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Common Fixed Points of Asymptotically Regular Mappings in Convex Metric Spaces with Application

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Abstract. In this paper, based upon Górnicki's work on fixed points of a continuous asymptotically regular self-mapping on a metric space, we establish common fixed point results for similar self-mappings and their average mappings on a convex metric space by using various types of contractive conditions. The closedness and convexity of the set of fixed points of a non- self- mapping is obtained here in the context of a uniformly convex hyperbolic space. We also apply our findings to solve Volterra type integral equations, demonstrating practical use of our work in mathematical analysis and its related fields.

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1. Introduction

Fixed point theory is an integral part of modern mathematics, offering essential tools and techniques for resolving various problems in nonlinear analysis, optimization, economics, and engineering. Over the past two decades, the development of this theory in metric-type spaces has garnered significant attention from researchers, especially its usefulness for solving many existence problems in nonlinear differential and integral equations with applications in engineering and applied sciences [1]. The relevance and applicability of

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fixed point theory continues to expand as new results and methodologies emerge.
Recently, Górnicki [2] generalized a well-known result of Reich [3] related to contractions on a complete metric space. This new result has been further extended by Bisht [4], Karapinar et al. [5], and Panja et al. [6] for discontinuous mappings and by Khan and Oyetunbi [7] for two discontinuous mappings satisfying a Lipschitz-Kannan type condition considered in [8]. These advancements highlight the dynamic nature of fixed point theory. Applications of fixed point theory have been widely explored in various settings. Specifically, the theory has been instrumental in developing iterative methods for solving existence and uniqueness problems in differential and integral equations, as well as systems of linear equations. These applications demonstrate the versatility of fixed point theory in addressing complex mathematical challenges. The theory's ability to ensure the existence and uniqueness of solutions makes it a valuable tool for tackling a wide range of real-world problems in mathematics, engineering and applied sciences [1, 9, 10].
In this paper, we aim to contribute to this growing body of knowledge by proving fixed point and common fixed point results for asymptotically regular mappings in hyperbolic spaces and convex metric spaces. Our work is built on a basic concept introduced by Browder and Petryshyn [11], namely, asymptotically regular mapping and explores its implications in more general spaces.

2. Preliminaries

Let (Q, d) be a metric space and $E : Q \to Q$ be a mapping. A point $q_0 \in Q$ is called fixed point of E if $Eq_0 = q_0$. The set of fixed points of E is denoted and defined as $Fix(E) := \{q_0 \in Q : Eq_0 = q_0\}$.

The concept of asymptotically regular mappings was introduced by Browder and Petryshyn [11]. A mapping $E : Q \to Q$ is called asymptotically regular at $x_0 \in Q$, if $\lim_{n\to\infty} d(E^n x_0, E^{n+1}x_0) = 0$. If E is asymptotically regular at each point of Q, then E is said to be asymptotically regular on Q. E is non-expansive if $d(Ex, Ey) \le d(x, y)$ for all $x, y \in Q$. Consider $Q = \mathbb{R}$, the set of real numbers with its usual metric $d(q, p) = |q - p|$. Suppose $E : Q \to Q$ is given by $Eq = \frac{q}{2}$. This function is both asymptotically regular and non-expansive. If E is a function on a metric space Q into itself, then the set $O(E, e) := \{E^n e : n = 0, 1, 2, 3, \cdots\}$ is called the orbit of E at the point $e \in Q$. E is called orbitally continuous at $p \in Q$ if for any sequence $\{q_k\} \subset O(E, q)$ for some $q \in Q$, $\lim_{k\to\infty} q_k = p$ entails that $\lim_{k\to\infty} Eq_k = Ep$. We say that E is orbitally continuous on Q if E is orbitally continuous at each point $p \in Q$. Clearly, continuity implies orbital continuity, but the converse is not true [4, 12, 13].
A self-mapping E on a metric space Q, is called k-continuous, $k = 1, 2, \cdots$

$$\text{if } \lim_{n\to\infty} E^{k-1}q_n = z \text{ implies that } \lim_{n\to\infty} E^k q_n = Ez.$$

Note that 1-continuity is equivalent to continuity, and for any $k = 1, 2, ...,$ k-continuity implies $(k + 1)$-continuity, while the converse is not true. Moreover, continuity of E and k-continuity of E are independent conditions when $k > 1$([13], Examples 1.2-1.5).
Górnicki [2] has obtained the following result.

Theorem 1. *([2], Theorem 2.6) Suppose that (Q,d) is a complete metric space and E is a continuous asymptotically regular self-mapping on Q satisfying the following condition:*

$$d(Eq,Ep) \leq \mu d(q,p) + \nu\{d(q,Eq) + d(p,Ep)\}, \forall q,p \in Q \tag{1}$$

where $0 \leq \mu < 1$, $0 \leq \nu < \infty$. Then E has a unique fixed point $x \in Q$ and $E^n q \to x$ for each $q \in Q$.

Khan and Oyetunbi [7] have obtained the following generalization of Theorem 1.

Theorem 2. *([7],Theorem 2.2) Suppose that (Q,d) is a complete metric space, E and F are asymptotically regular self-mappings on Q satisfying the following condition:*

$$d(Eq,Fp) \leq \mu d(q,p) + \nu\{d(q,Eq) + d(p,Fp)\}, \forall q,p \in Q \tag{2}$$

where $0 \leq \mu < 1$, $0 \leq \nu < \infty$. Suppose further that E and F are either k-continuous for some $k \geq 1$ or orbitally continuous. Then E and F have a unique common fixed point b. Furthermore, $\lim_{n\to\infty} E^n q = b = \lim_{n\to\infty} F^n q$ for any $q \in Q$.

Here is a result of Khan and Oyetunbi [7] without asymptotically regularity of the mappings.

Theorem 3. *[7, Theorem 2.6] Suppose that (Q,d) is a complete metric space and $E,F : Q \to Q$ are continuous mappings satisfying (2). Suppose that E and F have a common approximate fixed point sequence (i.e. there exists a sequence $\{q_n\} \subset Q$ such that $d(q_n, Eq_n) \to 0$ and $d(q_n, Fq_n) \to 0$ as $n \to \infty$). Then E and F have a unique common fixed point q. In particular, $q_n \to q$, as $n \to \infty$.*

Lemma 1 ([14],Lemma 2.1)**.** *Let $\{\tau_n\}$, $\{\phi_n\}$ and $\{\chi_n\}$ be three real sequences with $\tau_n \geq 0$ and $\phi_n \in (0,1)$. Suppose that*
(i)$\tau_{n+1} \leq (1-\phi_n)\tau_n + \phi_n \chi_n$
(ii) $\sum_{n=1}^{\infty} \phi_n = \infty$
(iii) $\lim sup_{n\to\infty} \chi_n \leq 0$ *or* $\sum_{n=1}^{\infty} \phi_n \chi_n$ *is convergent.*
Then $\lim_{n\to\infty} \tau_n = 0$.

Definition 1. *[14] Let $R^+ := \{f \in R \mid f \geq 0\}$. Define a function $\kappa : R^+ \to [0,1]$ satisfying the following properties:*

(i) $0 \leq \kappa(f) < 1$ *for all* $f > 0$.

(ii) $\lim_{n\to\infty} \kappa(f_n) = 1$ *implies* $\lim_{n\to\infty} f_n = 0$.

The collection of all functions $\kappa(f)$ is denoted by S.

Kohlenbouch [10] proposed the concept of a hyperbolic metric space as follows.

