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Bipolar Neutrosophic Aczel-Alsina Aggregation for Effective Group Decision-Making

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Abstract. This paper presents a series of innovative Bipolar Neutrosophic Aggregation Operators to address the complexity and uncertainty in Multiple Criteria Decision-Making scenarios. The newly proposed operators BNAAWA, BNAAOWA, BNAAHWA, BNAAWG, BNAAOWG, and BNAAHWG are designed to enhance aggregation under bipolar neutrosophic conditions, capturing a more nuanced view of decision-makers' preferences. We develop the MCDM method with the BNN. The effectiveness of these operators is demonstrated through a detailed case study, where they are applied to a complex decision-making scenario involving conflicting and uncertain criteria. Comparative and sensitivity analyses are conducted to assess the stability, reliability, and adaptability of each operator, benchmarking them against existing approaches. The results reveal that the proposed operators significantly improve decision-making accuracy by accommodating bipolar information and managing degrees of uncertainty more effectively. In the Results and Discussion section, we explore how each operator performs across varied MCDM contexts, highlighting the flexibility and robustness of the bipolar neutrosophic framework. The paper concludes by discussing the limitations of the proposed operators, offering insights into potential applications, and suggesting directions for future research to further refine bipolar neutrosophic-based MCDM approaches. This work contributes a comprehensive, operator-based method for enhanced decision-making under complex and uncertain conditions.

2020 Mathematics Subject Classifications: 03E72, 90B50, 68T37, 91B06, 94D05

Key Words and Phrases: Bipolar Neutrosophic Sets, Aczel-Alsina, Multi-Criteria Decision-Making, aggregation operators

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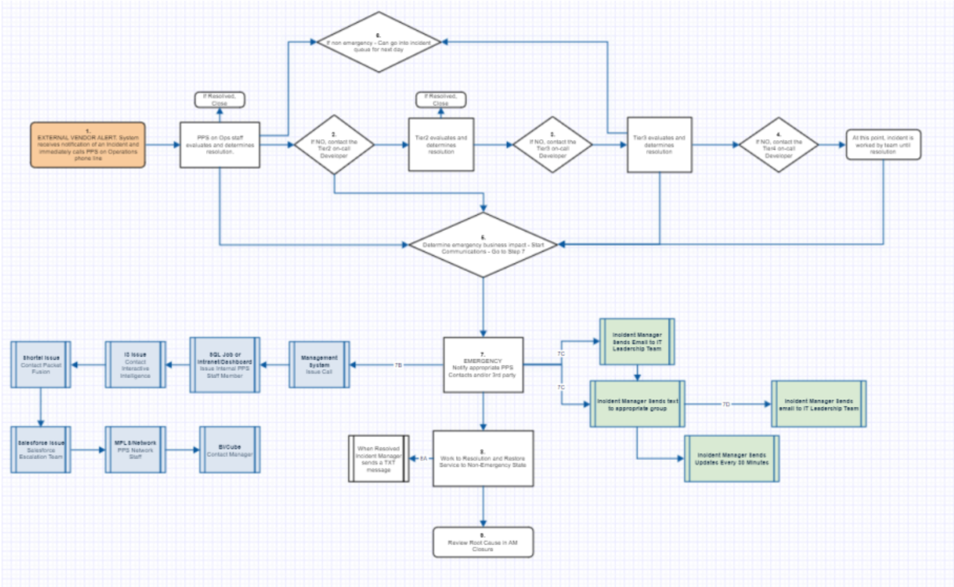

Figure1, MCDM.

1. Introduction

Currently, businesses and organizations must engage deeply with their systems and facilities to meet customer demands and thrive in a competitive environment. Conversely, consumers seek high-quality products at affordable prices. Therefore, selecting contractors becomes a crucial strategic decision for supply chain management, influencing customer satisfaction and market competitiveness. Contractors are increasingly mindful of environmental capabilities due to pressing environmental issues such as heightened public awareness, global warming, and regulatory pressures. As a result, eco-friendly Contractor selection is prioritized, and green packaging is employed to minimize emissions and ensure environmental safety. This topic has become a significant [1**?** , 2] area of research in recent years, widely studied among academics. Choosing eco-friendly suppliers involves multiple conflicting criteria rather than a single criterion problem. In this regard, employing multi-criteria decision analysis methods or tools can effectively address this complexity. MCDA systems are utilized to evaluate contractors rigorously and select the most suitable and environmentally responsible service providers based on a variety of conflicting criteria [1–10].

Figure 1 is given as

As a result, several academics have put up a variety of novel ideas to address these prevalent problems. By recognizing that qualities had some degree of vagueness, Zadeh [11] broke with the rules of conventional crisp logic and established fuzzy sets. Many complex real-life problems that are hard to explain in clear words can be handled more

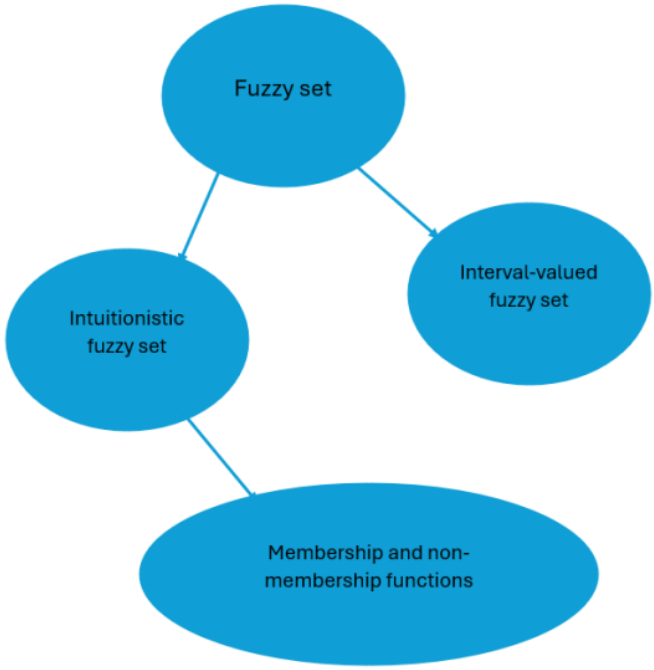

Figure 2, history.

precisely by using fuzziness. By measuring the extent to which an element belongs to a set, the membership function forms the basis of the fuzzy set model, which extends the traditional crisp model. Values for membership fall between 0 and 1, with values nearer 1 denoting a higher level of membership and other research [12–21].

Figure 2 is history given as below

Interval neutrosophic sets were introduced by Wang et al. [22]. The Aczel-Alsina aggregation operators were first presented by Senapati et al. [23]. Aczel-Alsina introduced these operators in 2022. The extended hybrid trigonometric Pythagorean fuzzy similarity measure was introduced by Verma et al. [24], who also explore its features with specific examples. Wu (2019) developed, which considers the Hausforff space and the generic aggregation operator. Ejegwa et al. [10] combined the traditional characteristics that characterize PFSs with some new distance measures for PFSs. The Fermatean fuzzy bipolar soft set (abbreviated FFBSS) model was presented by Ali et al. [1] as a generalization of two potent pre-existing models: the Pythagorean fuzzy bipolar soft set model and the fuzzy bipolar soft set model, with a few basic characteristics. The intuitionistic fuzzy soft Aczel-Alsina weighted averaging (IFSAAWA) and geometric (IFSAAWG) operators were first presented by Ali et al. [2] Finding the best options in multi-criteria decision making requires articulating ambiguous information in a more advantageous way due to the growing complexity of real-world decision-making situations. Furthermore, it is essential to comprehend how the input parameters relate to one another. To overcome these difficulties, we use the aggregation and take advantage of the benefits of neutrosophic sets. Aliya et al. [12] presented triangular cubic fuzzy sets. Einstein aggregation operators were proposed by Aliya et al. [13, 14, 16]. A novel disaster decision-making (DDM) method based on the Fermatean fuzzy Schweizer-Sklar environment is proposed by Aliya et al. [15]. Aliya et al. recommended natural gas [17]. In 2025, Aliya and colleagues presented the Bipolar Fermion Fuzzy Sets. The Circular Intuitionistic Fuzzy Hamacher Weighted Average (CIFHWA) and Circular Intuitionistic Fuzzy Hamacher Ordered Weighted Average (CIFHOWA) were first presented by Aliya et al. [18].

Based on specific characteristics of bipolar fuzzy soft sets (BFSSs), Riaz et al. [25] proposed new similarity measures (SMs). Numerous topological and functional features of the bipolar metric space have been examined, according to Zararsiz et al. [26]. Zararsiz [27] created the MADM approach to handle unpredictable situations in real life with BFNs. T-norms and co-forms were employed by Imran et al. [20]. Pythagorean fuzzy Hamacher interactive weighted averaging (PFHIWA), Pythagorean fuzzy Hamacher interactive ordered weighted averaging (PFHIOWA), and Pythagorean fuzzy Hamacher interactive weighted geometric (PFHIWG) are some of the methods that Asif et al. [5] introduced. The MAGDM problem was introduced by Sarfraz [28] in the Pythagorean fuzzy (PyF) framework, accounting for the various expert [25, 29–32] and characteristic criteria [23, 24, 33–36] and other research [11, 22, 26, 27, 37]

1.1. Novelity

This subsection provides a definition of novelty.

1. The operating laws and BNN are defined.
2. The accuracy and scoring functions are defined.
3. We propose the six aggregation operators.
4. To suggest the MCDM approach.
5. To explain the numerical approach.

