel functions are usually used in solutions of cylindrical coordinates Boundary value problems. Bessel functions are originated as solution of Bessel equation [2], that is,

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w = 0,$$

where ν is the order of Bessel Equation.

Bessel functions are of various kinds. The Bessel function of 1st kind is defined by:

$$J_\nu(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\Gamma(\nu+k+1)\Gamma(k+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$

Bessel function of 2nd kind is also reffered as **Weber Function** or **Neuman Function**, defined as:

$$Y_\nu(x) = \frac{J_{-\nu}(z)cos(\nu\pi) - J_{-\nu}(z)}{sin(\nu\pi)}.$$

Here is a 3rd kind named as **Hankel Function** defined as:

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z).$$

For more details about Bessel functions see [2].

Extending the Bessel functions, we have the generalized Bessel functions [3] defined as:

$$^{\tau}J_\nu^1(\lambda z) = e^{-\tau z} x^{\frac{\nu}{2}} J_\nu(\lambda\sqrt{z})$$

$$(\lambda, \tau \in \mathbb{R}^+)$$

$$^{\tau}J_\nu^2(\lambda z) = e^{\tau z} x^{\frac{-\nu}{2}} J_\nu(\lambda\sqrt{z}).$$

Cesarano and Assante [4] introduced the two-index cylinder generalized Bessel function

$$J_{n,\nu}(z) = \sum_{s=-\infty}^{\infty} J_{n-s}(z)J_{\nu-s}(z)J_s(z).$$

Bessel function has a wide variety of applications. It is commonly used in the solution of physical 2nd order differential equation problems [5]. Parand and Nikarya [6] attempted to solve fractional differential equations using Bessel function. Faisal *et.al.* [7] gives the solution of Schrodinger equation in a cylindrical function using Bessel function. More applications of Bessel functions can be seen in [8, 9].

Researchers are currently working in area of generalized Bessel functions including k, (s, k) , (p, k), q-Bessel functions and also properties of Bessel, modified Bessel functions and generalized Bessel functions. One can refer for the recent researches on generalized Special functions to [10–16].

Univalent function is defined as the function whose domain is meromorphic and injective. In other words, the function $g : D \to \mathbb{C}^*$ is univalent in D if and only if it is analytic in domain except at most one pole and

$$g(x_1) \neq g(x_2), \qquad (x_1, x_2 \in D, x_1 \neq x_2).$$

For more details about univalent functions see [17].

Let S be a set of univalent (meromorphic and injective domain) functions g in $\mathbb{U}$ where $\mathbb{U} := \{y \in \mathbb{C} : |y| < 1\}$ with $g(0) = 0$ and $g'(0) = 1$. Power expansion series [1] for such functions is of the form:

$$g(x) = \sum_{k=1}^{\infty} g_k x^k. \tag{1}$$

The class S is compact (locally bounded and closed) having the most classic example is **Koebe Function**, defined as:

$$k(y) = y(1-y)^{-2} = \frac{1}{4}\left[\left(\frac{1+y}{1-y}\right)^2 - 1\right] = \sum_{k=1}^{\infty} k y^k.$$

Here's involve the square of Cayley transformation $y \to \frac{1+y}{1-y} \in S$ as it is normalized.

A function is known as *starlike* if its mapping from $\mathbb{U}$ onto domain D is starlike. If the domain is *convex*, we can simply say it a convex function. Having a convex domain means that if line segment that joins any two points must lie in domain [18].

Let S^* and K be subclass of S where domain is starlike and convex with respect to origin respectively. Moreover it is known [19, 20] that $g(z) \in A$ is starlike if and only if

$$Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \qquad z \in \mathbb{U} \tag{2}$$

where A be the set of analytic (having complex derivatives) functions.

Taking it as of order η, we denote it $S^*(\eta)$ satisfies the condition:

$$Re\left(\frac{zg'(z)}{g(z)}\right) > \eta, \qquad z \in \mathbb{U}, \qquad 0 \leq \eta \leq 1. \tag{3}$$

Also convex function is characterized as $g(z) \in A$ satisfies the condition:

$$1 + Re\left(\frac{zg''(z)}{g'(z)}\right) > 0, \qquad z \in \mathbb{U}. \tag{4}$$

Similarly for order η, we can say

$$1 + Re\left(\frac{zg''(z)}{g'(z)}\right) > \eta, \qquad z \in \mathbb{U}, \quad 0 \leq \eta \leq 1. \tag{5}$$

As $S^*(\eta) \subset S^*(0) =: S^* \subset S$, $K(\eta) \subset K(0) =: K \subset S$ and $K(0) \subset S^*(0) \subset S$. For $\beta < 0$, $K(\eta) \nsubseteq S^*$.

Teodor and Hanaa [1] proved the inequality

$$(a)_m > a(a+\alpha)^{m-1}, \qquad a > 0, \qquad 0 \leq \alpha \leq \sqrt{2}$$

by partial derivative and variable extremum method for $m \in \mathbb{N}\backslash\{1, 2\}$ and used the inequality to examine the order of convexity as well as starlikeness of generalized Bessel function. Many other researchers [1, 21] have proved its classical form. Teodor and Hanaa [1] gave the graphical representations of generalized Bessel function, defined by:

$$W_{\xi,a} = x + \sum_{m=1}^{\infty} \frac{(-a)^m}{4^m (1)_m (\xi)_m} x^{m+1}, \qquad x \in \mathbb{U}$$

where $\left(\xi = c + \frac{d+2}{2}\right) \notin \{0, -1, -2, ...\}$ and Pochhammer's symbol defined by:

$$(a)_p = (a)(a+1)(a+2)...(a+p-1).$$

In this research paper, we discussed the starlikeness and convexity of order η for normalized form of generalized Bessel function. Also proved the following inequality by partial derivative and variable extremum method:

$$(q)_{r,k} > q(q+\beta)^{r-1}.$$

We also discussed some special cases and examples related to starlikeness and convexity of normalized form of generalized Bessel function and also discussed starlikeness and convexity conditons by using Silverman's theorem.

A well-known homogenous differential equation given explicitly by [22]

$$z^2y''(z) + zy'(z) + \frac{1}{k^2}(bz^2k - v^2)y(z) = 0,$$

gives solution as generalized Bessel k-function, with $k \in \mathbb{R}^+$ and $v > -k$.

The generalized Bessel function is defined as:

$$_kW_{r,b}(z) = \sum_{r=0}^{\infty} \frac{(-b)^r}{r!\Gamma_k(rk + v + k)} \left(\frac{z}{2}\right)^{2r+\frac{v}{k}}, \quad k \in \mathbb{R}^+, \quad v > -k, \quad b \in \mathbb{R}. \tag{6}$$

Γ_k stands for k-Gamma defined as:

$$\Gamma_k(a) = \int_0^{\infty} t^{a-1} e^{\frac{-t^k}{k}} dt, \quad Re(a) > 0.$$

Some basic properties of k-Gamma functions [23] are:

$$\Gamma_k(a) = k^{\frac{a}{k}-1}\Gamma\left(\frac{a}{k}\right)$$

$$\Gamma_k(a) = a\Gamma_k(a)$$

$$\Gamma_k(k) = 1.$$

Observe that if $k = 1$ and $b = 1$, the function reduced to classic Bessel function J_v

$$J_v(z) = \quad {}_1W_{v,1}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r + v + 1)} \left(\frac{z}{2}\right)^{2r+v}.$$

If $k = 1$ and $b = -1$, the function reduced to modified Bessel function I_v

$$I_v(z) = \quad {}_1W_{v,-1}(z) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r + v + 1)} \left(\frac{z}{2}\right)^{2r+v}.$$

k-digamma function [24] is defined as the logarithmic derivative of k-Gamma function, which is given as:

$$\psi_k(a) = \frac{\partial}{\partial a} \log \Gamma_k(a) = \frac{\Gamma'_k(a)}{\Gamma_k(a)}. \tag{7}$$

Some basic properties of k-digamma function [24, 25] are:

$$\psi_k(a+k) = \psi_k(a) + \frac{1}{a}$$
$$\psi_k(a) = \frac{\ln k}{k} + \frac{1}{k}\psi\left(\frac{x}{k}\right).$$