Definition 2. *A hyperbolic space is a triplet* (Q, d, η) *where* (Q, d) *is a metric space and* $\eta : Q\times Q\times[0,1] \to Q$ *satisfies the following conditions, for all* $\omega, \xi, \kappa, \iota \in Q$ *and* $\mu, x \in [0,1]$
$(\eta 1) : d(\iota, \eta(\omega, \xi, \mu)) \leq (1-\mu)d(\iota, \omega) + \mu d(\iota, \xi)$,
$(\eta 2) : d(\eta(\omega, \xi, \mu), \eta(\omega, \xi, x)) = |\mu - x| d(\omega, \xi)$,
$(\eta 3) : \eta(\omega, \xi, \mu) = \eta(\xi, \omega, 1-\mu)$,
$(\eta 4) : d(\eta(\omega, \iota, \mu), \eta(\xi, \kappa, \mu)) \leq (1-\mu)d(\omega, \xi) + \mu d(\iota, \kappa)$.

In case only $(\eta 1)$ is satisfied, then the Definition 2 of hyperbolic space coincides with the convex metric space introduced by Takahashi [15]. Clearly, a hyperbolic space is a convex metric space. In the case of a convex metric space, we shall replace η by W (a convex structure on Q) and denote it by (Q, d, W). A nonempty subset J of a convex metric space Q is convex if $W(p, q, \lambda) \in J$ for all $p, q \in J$ and $\lambda \in [0, 1]$.
In the context of a normed space Q, the natural convex structure on Q is given by

$$W(p, q; \lambda) = \lambda p + (1-\lambda)q,\ p, q \in Q \ and\ \lambda \in [0, 1].$$

If J is a convex subset of a normed space and $E : J \to J$, then the average mapping $E_\mu : J \to J$ is given by

$$E_\mu p = (1-\mu)p + \mu Ep, \quad \text{where } \mu \in (0, 1].$$

A hyperbolic space (Q, d, η) is said to be uniformly convex if for all $x, y, z \in Q$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that $d(\eta(x, y, 1/2), z) \leq (1-\delta)r$ whenever $d(x, z) \leq r$ and $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon r$.

A map $h : (0,\infty) \times (0, 2] \to (0, 1]$ which provides in the above definition, a $\delta = h(r, \varepsilon)$ for $r > 0$ and for a fixed $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call h monotone if it decreases with r (for a fixed ε). Let $\{q_n\}$ be a bounded sequence in a hyperbolic space Q. For $q \in Q$, we define a continuous functional $r(., \{q_n\}) : Q \to [0,\infty)$ by

$$r(q, \{q_n\}) = \lim_{n\to\infty} \sup d(q_n, q).$$

The aymptotic radius $\rho = r(\{q_n\})$ of $\{q_n\}$ is given by

$$\rho = \inf\{r(q, \{q_n\}) : q \in Q\}.$$

The asymptotic center of a bounded sequence $\{q_n\}$ with respect to a subset U of Q is defined as:

$$A_U(\{q_n\}) = \{q \in Q : r(q, \{q_n\}) \leq r(p, \{q_n\}) \text{ for any } p \in U\}.$$

If the asymptotic center is taken with respect to Q, then it is simply denoted by $A(\{q_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma ensures that this property also holds in complete uniformly convex hyperbolic spaces.

Lemma 2. *([16, Lemma 2.2]). Let (Q, d, η) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{q_n\}$ in Q has a unique asymptotic center with respect to any nonempty closed convex subset U of Q.*

Recall that a bounded sequence $\{q_n\}$ in Q is known as $\Delta -$ *convergent* to $q \in Q$ if q is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$of $\{q_n\}$. In this case, we write

$$\Delta - \lim_{n\to\infty} \{q_n\} = q.$$

We include the following lemmas of Berinde and Pacurar [17, 18] for a ready reference.

Lemma 3. *Let (Q,d,W) be a convex metric space. For all $x, y \in Q$ and any $\lambda \in [0,1]$, we have*

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y).$$

Proof. By the triangular inequality and $(\eta 1)$, we get

$$\begin{aligned} d(x,y) &\leq d(x, W(x,y;\lambda)) + d(W(x,y;\lambda), y) \\ &\leq \lambda d(x,x) + (1-\lambda)d(x,y) + \lambda d(x,y) + (1-\lambda)d(y,y) = d(x,y). \end{aligned}$$

Lemma 4. *Let (Q, d, W) be a convex metric space. For all $x, y \in Q$ and any $\lambda \in [0,1]$, we have*

$$d(x, W(x, y; \lambda)) = (1 - \lambda)d(x, y)$$

and

$$d(W(x, y; \lambda), y) = \lambda d(x, y).$$

Proof. By $(\eta 1)$, we get

$$d(x, W(x, y; \lambda)) \leq (1 - \lambda)d(x, y),$$

and

$$d(W(x, y; \lambda), y) \leq \lambda d(x, y).$$

If we had strict inequality in either of the above two inequalities, then, by Lemma 3, we would reach the contradiction

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) < d(x, y).$$

Lemma 5. *Let (Q, d, W) be a convex metric space and $E : Q \to Q$ be a mapping. Define the mapping $E_\lambda : Q \to Q$ by*

$$E_\lambda x = W(x, Ex; \lambda), \quad x \in Q.$$

Then for any $\lambda \in [0, 1)$,

$$Fix(E) = Fix(E_\lambda).$$

Proof. For $\lambda = 0$, $E_\lambda = E$ and the assertion is trivial. Assume $\lambda \in (0,1)$ and let $a \in Fix(E)$. This means $a = Ea$ and therefore it follows that

$$d(a, E_\lambda a) = d(a, W(a, Ea; \lambda)) \leq d(a, a) + (1-\lambda)d(a, Ea) = 0,$$

i.e., $a \in Fix(E_\lambda)$.

Conversely, assume that $a \in Fix(E_\lambda)$. This means that $d(a, Ea) = 0$, which implies

$$d(a, W(a, Ea; \lambda)) = 0.$$

By Lemma 4,

$$d(a, W(a, Ea; \lambda)) = (1-\lambda)d(a, Ea),$$

so it follows that

$$(1-\lambda).d(a, Ea) = 0,$$

which, in view of the fact that $(1-\lambda) \neq 0$, implies $d(a, Ea) = 0$. Hence $a \in Fix(E)$.

3. Common fixed points results

Following the notion introduced in Definition 1 by Huang and Qian [14], we extended Theorem 3 by replacing μ with a function $\kappa \in S$ and weakening the continuity hypothesis.

Theorem 4. *Suppose that (Q, d) is a complete metric space and E and F are asymptotically regular self-mappings on Q. Suppose that there exist a function $\kappa \in S$ and a constant $K \in [0, +\infty)$ satisfying the following condition:*

$$d(Eq, Fp) \leq \kappa(d(q,p))d(q,p) + K\{d(q, Eq) + d(p, Fp)\} \tag{3}$$

for all $q, p \in Q$.
Suppose that E and F have a common approximate fixed point sequence (i.e., there is a sequence $\{q_h\} \subset Q$, such that $d(q_h, Eq_h) \to 0$ and $d(q_h, Fq_h) \to 0$ as $h \to \infty$). Then E and F have a unique common fixed point p provided E and F are either k-continuous or orbitally continuous.
In particular, $\{q_h\} \to p$ as $h \to \infty$.