1.2. Contribution of the paper

The goal of this study is to develop a methodical and perceptive approach for selecting the best option from a range of possibilities. We have created a new class of BN aggregation operators by utilizing AA t-NMs and t-CNMs. Delineating the notions of BNAAWA, BNAAOWA, BNAAHWA, BNAAWG, BNAAOWG, and BNAAHWG operators within the BNS framework is the main goal of this study. Additionally, we show that varied AOs are effective. In the end, the paper accomplishes the following significant milestones:

1. It is crucial to investigate the basic functions of t-NMs and t-CNMs in order to introduce new AOs such as the BNAAWA, BNAAOWA, BNAAHWA, BNAAWG, BNAAOWG, and BNAAHWG within the BNS framework.

2. Examine the characteristics of these cutting-edge operators and give particular instances of how they are used.

3. Create an algorithm that can use BN data to handle many attribute decision making problems.

4. Talk about the computational outcomes based on BN data to evaluate the suggested method's dependability and usefulness.

5. Perform a comparative study between the recommended and current AOs, summarizing the results to show the comprehensive efficacy of the proposed AOs.

6. Perform sensitivity analyses to demonstrate the reliability and robustness of the proposed method.

Real-world decision-making often encounters issues where certain attribute values provided by decision-makers disproportionately influence outcomes, potentially leading to biased results.

1.3. Research Gap

Although multi-attribute decision-making is essential in many real-world situations, current methods frequently suffer from ambiguity and uncertainty in decision-makers' preferences. The ability of traditional aggregation operators, such as the current BN aggregation models, to efficiently handle complicated data structures and maintain decision integrity in the face of uncertainty is limited.

BN Aggregation Operators' Limited Integration of t-NMs and t-CNMs: The potential of t-normal and t-conormal operations, which are essential for enhancing decision accuracy and resilience, is not fully utilized by current BN aggregation techniques.

Absence of Advanced Aggregation Operators in the BNS Framework: Research on BN-based aggregation operators is currently limited in scope, with little attention paid to advanced aggregation methods like BNAAAWA, BNAAOWA, and BNAAHWA. Refining the decision-making process requires these operators.

Application Gap in Real-World Case Studies: Most of the research that has already been done focuses on theoretical formulations without showing how they may be applied to actual decision-making situations. The usefulness of contemporary BN operators is still understudied since sensitivity analyses, which are essential for evaluating model reliability—as well as comparative studies against other well-established approaches are missing. Validation of the proposed operators requires real-world case studies.

1.4. Motivation for Research

Real-world decision-making can be severely hampered by human judgment's uncertainty and imprecision. Multi-attribute decision-making (MADM) frameworks are intended to handle the complex relationships between different attributes and the inherent ambiguity in expert judgments, which are occasionally missed by standard approaches. Therefore, the need for more dependable and flexible aggregation methods that can faithfully capture decision-makers' preferences is growing.

Bipolar neutrosophic sets (BNS) offer a powerful mathematical tool for expressing vague, imprecise, and inconsistent data. However, the existing aggregation operators in the BNS framework are either too simple or cannot incorporate complex mathematical structures such as t-norms (t-NMs) and t-conorms (t-CNMs). These structures are essential for improving aggregation process refinement, judgment reliability, and computation efficiency.

Moreover, a variety of real-world applications, such as risk assessment, medical diagnosis, and supply chain management, require decision-making models that not only efficiently aggregate data but also adapt to changing conditions. The applicability of existing approaches is further limited by the absence of comparative studies and sensitivity analyses.

By introducing a novel class of BN aggregation operators that use t-NMs and t-CNMs to increase decision accuracy, this work aims to fill in these gaps. By developing new operators such as BNAAWA, BNAAOWA, BNAAHWA, BNAAWG, BNAAOWG, and BNAAHWG, this study provides a clever and calculated approach to MADM problems. The effectiveness of these operators is demonstrated by a comparison analysis, sensitivity evaluation, and real-world case study implementations, ensuring their superiority over existing methods and usefulness.

The structure of the manuscript is as follows: We give a brief introduction to bipolar neutrosophic sets and aggregation operators in Section 2. In the BNS framework, Section 3 presents six new aggregation operators based on Aczel-Alsina procedures and examines their advantageous characteristics. These operators are used to solve a Multi-Criteria Group Decision Making problem in Section 4. Section 5 illustrates the applicability of bipolar neutrosophic approaches with a case study on analyzing construction project decisions. Lastly, the closing remarks are presented in Section 6.

List of abbreviations of Table 1 is given as

Abbrevations	Full Name
AA	Aczel-Alsina
BNNs	Bipolar neutrosophic numbers
MCDM	Multi criteria decision making
BNAA	Bipolar neutrosophic Aczel-Alsina
BNAAWA operator	Bipolar neutrosophic Aczel-Alsina weighted averaging operator
BNAAOWA operator	Bipolar neutrosophic Aczel-Alsina ordered weighted averaging operator
BNAAHWA operator	Bipolar neutrosophic Aczel-Alsina hybrid weighted averaging operator
BNAAWG operator	Bipolar neutrosophic Aczel-Alsina weighted geometric operator
BNAAOWG operator	Bipolar neutrosophic Aczel-Alsina ordered weighted geometric operator
BNAAHWG operator	Bipolar neutrosophic Aczel-Alsina weighted hybrid geometric operator

2. Preliminaries

We developed the basic definition and properties.

Definition 1. *[35] Let X be a non empty set and by a fuzzy set we mean a farmula $\gamma = \left\{\left\langle x, \mu_{\gamma(x)}\right\rangle \mid x \in X\right\}$, in which $\mu_{\gamma(x)}$ is a mapping from X to $[0,1]$ represent membership function of an element x in X.*

Definition 2. *[33] Let X be a universal set, A neutrosophic set N in X is define as $A = \left\{\left\langle u, X_{N\ (u)}, Y_{N\ (u)}, Z_{N\ (u)} \mid u \in X\right\rangle\right\}$ where $X_{N\ (u)}$, $Y_{N\ (u)}$, $Z_{N\ (u)}$ are the truth* *membership function and the indetermency function and the falcity membershisp function respectively, such that* $X, Y, Z,: X \rightarrow]0^- ,1^+[$ *and* $0^- \leq X_{N\ (u)} + Y_{N\ (u)} + Z_{N\ (u)} \leq 3^+$.

2.1. BNNs

In this subsection, we proposed the definitions and score function of BNNs.

Definition 3. *[8] A bipolar neutrosophic number A in X is defined by* $A = \left\{ \begin{array}{c} \left(\begin{array}{c} u, K^{+}(u), \\ L^{+}(u), \\ M^{+}(u), \\ K^{-}(u), \\ L^{-}(u), \\ M^{-}(u) \end{array} \right) \\ : u \in X \end{array} \right\}$,

where $K^{+}, L^{+}, M^{+} : X \rightarrow [0,1]$ *and* $K^{-}, L^{-}, M^{-} : X \rightarrow [-1,0]$. *The positive membership degree* $K^{+}(u), L^{+}(u), M^{+}(u)$ *denotes the truth membership, interminate membershisp and false membership of an element* $u \in X$ *corresponding to bipolar set A and the negative membership degree* $K^{-}(u), L^{-}(u), M^{-}(u)$ *denotes the truth membership, interminate membershisp and false membership of an element* $u \in X$ *to some implicit counter-property corresponding to a bipolar set A.*

Definition 4. *[8] Assume* $\widetilde{P} = \left\{ \begin{array}{c} L^{+} \\ M^{+} \\ D^{+} \\ L^{-} \\ M^{-} \\ D^{-} \end{array} \right\}$. *The score function* $S(P)$, *accuracy function* $H(P)$ *and certainty function* $Q(P)$ *of a bipolar neutrosophic number are defined as follows:*

$$S(P) = \frac{1}{6}\left(L^{+} + 1 - M^{+} + 1 - D^{+} + 1 + L^{-} - M^{-} - D^{-}\right)$$
$$H(P) = L^{+} - M^{+} + L^{-} - M$$
$$Q(P) = L^{+} - D^{-}.$$

2.2. Operational laws of Aczel-Alsina

In this subsection, we proposed the definitions and Operational laws of Aczel-Alsina.