The function $z \rightarrow_k W_{v,b}(z) \notin A$. So, we define a function originating from ${}_kW_{v,b}(z)$ as:

$$\begin{aligned} {}_kh_{v,b}(z) &= \left(2\sqrt{k}\right)^{\frac{v}{k}} \Gamma_k(v+k)\ z^{-\frac{v}{2k}}\ {}_kW_{v,b}(\sqrt{\frac{z}{k}}) \\ &= \left(2\sqrt{k}\right)^{\frac{v}{k}} \Gamma_k(v+k)\ z^{-\frac{v}{2k}} \left[\sum_{r=0}^{\infty} \frac{(-b)^r \left(\frac{\sqrt{z}}{2\sqrt{k}}\right)^{2r+\frac{v}{k}}}{r!\ \Gamma_k(rk+v+k)}\right] \\ &= \sum_{r=0}^{\infty} \frac{(-b)^r\ z^r}{r!\ 4^r\ k^r\ (v+k)_{r,k}} \\ &= \sum_{r=0}^{\infty} \frac{(-b)^r\ z^r}{r!\ 4^r\ k^r\ (\xi)_{r,k}}, \end{aligned} \tag{8}$$

where $\xi = v + k \notin \{0, -1, -2, ...\}$.

2. Main Results

Keeping the above representations in mind, the normalized form of ${}_kh_{\xi,b}(z)$ is defined as:

Definition 1. *For $k \in \mathbb{R}^+$, $v > -k$, $b \in \mathbb{R}$, the normalized form of ${}_kh_{\xi,b}(z)$ is given by:*

$${}_kH_{\xi,b}(z) = z\ .\ {}_kh_{\xi,b}(z) = z + \sum_{r=1}^{\infty} \frac{(-b)^r\ z^{r+1}}{r!\ 4^r\ k^r\ (\xi)_{r,k}}, \tag{9}$$

where $\xi = v + k \in (0, +\infty)$.

We will be in need of the following Lemma in our research. At first, this was proved in classical form in [21], then proved by Bulboaca and Zayed in [1] by partial derivative and variable extremum technique. Now we are proving this Lemma for k-Pochammer by the technique as Bulboaca and Zayed. It is shown for $r \in \mathbb{N}\backslash\{1, 2\},\ q > 0,\ k \in \mathbb{R}^+,\ 0 \leq \beta \leq \sqrt{2}$, the inequality satisfied as follows:

Lemma 1. *If $q > 0,\ k \in \mathbb{R}^+,\ 0 \leq \beta \leq \beta_0 = \sqrt{2} \simeq 1.4142...$, and $r \in \mathbb{N}\backslash\{1, 2\}$, the results will sharp*

$$(q)_{r,k} > q(q+\beta)^{r-1}. \tag{10}$$

Proof. Take $f_k : (0, +\infty) \times [3, +\infty) \to \mathbb{R}$ be defined by:

$$f_k(q, rk) = \frac{\Gamma_k(q + rk)}{\Gamma_k(q + k)}(q + \beta)^{1-r} - 1, \tag{11}$$

where $0 \leq \beta \leq 2$. By simple computation,

$$\begin{aligned}
\frac{\partial}{\partial r}(f_k(q, rk)) &= \frac{1}{\Gamma_k(q+k)} \left[\frac{\partial}{\partial r} \left(\Gamma_k(q+rk)\ (q+\beta)^{1-r}\ - 1 \right) \right] \\
&= \frac{1}{\Gamma_k(q+k)} \left[\Gamma_k(q+rk) \frac{\partial}{\partial r}(q+\beta)^{1-r} + (q+\beta)^{1-r} \frac{\partial}{\partial r} \Gamma_k(q+rk) \right] \\
&= \frac{(q+\beta)^{1-r}}{\Gamma_k(q+k)} \left[-\Gamma_k(q+rk)\ ln(q+\beta) + \Gamma_k(q+rk)\ \psi_k(q+rk) \right],
\end{aligned}$$

where ψ_k is k-digamma function defined in eq. 7.

$$\begin{aligned}
\frac{\partial}{\partial r}(f_k(q, rk)) &= \frac{\Gamma_k(q+rk)\ (q+\beta)^{1-r}}{\Gamma_k(q+k)} \left[\ \psi_k(q+rk) - ln(q+\beta)\ \right] \\
&= \frac{\Gamma_k(q+rk)\ (q+\beta)^{1-r}}{\Gamma_k(q+k)}\ G_k(q, rk),
\end{aligned} \tag{12}$$

where $(q, rk) \in (0, \infty) \times [3, \infty)$ and

$$G_k(q, rk) = \psi_k(q + rk) - ln(q + \beta). \tag{13}$$

Using

$$\Gamma_k(z) = \int_0^{+\infty} t^{z-1}\ e^{-\frac{t^k}{k}}\ dt$$

implies that $\Gamma_k(q + k) > 0,\ \Gamma_k(q + rk) > 0$ for all $(q, rk) \in (0, \infty) \times [3, \infty)$.

As $\beta > 0$, then by eq 13, the sign of $\frac{\partial}{\partial r} f_k(q + rk)$ is same as of $G_k(q, rk)$. As it is well known by [26], for $x > 0$ and $0 < k \leq 1$, we have

$$\frac{1}{k}\ lny - \frac{1}{y} < \psi_k(y) < \frac{1}{k} lny. \tag{14}$$

Using eq. 14 in eq. 13, we get

$$G_k(q, rk)\ \geq\ \frac{1}{k}\ ln(q+rk) - \frac{1}{q+rk} - ln(q+\beta)\ =\ ln\frac{(q+rk)^{\frac{1}{k}}}{q+\beta} - \frac{1}{q+rk}, \qquad \beta > 0,\ r \geq 3.$$

$$G_k(q, rk)\ \geq\ ln\frac{(q+3k)^{\frac{1}{k}}}{q+\beta}\ -\ \frac{1}{q+3k}\ = I_k(q), \tag{15}$$

where $I_k(q) = ln\frac{(q+3k)^{\frac{1}{k}}}{q+\beta} - \frac{1}{q+3k}$.

By taking derivative of $I_k(q)$,

$$\begin{aligned}
I_k'(q) &= \frac{1}{\frac{(q+3k)^{\frac{1}{k}}}{q+\beta}} \left[\frac{d}{dq}\left(\frac{(q+3k)^{\frac{1}{k}}}{q+\beta}\right)\right] + \frac{1}{(q+3k)^2} \\
&= \frac{q+\beta}{(q+3k)^{\frac{1}{k}}} \left[\frac{(q+\beta)\ \frac{1}{k}\ (q+3k)^{\frac{1}{k}-1} - (q+3k)^{\frac{1}{k}}}{(q+\beta)^2}\right] + \frac{1}{(q+3k)^2} \\
&= \frac{1}{k(q+3k)} - \frac{1}{q+\beta} + \frac{1}{(q+3k)^2} \\
&= \frac{1}{q+3k}\left[\frac{1}{k} - \frac{q+3k}{q+\beta} + \frac{1}{q+3k}\right], \qquad q>0,\ 0\leq\beta\leq 2. \qquad (16)
\end{aligned}$$

And we have the following equivalences:

$$I_k'(q) < 0 \quad \Leftrightarrow \quad \frac{q+3k}{q+\beta} > \frac{1}{k} + \frac{1}{q+3k} \quad \Leftrightarrow \quad \beta < \frac{k(q+3k)^2}{q+4k} - q \qquad (17)$$

because

$$inf\left\{\frac{k(q+3k)^2}{q+4k} - q : q > 0\right\} = 2.$$

If $\beta \leq 2$, the inequality $I_k'(q) < 0$ is satisfied, for all $q > 0$. Hence function $I_k(q)$ is decreasing strictly on $(0, +\infty)$,

$$\Rightarrow I_k(q) > \lim_{x\to+\infty} I_k(y) = 0,\ q > 0. \qquad (18)$$

By combining eqs. 12 and 15

$$\frac{\partial}{\partial r}(f_k(q, rk)) > 0, \qquad (q, rk) \in (0, \infty) \times [3, \infty)$$

which implies $f_k(q, rk)$ is strictly increasing function on $r \in [3, +\infty)$, $q > 0$.