Proof. Take $q \in Q$. Define $q_h = E^h q$ and $p_h = F^h q$ for all $h \in \mathbb{N}$.
By (3), we have

$$d(E^{h+1}q, F^{h+1}q) = d(E(E^h q), F(F^h q)) \tag{4}$$

$$\begin{aligned} d(E^{h+1}q, F^{h+1}q) \quad &\leq \quad \kappa(d(E^h q, F^h q))d(E^h q, F^h q) \\ &\quad + K[d(E^h q, E^{h+1}q) + d(F^h q, F^{h+1}q)]. \end{aligned} \tag{5}$$

In view of the condition, $0 \leq \kappa(d(E^h q, F^h q)) \leq 1$, we consider two cases for $\limsup_{h\to\infty} \kappa(d(E^h q), F^h q)$.
Case 1:

$$\limsup_{h\to\infty} \kappa(d(E^h q, F^h q)) = 1.$$

In this case, there exists a subsequence $\{\kappa(d(E^{h_k}q, F^{h_k}q))\}$ of $\{\kappa(d(E^{h}q, F^{h}q))\}$ such that

$$\lim_{k\to\infty} \kappa(d(E^{h_k}q, F^{h_k}q)) = 1. \tag{6}$$

Now (6) implies on the basis of Definition 1 ,

$$\lim_{k\to\infty} d(E^{h_k}q, F^{h_k}q) = 0. \tag{7}$$

We prove that $\{E^{h_k}q\}$ is a Cauchy sequence.
Suppose on the contrary that $\{E^{h_k}q\}$ is not a Cauchy sequence. Then there exists $\epsilon_0 > 0$ and two integer sequences $\{h_{\tilde{k}(i)}\}, \{h_{k(i)}\}$ of $\{h_k\}$ with $h_{\tilde{k}(i)} > h_{k(i)} > i$ such that

$$d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q) \geq \epsilon_0, i = 1, 2, 3, \cdots \tag{8}$$

Consequently, we have

$$\epsilon_0 \leq d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q),$$

$$\epsilon_0 \leq d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i-1)}}q) + d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{k(i-1)}}q) + d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{k(i)}}q). \tag{9}$$

If $i \to \infty$ in (9), then by asymptotic regularity of E, we obtain

$$\liminf_{i\to\infty} d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{k(i-1)}}q) \geq \epsilon_0. \tag{10}$$

Now by (3), we have

$$d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q) \leq d(E^{h_{\tilde{k}(i)}}q, F^{h_{k(i)}}q) + d(F^{h_k(i)}q, E^{h_{k(i)}}q) \tag{11}$$

$$\begin{aligned} d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q) \quad \leq \quad & \kappa(d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{k(i-1)}}q))d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{k(i-1)}}q) \\ & + K[d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{\tilde{k}(i)}}q) + d(F^{h_k(i-1)}q, F^{h_{k(i)}}q)] \\ & + d(F^{h_k(i)}q, E^{h_{k(i)}}q). \end{aligned}$$

and

$$\begin{aligned} d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q) \quad \leq \quad & \kappa(d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{k(i-1)}}q))d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{k\tilde{i}}}q) \\ & + d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i-)}}q) + d(E^{h_k(i)}q, F^{h_{k(i)}}q)d(F^{h_k(i)}q, F^{h_{k(i-1)}}q) \\ & + K[d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{k\tilde{i}}}q) + d(F^{h_k(i-1)}q, F^{h_{k(i)}}q) \\ & + d(F^{h_k(i)}q, E^{h_{k(i)}}q)]. \end{aligned} \tag{12}$$

Dividing both sides of the inequality (12) by $d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)$ and comparing the new inequality with (8), we conclude that

$$\begin{aligned}1 \quad \leq \quad & \kappa(d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{k(i-1)}}q))(\frac{d(E^{h_{\tilde{k}(i-1)}}q, E^{h_{\tilde{k}}(i)}q)}{d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)}) \\ & +1 + \frac{d(E^{h_k(i)}q, F^{h_{k(i)}}q)}{d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)} + \frac{d(F^{h_k(i)}q, F^{h_{k(i-1)}}q)}{d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)} \\ & +K\frac{d(E^{h_{\tilde{k}(i-1)}}q, F^{h_{\tilde{k}(i)}}q) + d(F^{h_k(i-1)}q, F^{h_{k(i)}}q)}{d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)} \\ & +\frac{d(F^{h_k(i)}q, E^{h_{k(i)}}q)}{d(E^{h_{\tilde{k}(i)}}q, E^{h_{k(i)}}q)}. \end{aligned} \tag{13}$$

Let $i \to \infty$ in (13), using (7), (8), asymptotically regularity of E and F and the fact $0 \leq \kappa(.) \leq 1$, we deduce that

$$\lim_{i\to\infty} \kappa(d(E^{h_{\tilde{k}(i)}-1}q, F^{h_{k(i)}-1}q)) = 1.$$

Using Definition 1, we obtain

$$\lim_{i\to\infty} (d(E^{h_{\tilde{k}(i)}-1}q, F^{h_{k(i)}-1}q)) = 0. \tag{14}$$

In view of

$$d(E^{h_{\tilde{k}(i)}-1}q, E^{h_{k(i)}-1}q) \leq d(E^{h_{\tilde{k}(i)}-1}q, F^{h_{k(i)}-1}q) + d(F^{h_k(i)-1}q, E^{h_{k(i)}-1}q),$$

a combination of (7) and (14), gives

$$\lim_{i\to\infty} d(E^{h_{\tilde{k}(i)}-1}q, F^{h_{k(i)}-1}q) = 0.$$

It contradicts (10). Hence $\{q_{h_k}\} = \{E^{h_k}q\}$ is a Cauchy sequence.
Since Q is complete, $\{q_{h_k}\}$ converges to u in Q. Since

$$d(F^{h_k}q, u) \leq d(E^{h_k}q, F^{h_k}q) + d(E^{h_k}q, u),$$

therefore by (7), we conclude that $\{F^{h_k}q\}$ converges to u.
Suppose that E is orbitally continuous. Let $\{Eq_h\}$ be a sequence in the orbit of E at the point q. The orbital continuity of E implies that $\{Eq_{h_k}\}$ converges to Eu. By asymptotically regularity of E, we have

$$\lim_{k\to\infty} d(q_{h_{k+1}}, q_{h_k}) = \lim_{k\to\infty} d(E^{h_{k+1}}q, E^{h_k}q) = 0,$$

which gives

$$\lim_{k\to\infty} d(q_{h_{k+1}}, q_{h_k}) \leq \lim_{k\to\infty} \Big(d(q_{h_{k+1}}, Eq_{h_{k+1}}) + d(Eq_{h_{k+1}}, Eq_{h_k}) + d(Eq_{h_k}, q_{h_k})\Big).$$

Since

$$d(q_{h_{k+1}}, Eq_{h_{k+1}}) \to 0,$$

$$d(Eq_{h_k}, q_{h_k}) \to 0,$$

therefore,

$$\lim_{k\to\infty} d(q_{h_{k+1}}, q_{h_k}) = 0.$$

Hence

$$\lim_{k\to\infty} Eq_{h_k} = \lim_{k\to\infty} q_{h_{k+1}} = \lim_{k\to\infty} q_{h_k} = u$$

implies that $Eu = u$ by the uniqueness of limit.
Now suppose that E is k-continuous. In view of

$$d(q_{h_{k+j}}, q_{h_k}) \leq d(q_{h_{k+j}}, q_{h_{k+j-1}}) + \cdots + d(q_{h_{k+1}}, q_{h_k})$$

and

$$d(E^{h_k+j}q_{h_{k+j}}, E^{h_k}q_{h_k}) = d(E^{h_k+j}q, E^{h_k+j-1}q) + \cdots + d(E^{h_k+1}q, E^{h_k}q),$$

for all $j = 1, 2, 3, \cdots, k$ by asymptotic regularity of E, we get

$$\lim_{k\to\infty} q_{h_{k+j}} = \lim_{k\to\infty} q_{h_k} = u, j = 1, 2, 3, \cdots, k. \tag{15}$$

In particular,

$$\lim_{k\to\infty} E^{k-1}q_{h_k} = \lim_{k\to\infty} q_{h_{k+j-1}} = u. \tag{16}$$

As E is k-continuous, (16) gives

$$\lim_{k\to\infty} E^k q_{h_k} = Eu. \tag{17}$$

By (15), we have

$$\lim_{k\to\infty} E^k q_{h_k} = \lim_{k\to\infty} q_{h_{k+j}} = u. \tag{18}$$

A combination of (17) and (18), gives

$$Eu = u.$$

Similarly, we can prove that

$$Fu = u.$$

So u is a common fixed point of E and F.
Next, assume that v is another common fixed point of E and F with $u \neq v$.