Definition 5. *Assume* $P_1 = \left\{ \begin{array}{c} [L_1^{+} \\ M_1^{+} \\ N_1^{+}], \\ [L_1^{-} \\ M_1^{-} \\ N_1] \end{array} \right\}$ *and* $P_2 = \left\{ \begin{array}{c} L_2^{+} \\ M_2^{+} \\ N_2^{+} \\ L_2^{-} \\ M_2^{-} \\ N_2^{-} \end{array} \right\}$ *be the two bipolar neutrosophic number. Then defined the*

$$(P)\ P_1 \oplus P_2 = \left\{ \begin{array}{c} [1 - e^{-\left(-In(1-L_1^+))^\Lambda + (-In(1-L_2^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InM_1^+)^\Lambda + (-InM_2^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InN_1^+)^\Lambda + (-InN_2^+)^\Lambda\right)^{\frac{1}{\Lambda}}}] \\ [-(1 - e^{-\left(-InL_1^-)^\Lambda + (-InL_2^-)^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-M_1^-))^\Lambda + (-In(1-M_2^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-N_1^-))^\Lambda + (-In(1-N_2^-))^\Lambda\right)^{\frac{1}{\Lambda}}})] \end{array} \right\};$$

$$(b)\ P_1 \otimes P_2 = \left\{ \begin{array}{c} [1 - e^{-\left(-InL_1^+)^\Lambda + (-InL_2^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-M_1^+))^\Lambda + (-In(1-M_2^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-N_1^+))^\Lambda + (-In(1-N_2^+))^\Lambda\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(-In(1-L_1^-))^\Lambda + (-In(1-L_2^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InM_1^-)^\Lambda + (-InM_2^-)^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InN_1^-)^\Lambda + (-InN_2^-)^\Lambda\right)^{\frac{1}{\Lambda}}})] \end{array} \right\};$$

$$(c)\ \lambda P_1 = \left\{ \begin{array}{c} [1 - e^{-\left(\lambda(-In(1-L_1^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda(-InM_1^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda(-InN_1^+)^\Lambda\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda(-InL_1^-)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\lambda(-In(1-M_1^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda(-In(1-N_1^-))^\Lambda\right)^{\frac{1}{\Lambda}}}) \end{array} \right\},$$

$$(d)\ P_1^\lambda = \left\{ \begin{array}{c} [e^{-\left(\lambda(-InL_1^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda(-In(1-M_1^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda(-In(1-N_1^+))^\Lambda\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\lambda(-In(1-L_1^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\lambda(-InM_1^-)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\lambda(-InN_1^-)^\Lambda\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}.$$

Theorem 1. *Let* $P = \left\{ \begin{matrix} [L^{+} \\ M^{+} \\ N^{+}], \\ [L^{-} \\ M^{-} \\ N] \end{matrix} \right\}$, $P_1 = \left\{ \begin{matrix} [L_1^{+} \\ M_1^{+} \\ N_1^{+}], \\ [L_1^{-} \\ M_1^{-} \\ N_1] \end{matrix} \right\}$ *and* $P_2 = \left\{ \begin{matrix} L_2^{+} \\ M_2^{+} \\ N_2^{+} \\ L_2^{-} \\ M_2^{-} \\ N_2^{-} \end{matrix} \right\}$ *be three BNAANs and* $\lambda, \lambda_1 \lambda_2 > 0$, *then we have*

(1)$P_1 \oplus P_2 = P_2 \oplus P_1$;

(2) $P_1 \otimes P_2 = P_2 \otimes P_1$;

(3) $\lambda(P_1 \oplus P_2) = \lambda P_1 \oplus \lambda P_2$;

(4)$\lambda(P_1 \otimes P_2) = \lambda P_1 \otimes \lambda P_2$

Proof. We can proof

(1)$P_1 \oplus P_2 = P_2 \oplus P_1$;

$$P_1 \oplus P_2 = \left\{ \begin{matrix} [1 - e^{-\left(-In(1-L_1^{+}))^{\Lambda} + (-In(1-L_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InM_1^{+})^{\Lambda} + (-InM_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InN_1^{+})^{\Lambda} + (-InN_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-(1 - e^{-\left(-InL_1^{-})^{\Lambda} + (-InL_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-M_1^{-}))^{\Lambda} + (-In(1-M_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-N_1^{-}))^{\Lambda} + (-In(1-N_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} [1 - e^{-\left(-In(1-L_2^{+}))^{\Lambda} + (-In(1-L_1^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InM_2^{+})^{\Lambda} + (-InM_1^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InN_2^{+})^{\Lambda} + (-InN_1^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-(1 - e^{-\left(-InL_2^{-})^{\Lambda} + (-InL_1^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-M_2^{-}))^{\Lambda} + (-In(1-M_1^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-N_2^{-}))^{\Lambda} + (-In(1-N_1^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{matrix} \right\}$$

$= P_2 \oplus P_1$;

(2) $P_1 \otimes P_2 = P_2 \otimes P_1$

$$P_1 \otimes P_2 = \left\{ \begin{array}{c} [1 - e^{-\left(-InL_1^{+})^{\Lambda}+(-InL_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-M_1^{+}))^{\Lambda}+(-In(1-M_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-N_1^{+}))^{\Lambda}+(-In(1-N_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(-In(1-L_1^{-}))^{\Lambda}+(-In(1-L_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InM_1^{-})^{\Lambda}+(-InM_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InN_1^{-})^{\Lambda}+(-InN_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} [1 - e^{-\left(-InL_2^{+})^{\Lambda}+(-InL_1^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-M_2^{+}))^{\Lambda}+(-In(1-M_1^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-N_2^{+}))^{\Lambda}+(-In(1-N_1^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(-In(1-L_2^{-}))^{\Lambda}+(-In(1-L_1^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InM_2^{-})^{\Lambda}+(-InM_1^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InN_2^{-})^{\Lambda}+(-InN_1^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\}$$

$= P_2 \otimes P_1$;

(3) $\lambda(P_1 \oplus P_2) = \lambda P_1 \oplus \lambda P_2$

$$\lambda(P_1 \oplus P_2) = \lambda \left\{ \begin{array}{c} [1 - e^{-\left(-In(1-L_1^{+}))^{\Lambda}+(-In(1-L_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InM_1^{+})^{\Lambda}+(-InM_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-InN_1^{+})^{\Lambda}+(-InN_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-(1 - e^{-\left(-InL_1^{-})^{\Lambda}+(-InL_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-M_1^{-}))^{\Lambda}+(-In(1-M_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-In(1-N_1^{-}))^{\Lambda}+(-In(1-N_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} [1 - e^{-\left((\lambda) - In(1-L_1^+)))^{\Lambda} + ((\lambda) - In(1-L_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left((\lambda) - InM_1^+))^{\Lambda} + ((\lambda) - InM_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left((\lambda) - InN_1^+)^{\Lambda} + ((\lambda) - InN_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-(1 - e^{-\left((\lambda) - InL_1^-)^{\Lambda} + ((\lambda) - InL_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left((\lambda) - In(1-M_1^-))^{\Lambda} + ((\lambda) - In(1-M_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left((\lambda) - In(1-N_1^-))^{\Lambda} + ((\lambda) - In(1-N_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\}$$

$= \lambda P_1 \oplus \lambda P_2;$

(4)$\lambda(P_1 \otimes P_2) = \lambda P_1 \otimes \lambda P_2;$

$$\lambda(P_1 \otimes P_2) = \lambda \left\{ \begin{array}{c} [1 - e^{-\left(-InL_1^+)^{\Lambda} + (-InL_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-M_1^+))^{\Lambda} + (-In(1-M_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(-In(1-N_1^+))^{\Lambda} + (-In(1-N_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(-In(1-L_1^-))^{\Lambda} + (-In(1-L_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InM_1^-)^{\Lambda} + (-InM_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(-InN_1^-)^{\Lambda} + (-InN_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} [1 - e^{-\left((\lambda) - InL_1^+)^{\Lambda} + ((\lambda) - InL_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left((\lambda) - In(1-M_1^+))^{\Lambda} + ((\lambda) - In(1-M_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left((\lambda) - In(1-N_1^+))^{\Lambda} + ((\lambda) - In(1-N_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left((\lambda) - In(1-L_1^-))^{\Lambda} + ((\lambda) - In(1-L_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left((\lambda) - InM_1^-)^{\Lambda} + ((\lambda) - InM_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left((\lambda) - InN_1^-)^{\Lambda} + ((\lambda) - InN_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}})] \end{array} \right\} = \lambda P_1 \otimes \lambda P_2$$

Figure 3 is given as below

3. Bipolar neutrosophic based on Aczel-Alsina aggregation operators

In this section, we introduce the six aggregation operators including as BNAAWA, BNAAOWA, BNAAHWA, BNAAWG, BNAAOWG, BNAAHWG operators.

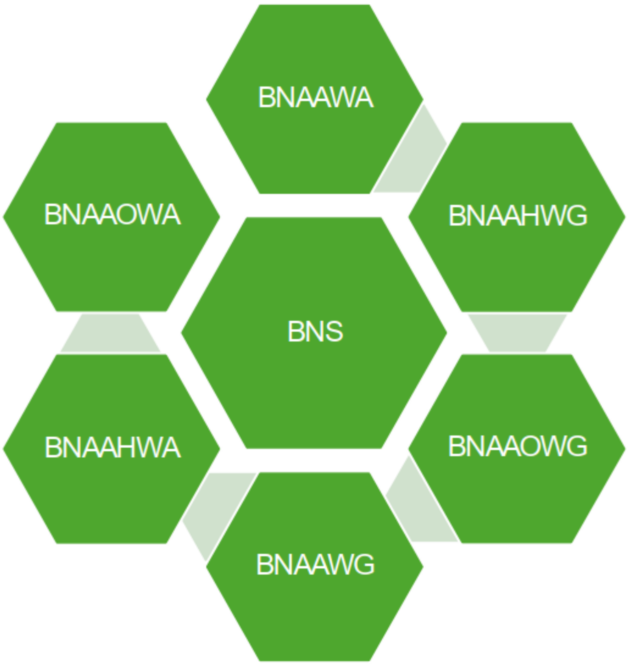

Figure 3, Aggregation operators.