$$\begin{aligned}
f_k(q, rk) \geq f_k(q, 3k) &= \frac{\Gamma_k(q+3k)}{\Gamma_k(q+k)}(q+\beta)^{-2} - 1 \\
&= \frac{\Gamma_k(q+3k)}{\Gamma_k(q+k)(q+\beta)^2} - 1
\end{aligned}$$

$$= \frac{(q+2k)(q+k)\Gamma_k(q+k)}{\Gamma_k(q+k)(q+\beta)^2} - 1$$
$$= \frac{(q+2k)(q+k)}{(q+\beta)^2} - 1. \tag{19}$$

Since $\frac{(q+2k)(q+k)}{(q+\beta)^2} - 1 > 0$ under conditions $q > 0,\ \beta > 0$, Therefore, we have

$$(q+2k)(q+k) > (q+\beta)^2$$
$$|q+\beta| < \sqrt{(q+2k)(q+k)}$$
$$\beta < \sqrt{(q+2k)(q+k)} - q = g_k(q). \tag{20}$$

Taking derivative of $g_k(q) = \sqrt{(q+2k)(q+k)} - q$ gives

$$g_k'(q) = \frac{1}{2}[(q+2k)(q+k)]^{-\frac{1}{2}} \frac{d}{da}[(q+2k)(q+k)] - 1$$
$$= \frac{(q+2k)(q+k)}{2\sqrt{(q+2k)(q+k)}} - 1.$$

By easy computation, we have

$$g_k'(q) = \frac{k^2}{2\sqrt{(q+2k)(q+k)}(2q+3k+2\sqrt{(q+2k)(q+k)})} > 0, \qquad q > 0,\ \beta \in \mathbb{R}^+$$

g_k is strictly increasing on $(0, +\infty)$, hence

$$\lim_{x\to 0^+} g_k(q) = \sqrt{2}k < g_k(q) < \lim_{x\to +\infty} g_k(q) = \frac{3k}{2}. \tag{21}$$

Consequently if $\beta \leq \sqrt{2}$

$$f_k(q, rk) > 0, \qquad q > 0,\ r \geq 3,\ k \in \mathbb{R}^+. \tag{22}$$

Left side of inequality 14 could be improved as:

$$inf \left\{ \psi_k(y) - \frac{1}{k} ln(y) + \frac{1}{y} : y > 0 \right\} = 0.$$

As $y \longmapsto \psi_k(y) - \frac{1}{k}ln(y) + \frac{1}{y}$ is decreasing and positive on $(0, +\infty)$ and by inequality 14, we have

$$\lim_{y\to+\infty} \left(\psi_k(y) - \frac{1}{k} ln(y) + \frac{1}{y} \right) = 0. \tag{23}$$

Using eqs. 15 , 19 and 21, we conclude that the value $\sqrt{2}k$ is maximum possible to holds the inequality 22 for sharp results that completes the proof.

3. Starlikeness and Convexity of Order η

In this section, the following Theorems provide the starlikeness and convexity of order η with improved results for generalized Bessel function ${}_kH_{\xi,b}$.

Theorem 1. *Assume $\xi > 0$ and let $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ with*

$$0 < |b| < \frac{4k\xi}{1+\xi} =: b_*. \tag{24}$$

If

$$\eta \leq 1 - \frac{|b|}{\xi(4k - |b|) - |b|} =: \eta_* \tag{25}$$

then ${}_kH_{\xi,b} \in S^*(\eta)$.

Proof. By considering k-form of the condition in [20], we have

$$\left| \frac{z({}_kH_{\xi,b}(z))'}{{}_kH_{\xi,b}(z)} - 1 \right| < 1 - \eta, \qquad \eta \leq 1. \tag{26}$$

Taking

$$\begin{aligned}
\left| ({}_kH_{\xi,b}(z))' - \frac{{}_kH_{\xi,b}(z)}{z} \right| &= \left| \left(z + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^{r+1}}{r! \ 4^r \ k^r \ (\xi)_{r,k}} \right)' - \frac{z + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^{r+1}}{r! \ 4^r \ k^r \ (\xi)_{r,k}}}{z} \right| \\
&= \left| 1 + \sum_{r=1}^{\infty} \frac{(r+1)(-b)^r \ z^r}{r! \ 4^r \ k^r \ (\xi)_{r,k}} - \frac{z \left[1 + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^r}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right]}{z} \right| \\
&= \left| \sum_{r=1}^{\infty} \frac{(r+1) \ (-b)^r \ z^r}{4^r \ k^r \ r! \ (\xi)_{r,k}} - \sum_{r=1}^{\infty} \frac{(-b)^r \ z^r}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right| \\
&= \left| \sum_{r=1}^{\infty} \frac{(r) \ (-b)^r \ z^r}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right| < \sup_{\theta \in 2\pi} \left| \sum_{r=1}^{\infty} \frac{(r) \ (-b)^r \ e^{i\theta r}}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right| \\
&\leq \sum_{r=1}^{\infty} \frac{(r) \ |b|^r}{4^r \ k^r \ r! \ \frac{\Gamma_k(\xi + rk)}{\Gamma_k(\xi)}} \ , \qquad |z| \leq 1 \\
&\leq \frac{\Gamma_k(\xi + k)}{\xi} \sum_{r=1}^{\infty} \frac{|b|^r \ (r)}{4^r \ k^r \ \Gamma_k(r+k)\Gamma_k(\xi + rk)}.
\end{aligned} \tag{27}$$

Letting the function

$$\sigma_k(a) = \frac{a}{\Gamma_k(a+k)\Gamma_k(\xi + ak)}, \qquad r \in \mathbb{N}. \tag{28}$$

By easy computation

$$\begin{aligned}\sigma_k(r+1)-\sigma_k(r) &= \frac{r+1}{\Gamma_k(r+1+k)\Gamma_k(\xi+(r+1)k)}-\frac{r}{\Gamma_k(r+k)\Gamma_k(\xi+rk)}\\ &= \frac{r+1-(r)(r+k)(\xi+rk)}{(r+k)(\xi+rk)\Gamma_k(r+k)\Gamma_k(\xi+rk)}\\ &= \frac{1+r-rk(r^2-\xi)-r^2(k^2+\xi)}{\Gamma_k(r+k+1)\Gamma_k(\xi+rk+k)} \qquad (29)\end{aligned}$$

which implies that

$$\sigma_k(r+1)-\sigma_k(r)<0,\ r\in\mathbb{N}. \qquad (30)$$

Hence the function is strictly decreasing, so:

$$\frac{a}{\Gamma_k(a+k)\Gamma_k(\xi+ak)}\leq\sigma_k(1)=\frac{1}{\Gamma_k(1+k)\Gamma_k(\xi+k)}.$$

By the inequality 27, we get:

$$\begin{aligned}\left|({}_kH_{\xi,b}(z))'-\frac{{}_kH_{\xi,b}(z)}{z}\right| &\leq \frac{\Gamma_k(\xi+k)}{\xi}\sum_{r=1}^{\infty}\frac{|b|^r}{4^r\ k^r\ \Gamma_k(1+k)\Gamma_k(\xi+k)}\\ &= \frac{1}{\xi}\sum_{r=1}^{\infty}\left(\frac{|b|}{4k}\right)^r\\ &= \frac{|b|}{\xi(4k-|b|)}\\ \Rightarrow \quad \left|({}_kH_{\xi,b}(z))'-\frac{{}_kH_{\xi,b}(z)}{z}\right| &\leq \frac{|b|}{\xi(4k-|b|)}, \qquad z\in\mathbb{U},\ k\in\mathbb{R}^+. \qquad (31)\end{aligned}$$

Now by assuming the above inequality $\frac{|b|}{\xi(4k-|b|)}>0$, equivalence to $0<|b|<4k$ which holds according to 35 and the assumption

$$0<|b|<\frac{4k\xi}{1+\xi} \qquad (32)$$

The case $|b|=0$ is the trivial case gives the identity function.