Definition 1 (i) applied to (3) gives

$$d(u, v) = d(Eu, Fv) \leq \kappa(d(u, v))d(u, v) + K[d(u, Eu) + d(v, Fv)].$$

$$d(u, v) \leq \kappa(d(u, v))d(u, v).$$

$$d(u, v) < d(u, v),$$

which is a contradiction. Therefore common fixed point of E and F is unique.

Case 2:

$$\limsup_{h\to\infty} \kappa(d(E^h q, F^h q)) < 1.$$

In this case, there exists $\sigma \in (0, 1)$ such that

$$0 < \kappa(d(E^h q, F^h q)) < \sigma.$$

By (5) we have

$$(d(E^{h+1}q, F^{h+1}q)) \leq \sigma d(E^h q, F^h q) + K(d(E^h q, E^{h+1}q) + d(F^h q, F^{h+1}q)). \tag{19}$$

Let

$$u_h = d(E^h q, F^h q),\ v_h = 1 - \sigma,\ w_h = \frac{K(d(E^h q, E^{h+1}q) + d(F^h q, F^{h+1}q))}{1 - \sigma}.$$

By (19), we have

$$u_{h+1} \leq (1 - v_h)u_h + v_h w_h, \forall h \in \mathbb{N}.$$

Since E and F are asymptotically regular on Q, we conclude

$$\lim_{h\to\infty} w_h = 0.$$

Moreover,

$$\sum_{h=1}^{\infty} v_h = \sum_{h=1}^{\infty} (1 - \sigma) = \infty.$$

By Lemma 1,

$$\lim_{h\to\infty} (E^h q, F^h q) = 0.$$

Hence for any subsequence $\{h_{k(i)}\}$ of $\{h_k\}$, we have

$$\lim_{i\to\infty} (E^{h_{k(i)}} q, F^{h_{(i)}} q) = 0.$$

Thus (7) holds. The rest of proof is the same as in Case I.

Next, we present a numerical example to illustrate Theorem 4.

Example 1. *Consider $Q = [0, 1]$, equipped with the metric d defined by $d(q, p) = |q - p|$. Define self-mappings on Q:*

$$Eq = \frac{q}{2} \quad \textit{and} \quad Fq = \frac{q}{3} \quad \textit{for all } q \in Q.$$

For $q = \frac{1}{2} \in Q$,

$$\lim_{k\to\infty} d\Big(E^k\Big(\frac{1}{2}\Big), E^{k+1}\Big(\frac{1}{2}\Big)\Big) = \lim_{k\to\infty}\Big|\frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}\Big| = 0$$

implies that E *is asymptotically regular. Similarly, it can be demonstrated that* F *is asymptotically regular. Furthermore, the mappings* E *and* F *are orbitally continuous. Next, we establish that* E *and* F *satisfy condition* (3) *with the parameter* $K = 1$. *Now define* $\kappa : \mathbb{R}^+ \to [0,1]$ *by*

$$\kappa(t) = \begin{cases} \frac{1}{2} + \frac{1}{2}t & \text{if } t \in [0,1], \\ 1 & \text{if } t \in (1, +\infty). \end{cases}$$

Then $\kappa \in S$, *for any* $q, p \in Q$, *since* $|q - p| \leq 1$. *Consider* (3) *in the form*

$$\begin{aligned} &\kappa(d(q,p))d(q,p) + d(q,Eq) + d(p,Fp) - d(Eq,Fp) \\ &= \frac{1}{2}(1 + |q-p|)|q-p| + \Big|q - \frac{q}{2}\Big| + \Big|p - \frac{p}{3}\Big| - \Big|\frac{q}{2} - \frac{p}{3}\Big| \\ &= \frac{1}{2}(1 + |q-p|)|q-p| + \frac{q}{2} + \frac{2p}{3} - \Big|\frac{q}{2} - \frac{p}{3}\Big|. \end{aligned} \tag{20}$$

Now, there are two cases to consider for (20)*:*
Case i. *If* $q \geq p$, *then we have:*

$$\begin{aligned} \frac{1}{2}(1 + (q-p))(q-p) + \frac{q}{2} + \frac{2p}{3} - \Big(\frac{q}{2} - \frac{p}{3}\Big) &= \frac{1}{2}(q-p) + \frac{1}{2}(q-p)^2 + \frac{q}{2} + \frac{2p}{3} - \frac{q}{2} + \frac{p}{3} \\ &= \frac{1}{2}(q+p) + \frac{1}{2}(q-p)^2 \geq 0; \end{aligned}$$

thus

$$d(Eq,Fp) \leq \kappa(d(q,p))d(q,p) + d(q,Eq) + d(p,Fp).$$

Case ii. *if* $q < p$, *then we have:*

$$\begin{aligned} \frac{1}{2}(1 + (p-q))(p-q) + \frac{q}{2} + \frac{2p}{3} - \Big|\frac{q}{2} - \frac{p}{3}\Big| &= \begin{cases} \frac{1}{2}(p-q) + \frac{1}{2}(p-q)^2 + p & \text{if } \frac{q}{3} \geq \frac{p}{3}, \\ \frac{1}{2}(p-q) + \frac{1}{2}(p-q)^2 + q + \frac{p}{3} & \text{if } \frac{q}{2} < \frac{p}{3} \end{cases} \\ &\geq 0; \end{aligned}$$

hence

$$d(Eq,Fp) \leq \kappa(d(q,p))d(q,p) + d(q,Eq) + d(p,Fp).$$

Therefore E *and* F *satisfy* (3) *with* $K = 1$. *Next, we show that* E *and* F *have a common approximate fixed point sequence. Consider the sequence* $\{q_k\} \subset Q$ *where* $q_k = \frac{1}{k}$.

$$d(q_k, Eq_k) = \Big|\frac{1}{k} - \frac{1}{2k}\Big| = \frac{1}{2k} \to 0 \text{ as } k \to \infty,$$

and

$$d(q_k, Fq_k) = \left|\frac{1}{k} - \frac{1}{3k}\right| = \frac{1}{3k} \to 0 \ as\ k \to \infty.$$

Hence, E and F have a common approximate fixed point sequence. Since E and F are both continuous (and thus k-continuous), therefore by Theorem 4, they have a unique common fixed point $q = 0$. In particular, $\{q_k\} \to 0$ as $k \to \infty$.

We now present a partial extension of Theorem 2 for a function which is not asymptotically regular and is defined on a closed and convex subset of a uniformly convex hyperbolic space.

