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Pretopological Spaces Induced by Rough Sets and Their Applications

A. A. Azzam[1,2,*], R. Mareay[3], Gehad M. Abd-Elhamed[1,4], M. Aldawood[1], Manal E. Ali[5]

[1] *Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt*
[3] *Department of Mathematics, Faculty of Science, Kafrelsheikh University, Kafrelsheikh 33516, Egypt*
[4] *Department of Mathematics,College of Girls, Ain Shams University, Egypt*
[5] *Department of Physics and Engineering Mathematics, Faculty of Engineering, Kafrelsheikh University, Kafrelsheikh, 33516, Egypt*

Abstract. In this paper, we generate pretopological spaces from a binary relation. We introduce a new approximation space by using pretopological concepts. Some properties and the comparison among different types of lower approximation and the upper approximation are studied. We introduce an application of pretopological spaces in rough approximation. some generalizations of rough sets concepts based on pretopological space are introduced.

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1. Introduction

There are a huge amount of information based on technology and there is a need high accurate tools for discovering their valuable knowledge. The field of information technology is an important filed and has attracted the researchers in many fields. The topological generalizations of rough set theory based on concepts of near open sets are presented in many researches. Abu-donia [1] introduced new kinds of rough set approximations via multi knowledge base,this means family of finite number of (reflexive, tolerance, dominance, equivalence) relations by two techniques.Abu-donia, A.S. Salama [2] extended

*Corresponding author.
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Email addresses: azzam0911@yahoo.com (A. A. Azzam), roshdeymareay@sci.kfs.edu.eg (R. Mareay), gehadmahfood@gmail.com (G. M. Abd-Elhamed), m.aldawood@psau.edu.sa (M. Aldawood), manal.ali@eng.kfs.edu.eg (M. E. Ali)

Pawlak's rough set model to a topological structure, where the set approximations are defined by the topological concept $\delta\beta$-open sets. In [3], Al-shami and Mhemdi introduced the concept of overlapping containment rough neighborhoods and their corresponding generalized approximation spaces, which were then applied to computational problems. Finally, Kaur et. al. [4] presented a novel multi-ideal nano-topological model aimed at improving the diagnosis and treatment of dengue. Collectively, these manuscripts contribute significantly to the theoretical foundations and practical implementations of rough set theory, particularly in the medical domain.

A. Galton [5] applied topological concepts to the problem of describing motion in discrete space space. Tareq M. Al-shami and et al. [6] studied the concept of primal soft topology based on the soft primal, which is a complementary concept of a soft grill. R. Mareay and et al.[7] putted fourth some concepts of topological near open sets and a new approximation structure based on the topological near open sets is introduced. New models of intuitionistic fuzzy set approximation space depending on covering are defined via neighborhood concept [8]. An attributes reduction method is introduced [9] based on constructing a weighted pre-topology that represents the information system under consideration. K. Y. Qin, Z. Pei [10] introduced the discussion of the relationship between fuzzy topologies and fuzzy rough set models and the axiom of fuzzy topology. K. Y. Qin [11] discussed the relationship between generalized rough sets based on reflexive and transitive relations and the topologies on the universe which is not limited to be finite. A. S. Salama [12] introduced new pre-topological approximations, pre-topological measures and a new method of data decomposition to avoid the necessity of reasoning from data with missing attribute values.

The concept of near open sets is an accurate and applicable tool for dealing with data. In 1989, Wiweger [13] introduced the concept of topological rough set. This concept was the basic start point for many researchers in generalization of rough set. Wiweger's generalization defined approximation space by using the interior and closure operators which are define on the the topological spaces. M. E. Abd El-Monsef et. al. [14] introduced β-open set concept. This concept has been used by many researchers in generalization of rough set. Since The closure function has an idempotent property in topology, so the the classical topology is not more adequate. The formalism of pretopology comes from usual topology with weaker axioms. The applications of Pretopology in the problems of social sciences find their foundations [15, 16]. In 1975, the first definition of pretopology space [17] was given by Marcel Brissaud. This definition of Marcel Brissaud, who is known as "pretopology's father", is based on Čech closure operator [18], works of Frochet spaces [19], and closure axioms of Kuratowski [20]. Based on these works, many of important theories in pretopology have been developed during the 1970's and 1980's such as Marcel Brissaud [21–24], Jean-Paul Auray [25, 26], Nicolas Nicoloyannis [27], Gerard Duru [28, 29], Michel Lamure [30], and Hubert Emptoz [31], and et. al. [32–34]. Some concepts of pretopological spaces are investigated during this paper. Different kinds of pretopological L_{apprs} and U_{apprs} are introduced.