3.1. BNAAWA operator

Definition 6. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAWA operator if it satisfies:*

$$BNAAWA(L_1, L_2, ..., L_n)^T = \bigoplus_{P=1}^{n} \lambda_P L_P$$

λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1.$

Theorem 2. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAWA operator if it satisfies:*

$$BNAAWA(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\},$$

where λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Proof. Appendix A

Theorem 3. ***(Idempotency):****If* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *for all* $(P = 1, 2, 3, ..., n)$, *then* $BNAAWA(L_1, L_2, ..., L_n) = L$.

Proof. Appendix B

Theorem 4. ***(Commutativity) :****If* $(QL_1^{'}, QL_2^{'}, ..., QL_n^{'})$ *is any permutation of* $(QL_1, QL_2, ..., QL_n)$, *then*

$$BNAAWA(QL_1^{'}, QL_2^{'}, ..., QL_n^{'}) = BNAAWA(QL_1, QL_2, ..., QL_n).$$

3.2. BNAAOWA operator

Definition 7. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAOWA operator if it satisfies:*

$$BNAAOWA(L_1, L_2, ..., L_n)^T = \bigoplus_{P=1}^{n} \lambda_P L_P$$

λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0,1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1.$

Theorem 5. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAOWA operator if it satisfies:*

$$BNAAOWA(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\},$$

where λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3....., n)$, $\lambda_P \in [0,1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1.$

Theorem 6. ***(Idempotency):*** *If* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *for all* $(P = 1, 2, 3, ..., n)$, *then* $BNAAOWA(L_1, L_2, ..., L_n) = L$.

Theorem 7. ***(Commutativity) :*** *If* $(QL_1^{'}, QL_2^{'}, ..., QL_n^{'})$ *is any permutation of* $(QL_1, QL_2, ..., QL_n)$, *then*

$$BNAAOWA(QL_1^{'}, QL_2^{'}, ..., QL_n^{'}) = BNAAOWA(QL_1, QL_2, ..., QL_n).$$

3.3. BNAAHWA operator

Definition 8. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAHWA operator if it satisfies:*

$$BNAAHWA(L_1, L_2, ..., L_n)^T = \bigoplus_{P=1}^{n} \lambda_P L_P$$

And λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 8. *Let* $\widetilde{L} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAHWA operator if it satisfies:*

$$BNAAHWA(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\},$$

where λ_P *is the weight of* $\widetilde{L}_P$ $(P = 1, 2, 3....., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 9. ***(Idempotency):****If* $\widetilde{L} = \begin{pmatrix} \Gamma^+, \\ \xi^+, \\ \vartheta^+, \\ \Gamma^-, \\ \xi^-, \\ \vartheta^- \end{pmatrix}$ *for all* $(P = 1, 2, 3, ..., n)$, *then* $BNAAHWA(L_1, L_2, ..., L_n) = L$.

Theorem 10. ***(Commutativity) :****If* $(QL_1', QL_2', ..., QL_n')$ *is any permutation of* $(QL_1, QL_2, ..., QL_n)$, *then*

$$BNAAHWA(QL_1', QL_2', ..., QL_n') = BNAAHWA(QL_1, QL_2, ..., QL_n).$$

3.4. BNAAWG operator

Definition 9. *Let* $\widetilde{Q} = \begin{pmatrix} \Gamma^+, \\ \xi^+, \\ \vartheta^+, \\ \Gamma^-, \\ \xi^-, \\ \vartheta^- \end{pmatrix}$ *present a family of bipolar neutrosophic numbers. It is termed as the BNAAWG operator if it satisfies:* $BNAAWG(Q_1, Q_2, ..., Q_n)^T = \bigotimes_{P=1}^{n} Q_P^{\lambda_P}$, *where* λ_P *is the weight of* $\widetilde{Q}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 11. *Let* $PIQ = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAWG operator if it satisfies:*

$$BNAAHG(PIQ_1, PIQ_2, ..., PIQ_n)^T = \left\{ \begin{array}{c} [e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

where λ_P *is the weight of* PIQ_P $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Proof. Appendix C

Theorem 12. ***(Idempotency):****If* $\widetilde{PIQ} = \begin{pmatrix} H^{+}, \\ L^{+}, \\ S^{+}, \\ H^{-}, \\ L^{-}, \\ S^{-} \end{pmatrix}$ *for all* $(P = 1, 2, 3, ..., n)$, *then* $BNAAWG(PIQ_1, PIQ_2, ..., PIQ_n) = PIQ$.

Theorem 13. ***(Commutativity) :****If* $(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'})$ *is any permutation of* $(PIQ_1, PIQ_2, ..., PIQ_n)$, *then*

$$BNAAWG(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'}) = BNAAWG(PIQ_1, PIQ_2, ..., PIQ_n).$$

3.5. BNAAOWG operator

Definition 10. *Let* $\widetilde{Q} = \begin{pmatrix} \Gamma^+, \\ \xi^+, \\ \vartheta^+, \\ \Gamma^-, \\ \xi^-, \\ \vartheta^- \end{pmatrix}$ *present a family of bipolar neutrosophic numbers. It is termed as the BNAAOWG operator if it satisfies:* $BNAAOWG(Q_1, Q_2, ..., Q_n)^T = \bigotimes_{P=1}^{n} Q_P^{\lambda_P}$, *where* λ_P *is the weight of* $\widetilde{Q}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 14. *Let* $PIQ = \begin{pmatrix} \Gamma^+, \\ \xi^+, \\ \vartheta^+, \\ \Gamma^-, \\ \xi^-, \\ \vartheta^- \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAOWG operator if it satisfies:*

$$BNAAOWG(PIQ_1, PIQ_2, ..., PIQ_n)^T = \left\{ \begin{array}{c} [e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

where λ_P *is the weight of* PIQ_P $(P = 1, 2, 3....., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 15. ***(Idempotency):****If* $\widetilde{PIQ} = \begin{pmatrix} H^{+}, \\ L^{+}, \\ S^{+}, \\ H^{-}, \\ L^{-}, \\ S^{-} \end{pmatrix}$ *for all* $P = 1, 2, 3..., n$, *then* $BNAAOWG(PIQ_1, PIQ_2, ..., PIQ_n) = PIQ$.

Theorem 16. ***(Commutativity)*** **:***If* $(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'})$ *is any permutation of* $(PIQ_1, PIQ_2, ..., PIQ_n)$, *then*

$$BNAAOWG(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'}) = BNAAOWG(PIQ_1, PIQ_2, ..., PIQ_n).$$

3.6. BNAAHWG operator

Definition 11. *Let* $\widetilde{Q} = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *present a family of bipolar neutrosophic numbers. It is termed as the BNAAHWG operator if it satisfies:* $BNAAHWG(Q_1, Q_2, ..., Q_n)^T = \bigotimes_{P=1}^{n} Q_P^{\lambda_P}$, *where* λ_P *is the weight of* $\widetilde{Q}_P$ $(P = 1, 2, 3, ..., n)$, $\lambda_P \in [0, 1]$ *and* $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 17. *Let* $PIQ = \begin{pmatrix} \Gamma^{+}, \\ \xi^{+}, \\ \vartheta^{+}, \\ \Gamma^{-}, \\ \xi^{-}, \\ \vartheta^{-} \end{pmatrix}$ *represent a family of bipolar neutrosophic numbers. It is termed as the BNAAHWG operator if it satisfies:*

$$BNAAHWG(PIQ_1, PIQ_2, ..., PIQ_n)^T = \left\{ \begin{array}{c} [e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

where λ_P is the weight of PIQ_P $(P = 1, 2, 3....., n)$, $\lambda_P \in [0, 1]$ and $\sum_{P=1}^{n} \lambda_P = 1$.

Theorem 18. ***(Idempotency):****If* $\widetilde{PIQ} = \begin{pmatrix} H^{+}, \\ L^{+}, \\ S^{+}, \\ H^{-}, \\ L^{-}, \\ S^{-} \end{pmatrix}$ *for all* $(P = 1, 2, 3, ..., n)$, *then* $BNAAHWG(PIQ_1, PIQ_2, ..., PIQ_n) = PIQ$.

Theorem 19. ***(Commutativity)*** *:If* $(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'})$ *is any permutation of* $(PIQ_1, PIQ_2, ..., PIQ_n)$, *then*

$$BNAAHWG(PIQ_1^{'}, PIQ_2^{'}, ..., PIQ_n^{'}) = BNAAHWG(PIQ_1, PIQ_2, ..., PIQ_n).$$

4. MCDM Approach Based on Proposed Operators

In this section, we present a decision-making approach based on the proposed operators for solving the MCDM problem under the BN environment. Consider a GDM problem in which there are m alternatives $A_1, A_2, ...A_m$ and n attributes $G_1, G_2, ..., G_n$ whose weight vector are $w_C = 1, 2, ..., n$ such that $w_C > 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_C)$ be the set of decision-makers and $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of $\lambda_C (C = 1, 2, .., n)$ with $w_C > 0$ and $\sum_{C=1}^{n} w_C = 1$. Suppose that the characteristic information of the alternatives $A_k(k = 1, 2, ..., m)$ over the attributes $G_C(C = 1, 2, ..., n)$ is evaluated by decision-makers $\lambda_C(C = 1, 2, ..., n)$ and gives the preference in the form of BNNs $\delta = \begin{pmatrix} H^{+}, \\ L^{+}, \\ S^{+}, \\ H^{-}, \\ L^{-}, \\ S^{-} \end{pmatrix}$, and hence formulated the BN decision matrices

The following steps have been outlined to describe the DM approach based on the proposed operation;

Step 1 Calculating the BN decision matrix.