Now taking the other part, we have

$$\left|\frac{{}_kH_{\xi,b}(z)}{z}\right|=\left|1+\sum_{r=1}^{\infty}\frac{(-b)^r\ z^r}{r!\ 4^r\ k^r\ (\xi)_{r,k}}\right|.$$

By using triangular inequality and the modulus theorem:

$$\left|\frac{{}_kH_{\xi,b}(z)}{z}\right|>1-\sup_{\theta\in 2\pi}\left|\sum_{r=1}^{\infty}\frac{(-b)^r\ e^{i\theta r}}{4^r\ k^r\ r!\ (\xi)_{r,k}}\right|$$

$$\geq 1 - \sum_{r=1}^{\infty} \frac{|b|^r}{4^r \ k^r \ r! \ (\xi)_{r,k}}$$
$$= 1 - \sum_{r=1}^{\infty} \frac{|b|^r \ \Gamma_k(\xi)}{4^r \ k^r \ \Gamma_k(\xi + rk) \ \Gamma_k(r+k)}$$
$$= 1 - \frac{\Gamma_k(\xi + k)}{\xi} \sum_{r=1}^{\infty} \frac{|b|^r}{4^r \ k^r \ \Gamma_k(\xi + rk) \ \Gamma_k(r+k)}.$$

As we know that $\frac{1}{\Gamma_k(\xi+rk) \ \Gamma_k(r+k)}$ is strictly decreasing, we have

$$\left| \frac{{}_kH_{\xi,b}(z)}{z} \right| > 1 - \frac{\Gamma_k(\xi + k)}{\xi} \sum_{r=1}^{\infty} \frac{|b|^r}{4^r \ k^r \ \Gamma_k(\xi + k) \ \Gamma_k(1+k)}$$
$$= 1 - \frac{1}{\xi} \sum_{r=1}^{\infty} \left(\frac{|b|}{4k} \right)^r$$
$$= 1 - \frac{|b|}{\xi(4k - |b|)}$$
$$= \frac{\xi(4k - |b|) - |b|}{\xi(4k - |b|)} \tag{33}$$

where

$$\frac{\xi(4k - |b|) - |b|}{\xi(4k - |b|)} > 0. \tag{34}$$

Above equation holds because $\xi > 0$ and

$$|b| < \min \left\{ 4; \frac{4k\xi}{1+\xi} \right\} = \frac{4k\xi}{1+\xi}. \tag{35}$$

Since

$$\left| \frac{z({}_kH_{\xi,b}(z))'}{{}_kH_{\xi,b}(z)} - 1 \right| = \left| ({}_kH_{\xi,b}(z))' - \frac{{}_kH_{\xi,b}(z)}{z} \right| \left| \frac{z}{{}_kH_{\xi,b}(z)} \right|$$
$$< \frac{|b|}{\xi(4k - |b|)} \ \times \ \frac{\xi(4k - |b|)}{\xi(4k - |b|) - |b|}$$
$$< \frac{|b|}{\xi(4k - |b|) - |b|} \leq 1 - \eta$$

which gives

$$\eta \leq 1 - \frac{|b|}{\xi(4k - |b|) - |b|}. \tag{36}$$

Finally from inequality 26, it is proved that ${}_kH_{\xi,b} \in S^*(\eta)$.

Theorem 2. *Assume* $\xi > \frac{1}{2}$ *and let* $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ *with*

$$0 < |b| < \frac{4k\xi}{2+\xi} =: b_c. \tag{37}$$

If

$$\eta \leq 1 - \frac{2|b|}{\xi(4k - |b|) - 2|b|} =: \eta_* \tag{38}$$

then ${}_kH_{\xi,b} \in \mathrm{K}(\eta)$.

Proof. By considering k-form of the condition in [20], we have

$$\left| \frac{z({}_kH_{\xi,b}(z))''}{({}_kH_{\xi,b}(z))'} \right| < 1 - \eta, \qquad \eta \leq 1. \tag{39}$$

Taking the part:

$$\begin{aligned}
\left| z({}_kH_{\xi,b}(z))'' \right| &= \left| z \left(z + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^{r+1}}{r! \ 4^r \ k^r \ (\xi)_{r,k}} \right)'' \right| \\
&= \left| z \left(1 + \sum_{r=1}^{\infty} \frac{(r+1)(-b)^r \ z^r}{r! \ 4^r \ k^r \ (\xi)_{r,k}} \right)' \right| \\
&= \left| z \left(\sum_{r=1}^{\infty} \frac{(r+1) \ (r) \ (-b)^r \ k^r \ z^{r-1}}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right) \right| \\
&= \left| \sum_{r=1}^{\infty} \frac{(r+1) \ (r) \ (-b)^r \ z^r}{4^r \ k^r \ r! \ (\xi)_{r,k}} \right|.
\end{aligned}$$

By using maximum modulus theorem:

$$\begin{aligned}
\left| z({}_kH_{\xi,b}(z))'' \right| &= \left| \sum_{r=1}^{\infty} \frac{(r+1)(r)(-b)^r \ z^r}{4^r k^r r! (\xi)_{r,k}} \right| \\
&< \sup_{\theta \in 2\pi} \left| \sum_{r=1}^{\infty} \frac{(r+1)(r)(-b)^r \ e^{i\theta r}}{4^r \ k^r r! (\xi)_{r,k}} \right| \\
&\leq \sum_{r=1}^{\infty} \frac{(r+1)(r)|b|^r}{4^r k^r r! \frac{\Gamma_k(\xi + rk)}{\Gamma_k(\xi)}}, \qquad |z| \leq 1 \\
&= \frac{\Gamma_k(\xi + k)}{\xi} \sum_{r=1}^{\infty} \frac{|b|^r (r)(r+1)}{4^r k^r \Gamma_k(r+k) \Gamma_k(\xi + rk)}.
\end{aligned} \tag{40}$$

Letting the function

$$\varrho_k(a) = \frac{(a)(a+1)}{\Gamma_k(a+k)\Gamma_k(\xi + ak)}, \qquad r \in \mathbb{N}. \tag{41}$$

By easy computation

$$\begin{aligned}\varrho_k(r+1)-\varrho_k(r) &= \frac{(r+1)(r+2)}{\Gamma_k(r+1+k)\Gamma_k(\xi+(r+1)k)} - \frac{(r)(r+1)}{\Gamma_k(r+k)\Gamma_k(\xi+rk)} \\ &= \frac{r+1}{\Gamma_k(r+k)\Gamma_k(\xi+rk)}\left[\frac{r+2}{(r+k)(\xi+rk)} - r\right] \\ &= \frac{r+1}{\Gamma_k(r+k)\Gamma_k(\xi+rk)}\left[\frac{r+2-(r)(r+k)(\xi+rk)}{(r+k)(\xi+rk)}\right] \\ &= -\frac{r^2\left(\xi+k^2\right)+\xi kr+kr^3-r-2}{\Gamma_k(r+1+k)\Gamma_k(\xi+rk+k)}\end{aligned} \tag{42}$$

which implies that:

$$\varrho_k(r+1)-\varrho_k(r) < 0, \ r\in\mathbb{N}. \tag{43}$$

Hence the function is decreasing strictly, so:

$$\frac{(a)(a+1)}{\Gamma_k(a+k)\Gamma_k(\xi+ak)} \leq \varrho_k(1) = \frac{2}{\Gamma_k(1+k)\Gamma_k(\xi+k)}.$$

By the inequality 40, we get:

$$\begin{aligned}\left|z({}_kH_{\xi,b}(z))''\right| &< \frac{\Gamma_k(\xi+k)}{\xi}\sum_{r=1}^{\infty}\frac{2\ |b|^r}{4^r\ k^r\ \Gamma_k(1+k)\Gamma_k(\xi+k)} \\ &= \frac{2}{\xi}\sum_{r=1}^{\infty}\left(\frac{|b|}{4k}\right)^r \\ &= \frac{2|b|}{\xi(4k-|b|)}\end{aligned}$$

$$\left|z({}_kH_{\xi,b}(z))''\right| < \frac{2|b|}{\xi(4k-|b|)}, \qquad z\in\mathbb{U},\ k\in\mathbb{R}^+. \tag{44}$$

Now by assuming the above inequality 44 is greater than 0, gives

$$0 < |b| < 4k, \tag{45}$$

which holds because of our assumption. The case $|b|=0$ is the trivial case gives the identity function.