In this paper, we give and investigate some concepts of pretopological spaces. Different types of pretopological lower and upper approximations are introduced. This paper give the relationship within distinct types of pretopological L_{apprs} and U_{apprs}. Some generalizations of rough theory concepts depending on pretopological space are introduced.

2. Preliminary of pretopological concepts and rough theory

In this part, we give some primary concepts of pretopological spaces and rough theory.

Definition 2.1. *[20] Assume that $X \neq \emptyset$. Then the operator $cl : P(X) \to P(X)$ is called Kuratowski closure if the following properties are hold:*

i. $cl(\emptyset) = \emptyset$,

ii. If $A_1 \subseteq X$, *then* $A_1 \subseteq cl(A_1)$,

iii. $\forall A_1, A_2 \subseteq X$, $cl(A_1 \cup A_2) = cl(A_1) \cup cl(A_2)$,

iv. $\forall A_1 \subseteq X$, $cl(cl(A_1)) = cl(A_1)$, .

Definition 2.2. *[17] For any a nonempty set X, the operator $l : P(X) \to P(X)$ is called pseudo closure if the following properties are hold:*

i. $l(\emptyset) = \emptyset$,

ii. $\forall A_1 \subseteq X$, $A_1 \subseteq l(A_1)$,

Definition 2.3. *[17] If a mapping $l : P(X) \to P(X)$ is pseudo closure and X is any set. Then, (X, l) is called a pretopological space.*

Definition 2.4. *[17] Let (X, l) be pretopological space and interior function . Then, $i(A) = (A(l(A^c)))^c$*

Definition 2.5. *[17] Assume that (X, l) is pretopological space and $i : P(X) \to P(X)$ is a mapping. Then, i is called interior mapping if the following are satisfied:*

i. $i(X) = X$,

ii. If $A_1 \subseteq X$. *Then,* $i(A_1) \subset A_1$.

Definition 2.6. *[17] If $A_1 \subseteq X$ and (X, l) is pretopological space. Then, A_1 is said to be closed if and only if $l(A_1) = A_1$.*

Definition 2.7. *[17] Assume that (X, l) be pretopological space, $A_1 \subseteq X$. Then, A_1 is said to be open if and only if $i(A_1) = A_1$.*

2.1. Basic concepts of the Pawlakś rough theory

Assume that R is an equivalence relation on a nonempty set X. Then, $X/R = \{Y_1, Y_2, Y_3, ..., Y_m\}$ is a partition on X, where R is an equivalence which generate the equivalence classes $Y_1, Y_2, Y_3, ..., Y_m$. .

Definition 2.8. *[35] Assume that R is an equivalence relation on a nonempty set X. For any $A_1 \subseteq X$, the set $\underline{R}(A_1) = \cup\{Y_i \in X/R : Y_i \subseteq A_1\}$ is called L_{appros} of A_1 and the set $\overline{R}(A_1) = \cup\{Y_i \in X/R : Y_i \cap A_1 \neq \emptyset\}$ is called U_{appros} of A_1.*

Proposition 2.1. *[35] Assume that $K = (X, R)$ is an approximation structure. Then, the following properties are hold,for $X_1, X_2 \subseteq X$:*

(iL) $\underline{R}(X) = X$;

(iH) $\overline{R}(X) = X$;

(iiL) $\underline{R}(\emptyset) = \emptyset$;

(iiH) $\overline{R}(\emptyset) = \emptyset$;

(iiiL) $\underline{R}(X_1) \subseteq X_1$;.

(iiiH) $X_1 \subseteq \overline{R}(X_1)$.