Theorem 5. *Let J be a nonempty closed and convex subset of complete uniformly convex hyperbolic space Q with monotone modulus of convexity η. Let $E : J \to Q$ be a mapping satisfying*

$$d(Eq, Ep) \leq \mu d(q, p) + \nu\{d(q, Eq) + d(p, Ep)\},\ for\ all\ q, p \in J, \tag{21}$$

where $0 \leq \mu < 1$, $0 \leq \nu < \infty$ with $\mu + 2\nu blue{<}1$. Let $\{q_k\}$ be a bounded sequence in J such that $\lim_{k\to\infty} d(q_k, Eq_k) = 0$ and $\Delta - \lim_{k\to\infty} q_k = q^*$. Then E has a unique fixed point q^*.

Proof. Let $\{q_k\}$ be any bounded sequence in J. Since J is a subset of Q which is complete convex hyperbolic space with monotone uniform convexity, therefore by Lemma 2, $\{q_k\}$ has a unique asymptotic center in J.
But $\Delta - \lim_{k\to\infty} q_k = q^*$. So $A(\{q_k\}) = q^*$. Using (21), we get

$$\begin{aligned} d(q_k, Eq^*) &\leq d(q_k, Eq_k) + d(Eq_k, Eq^*). \\ &\leq d(q_k, Eq_k) + \mu d(q_k, q^*) + \nu[d(q_k, Eq_k) + d(q^*, Eq^*)]. \\ &\leq (1 + \nu)d(q_k, Eq_k) + (\mu + \nu)d(q_k, q^*) + \nu d(q_k, Eq^*). \end{aligned}$$

Thus

$$\begin{aligned} d(q_k, Eq^*) &\leq \frac{1+\nu}{1-\nu} d(q_k, Eq_k) + \frac{\mu+\nu}{1-\nu} d(q_k, q^*) \\ &= \frac{1+\nu}{1-\nu} d(q_k, Eq_k) + \frac{\mu+2\nu-\nu}{1-\nu} d(q_k, q^*) \\ &\leq \frac{1+\nu}{1-\nu} d(q_k, Eq_k) + d(q_k, q^*). \end{aligned} \tag{22}$$

Since $\lim_{k\to\infty} d(q_k, Eq_k) = 0$, taking $\limsup$ on both sides of inequality (22), we obtain

$$r(Eq^*, q_k) = \limsup_{k\to\infty} d(q_k, Eq^*) \leq \limsup_{k\to\infty} d(q_k, q^*) = r(q^*, \{q_k\}).$$

We know that asymptotic center of sequence $\{q_k\}$ is unique, so $Eq^* = q^*$; thus $q^* \in Fix(E)$. The uniqueness of fixed point follows by (21).

Example 2. *Let* $Q=\mathbb{R}^2$ *and* $d^*:\mathbb{R}^2\times\mathbb{R}^2\to[0,\infty)$ *be defined by*

$$d^*(\bar{q},\bar{p})=\sqrt{(q_1-p_1)^2+(q_1^2-q_2-p_1^2+p_2)^2}, \tag{23}$$

where $\bar{q}=(q_1,q_2),\bar{p}=(p_1,p_2)\in\mathbb{R}^2$. *Then* $(\mathbb{R}^2,d^*)$ *is not a metric space in the classical sense but is a Hardamard space (see [19]). Hence* $(\mathbb{R}^2,d^*)$ *is a complete uniformly convex hyperbolic space (for details, see Example 2.1 in [20]). Let* $J=[0,1]\times[0,1]$.
Define $E:J\to Q$ *as* $E\bar{q}=(q_1,q_1^2+q_2)$, *where* $\bar{q}=(q_1,q_2)$.
For any $\mu\in\left[\frac{2}{5},1\right)$ *and* $0\leq\nu<\infty$ *with* $\mu+2\nu<1$, *we need to verify the following, for any* $\bar{q},\bar{p}\in J$,

$$\Gamma:=\mu d^*(\bar{q},\bar{p})+\nu[d^*(\bar{q},E\bar{q})+d^*(\bar{p},E\bar{p})]-d^*(E\bar{q},E\bar{p})\geq 0. \tag{24}$$

Thus, for any $\bar{q}=(q_1,q_2),\bar{p}=(p_1,p_2)\in J$, *by using* (23) *in* (24), *we get*

$$\begin{aligned}\Gamma &= d^*((q_1,q_2),(p_1,p_2))+d^*((q_1,q_2),(q_1,q_1^2+q_2))+d^*((p_1,p_2),(p_1,p_1^2+p_2))\\ &\quad -d^*((q_1,q_1^2+q_2),(p_1,p_1^2+p_2))\\ &= \mu\sqrt{(q_1-p_1)^2+(q_1^2-q_1-p_2^2+p_2)^2}+\nu\Big(\sqrt{(q_1-q_1)^2+(q_1^2-q_2-q_1^2+(q_1^2+q_2))^2}\\ &\quad +\sqrt{(p_1-p_1)^2+(p_1^2-p_2-p_1^2+(p_1^2+p_2))^2}\Big)\\ &\quad -\sqrt{(q_1-p_1)^2+(q_1^2-(q_1^2+q_2)-p_1^2+(p_1^2+p_2))^2}\\ &= \mu\sqrt{(q_1-p_1)^2+((q_2-p_2)+(q_1^2-p_1^2))^2}+\nu(q_1^2+p_1^2)-\sqrt{(q_1-p_1)^2+(q_2-p_2)^2}\\ &\geq 0\end{aligned}$$

Hence (24) *holds. Therefore* E *satisfies* (21) *with* $\mu\in\left[\frac{2}{5},1\right)$ *and* $0\leq\nu<\infty$ *where* $\mu+2\nu<1$. *For any* $q_k\in J$, *define* $q_k=\left(\frac{1}{k},\frac{1}{k}\right)$. *Then* $Eq_k=\left(\frac{1}{k},\frac{1}{k^2}+\frac{1}{k}\right)$, *so from* (23), *we get that*

$$\begin{aligned}d^*(q_k,Eq_k) &= d^*(\frac{1}{k},\frac{1}{k^2}+\frac{1}{k})\\ &= \sqrt{\Big(\frac{1}{k}-\frac{1}{k}\Big)^2+\Big(\frac{1}{k^2}-\frac{1}{k}-\frac{1}{k^2}+\Big(\frac{1}{k^2}+\frac{1}{k}\Big)\Big)^2}=\frac{1}{k^2}.\end{aligned}$$

It follows that $d^*(q_k,Eq_k)\to 0$ *as* $k\to\infty$. *All the assumptions of Theorem 5 hold and hence, the mapping* E *has a unique fixed point at* $(0,0)$.

Khan [21] has studied some properties of the set of fixed points of a $*$-nonexpansive mapping in the context of strictly convex Banach spaces. In the following result, we establish the closedness and convexity of the set of fixed points of a contractive-type mapping on a convex metric space.

Theorem 6. *Let* J *be a closed and convex subset of a convex metric space* Q *and* $E:J\to Q$ *be a mapping satisfying (21). Then* $Fix(E)$ *is closed and convex.*

Proof. If $Fix(E) = \emptyset$, then nothing to show, since empty set is closed and convex. Now assume that $Fix(E) \neq \emptyset$. We first show that $Fix(E)$ is closed. Let $\{q_k\}$ be a sequence in $Fix(E)$ such that $\{q_k\}$ converges to a point u in J. We show that $u \in Fix(E)$. By (21) , we obtain

$$d(q_k, Eu) = d(Eq_k, Eu) \leq \mu d(q_k, u) + \nu[d(q_k, Eq_k) + d(u, Eu)].$$

Implies

$$d(q_k, Eu) \leq \mu d(q_k, u) + \nu[d(u, q_k) + d(q_k, Eu)].$$

Thus

$$\begin{aligned} d(u, Eu) &\leq d(u, q_k) + d(q_k, Eu) \\ &\leq d(u, q_k) + \mu d(q_k, u) + \nu[d(q_k, Eq_k) + d(u, Eu)] \\ &= (1 + \mu)d(q_k, u) + \nu d(u, Eu); \end{aligned}$$

hence

$$d(u, Eu) \leq \frac{1 + \mu}{1 - \nu} d(q_k, u) \tag{25}$$

As $\mu \in [0, 1)$ and $\mu + 2\nu < 1$, that is, $\nu < 1$ and since $\{q_k\} \to u$ as $k \to \infty$, therefore $\lim_{k\to\infty} d(q_k, u) = 0$. From (25), we get $u \in Fix(E)$ as desired.