Now the other part, we have

$$\left|({}_kH_{\xi,b}(z))'\right| = \left|\left(z+\sum_{r=1}^{\infty}\frac{(-b)^r\ z^{r+1}\Gamma_k(\xi)}{k^r\ 4^r\ \Gamma_k(r+k)\ \Gamma_k(\xi+rk)}\right)'\right|$$

$$= \left| 1 + \sum_{r=1}^{\infty} \frac{(-b)^r \ z^r \ (r+1)\Gamma_k(\xi+k)}{\xi \ k^r \ 4^r \ \Gamma_k(r+k) \ \Gamma_k(\xi+rk)} \right|.$$

By using triangular inequality and the modulus theorem:

$$\left|({}_kH_{\xi,b}(z))'\right| > |e^{i\theta r}| - \sup_{\theta \in 2\pi} \left| \sum_{r=1}^{\infty} \frac{(r+1) \ (-b)^r \ e^{i\theta r} \ \Gamma_k(\xi+k)}{4^r \ k^r \ \Gamma_k(r+k) \ \Gamma_k(\xi+rk)} \right|$$

$$\geq 1 - \frac{\Gamma_k(\xi+k)}{\xi} \sum_{r=1}^{\infty} \frac{(r+1) \ |b|^r}{4^r \ k^r \ \Gamma_k(\xi+rk) \ \Gamma_k(r+k)}.$$

As we know that $\frac{1}{\Gamma_k(\xi+rk) \ \Gamma_k(r+k)}$ is strictly decreasing, we have

$$\left|({}_kH_{\xi,b}(z))'\right| > 1 - \frac{\Gamma_k(\xi+k)}{\xi} \sum_{r=1}^{\infty} \frac{2 \ |b|^r}{4^r \ k^r \ \Gamma_k(\xi+k) \ \Gamma_k(1+k)}$$

$$= 1 - \frac{2}{\xi} \sum_{r=1}^{\infty} \left(\frac{|b|}{4k} \right)^r$$

$$= 1 - \frac{2|b|}{\xi(4k-|b|)}$$

$$= \frac{\xi(4k-|b|) - 2|b|}{\xi(4k-|b|)} \tag{46}$$

where

$$\frac{\xi(4k-|b|) - 2|b|}{\xi(4k-|b|)} > 0. \tag{47}$$

Above equation holds because $\xi \geq 0$ and

$$|b| < \min \left\{ 4; \frac{4k\xi}{2+\xi} \right\} = \frac{4k\xi}{2+\xi}. \tag{48}$$

Since

$$\left| \frac{z({}_kH_{\xi,b}(z))''}{({}_kH_{\xi,b}(z))'} \right| = \left| z({}_kH_{\xi,b}(z))'' \right| \ \left| \frac{1}{({}_kH_{\xi,b}(z))'} \right|$$

$$< \frac{2|b|}{\xi(4k-|b|)} \ \times \ \frac{\xi(4k-|b|)}{\xi(4k-|b|) - 2|b|}$$

$$< \frac{2|b|}{\xi(4k-|b|) - 2|b|} \leq 1 - \eta$$

which gives

$$\eta \leq 1 - \frac{2|b|}{\xi(4k-|b|) - 2|b|}. \tag{49}$$

Finally from inequality 39, it is proved that ${}_kH_{\xi,b} \in K(\eta)$.

4. Examples

This section illustrates some examples to check convexity and starlikeness of different orders:

Example 1. *Taking* $\xi = 2.2565$, $b = 0.0253$, $\eta = 0$ *and* $k = 0.1$. *To check* ${}_{0.1}H_{2.2565,0.0253} \in S^*(0)$ *or* ${}_{0.1}H_{2.2565,0.0253} \in \mathrm{K}(0)$ *, we will check the sufficient conditions for starlikeness and convexity, that is:*
For $\xi > 0$ *and let* $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ *with*

$$0 < |b| < \frac{4k\xi}{1+\xi} =: b_*. \tag{50}$$

If

$$\eta \leq 1 - \frac{|b|}{\xi(4k-|b|)-|b|} =: \eta_* \tag{51}$$

then ${}_kH_{\xi,b} \in S^*(\eta)$. *Above parameter values satisfy the condition;*

$$0 < |0.0253| < \frac{4 \times 0.1(2.2565)}{2.2565+1} = 0.277169.$$

and

$$0 \leq 1 - \frac{|0.0253|}{2.2565(4 \times 0.1 - |0.0253|) - |0.0253|} = 0.969154.$$

Both conditions satisfied, hence ${}_{0.1}H_{2.2565,0.0253} \in S^*(0)$.
Now check the conditions for convexity: For $\xi > \frac{1}{2}$ *and let* $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ *with*

$$0 < |b| < \frac{4k\xi}{2+\xi} =: b_c. \tag{52}$$

If

$$\eta \leq 1 - \frac{2|b|}{\xi(4k-|b|)-2|b|} =: \eta_* \tag{53}$$

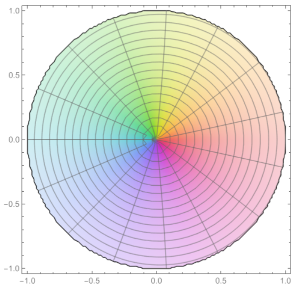
then ${}_kH_{\xi,b} \in \mathrm{K}(\eta)$. *Above parameter values satisfy the condition;*

$$0 < |0.0253| < \frac{4 \times 0.1(2.2565)}{2.2565+2} = 0.212052.$$

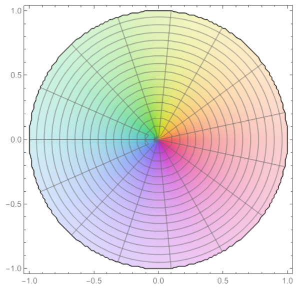
and

$$0 \leq 1 - \frac{2|0.0253|}{2.2565(4(0.1) - |0.0253|) - 2|0.0253|} = 0.936345.$$

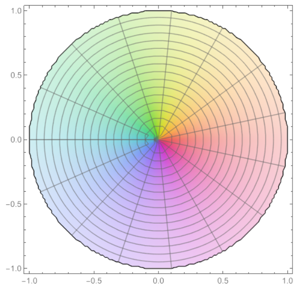
Both conditions satisfied, hence ${}_{0.1}H_{2.2565,0.0253} \in \mathrm{K}(0)$.



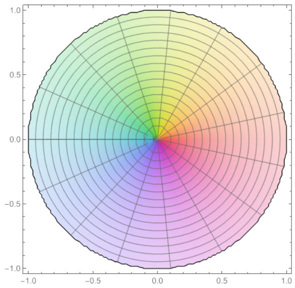
(a) ${}_{0.1}H_{2.2565,0.0253}$



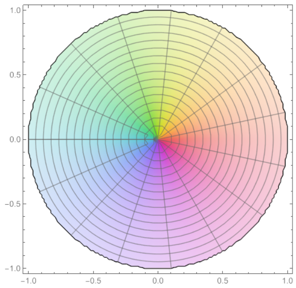
(b) ${}_{0.4}H_{2.2565,0.0253}$



(c) ${}_{0.8}H_{2.2565,0.0253}$



(d) ${}_{1}H_{2.2565,0.0253}$



(e) $H_{2.2565,0.0253}$

Figures are the illustrations of the Example 1 which show the contour plot of generalized Bessel k-function for different values of k. It can be observed that the graphs show the symmetrical behaviour and provide a way to check the accuracy of our results i.e., as k approaches to 1, the graphical behaviour will approach the classical form of generalized Bessel function. Exactly at $k = 1$, the graphs are same.