(ivL) $\underline{R}(X_1 \cap X_2) = \underline{R}(X_1) \cap \underline{R}(X_2)$;

(ivH) $\overline{R}(X_1 \cup X_2) = \overline{R}(X_1) \cup \overline{R}(X_2)$;

(v) $\underline{R}(X_1^c) = [\overline{R}(X_1)]^c$, *where* (X_1^c) *is the complement of* X_1;

(viL) $\underline{R}(\underline{R}(X_1)) = \underline{R}(X_1)$;

(viH) $\overline{R}(\overline{R}(X_1)) = \overline{R}(X_1)$;

(viiL) $X_1 \subseteq X_2 \Rightarrow \underline{R}(X_1) \subseteq \underline{R}(X_2)$;

(viiH) $X_1 \subseteq X_2 \Rightarrow \overline{R}(X_1) \subseteq \overline{R}(X_2)$;

(viiiL) $\underline{R}(\underline{R}(X_1))^c = (\underline{R}(X_1))^c$;

(viiiH) $\overline{R}(\overline{R}(X_1))^c = (\overline{R}(X_1))^c$;

(ixL) $\underline{R}(X_1) \cup \underline{R}(X_2) \subseteq \underline{R}(X_1 \cup X_2)$;

(ixH) $\overline{R}(X_1 \cap X_2) \subseteq \overline{R}(X) \cap \overline{R}(X_2)$;

3. Rough pretopological approximation space

This section introduces the application of pretopological space in rough approximation space. Let R be a binary relation which defined on a finite set X. Suppose that $R(x)$ is a neighborhood of x which is defined by $R(x) = \{y \in: (x, y) \in R\}$ and $R^{-1}(x) = \{y \in X : (y, x) \in R\}$. Hence, we will define the pseudo-closure $\Gamma_d(.)$ and The interior function $i_d(.)$ as follow:

Definition 3.1. *For any a nonempty set X, and R is a binary relation defined on R. Suppose $\Gamma_d(.) : P(X) \to P(X)$ defined by $\Gamma_d(A_1) = \{x \in X : R(x) \cap A_1 \neq \emptyset\} \cup A_1$, $\forall A_1 \subseteq X$. Then $\Gamma_d(.)$ is called pseudo closure if the following properties are hold:*

i. $\Gamma_d(\emptyset) = \emptyset$,

ii. $\forall A_1 \subseteq X$, $A_1 \subseteq \Gamma_d(A_1)$,

Hence (X, Γ_d) is called pretopological space

Definition 3.2. *Consider (X, Γ_d) is pretopological space. Suppose that $i_d(.) : P(X) \to P(X)$ defined by $i_d(A_1) = \{x \in X : R(x) \cap A_1 \neq \emptyset\}$, $\forall A_1 \subseteq X$. Then $i_d(.)$ is called interior function if the following properties are hold:*

i. $i_d(\emptyset) = \emptyset$,

ii. $\forall A_1 \subseteq X$, $i_d(A_1) \subseteq A_1$,

The pretopological space, represented by $\Gamma_d; i_d$, is created from the pseudo-closure function that depends on $R(x)$ and is called the pseudo-closure of descendants. Likewise, we refer to the pretopological space, represented by $\Gamma_a; i_a$, that is produced by the pseudo-closure function based on $R^{-1}(x)$ by pseudo-closure of ascendants.

Definition 3.3. *Suppose that (X, Γ_d) is pretopological space. Then, we define the pretopological L_{apprs} and pretopological U_{apprs} of a subset $A \subseteq X$ as the following:*
$i_d(A) = \{x \in X : R(x) \subseteq A\}$, $\forall A \subseteq X$
$\Gamma_d(A) = \{x \in X : R(x) \cap A \neq \emptyset\} \cup A$, $\forall A \subseteq X$
$i_a(A) = \{x \in X : R^{-1}(x) \subseteq A\}$, $\forall A \subseteq X$
$\Gamma_a(A) = \{x \in X : R^{-1}(x) \cap A \neq \emptyset\} \cup A$, $\forall A \subseteq X$.

The approximation structure (X, R) is called pretopological approximation space.

Definition 3.4. *Assume that (X, R) is a pretopological approximation structure. Then $\forall A \subseteq X$:*

i. *If $A \subseteq \Gamma_a(i_d(A))$, then A is semi rough (S_{ad}-rough) ,*

ii. *If $A \subseteq \Gamma_d(i_a(A))$, then A is prerough (P_{ad}-rough) ,*

iii. *If $A \subseteq \Gamma_a(i_d(\Gamma_a(A)))$, then A is semi-prerough (β_{ad}-rough) ,* *iv.* *If* $A \subseteq i_d(\Gamma_a(i_d(A)))$*, then* A *is* α*-rough (*α_{ad}*-rough)* ,

v. *If* $A \subseteq \Gamma_a(i_d(A)) \cup i_d(\Gamma_a(A))$*, then* A *is* γ*-rough (*γ_{ad}*-rough)* .