Next, we show that $Fix(E)$ is convex. Let $q, p \in Fix(E)$ and $\alpha \in [0, 1]$. For any $z \in J$, assume that $z = \eta(q, p, \alpha)$. Now we show that $z \in Fix(E)$, that is, $z = Ez$ or $E(\eta(q, p, \alpha)) = \eta(q, p, \alpha)$. Thus

$$\begin{aligned} d(q, Ez) &= d(Eq, Ez) \leq \mu d(q, z) + \nu[d(q, Eq) + d(z, Ez)]. \\ &\leq \mu d(q, z) + \nu[d(z, q) + d(q, Ez)]. \\ &\leq (\mu + \nu)d(q, z) + \nu d(q, Ez). \end{aligned}$$

Hence

$$d(q, Ez) \leq \frac{(\mu + \nu)}{1 - \nu} d(q, z) \leq d(q, z). \tag{26}$$

Similarly,

$$d(p, Ez) \leq d(p, z). \tag{27}$$

Using (26), (27) and (η1), we get

$$\begin{aligned} d(q, p) &\leq d(q, Ez) + d(Ez, p). \\ &\leq d(q, z) + d(z, p). \\ &\leq d(q, \eta(q, p, \alpha)) + d(\eta(q, p, \alpha), p) \\ &\leq (1 - \alpha)d(q, q) + \alpha d(q, p) + (1 - \alpha)d(q, p) + d(p, p) \end{aligned}$$

$$= \quad d(p,q)$$

Thus, we conclude by (26)) and (27) that $d(q,Ez) = d(q,z)$ and $d(p,Ez) = d(p,z)$ because if $d(q,Ez) < d(q,z)$ or $d(p,Ez) < d(p,z)$, then we will obtain a contradiction $d(p,q) < d(p,q)$. Hence $Ez = z$, that is, $E\eta((q,p,\alpha)) = \eta(q,p,\alpha)$ for all $q,p \in Fix(E)$ and $\alpha \in [0,1]$. Therefore $Fix(E)$ is convex.

Based on Lemma 5, we introduce the contractive condition for self-mappings T_λ and S_λ on Q as follows:

$$d(T_\lambda x, S_\lambda y) \leq M(d(x,y)) + K\{d(x,T_\lambda x) + d(y,S_\lambda y)\} \tag{28}$$

where $0 \leq M < 1$, $1 < K < \infty$ and $M + 2K \leq 1$.
An extension of Theorem 2, is obtained by replacing the maps E and F on a metric space with their, respective, average mappings, in the context of a convex metric space, as follows.

Theorem 7. *Let (Q,d,W) be a complete convex metric space. Assume that T_λ and S_λ are asymptotically regular self-mappings on Q satisfying (28). If T_λ and S_λ are orbitally continuous or k-continuous for some $k \geq 1$, then T_λ and S_λ have a unique common fixed point p. Furthermore,* $\lim_{n\to\infty} T_\lambda^n x = p = \lim_{n\to\infty} S_\lambda^n x$ *for any $x \in Q$.*

Proof. **Step 1:** We note that the Picard iterations of T_λ and S_λ form the Krasnoselskij iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = T_\lambda x_n = (1-\mu)x_n + \mu T_\lambda x_n \text{ and } x_n = S_\lambda x_{n-1} \quad n = 0,1,2,\cdots \tag{29}$$

Using (28), we get

$$d(x_{n+1},x_n) \leq Md(x_n,x_{n-1}) + K\{d(x_n,x_{n+1}) + d(x_{n-1},x_n)\};$$

thus

$$d(x_{n+1},x_n) \leq \frac{(M+K)}{1-K} d(x_n,x_{n-1})$$

$$d(x_{n+1},x_n) \leq \alpha d(x_n,x_{n-1}), \qquad \left(\alpha := \frac{M+K}{1-K}\right), \tag{30}$$

which inductively implies that

$$d(T_\lambda^{n+1}x, S_\lambda^n x) = d(x_{n+1},x_n) \leq \alpha^n d(x_1,x_0). \tag{31}$$

This implies

$$d(T_\lambda^{n+1}x, S_\lambda^n x) \to 0 \text{ as } n \to \infty.$$

Step 2: We now prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Suppose on contrary that $\{x_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exisst $\epsilon > 0$ and sequences of numbers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ for $k = 1, 2, 3, \cdots$ such that

$$d(T_\lambda^{m(k)}, T_\lambda^{n(k)}) \geq \epsilon. \tag{32}$$

Assume that $m(k) \geq n(k)$. So by (31), we have $d(T_\lambda^{m(k)-1}x, T_\lambda^{n(k)}x) < \epsilon$.
Hence

$$\begin{aligned} \epsilon \leq d(T_\lambda^{m(k)}x, T_\lambda^{n(k)}x) &\leq d(T_\lambda^{n(k)}x, T_\lambda^{n(k)-1}x) + d(T_\lambda^{m(k)-1}x, T_\lambda^{n(k)}x) \\ &< d(T_\lambda^{m(k)}x, T_\lambda^{m(k)-1}x) + \epsilon. \end{aligned} \tag{33}$$

Letting $k \to \infty$ and using asymptotically regularity of T_λ, we get

$$\lim_{k\to\infty} d(T_\lambda^{m(k)}x, T_\lambda^{n(k)}x) = \epsilon.$$

Now the following inequality and asymptotic regularity of T_λ

$$\begin{aligned} d(T_\lambda^{m(k)-1}x, T_\lambda^{n(k)-1}x) &\leq d(T_\lambda^{m(k)-1}x, T_\lambda^{m(k)}x) + d(T_\lambda^{m(k)}x, T_\lambda^{m(k)}x) \\ &\quad + d(T_\lambda^{n(k)}x, T_\lambda^{n(k)-1}x) \end{aligned}$$

imply

$$\lim_{k\to\infty} d(T_\lambda^{m(k)-1}x, T_\lambda^{n(k)-1}x) = \epsilon.$$

From (29), we get

$$d(T_\lambda^{m(k)}x, T_\lambda^{n(k)}x) \leq d(T_\lambda^{m(k)}x, S_\lambda^{n(k)}x) + d(S_\lambda^{n(k)}x, T_\lambda^{n(k)}x)$$

and

$$\begin{aligned} d(T_\lambda^{m(k)}x, T_\lambda^{n(k)}x) &\leq d(S_\lambda^{n(k)}x, T_\lambda^{n(k)}x) + Md(T_\lambda^{m(k)-1}x, T_\lambda^{n(k)-1}x) \\ &\quad + K[d(T_\lambda^{m(k)-1}x, T_\lambda^{m(k)}x) + d(S_\lambda^{n(k)-1}x, S_\lambda^{n(k)}x)] \\ &\quad + Md(T_\lambda^{n(k)-1}x, S_\lambda^{n(k)-1}x). \end{aligned}$$

Letting $k \to \infty$, it follows by (30) (32) and (33) that $\epsilon \leq M\epsilon$. This is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Given that Q is complete, $\{x_n\}$ converges to $p \in Q$. Moreover,

$$d(S_\lambda^n x, p) \leq d(S_\lambda^n x, T_\lambda^n x) + d(T_\lambda^n x, p),$$

so it follows from Step 1 that $T_\lambda^n x$ and $S_\lambda^n x$ converge to $p \in Q$.