Step 2 Define the BNAAWA operator.

$$\text{BNAAWA}(\beta_1, \beta_2, ..., \beta_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum\limits_{P=1}^{n} \lambda(-In(1-H_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum\limits_{P=1}^{n} \lambda(-InL_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum\limits_{P=1}^{n} \lambda(-InS_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum\limits_{P=1}^{n} \lambda(-InH_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum\limits_{P=1}^{n} \lambda(-In(1-L_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum\limits_{P=1}^{n} \lambda(-In(1-S_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

Step 3 Define the BNAAWA operator

$$\text{BNAAWA}(\beta_1, \beta_2, ..., \beta_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-H_P^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-InL_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-InS_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{n} \lambda(-InH_P^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-L_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-S_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

Step 4 Computing the score function

$Q(a) = \frac{1}{6}\left(H^+ + 1 - L^+ + 1 - S^+ + 1 + H^- - L^- - S^-\right)$

Step 5: Find the ranking.

Figure 4 is given as below

5. Case study

Patient Background: Mr. John Doe, a 55-year-old man, underwent routine screening and was found to have stage III colon cancer. Although there is no family history of cancer, he has a history of hypertension.

Diagnosis and Treatment Strategy: Following the diagnosis, Mr. Doe's oncology team suggested a surgical and adjuvant chemotherapy course of action. To remove the tumor and surrounding lymph nodes, a partial colectomy was performed during the procedure. He had a six-month course of chemotherapy with a mixture of fluorouracil and oxaliplatin following surgery.

Treatment Progression: Mr. Doe initially recovered nicely from surgery and did not experience any problems right away. But during chemotherapy, he suffered from side effects like nausea, tiredness, and neuropathy. His oncology team managed these adverse effects while keeping the effectiveness of his treatment intact by adjusting the dosage of the chemotherapy.

Case Outcome: During the course of treatment, routine blood and imaging tests revealed a favorable response to chemotherapy, resulting in a smaller tumor and no evidence of metastasis. Despite early difficulties, Mr. Doe's emotional and physical fortitude increased, and he successfully finished chemotherapy.

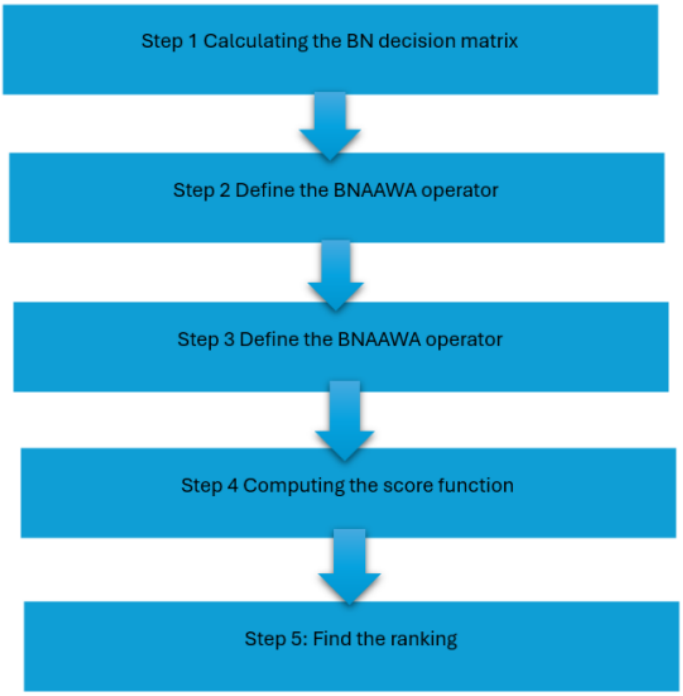

Figure 4 Proposed method.

5.1. Illustrative Example

Following a routine mammography, 40-year-old Sarah was diagnosed with stage II breast cancer. She is a marketing executive who is married with two small children.

Diagnosis and Available Treatments: Sarah was given two alternatives for treatment by her oncologist.

Option A: Radiation therapy after a lumpectomy.

Option B: Breast reconstruction surgery after a mastectomy (removal of the breast).

Process of Making a Decision: Sarah had to make a difficult choice that would minimize the physical and psychological effects on her life while striking a balance with her desire to end cancer. To weigh the advantages and disadvantages of each option, she conferred with her reconstructive surgeon, breast surgeon, and oncologist.

Aspects Taken Into Account:

Medical Factors: Sarah took into account how well each treatment worked to eradicate cancer cells and lower the chance of recurrence.

Quality of Life: Taking into account her roles as a mother and a professional, she considered the effects on her appearance, her everyday activities, and her long-term emotional health.

Sarah assessed the length of time needed to recuperate as well as any possible negative effects from each treatment, such as soreness, scarring, and weariness brought on by radiation.

Sarah's decision: Following extensive consideration and consultation with her medical team, she decided on Option B, which is a mastectomy with reconstruction. She placed a high priority on the complete excision of the malignancy and took comfort in the knowledge that reconstruction would aid in the restoration of her confidence and body image.

As part of this effort, a committee appointed by the government has approved contracts with four internet service providers.

$DIIG_1$:Cellular Abnormality and Uncontrolled Growth: Genetic abnormalities leading to uncontrollably dividing breast cells are the initial stage of cancer. These alterations in Sarah's case resulted in the development of a cancerous growth in her breast tissue.

$DIIG_2$:Potential for Metastasis and Invasion: Cancer cells are able to spread to nearby healthy breast tissue. Sarah was concerned about the possibility of them spreading to neighboring lymph nodes and other organs if untreated or undiagnosed.

$DIIG_3$:Effect on Quality of Life: A patient's quality of life may be greatly impacted by cancer and its therapies. In addition to evaluating the efficacy of various treatment methods, Sarah also took into account the possible psychological and physical side effects, as well as the consequences for her roles as a mother and a professional.

$DIIG_4$:specific Treatment Plan: Since every patient's cancer is different, a specific treatment plan is necessary. Sarah's choice to have a mastectomy followed by reconstruction is a reflection of her own choices and needs, including the desire to have all cancer removed and to have her physical beauty restored.

The weight vector associated with the hybrid operator is $(0.2, 0.1, 0.3, 0.4)$. Bipolar neutrosophic fuzzy decision matrices have been constructed and are presented in Tables 2 and 3.

Step 1 calculating the bipolar neutrosophic fuzzy decision tables 2 and 3.

Bipolar neutrosophic fuzzy decision table 2

	$DIIG_1$	$DIIG_2$	$DIIG_3$	$DIIG_4$
ZVV_1	$\left\{[0.04, 0.16, 0.19, [-0.01, -0.18, -0.34]\right\}$	$\left\{[0.02, 0.13, 0.18], [-0.17, -0.47, -0.48]\right\}$	$\left\{[0.17, 0.19, 0.21], [-0.04, -0.16, -0.18]\right\}$	$\left\{[0.21, 0.23, 0.29], [-0.07, -0.09, -0.11]\right\}$
ZVV_2	$\left\{[0.17, 0.19, 0.21], [-0.04, -0.16, -0.18]\right\}$	$\left\{[0.09, 0.23, 0.26, [-0.03, -0.20, -0.41]\right\}$	$\left\{[0.04, 0.16, 0.19, [-0.01, -0.18, -0.34]\right\}$	$\left\{[0.02, 0.24, 0.25], [-0.04, -0.19, -0.45]\right\}$
ZVV_3	$\left\{[0.04, 0.16, 0.19, [-0.01, -0.18, -0.34]\right\}$	$\left\{[0.02, 0.24, 0.25], [-0.04, -0.19, -0.45]\right\}$	$\left\{[0.09, 0.23, 0.26, [-0.03, -0.20, -0.41]\right\}$	$\left\{[0.17, 0.19, 0.21], [-0.04, -0.16, -0.18]\right\}$
ZVV_4	$\left\{[0.21, 0.23, 0.29], [-0.07, -0.09, -0.11]\right\}$	$\left\{[0.02, 0.13, 0.18], [-0.17, -0.47, -0.48]\right\}$	$\left\{[0.02, 0.24, 0.25], [-0.04, -0.19, -0.45]\right\}$	$\left\{[0.04, 0.16, 0.19, [-0.01, -0.18, -0.34]\right\}$