Example 2. *Taking $\xi = 2.5$, $b = 1.2$, $\eta = 0.32$ and $k = 0.6$. To check ${}_{0.6}H_{2.5,1.2} \in S^*(0.32)$ or ${}_{0.6}H_{2.5,1.2} \in \mathrm{K}(0.32)$.*

We will check the sufficient conditions for starlikeness and convexity, that is: For $\xi > 0$ and let $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ with

$$0 < |b| < \frac{4k\xi}{1+\xi} =: b_*. \tag{54}$$

If

$$\eta \leq 1 - \frac{|b|}{\xi(4k - |b|) - |b|} =: \eta_* \tag{55}$$

then ${}_kH_{\xi,b} \in S^*(\eta)$. Above parameter values satisfy the condition;

$$0 < |1.2| < \frac{4 \times 0.6(2.5)}{2.5+1} = 1.71429.$$

Now check for:

$$0.32 \leq 1 - \frac{|1.2|}{2.5(4 \times 0.6 - |1.2|) - |1.2|} = 0.333333.$$

Both conditions satisfied, hence ${}_{0.6}H_{2.5,1.2} \in S^*(0.32)$.

Now check the conditions for convexity: For $\xi > \frac{1}{2}$ and let $b \in \mathbb{C}^*$, $k \in \mathbb{R}^+$ with

$$0 < |b| < \frac{4k\xi}{2+\xi} =: b_c. \tag{56}$$

If

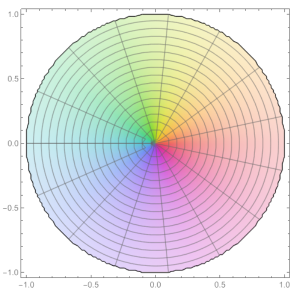
$$\eta \leq 1 - \frac{2|b|}{\xi(4k - |b|) - 2|b|} =: \eta_* \tag{57}$$

then ${}_kH_{\xi,b} \in K(\eta)$. Above parameter values do not satisfy the conditions;

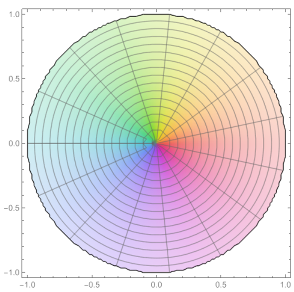
$$0 < |1.2| < \frac{4 \times 0.6(2.5)}{2.5+2} = 1.3333.$$

$$0.32 \nleq 1 - \frac{2 \times 2.5}{2.5(4 \times 0.6 - |1.2|) - 2|1.2|} = -3.$$

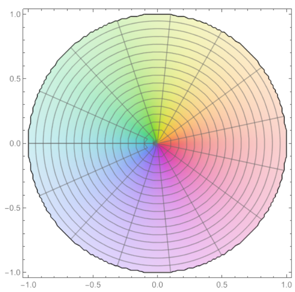
Hence ${}_{0.6}H_{2.5,1.2} \notin K(0.32)$.



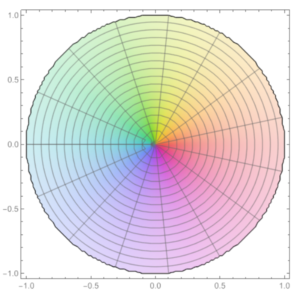
(f) ${}_{0.6}H_{2.5,1.2}$



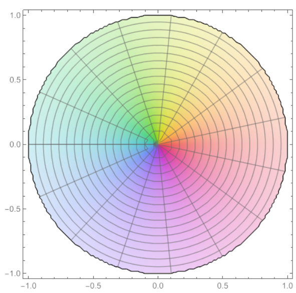
(g) ${}_{0.8}H_{2.5,1.2}$



(h) ${}_{0.9}H_{2.5,1.2}$



(i) ${}_{1}H_{2.5,1.2}$



(j) $H_{2.5,1.2}$

Figures are the illustrations of the Example 2 which show the contour plot of generalized Bessel k-function for different values of k. It can be observed that the graphs show the symmetrical domains with respect to real axis and provide a way to check the accuracy of our results i.e., as k approaches to 1, the graphical behaviour will approach the classical form of generalized Bessel function. Exactly at $k = 1$, the graphs are same.

5. Order of Starlikeness and Convexity by Silverman's Theorem

In this section, the Theorem describes the sufficient condition for starlikeness and convexity of order η by using the Lemma 1 and a result from by Silverman ([27], Theorem 1).

Theorem 3. *Let*

$$
\begin{aligned}
{}_kX_{\xi,b} = &\left(1-\frac{|b|}{4\xi}-\frac{|b|^2}{32\xi(\xi+k)}-\frac{|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4\xi+4\sqrt{2}\xi+4\sqrt{2}k+8k-|b|)}\right)\eta\\
&+\frac{|b|}{2\xi}+\frac{3|b|^2}{32\xi(\xi+k)}+\frac{3|b|^3}{64\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(\xi+\sqrt{2}\xi+\sqrt{2}k+2k-|b|)}\\
&+\frac{|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4\xi+4\sqrt{2}\xi+4\sqrt{2}k+8k-|b|)}-1
\end{aligned}
$$

where $\xi > 2$ *and* $b \in \mathbb{C}^*$. *If there exists* $\eta < 1$ *such that*

$$
{}_kX_{\xi,b} \leq 0, \tag{58}
$$

then ${}_kH_{\xi,b} \in S^*(\eta)$.

Proof. From Silverman's [27] well-known result, if f is in form of eq. 1 and satisfies $\sum_{r=2}^{\infty}(r-\eta)|f_k| \leq 1-\eta$, then $f \in S^*(\eta)$. According to eq. 9, it is sufficient to show:

$$
\begin{aligned}
B_1 &:= \sum_{r=2}^{\infty}(r-\eta)\left|\frac{(-b)^{r-1}}{4^{r-1}(1)_{r-1,k}(\xi)_{r-1,k}}\right| \leq 1-\eta, \qquad \xi>0,\ b\in\mathbb{C}^*\\
&= \sum_{r=2}^{\infty}\frac{(r-\eta)|b|^{r-1}}{4^{r-1}(1)_{r-1,k}(\xi)_{r-1,k}}\\
&= \sum_{r=1}^{\infty}\frac{(r+1-\eta)|b|^{r}}{4^{r}(1)_{r}(\xi)_{r,k}}\\
&= \frac{(2-\eta)|b|}{4\xi}+\frac{(3-\eta)|b|^2}{8(\xi)_{2,k}(1)_{2,k}}+\sum_{r=3}^{\infty}\frac{(1-\eta)|b|^r}{4^r(1)_{r,k}(\xi)_{r,k}}+\sum_{r=3}^{\infty}\frac{(r)|b|^r}{4^r(1)_{r,k}(\xi)_{r,k}}\\
&= \frac{(2-\eta)|b|}{4\xi}+\frac{(3-\eta)|b|^2}{8(\xi)_{2,k}(1)_{2,k}}+(1-\eta)\sum_{r=3}^{\infty}\frac{|b|^r}{4^r(1)_{r,k}(\xi)_{r,k}}+\sum_{r=3}^{\infty}\frac{(r)|b|^r}{4^r(1)_{r,k}(\xi)_{r,k}}.
\end{aligned}
$$

By simple mathematical induction, we have $r \leq \left(\frac{3}{64}\right)4^r$ for all $r \in \mathbb{N}\backslash[1,2]$. By using Lemma, we have

$$
(1)_{r,k} > (1+\beta)^{r-1}, \qquad r\in\mathbb{N}\backslash[1,2], \qquad 0\leq\beta\leq\sqrt{2}.
$$

Since

$$
max(1+\beta)^{r-1} : 0\leq\beta\leq\sqrt{2} = (1+\sqrt{2})^{r-1}
$$

follows that:

$$(\xi)_{r,k} \geq \xi(\xi+\sqrt{2})^{r-1}, \xi > 0. \tag{59}$$

As by eq. 58, ${}_kX_{\xi,b}(\eta) \leq 0$, we have

$$B_1 \leq \frac{(2-\eta)|b|}{4\xi} + \frac{(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{3|b|}{64\xi}\sum_{r=3}^{\infty}\frac{|b|^{r-1}}{(1+\sqrt{2})^{r-1}(\xi+\sqrt{2}k)^{r-1}} + \frac{(1-\eta)|b|}{4\xi}\sum_{r=3}^{\infty}\frac{|b|^{r-1}}{4^{r-1}(1+\sqrt{2})^{r-1}(\xi+\sqrt{2}k)^{r-1}}.$$

After simplifications, we get:

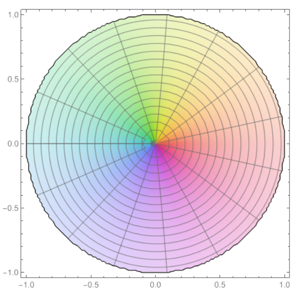
$$\begin{aligned} B_1 &= \frac{(2-\eta)|b|}{4\xi} + \frac{(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{3|b|^3}{64\xi(1+\sqrt{2})(\xi+\sqrt{2}k)((1+\sqrt{2})(\xi+\sqrt{2}k)-|b|)} \\ &+ \frac{(1-\eta)|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4(1+\sqrt{2})(\xi+\sqrt{2}k)-|b|)} \\ &= \frac{(2-\eta)|b|}{4\xi} + \frac{(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{3|b|^3}{64\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(\xi+\xi\sqrt{2}+\sqrt{2}k+2k-|b|)} \\ &+ \frac{(1-\eta)|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4\xi+4\sqrt{2}\xi+4\sqrt{2}k+8k-|b|)} \leq 1-\eta. \end{aligned} \tag{60}$$

This completes the proof.