Step 3: T_λ and S_λ have a unique common fixed point p. Let T_λ be k-continuous. Since $\lim_{n\to\infty} T_\lambda^{n-1}x = p$. So by k-continuity of T_λ, $\lim_{n\to\infty} T_\lambda^{n}x = T_\lambda p$. By uniqueness of limit , $T_\lambda p = p$.
Analogically, suppose that T_λ is orbital continuous . Since $\lim_{n\to\infty} x_n = p$, orbital continuity of T_λ implies that

$$\lim_{n\to\infty} T_\lambda x_n = T_\lambda p.$$

This yield $T_\lambda p = p$. Similarly, $\lim_{n\to\infty} S_\lambda^n x = p$, we have that $Sp = p$. Hence $p \in Fix(T_\lambda) \cap Fix(S_\lambda)$.
Uniqueness: Assume that $p, q \in Fix(T_\lambda) \bigcap Fix(S_\lambda)$ and $q_1 \neq p$. Let $x = p$ and $y = q_1$. Then (28) becomes $d(p, q_1) \leq Md(p, q_1)$. It is a contradiction. Hence p is a unique common fixed point of T_λ and S_λ.

In the light of Lemma 5, we define Zemfirescue mapping in terms of T_λ:

Definition 3. *Let (Q, d, W) be a complete convex metric space and $T_\lambda : Q \to Q$ satisfies the following conditions.*

(i) $d(T_\lambda x, T_\lambda y) \leq ad(x, y), \qquad 0 < a < 1,$

(ii) $d(T_\lambda x, T_\lambda y) \leq b\Big((d(x, T_\lambda x) + d(y, T_\lambda y)\Big), \qquad 0 < b < \frac{1}{2}$

(iii) $d(T_\lambda x, T_\lambda y) \leq c\Big(d(x, T_\lambda y) + d(y, T_\lambda x)\Big), \qquad 0 < c < \frac{1}{2}.$

An analogue of Theorem 1 by Zamfiresuc [22] for the mapping T_λ is presented below.

Theorem 8. *Let (Q, d, W) be a complete convex metric space and $T_\lambda : Q \to Q$ be Zamfirescue asymptotically regular map. Then,*

(i) $Fix(T_\lambda) = \{p\}$, *and*

(ii) the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process $x_{n+1} = W(x_n, T_\lambda x_n; \lambda)$, $n \geq 0$ converges to p, for $x_0 \in Q$.

Proof. We note that the Picard iterations of T_λ actually form the Kranoselskij iterative process $\{x_n\}_{n=0}^{\infty}$. Now, choose $x_0 \in Q$ arbitrarily and fix integer $n \geq 0$.
Consider $x = T_\lambda^n x_0$ and $y = T_\lambda^{n+1} x_0$, we obtain by Definition 3(i), with $\delta < 1$

$$d(T_\lambda^{n+1} x_0, T_\lambda^{n+2} x_0) \leq \delta d(T_\lambda^{n} x_0, T_\lambda^{n+1} x_0)$$

and by Definition 3(ii)

$$d(T_\lambda^{n+1} x_0, T_\lambda^{n+2} x_0) \leq b\Big(d(T_\lambda^{n} x_0, T_\lambda^{n+1} x_0) + d(T_\lambda^{n+1} x_0, T_\lambda^{n+2} x_0)\Big)$$

which implies

$$d(T_\lambda^{n+1} x_0, T_\lambda^{n+2} x_0) \leq \frac{b}{1-b} d(T_\lambda^{n} x_0, T_\lambda^{n+1} x_0).$$

Thus

$$d(T_\lambda^{n+1}x_0, T_\lambda^{n+2}x_0) \leq \delta d(T_\lambda^n x_0, T_\lambda^{n+1}x_0) \qquad \text{where } \delta := \frac{b}{1-b}.$$

Similarly, with condition (iii) of Definition 3, we obtain

$$d(T_\lambda^{n+1}x_0, T_\lambda^{n+2}x_0) \leq c\Big(d(T_\lambda^n x_0, T_\lambda^{n+2}x_0) + d(T_\lambda^{n+1}x_0, T_\lambda^{n+1}x_0)\Big).$$

Hence

$$d(T_\lambda^{n+1}x_0, T_\lambda^{n+2}x_0) \leq \delta d(T_\lambda^n x_0, T_\lambda^{n+1}x_0).$$

This inequality is true for every n. So that $\{T_\lambda^n(x_0)\}_{n=0}^\infty$ is a Cauchy sequence and therefore converges to some point $p \in Q$.

(i) We now prove that p is a fixed point of T_λ. Suppose $T_\lambda p \neq p$. We consider the ball

$$B = \{p \in Q : d(p, x) \leq \frac{1}{4} d(p, T_\lambda p)\}.$$

Observe that $d(x, T_\lambda p) \geq \frac{3}{4} d(p, T_\lambda p)$ for every point $p \in B$. So there exists a number N such that $T_\lambda^n x_0 \in B$ for each $n \geq N$. Now taking $x = T_\lambda^N x_0$ and $y = p$. We must have one of the following situations:

(a) $d(T_\lambda^{n+1}x_0, T_\lambda p) \leq a d(T_\lambda^N x_0, p)$, which sets a contradiction as follows

$$d(T_\lambda^n x_0, p) \leq \frac{1}{4} d(p, T_\lambda p) < d(T_\lambda^{n+1}x_0, T_\lambda p),$$

(b) $d(T_\lambda^n x_0, T_\lambda p) \leq b\Big(d(T_\lambda^n x_0, T_\lambda^{n+1}x_0) + d(p, T_\lambda p)\Big)$, contradicting

$$\begin{aligned} b\Big(d(T_\lambda^n x_0, T_\lambda^{n+1}x_0) + d(p, T_\lambda p)\Big) &< \frac{1}{2}(d(T_\lambda^n x, p) + d(p, T_\lambda^{n+1}x_0) + d(p, T_\lambda p)) \\ &\leq \frac{3}{4} d(p, T_\lambda p) \\ &\leq d(T_\lambda^{n+1}x_0, T_\lambda p) \end{aligned}$$

(c) $d(T_\lambda^{n+1}x_0, T_\lambda p) \leq c\Big(d(T_\lambda^n x_0, T_\lambda(p)) + d(T_\lambda^{n+1}x_0, p)\Big)$ contradicting

$$\begin{aligned} c\Big(d(T_\lambda^n x_0, T_\lambda(p)) + d(T_\lambda^{n+1}x_0, p)\Big) &< \frac{1}{2}(d(T_\lambda^n x_0, p) + d(p, T_\lambda p) + d(T_\lambda^{n+1}x_0, p)) \\ &\leq \frac{3}{4} d(p, T_\lambda, p) \\ &\leq d(T_\lambda^{n+1}x_0, T_\lambda p) \end{aligned}$$

Thus $T_\lambda p = p$. Now we show that this fixed point p is unique. Suppose that this is not true .
Let $T_\lambda p^{'} = p^{'}$ for some point $p^{'} \neq p \in Q$. Then

$$d(T_\lambda p, T_\lambda p^{'}) = d(p, p^{'})$$

$$d(T_\lambda p, T_\lambda p^{'}) > d(p, T_\lambda p) + d(p^{'}, T_\lambda p^{'})$$

$$d(T_\lambda p, T_\lambda p^{'}) = \frac{1}{2}(d(p, T_\lambda p^{'}) + d(p^{'}, T_\lambda(p)))$$

So that none of the three conditions of Zamfirescue map is satisfied by the points p and $p^{'}$. This is a contradiction. Hence T_λ has a unique fixed point.