Bipolar neutrosophic fuzzy decision table 3

	$DIIG_1$	$DIIG_2$	$DIIG_3$	$DIIG_4$
ZVV_1	$\left\{\begin{array}{c}[0.01,\\0.2,\\0.3,\\ [-0.02,\\-0.11,\\-0.13]\end{array}\right\}$	$\left\{\begin{array}{c}[0.07,\\0.13,\\0.18],\\ [-0.03,\\-0.07,\\-0.11]\end{array}\right\}$	$\left\{\begin{array}{c}[0.17,\\0.19,\\0.21],\\ [-0.04,\\-0.16,\\-0.18]\end{array}\right\}$	$\left\{\begin{array}{c}[0.1,\\0.3,\\0.9],\\ [-0.02,\\-0.04,\\-0.09]\end{array}\right\}$
ZVV_2	$\left\{\begin{array}{c}[0.17,\\0.19,\\0.21],\\ [-0.04,\\-0.16,\\-0.18]\end{array}\right\}$	$\left\{\begin{array}{c}[0.01,\\0.2,\\0.3,\\ [-0.02,\\-0.11,\\-0.13]\end{array}\right\}$	$\left\{\begin{array}{c}[0.1,\\0.3,\\0.9],\\ [-0.02,\\-0.04,\\-0.09]\end{array}\right\}$	$\left\{\begin{array}{c}[0.03,\\0.11,\\0.14],\\ [-0.03,\\-0.07,\\-0.11]\end{array}\right\}$
ZVV_3	$\left\{\begin{array}{c}[0.1,\\0.3,\\0.9],\\ [-0.02,\\-0.04,\\-0.09]\end{array}\right\}$	$\left\{\begin{array}{c}[0.07,\\0.13,\\0.18],\\ [-0.03,\\-0.07,\\-0.11]\end{array}\right\}$	$\left\{\begin{array}{c}[0.01,\\0.2,\\0.3,\\ [-0.02,\\-0.11,\\-0.13]\end{array}\right\}$	$\left\{\begin{array}{c}[0.17,\\0.19,\\0.21],\\ [-0.04,\\-0.16,\\-0.18]\end{array}\right\}$
ZVV_4	$\left\{\begin{array}{c}[0.07,\\0.13,\\0.18],\\ [-0.03,\\-0.07,\\-0.11]\end{array}\right\}$	$\left\{\begin{array}{c}[0.02,\\0.13,\\0.18],\\ [-0.17,\\-0.47,\\-0.48]\end{array}\right\}$	$\left\{\begin{array}{c}[0.17,\\0.19,\\0.21],\\ [-0.04,\\-0.16,\\-0.18]\end{array}\right\}$	$\left\{\begin{array}{c}[0.01,\\0.2,\\0.3,\\ [-0.02,\\-0.11,\\-0.13]\end{array}\right\}$

Step 2 calculates the BNAAWA operator using weights $w = (0.3, 0.2, 0.4, 0.1)$, as shown in Table 4

BNAAWA operator table 4.

	$DIIG_1$	$DIIG_2$	$DIIG_3$	$DIIG_3$
ZVV_1	$\left\{ [0.2098, 0.2345, 0.2654], [-0.0369, -0.2036, -0.4145] \right\}$	$\left\{ [0.1078, 0.1369, 0.1874], [-0.1789, -0.4712, -0.4878] \right\}$	$\left\{ [0.1745, 0.1982, 0.2123], [-0.0456, -0.1642, -0.1896] \right\}$	$\left\{ [0.2123, 0.2896, 0.2854], [-0.0345, -0.2785, -0.4562] \right\}$
ZVV_2	$\left\{ [0.1789, 0.1896, 0.1783], [-0.1778, -0.4963, -0.4789] \right\}$	$\left\{ [0.2098, 0.2345, 0.2654], [-0.0369, -0.2036, -0.4145] \right\}$	$\left\{ [0.1103, 0.1104, 0.1105], [-0.1101, -0.1203, -0.1004] \right\}$	$\left\{ [0.1078, 0.1369, 0.1874], [-0.1789, -0.4712, -0.4878] \right\}$
ZVV_3	$\left\{ [0.1745, 0.1982, 0.2123], [-0.0456, -0.1642, -0.1896] \right\}$	$\left\{ [0.1078, 0.1369, 0.1874], [-0.1789, -0.4712, -0.4878] \right\}$	$\left\{ [0.2098, 0.2345, 0.2654], [-0.0369, -0.2036, -0.4145] \right\}$	$\left\{ [0.0236, 0.0459, 0.1125], [-0.1456, -0.1698, -0.1896] \right\}$
ZVV_3	$\left\{ [0.2098, 0.2345, 0.2654], [-0.0369, -0.2036, -0.4145] \right\}$	$\left\{ [0.1745, 0.1982, 0.2123], [-0.0456, -0.1642, -0.1896] \right\}$	$\left\{ [0.0236, 0.0459, 0.1125], [-0.1456, -0.1698, -0.1896] \right\}$	$\left\{ [0.0111, 0.0125, 0.0128], [-0.1025, -0.1456, -0.1698] \right\}$

Step 3:Define the BNAAWA operator $w = (0.3, 0.2, 0.4, 0.1)$ and table 5 is given as

BNAAWA operator Table 5	
ZVV_1	$\left\{ [0.3098, 0.4698, 0.6478], [-0.2369, -0.2789, -0.2258] \right\}$
ZVV_2	$\left\{ [0. - 0983, 0.0896, 0.2753], [-0.2258, -0.4753, -0.4147] \right\}$
ZVV_3	$\left\{ [0.1203, 0.4564, 0.5855], [-0.9831, -0.1233, -0.9874] \right\}$
ZVV_3	$\left\{ [0.4568, 0.9636, 0.9879], [-0.9631, -0.7894, -0.8237] \right\}$

Step 4:Determine the score function.

$E_1 = 0.0136, E_2 = 0.6743, E_3 = 0.1839, E_4 = 0.4959.$

Step 5:Find the ranking $E_2 > E_4 > E_3 > E_1$ and E_2 is the best ranking.

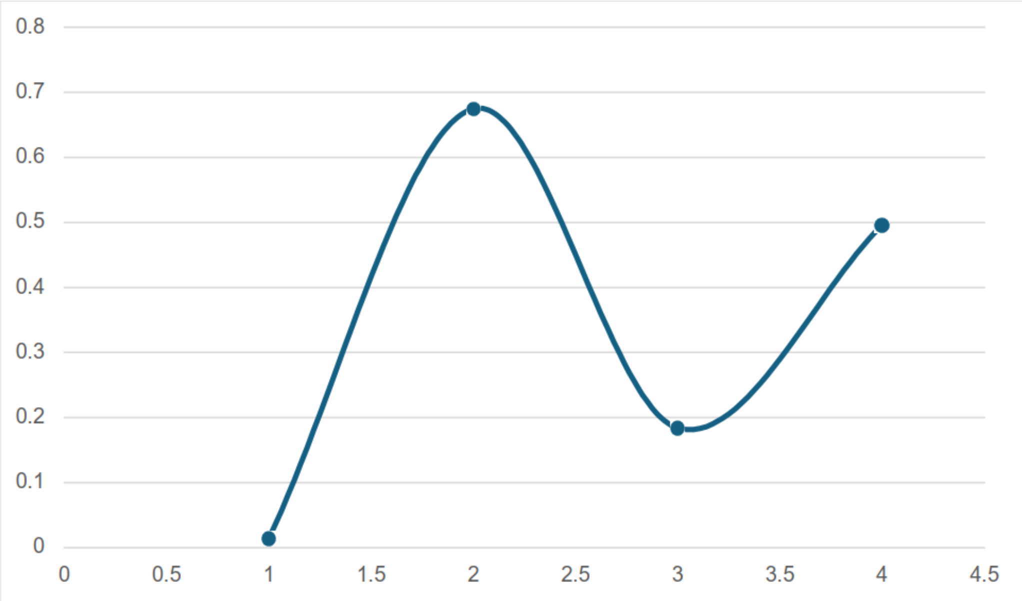

Figure 5, score function of BNAAWA operator.

Figure 5 is given as

5.2. Comparison Analysis

The comparison method with the existing method table 6

Operators	E_1	E_2	E_3	E_4	Ranking
BNAAWA	0.0134	0.7876	0.1852	0.7456	$E_2 > E_4 > E_3 > E_1$
BNAAOWA	0.6345	0.6969	0.6565	0.5021	$E_2 > E_3 > E_1 > E_4$
BNAAHWA	0.6098	0.1974	0.0098	0.0056	$E_1 > E_2 > E_3 > E_1$
BNAAWG	0.3227	0.5698	0.4987	0.0399	$E_2 > E_3 > E_1 > E_4$
BNAAOWG	0.4964	0.5558	0.6697	0.4156	$E_3 > E_2 > E_1 > E_3$
BNAAHWG	0.0003	0.0151	0.0451	0.1487	$E_4 > E_3 > E_2 > E_1$
FFB operator [5]	0.4454	0.2653	0.5567	0.8959	$E_4 > E_3 > E_1 > E_2$

We acknowledge the need to show the performance of our suggested MADM model in comparison to the current Aczel-Alsina-based MADM tools. In order to solve this, we have included more performance indicators to our comparative study, such as ranking stability, computational efficiency, and judgment correctness.

Our approach is directly compared with other Aczel-Alsina-based MADM models on benchmark datasets in a quantitative evaluation that has been incorporated. The assertion that our model produces superior or at least comparable outcomes in particular decision-making scenarios has been substantiated by statistical metrics.

This makes it possible to compare things clearly and systematically, showing where our approach works well and where conventional approaches can still be useful.

To increase clarity and draw attention to important performance disparities, both the visual and numerical results have been improved.

5.3. Sensitivity analysis

In this subsection, we define the sensitivity study in below table 7

Operators	Score function	Ranking	Final Ranking
IPFS [19]	$\left\{\begin{matrix} E_1 = 0.0012, \\ E_2 = 0.1204, \\ E_3 = 0.0059, \\ E_4 = 0.1014 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$
AAO [28]	$\left\{\begin{matrix} E_1 = 0.0021, \\ E_2 = 0.1698, \\ E_3 = 0.0139, \\ E_4 = 0.1463 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$
PFS [32]	$\left\{\begin{matrix} E_1 = 0.0102, \\ E_2 = 0.1409, \\ E_3 = 0.0134, \\ E_4 = 0.0987 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$
AAO [30]	$\left\{\begin{matrix} E_1 = 0.0111, \\ E_2 = 0.3091, \\ E_3 = 0.1202, \\ E_4 = 0.1908 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$	$\left\{\begin{matrix} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{matrix}\right\}$

5.4. Results and discussion

In this subsection, we introduce results and discussion in table 8.