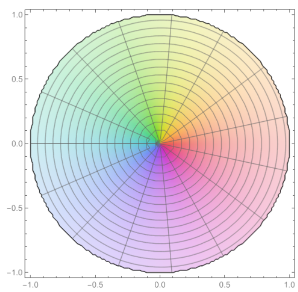
Example 3. *Some special cases of Theorem 3 gives the following different situations:*

Case-1

For $b = 0.1$, $\eta = 0.1$ *and* $\xi = 2.05166$.



(k) ${}_{0.8}H_{2.05166,0.1}$

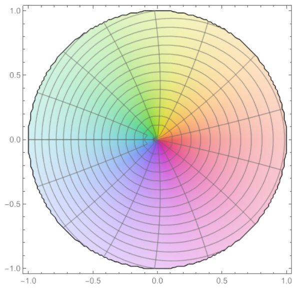


(l) ${}_{0.9}H_{2.05166,0.1}$

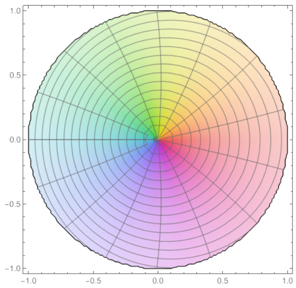
This shows that the behaviour of the graph is same whether we change the values of k. *Exactly at* $k = 1$, *the conditions and graph approaches to classical form which gives the accuracy of our results.*

Case-2

For $b = 1.2$, $\eta = 0.2$ *and* $\xi = 2.8566$.



(m) ${}_{0.8}H_{2.8566,1.2}$



(n) ${}_{0.9}H_{2.8566,1.2}$

This shows that the behaviour of the graph is same whether we change the values of k. *Exactly at* $k = 1$, *the conditions and graph approaches to classical form which gives the accuracy of our results.*

Theorem 4. *Let*

$$
\begin{aligned}
{}_kY_{\xi,b} = &\Bigg(1 - \frac{|b|}{2\xi} - \frac{3|b|^2}{32\xi(\xi+k)} - \frac{3|b|^3}{64\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(\xi+\sqrt{2}\xi+\sqrt{2}k+2k-|b|)} \\
&- \frac{|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4\xi+4\sqrt{2}\xi+4\sqrt{2}k+8k-|b|)}\Bigg)\eta \\
&+ \frac{|b|}{\xi} + \frac{9|b|^2}{32\xi(\xi+k)} + \frac{15|b|^3}{64\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(\xi+\sqrt{2}\xi+\sqrt{2}k+2k-|b|)} \\
&+ \frac{|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(4\xi+4\sqrt{2}\xi+4\sqrt{2}k+8k-|b|)} - 1
\end{aligned}
$$

where $\xi > 3$ *and* $b \in \mathbb{C}^*$. *If* $\eta < 1$ *then*

$$ {}_kY_{\xi,b} \leq 0, \tag{61} $$

Hence ${}_kH_{\xi,b} \in \mathrm{K}(\eta)$.

Proof. From Silverman's [27] well-known result, if f is in form of eq. 1 and satisfies $\sum\limits_{r=2}^{\infty} r(r-\eta)|f_k| \leq 1-\eta$, then $f \in K(\eta)$. According to eq. 9, it is sufficient to show:

$$
\begin{aligned}
B_2 &:= \sum_{r=2}^{\infty} r(r-\eta)\left|\frac{(-b)^{r-1}}{4^{r-1}(1)_{r-1,k}(\xi)_{r-1,k}}\right| \leq 1-\eta, \qquad \xi > 0,\ b \in \mathbb{C}^* \\
&= \sum_{r=2}^{\infty} \frac{r(r-\eta)|b|^{r-1}}{4^{r-1}(1)_{r-1,k}(\xi)_{r-1,k}}
\end{aligned}
$$

$$= \sum_{r=1}^{\infty} \frac{(r+1)(r+1-\eta)|b|^r}{4^r (1)_{r,k} (\xi)_{r,k}}$$
$$= \frac{2(2-\eta)|b|}{4\xi} + \frac{3(3-\eta)|b|^2}{16(\xi)_{2,k}(1)_{2,k}} + \sum_{r=3}^{\infty} \frac{(r+1)(r+1-\eta)|b|^r}{4^r (1)_{r,k} (\xi)_{r,k}}$$
$$= \frac{(2-\eta)|b|}{2\xi} + \frac{3(3-\eta)|b|^2}{32\xi(\xi+1)} + \sum_{r=3}^{\infty} \frac{r^2|b|^r}{4^r (1)_{r,k} (\xi)_{r,k}} + (2-\eta)\sum_{r=3}^{\infty} \frac{r|b|^r}{4^r (1)_{r,k} (\xi)_{r,k}}$$
$$+ (1-\eta)\sum_{r=3}^{\infty} \frac{|b|^r}{4^r (1)_{r,k} (\xi)_{r,k}}.$$

By simple mathematical induction, we have $r^2 \leq \left(\frac{9}{64}\right) 4^r$ and $r \leq \left(\frac{9}{64}\right) 4^r$ for all $r \in \mathbb{N}\backslash[1,2]$. By using Lemma, we have

$$(1)_{r,k} > (1+\beta)^{r-1}, \qquad r \in \mathbb{N}\backslash[1,2], \qquad 0 \leq \beta \leq \sqrt{2}.$$

Since

$$max(1+\beta)^{r-1} : 0 \leq \beta \leq \sqrt{2} = (1+\sqrt{2})^{r-1}$$

follows that:

$$(\xi)_{r,k} \geq \xi(\xi+\sqrt{2})^{r-1}, \xi > 0. \tag{62}$$

As by eq. 61, ${}_kY_{\xi,b}(\eta) \leq 0$, we have

$$B_2 \leq \frac{(2-\eta)|b|}{2\xi} + \frac{3(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{(9+3(2-\eta))|b|}{64\xi} \sum_{r=3}^{\infty} \frac{|b|^{r-1}}{(1+\sqrt{2})^{r-1}(\xi+\sqrt{2}k)^{r-1}}$$
$$+ \frac{(1-\eta)|b|}{4\xi} \sum_{r=3}^{\infty} \frac{|b|^{r-1}}{4^{r-1}(1+\sqrt{2})^{r-1}(\xi+\sqrt{2}k)^{r-1}}.$$

After simplifications, we get:

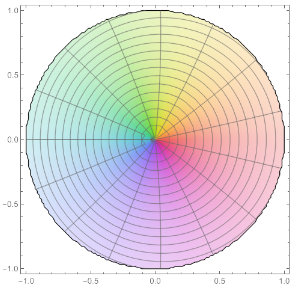
$$B_2 = \frac{(2-\eta)|b|}{2\xi} + \frac{3(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{3(5-\eta)}{64\xi} . \frac{|b|^3}{(1+\sqrt{2})(\xi+\sqrt{2}k)(2k+\sqrt{2}k+\xi+\xi\sqrt{2}-|b|}$$
$$+ \frac{(1-\eta)|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(8k+4\sqrt{2}k+4\xi+4\xi\sqrt{2}-|b|}$$
$$= \frac{(2-\eta)|b|}{2\xi} + \frac{3(3-\eta)|b|^2}{32\xi(\xi+k)} + \frac{3(5-\eta)}{64\xi} . \frac{|b|^3}{(1+\sqrt{2})(\xi+\sqrt{2}k)(2k+\sqrt{2}k+\xi+\xi\sqrt{2}-|b|}$$
$$+ \frac{(1-\eta)|b|^3}{16\xi(1+\sqrt{2})(\xi+\sqrt{2}k)(8k+4\sqrt{2}k+4\xi+4\xi\sqrt{2}-|b|} \leq 1-\eta. \tag{63}$$

This completes the proof, hence ${}_kH_{\xi,b} \in K(\eta)$.