(ii) Obvious.

It is natural to ask, when the assumption that T_λ is asymptotically regular in Theorem 7, is satisfied. For an affirmative answer to this question, we need the following useful result.

Theorem 9. *[9, Theorem 5.2.7] Let J be a nonempty convex subset of a normed space Q and T be non-expansive map on J. If for $x_0 \in J$, $\{T_\lambda^n x_0\}$ is bounded, then the average map T_λ is asymptotically regular at x_0.*

For the existence of fixed points, apart from the other conditions, Huang and Qian ([14, Theorem 2.5]) have imposed continuity condition on the mappings already satisfying contractive condition similar to (1). In the result to follow, we employ weak requirement of nonexpansiveness to get nonexpansive version of Górnicki result for the average mapping.

Theorem 10. *Let J be a nonempty convex subset of a normed space Q and $T : J \to J$ be non-expansive map satisfying (1). If for $x_0 \in J$, $\{T_\lambda^n x_0\}$ is bounded, then T_λ has a unique fixed point $p \in Q$. Moreover, $\{x_n\}_{n=0}^{\infty}$, the Krasnoselskij iterative sequence converges to p.*

Proof.
The map T_λ is asymptotically regular at x_0 by Theorem 9. Now the rest of the proof is similar to that of Theorem 1.

Remark 1.

(1) Theorem 2.5 of Khan and Oyetuabi [7], holds in a convex metric space with the same proof.

(2) Theorem 10 provides a non-expansive version of Theorem 1 with a very simple proof.

4. Application: Volterra-Type Integral Equations

In this section, we investigate the existence and uniqueness of the common solution of Volterra-type integral equations, utilizing the common fixed point result established in Theorem 4.

Volterra-type integral equations play a significant role in various fields, including physics, biology, and engineering, in view of their capability to model systems with memory effects. These equations are crucial for capturing the dynamics of processes where the future state depends on the entire history of the system, such as in viscoelastic materials, population dynamics, and heat conduction. The importance of Volterra-type integral equations lies in their ability to provide insights into the existence and uniqueness of solutions to complex mathematical problems. By applying fixed point theorems, researchers can establish stability and convergence of these solutions; thereby addressing fundamental challenges in both theoretical and applied mathematics. The reader interested in this matter, is referred to the recent literature developed in [23–25].

Let Q be the space of continuous functions on $[0, T]$ equipped with the supremum norm. That is,

$$Q := \{u : [0, T] \to \mathbb{R} : u \text{ is continuous}\}.$$

Define a metric $d : Q \times Q \to \mathbb{R}_+$ by

$$d(u, v) = ||u - v||_\infty = \sup_{t\in[0,T]} |u(t) - v(t)|, \quad u, v \in Q.$$

Then (Q, d) is a complete metric space.
We consider the following Volterra-type integral equations formulated as a common fixed point problem of the following nonlinear mappings:

$$u(t) = \int_0^t K_1(t, s, u(s))ds \tag{34}$$

$$v(t) = \int_0^t K_2(t, s, v(s))ds \tag{35}$$

for all $t, s \in [0, T]$, where the kernels $K_1(t, s, u(s))$ and $K_2(t, s, v(s))$ are known function. We will find the solution of (34) and (35).

Now, we prove the following theorem to ensure the existence of common solution of the integral equations (34) and (35).

Theorem 11. *Assume that the following conditions are satisfied:*

(a) $K_1, K_2 : [0, T] \times [0, T] \times Q \to \mathbb{R}_+$,

(b) define the mappings E and F as follows:

$$(Eu)(t) = \int_0^t K_1(t, s, u(s))ds \tag{36}$$

$$(Fv)(t) = \int_0^t K_2(t, s, v(s))ds. \tag{37}$$

Assume further that

(i) there exists a continuous functions $\tau : [0,T] \times [0,T] \to \mathbb{R}_+$ *and a continuous and nondecreasing function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\alpha(0) = 0$ *and* $\alpha(t) < t$ *for* $t > 0$ *satisfying*

$$\Big|K_1(t,s,u(s)) - K_2(t,s,v(s))\Big| \leq \tau(t,s)\alpha(|u-v|)$$

for $t, s \in [0,T]$ *and* $u(s), v(s) \in Q$.

(ii) $\sup\limits_{t\in[0,T]} \int_0^t \tau(t,s)ds \leq \omega$, *for some* $\omega \in (0,1)$.

Then E and F have a unique common solution.

Proof. Let $u, v \in Q$. By assumptions (i) and (ii), we have

$$\begin{aligned}
d(Eu,Fv) &= ||Eu - Fv||_\infty = \sup_{t\in[0,T]} |(Eu)(t) - (Fv)(t)| \\
&= \sup_{t\in[0,T]} \Big| \int_0^t K_1(t,s,u(s))ds - \int_0^t K_2(t,s,v(s))ds \Big| \\
&= \sup_{t\in[0,T]} \Big| \int_0^t \Big(K_1(t,s,u(s)) - K_2(t,s,v(s))\Big)ds \Big| \\
&\leq \sup_{t\in[0,T]} \int_0^t \Big|K_1(t,s,u(s)) - K_2(t,s,v(s))\Big|ds \\
&\leq \sup_{t\in[0,T]} \int_0^t \tau(t,s)\alpha(|u-v|)ds \\
&\leq \sup_{t\in[0,T]} \int_0^t \tau(t,s)|u-v|ds \\
&\leq ||u-v||_\infty \sup_{t\in[0,T]} \int_0^t \tau(t,s)ds \\
&\leq \omega||u-v||_\infty \\
&= \omega\Big(||u - v + (E(u) - u) - (E(u) - u) + (v - F(v)) - (v - F(v))||_\infty\Big) \\
&\leq \omega\Big(2||u-v||_\infty| + ||u - E(u)||_\infty + ||v - F(v)||_\infty + ||E(u) - F(v)||_\infty\Big);
\end{aligned}$$

thus

$$d(Eu,Fv) \leq \frac{2\omega}{1-\omega} d(u,v) + \frac{\omega}{1-\omega}\Big(d(u,Eu) + d(v,Fv)\Big)$$

Letting $\kappa(d(u,v)) := \frac{2\omega}{1-\omega}$, it follows that the mappings E and F satisfy (3). Hence by Theorem 4, there exists a unique common fixed point of E and F which is the common solution of the Voterra-type integral equations (34) and (35).

5. Conclusions

We have extended the existing body of knowledge on fixed point theory for asymptotically regular mappings on a metric space by proving new common fixed point results on convex metric spaces. Our work verifies that under certain contractive conditions, asymptotically regular self-mappings not only possess unique fixed points but also exhibit properties of closedness and convexity for their fixed point sets. We have also demonstrated that replacing standard mappings with their average mappings in a convex metric space context retains their fixed point properties. Furthermore, we have applied our theoretical findings to solve integral equations, showcasing the practical utility of our theorems in real-world problems. These results are pivotal in advancing the understanding of fixed point theory in more general and complex spaces, offering new avenues for future research and applications in various mathematical and practical fields.
Looking ahead, several interesting directions arise for future exploration:

(i) Establishing an analogue of Theorem 5 in a convex metric space.

(ii) Finding an analogue of Theorem 6 for a quasi-nonexpansive map.

(iii) Addressing the question posed before Theorem 9 in the context of a convex metric space.

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