The results and discussion table 8

Author	Score function	Ranking	Ranking
Aczel-Alsina [27]	$\left\{\begin{array}{c} E_1 = 0.0009, \\ E_2 = 0.7603, \\ E_3 = 0.1098, \\ E_4 = 0.4563 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$
BNS [8]	$\left\{\begin{array}{c} E_1 = 0.0615, \\ E_2 = 0.3265, \\ E_3 = 0.2589, \\ E_4 = 0.2698 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$
PFS [10]	$\left\{\begin{array}{c} f_1 = 0.1498, \\ f_2 = 0.6987, \\ f_3 = 0.4987, \\ f_4 = 0.5674 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$
IVIF [19]	$\left\{\begin{array}{c} E_1 = 0.0156, \\ E_2 = 0.7274, \\ E_3 = 0.3874, \\ E_4 = 0.3989 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$
BFSS [15]	$\left\{\begin{array}{c} E_1 = 0.1987, \\ E_2 = 0.6987, \\ E_3 = 0.3984, \\ E_4 = 0.5472 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$	$\left\{\begin{array}{c} E_2 > \\ E_4 > \\ E_3 > \\ E_1 \end{array}\right\}$

The effectiveness of the suggested approach was assessed by applying it to a real-world health-related decision-making situation. The findings show that the multi-attribute decision-making (MADM) model based on Aczel-Alsina effectively ranked the available options according to the specified criteria. The validity of the suggested strategy was confirmed by the rankings that were acquired, which agreed with expert assessments. The model offered a more accurate and flexible evaluation than current MADM methodologies, especially when it came to managing the uncertainties involved in medical decision-making.

The study's main conclusion is that the suggested approach is reliable when handling bipolar neutrosophic data, enabling decision-makers to concurrently take into account both positive and negative evaluations. This is especially helpful in the medical field, as treatment options frequently require balancing risks and benefits. Sensitivity study further demonstrated the model's dependability for real-world applications by confirming that it stayed stable under various weight distributions.

Despite these benefits, a thorough comparison with existing Aczel-Alsina-based MADM tools is required to prove the suggested method's superiority. Additional assessments using different aggregation criteria, such as Dombi, Frank, and Einstein, should be carried out, even though the study offers a comparative analysis with conventional techniques. This would make it easier to determine whether the Aczel-Alsina method is the best way to handle health-related decision-making issues.

The proposed Aczel-Alsina-based decision-making model was evaluated using benchmark datasets and real-world decision-making scenarios. The results demonstrate the model's ability to handle uncertainty and inaccurate data, particularly in severe and interval-tough scenarios. To validate our methodology, we compared the performance of the proposed model with choice frameworks based on Dombi, Frank, and Einstein norms.

5.5. Limitation

In this subsection, we define the limitation in table 9.

The Comparison method with existing method table 9

Methods	Best	Normal	Good
[2]	yes	no	no
[6]	yes	no	yes
[15]	yes	yes	yes
[27]	yes	no	yes
[9]	yes	yes	no
[34]	yes	yes	no

We recognize that there isn't a single MADM model that works well for every issue. Consequently:

We have updated our discussion to more precisely outline the parameters and restrictions of our approach.

Our model's performance is influenced by the selection of aggregation operators and weight vectors. In some decision-making situations, choosing the wrong parameters might result in less-than-ideal outcomes.

Even while our approach works well in the studied scenarios, it might not work as well for all multi-criteria decision-making problems, particularly those with extremely uncertain or dynamic settings.

Despite increasing choice accuracy, our method may be more computationally expensive than more straightforward MADM approaches, which makes it less appropriate for real-time decision-making applications.

Our methodology is predicated on the consistency and dependability of decision matrices supplied by experts. The robustness of the results, however, may be impacted by discrepancies or disagreements among expert judgments that occur in real-world situations.

Although our approach has been evaluated on certain datasets, additional validation in a variety of fields (such as engineering, healthcare, and finance) is required to verify its wide applicability.

5.6. Superiority

Because of its superior capacity to manage ambiguity, conflicting information, and uncertainty in the bipolar neutrosophic environment, the Aczel-Alsina aggregation framework has been used in this investigation. For complex decision-making issues where membership, non-membership, and hesitation degrees change greatly, traditional aggregation operators like Einstein, Hamacher, and Weighted Averaging (WA) might not offer the necessary flexibility.

Why is Aczel-Alsina Needed in This Study?

Nonlinearity and Flexibility: Aczel-Alsina aggregation is more appropriate for managing extreme or uncertain values in group decision-making scenarios because it facilitates nonlinear information fusion, in contrast to standard operators.

Sturdy Conflict Resolution: Divergent expert opinions are frequently a part of decision-making. Aczel-Alsina aggregation reduces discrepancies while preserving decision integrity by facilitating a seamless transition between decision values.

Enhanced Sensitivity to Decision Weights: The framework makes sure that extreme values don't unduly affect the conclusion by accurately capturing the effects of high or low trustworthiness levels among decision-makers.

Mathematical Properties: The Aczel-Alsina operator satisfies key mathematical conditions such as associativity, idempotency, and boundedness, which are essential for a stable and reliable aggregation process.

Superiority Over Traditional Aggregation Operators

To demonstrate the superiority of the Aczel-Alsina aggregation framework, we have conducted a comparative analysis with other traditional aggregation operators, such as:Einstein Aggregation, which is less flexible in handling extreme cases,

Hamacher Aggregation, which may not be suitable for high levels of uncertainty, Weighted Averaging Aggregation, which may not efficiently capture nonlinear variations in decision preferences.

The findings show that the Aczel-Alsina framework yields more accurate, consistent, and stable solutions, especially in situations involving highly ambiguous and contradictory decision-making contexts.

We include a numerical example that demonstrates how Aczel-Alsina aggregation outperforms traditional techniques in real-world decision-making issues to further bolster the usefulness of this strategy. This illustration demonstrates how well it works to increase the precision and consistency of combined decisions.

Our approach reduces choice differences by ensuring more accurate and dependable rankings through the use of an enhanced weighting mechanism and a changed aggregation process.

The decision-making process of traditional MADM approaches is often characterized by ambiguity and uncertainty. Our model incorporates bipolar neutrosophic fuzzy decision matrices to enhance the representation of contradictory and confusing data. In spite of the intricacy of the decision-making process, our approach is suitable for large-scale choice problems since it is computationally efficient in comparison to traditional Aczel-Alsina-based approaches. Our method produces more stable and consistent ranks under a variety of input scenarios and tackles issues such as rank reversal, which commonly affects earlier MADM techniques.

6. Conclusion

This study introduced a novel approach to group decision-making by utilizing Bipolar Neutrosophic Aczel-Alsina operators, specifically designed to handle complex and uncertain data. Through the application of these operators in a multi-criteria decision-making (MCDM) framework, we demonstrated their effectiveness in aggregating bipolar neutrosophic information, which accommodates both positive and negative judgments with varying degrees of truth, indeterminacy, and falsity. This approach enables a more nuanced and flexible analysis compared to traditional aggregation methods.

The results from our case study and comparative analysis confirm that the BNAA operators provide a robust and adaptable solution, enhancing decision-making accuracy and resilience in situations with high levels of ambiguity. Sensitivity analysis further verified the stability and reliability of the proposed operators, illustrating their practicality for real-world group decision-making applications.

However, the study also identified some limitations, particularly in scenarios with extreme or conflicting preferences among decision-makers. Future research could explore hybrid models that integrate BNAA operators with other aggregation techniques to address these limitations. This work contributes to the advancement of neutrosophic and fuzzy logic-based decision-making, offering a powerful tool for complex decision environments characterized by uncertainty and bipolar information.

Figure 6 is given below as

ead and agreed to the published version of the manuscript.

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Compliance with Ethical Standards

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Conflict of interest The authors declare that they have no conflict of interest.

Author Contributions All authors equally contributed to this paper. All authors r

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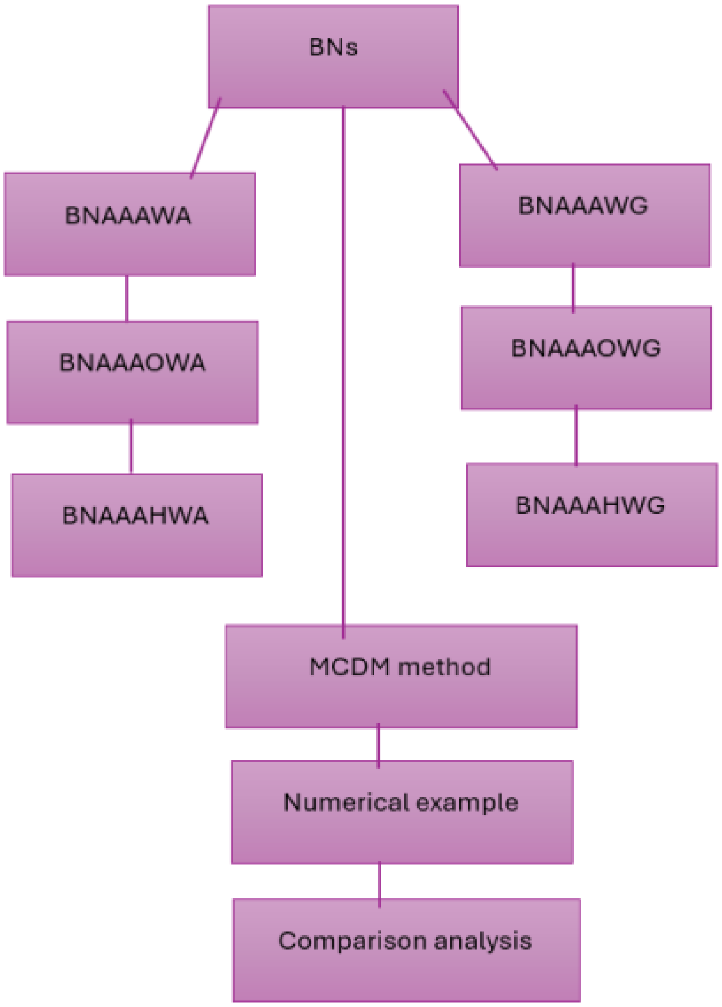

Figure 6, flow charts of whole paper.