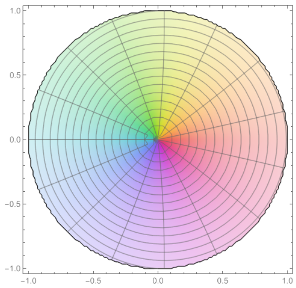
Example 4. *Some special cases of Theorem 4 gives the following different situations:*

Case-1

For $b = 1.2$, $\eta = 0.3$ *and* $\xi = 5.87545$.



(o) ${}_{0.8}H_{5.87545,1.2}$

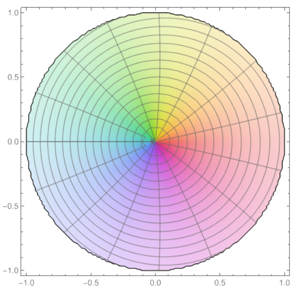


(p) ${}_{0.9}H_{5.87545,1.2}$

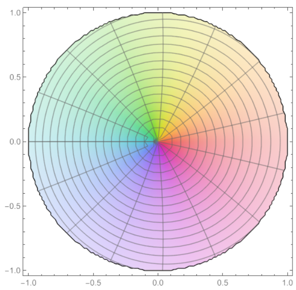
This shows that the behaviour of the graph is same whether we change the values of k. Exactly at $k = 1$, the conditions and graph approaches to classical form which shows the accuracy of our results.

Case-2

For $b = 1.2$, $\eta = 0.4$ and $\xi = 4.7998$.



(q) ${}_{0.9}H_{4.7998,1.2}$



(r) ${}_{0.9}H_{4.7998,1.2}$

This shows that the behaviour of the graph is same whether we change the values of k. Exactly at $k = 1$, the conditions and graph approaches to classical form which shows the accuracy of our results.

Conclusions

In our current findings, we have discussed the geometrical interpretation for different values of k.

(i) The generalization of Pochammer's symbol in the form of inequality $(q)_{r,k} > q(q+\beta)^{r-1}$ is proved by using the generalization of Lemma [1]

(ii) Convexity and Starlikeness of order η for generalized Bessel function. Theorem 1 in our paper is the generalized form of theorem 2.1 from [1], and Theorem 2 is generalization of Theorem 2.2 in [1]. It has given the sufficient conditions for finding the order of starlikeness and Convexity respectively.

(iii) Convexity and Starlikeness of order η by Silverman's theorem for generalized Bessel function is proved. Theorem 3 in our paper is the extended form of Theorem 4.1 from [1], and Theorem 4 is generalization of Theorem 4.2 in [1]. It gives sufficient condition on ξ ,b and its proof uses the Lemma 1.

(iv) For the reader's help, illustrated some examples with graphs to estimate our approach. There is a comparison between different graphs with different values of k. By observing them, it is concluded that the behaviour of the graphs is same whether the values of k changes and approaches to the graph of classical function as k approaches to 1.

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References

[1] T Bulboacă and H M Zayed. Analytical and geometrical approach to the generalized bessel function. *Journal of Inequalities and Applications*, 2024(1):51, 2024.

[2] G Dattoli and A Torre. Theory and applications of generalized bessel functions. Aracne Rome, 1996.

[3] H Khosravian-Arab, M Dehghan, and M R Eslahchi. Generalized bessel functions: Theory and their applications. *Mathematical Methods in the Applied Sciences*, 40(18):6389–6410, 2017.

[4] C Cesarano and D Assante. A note on generalized bessel functions. *International Journal of Mathematical Models and Methods in Applied Sciences*, 7(6):625–629, 2013.

[5] J Niedziela. Bessel functions and their applications. *University of Tennessee-Knoxville,(Dated: October 29, 2008)*, 2008.

[6] K Parand and M Nikarya. Application of bessel functions for solving differential and integro-differential equations of the fractional order. *Applied Mathematical Modelling*, 38(15-16):4137–4147, 2014.

[7] F A Idris, A L Buhari, and T U Adamu. Bessel functions and their applications: Solution to schrödinger equation in a cylindrical function of the second kind and hankel functions. *International Journal Novel Research in Physics Chemistry & Mathematics*, 3:17–31, 2016.

[8] Robert Reynolds and Allan Stauffer. Double integral involving the product of the bessel function of the first kind and modified bessel function of the second kind: Derivation and evaluation. *European Journal of Pure and Applied Mathematics*, 15(3):856–863, 2022.

[9] Robert Reynolds and Allan Stauffer. Triple integral involving the bessel-integral function jiv (z): Derivation and evaluation. *European Journal of Pure and Applied Mathematics*, 15(3):916–923, 2022.

[10] B A Frasin and I Aldawish. On subclasses of uniformly spiral-like functions associated with generalized bessel functions. *Journal of Function Spaces*, 2019(1):1329462, 2019.

[11] Syed Ali Haider Shah and Shahid Mubeen. Expressions of the laguerre polynomial and some other special functions in terms of the generalized meijer g-functions. *AIMS Math*, 6:11631–11641, 2021.

[12] R M Ali, S K Lee, and S R Mondal. Inequalities on an extended bessel function. *Journal of Inequalities and Applications*, 2018:1–22, 2018.

[13] Á Baricz. Generalized bessel functions of the first kind. *Lecture Notes in Mathematics/Springer-Verlag*, 2010.

[14] Á Baricz. Functional inequalities for galué's generalized modified bessel functions. *Journal of Mathematical Inequalities*, 1(2):183–193, 2007.

[15] R Ali, M Kamran, and I Nayab. Some results of generalized k-fractional integral operator with k-bessel function. *Turkish Journal of Science*, 5(3):157–169, 2020.

[16] H M Srivastava. An introductory overview of bessel polynomials, the generalized bessel polynomials and the q-bessel polynomials. *Symmetry*, 15(4):822, 2023.

[17] D Girela. Basic theory of univalent functions. In *Conference Paper. Universidad de Málaga*, 2013.

[18] D K Thomas, N Tuneski, and A Vasudevarao. *Univalent Functions: A Primer*, volume 69. Walter de Gruyter GmbH & Co KG, 2018.

[19] S R Mondal and A Swaminathan. Geometric properties of generalized bessel functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 35(1), 2012.

[20] H M Zayed and T Bulboacă. Normalized generalized bessel function and its geometric properties. *Journal of Inequalities and Applications*, 2022(1):158, 2022.

[21] Á Baricz and S Ponnusamy. Starlikeness and convexity of generalized bessel functions. *Integral Transforms and Special Functions*, 21(9):641–653, 2010.

[22] S R Mondal and M S Akel. Differential equation and inequalities of the generalized k-bessel functions. *Journal of Inequalities and Applications*, 2018:1–14, 2018.

[23] R Diaz and E Pariguan. On hypergeometric functions and pochhammer k-symbol. *arXiv preprint math/0405596*, 2004.

[24] S Ahmed. Some integrals involving k-gamma and k-digamma function. *Journal of the Egyptian Mathematical Society*, 28(1):39, 2020.

[25] E M R A H Yildirim and I N C I Ege. On k-analogues of digamma and polygamma functions. *Journal of Classical Analysis*, 13(2):123–131, 2018.

[26] K Nantomah, F Merovci, and S Nasiru. Some monotonicity properties and inequalities for the (p, k)-gamma function. *Kragujevac Journal of Mathematics*, 42(2):287–297, 2018.

[27] H Silverman. Univalent functions with negative coefficients. *Proceedings of the American Mathematical Society*, 51(1):109–116, 1975.