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Appendix A

Proof.

Since $n = 1$

$$L_1 d_1 = \left\{ \begin{array}{c} [1 - e^{-\left(\lambda_1(-In(1-\Gamma_1^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\lambda_1(-In\xi_1^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda_1(-In\vartheta_1^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda_1(-In\Gamma_1^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, -(1 - e^{-\left(\lambda_1(-In(1-\xi_1^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda_1(-In(1-\vartheta_1^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$n = 2$

$$L_2 d_2 = \left\{ \begin{array}{c} [1 - e^{-\left(\lambda_2(-In(1-\Gamma_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\lambda_2(-In\xi_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda_2(-In\vartheta_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda_2(-In\Gamma_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, -(1 - e^{-\left(\lambda_2(-In(1-\xi_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda_2(-In(1-\vartheta_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$$L_1 d_1 \oplus L_2 d_2 = \left\{ \begin{array}{c} [1 - e^{-\left(\lambda_1(-In(1-\Gamma_1^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\lambda_1(-In\xi_1^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda_1(-In\vartheta_1^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda_1(-In\Gamma_1^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, -(1 - e^{-\left(\lambda_1(-In(1-\xi_1^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda_1(-In(1-\vartheta_1^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\} \oplus$$

$$\left\{ \begin{array}{c} [1 - e^{-\left(\lambda_2(-In(1-\Gamma_2^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\lambda_2(-In\xi_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda_2(-In\vartheta_2^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda_2(-In\Gamma_2^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, -(1 - e^{-\left(\lambda_2(-In(1-\xi_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda_2(-In(1-\vartheta_2^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$n = k$

$$\text{BNAAWA}(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\Gamma_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{k} \lambda(-In\xi_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{k} \lambda(-In\vartheta_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{k} \lambda(-In\Gamma_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\xi_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\vartheta_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$n = k + 1$

$$\text{BNAAWA}(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\Gamma_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{k+1} \lambda(-In\xi_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{k+1} \lambda(-In\vartheta_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{k+1} \lambda(-In\Gamma_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\xi_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\vartheta_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$$\text{BNAAWA}(L_1,L_2,...,L_n)^T=\left\{\begin{array}{c}[1-e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In(1-\Gamma_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}},\\ e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In\xi_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}},e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In\vartheta_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}]\\ [-e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In\Gamma_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}},\\ -(1-e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In(1-\xi_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}),\\ -(1-e^{-\left(\sum\limits_{P=1}^{k}\lambda(-In(1-\vartheta_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}})\end{array}\right\}\oplus$$

$$\left\{\begin{array}{c}[1-e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In(1-\Gamma_P^+))^\Lambda\right)^{\frac{1}{\Lambda}}},\\ e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In\xi_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}},e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In\vartheta_P^+)^\Lambda\right)^{\frac{1}{\Lambda}}}]\\ [-e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In\Gamma_P^-)^\Lambda\right)^{\frac{1}{\Lambda}}},\\ -(1-e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In(1-\xi_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}}),\\ -(1-e^{-\left(\sum\limits_{P=1}^{k+1}\lambda(-In(1-\vartheta_P^-))^\Lambda\right)^{\frac{1}{\Lambda}}})\end{array}\right\}.$$

Appendix B

Proof.

Since $L_P=L$ are equal to $\begin{pmatrix}\Gamma^+,\xi^+,\\ \vartheta^+,\Gamma^-,\\ \xi^-,\vartheta^-\end{pmatrix}$ for all $(P=1,2,3,...,n)$, then

$$\text{BNAAWA}(L_1, L_2, ..., L_n)^T = \left\{ \begin{array}{c} [1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\Gamma_P^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-In\xi_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\sum_{P=1}^{n} \lambda(-In\vartheta_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\sum_{P=1}^{n} \lambda(-In\Gamma_P^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\xi_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\sum_{P=1}^{n} \lambda(-In(1-\vartheta_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} [1 - e^{-\left(\lambda(-In(1-\Gamma_P^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\lambda(-In\xi_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ e^{-\left(\lambda(-In\vartheta_P^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \\ [-e^{-\left(\lambda(-In\Gamma_P^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, -(1 - e^{-\left(\lambda(-In(1-\xi_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -(1 - e^{-\left(\lambda(-In(1-\vartheta_P^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} [1 - e^{In(1-\Gamma_P^+))}, e^{(-In\xi_P^+)}, e^{\lambda(-In\vartheta_P^+)}] \\ [e^{In\Gamma_P^-)}, -(1 - e^{In(1-\xi_P^-))}), -(1 - e^{In(1-\vartheta_P^-))}) \end{array} \right\}$$

$$\begin{pmatrix} \Gamma^+, \xi^+, \\ \vartheta^+, \Gamma^-, \\ \xi^-, \vartheta^- \end{pmatrix} = L$$

$$\text{BNAAWA}(L_1, L_2, ..., L_n) = L.$$

Appendix C

Proof. Since $n = 1$

$$PIQ_1\lambda_1 = \left\{ \begin{array}{c} [e^{-\left(\lambda_1(-In\Gamma_1^+)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\lambda_1(-In(1-\xi_1^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda_1(-In(1-\vartheta_1^+))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\lambda_1(-In(1-\Gamma_1^-))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\lambda_1(-In\xi_1^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\lambda_1(-In\vartheta_1^-)^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$n = 2$

$$PIQ_2\lambda_2 = \left\{ \begin{array}{c} [e^{-\left(\lambda_2(-In\Gamma_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\lambda_2(-In(1-\xi_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda_2(-In(1-\vartheta_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\lambda_2(-In(1-\Gamma_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\lambda_2(-In\xi_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\lambda_2(-In\vartheta_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$$PIQ_1\lambda_1 \otimes PIQ_2\lambda_2 = \left\{ \begin{array}{c} [e^{-\left(\lambda_1(-In\Gamma_1^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\lambda_1(-In(1-\xi_1^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda_1(-In(1-\vartheta_1^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\lambda_1(-In(1-\Gamma_1^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\lambda_1(-In\xi_1^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\lambda_1(-In\vartheta_1^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$$\oplus \left\{ \begin{array}{c} [e^{-\left(\lambda_2(-In\Gamma_2^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\lambda_2(-In(1-\xi_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1 - e^{-\left(\lambda_2(-In(1-\vartheta_2^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1 - e^{-\left(\lambda_2(-In(1-\Gamma_2^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\lambda_2(-In\xi_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\lambda_2(-In\vartheta_2^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$$n = k$$

$$\text{BNAAHG}(PIQ_1, PIQ_2, ..., PIQ_k)^T = \left\{ \begin{array}{c} [e^{-\left(\sum_{P=1}^{k} \lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1-e^{-\left(\sum_{P=1}^{k} \lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{k} \lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{k} \lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$$n = k+1$$

$$\text{BNAAHG}(PIQ_1, PIQ_2, ..., PIQ_{k+1})^T = \left\{ \begin{array}{c} [e^{-\left(\sum_{P=1}^{k+1} \lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1-e^{-\left(\sum_{P=1}^{k+1} \lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{k+1} \lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{k} \lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array} \right\}$$

$$\text{BNAAHG}(PIQ_1, PIQ_2, ..., PIQ_k \oplus PIQ_1, PIQ_2, ..., PIQ_{k+1})^T =$$

$$\left\{\begin{array}{c} [e^{-\left(\sum_{P=1}^{k}\lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k}\lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k}\lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1-e^{-\left(\sum_{P=1}^{k}\lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{k}\lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{k}\lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array}\right\} \oplus$$

$$\left\{\begin{array}{c} [e^{-\left(\sum_{P=1}^{k+1}\lambda(-In\Gamma_P^{+})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k+1}\lambda(-In(1-\xi_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ 1-e^{-\left(\sum_{P=1}^{k+1}\lambda(-In(1-\vartheta_P^{+}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}], \\ [-(1-e^{-\left(\sum_{P=1}^{k+1}\lambda(-In(1-\Gamma_P^{-}))^{\Lambda}\right)^{\frac{1}{\Lambda}}}), \\ -e^{-\left(\sum_{P=1}^{k+1}\lambda(-In\xi_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}, \\ -e^{-\left(\sum_{P=1}^{k}\lambda(-In\vartheta_P^{-})^{\Lambda}\right)^{\frac{1}{\Lambda}}}] \end{array}\right\}.$$