In the pretopological approximation space (X, R), the set family of all S_{ad}-rough (resp. P_{ad}-rough, β_{ad}-rough, α_{ad}-rough and γ_{ad}-rough) is denoted by $FS_{ad}(X)$(resp. $FP_{ad}(X)$, $F\beta_{ad}(X)$, $F\alpha_{ad}(X)$, and $F\gamma_{ad}(X)$).

The complement of the sets $S_{ad}(X)$(resp. $P_{ad}(X)$, $\beta_{ad}(X)$, $\alpha_{ad}(X)$, and $\gamma_{ad}(X)$) in (X, R) is called S^c_{ad}-rough (resp. P^c_{ad}-rough, β^c_{ad}-rough, α^c_{ad}-rough and γ^c_{ad}-rough) and is denoted by FS^c_{ad}-rough (resp. FP^c_{ad}-rough, $F\beta^c_{ad}$-rough, $F\alpha^c_{ad}$-rough and $F\gamma^c_{ad}$-rough).

Proposition 3.1. *If* (X, R) *is a pretopological approximation structure. Then, the following properties are satisfied:*

i. $F\alpha_{ad}(X) \subseteq FS_{ad}(X) \subseteq F\gamma_{ad}(X) \subseteq F\beta_{ad}(X)$,

ii. $F\alpha_{ad}(X) \subseteq FP_{ad}(X) \subseteq F\gamma_{ad}(X) \subseteq F\beta_{ad}(X)$,

proof *It's clear from the above definition.* □

Example 3.1. *Consider* $X = \{a, b, c, d\}$ *is the universe set. Suppose that* R *is a binary relation defined on* X *by* $R = \{(a, a), (a, d), (a, c), (b, b), (b, d), (c, d), (c, a), (c, b), (d, a)\}$. *Hence* $R(a) = \{a, c, d\}, R(b) = \{b, d\}, R(c) = \{a, b, d\}, R(d) = \{a\}$ *and* $R^{-1}(a) = \{a, c, d\}, R^{-1}(b) = \{b, c\}, R^{-1}(c) = \{a\}, R^{-1}(d) = \{a, b, c\}$. *Therefore,* $FS_{ad}(X) = F\alpha_{ad}(X) = \{X, \emptyset, \{a, d\}, \{d\}, \{a\}, \{b, d\}, \{a, c, d\}, \{a, b, d\}\}$, $FP_{ad}(X) = F\beta_{ad}(X) = F\gamma_{ad}(X) = \{X, \emptyset, \{a, b, d\}, \{a, c, d\}, \{a, b, c\}, \{a, c\}, \{a, d\}, \{a, b\}, \{b, d\}, \{a\}, \{d\}\}$.

Definition 3.5. *Assume that* (X, R) *is a pretopological approximation structure,* $A \subseteq X$. *Then, we denote the general lower of* A *by* $\underline{\mu}_{ad}(A)$ *for all* $\mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *and is defined by* $\underline{\mu}_{ad}(A) = \cup\{G \in F\mu_{ad} : G \subseteq A\}$.

Definition 3.6. *Assume that* (X, R) *is a pretopological approximation structure,* $A \subseteq X$. *Then, we denote the general upper of* A *by* $\overline{\mu}_{ad}(A)$ *for all* $\mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *and is defined by* $\overline{\mu}_{ad}(A) = \cap\{H \in F\mu^c_{ad} : A \subseteq H\}$.

Definition 3.7. *Suppose that* (X, R) *is a pretopological approximation structure and* $A \subseteq X$. *Hence,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *the pretopological general lower and the pretopological general upper approximations are defined as* $i_{\mu ad}(A) = \underline{\mu}_{ad}(A)$, $\Gamma_{\mu ad}(A) = \overline{\mu}_{ad}(A)$.

Proposition 3.2. *Assume that* (X, R) *is a pretopological approximation space,* R *is a binary relation on* X. *Then, for any* $A \subseteq X$ *the following properties are satisfied :*

i. $i_d(A) \subseteq i_{\alpha ad} \subseteq i_{Sad} \subseteq i_{\gamma ad} \subseteq i_{\beta ad} \subseteq A \subseteq \Gamma_{\beta ad} \subseteq \Gamma_{\gamma ad}(A) \subseteq \Gamma_{Sad}(A) \subseteq \Gamma_{\alpha ad} \subseteq \Gamma_d(A)$,

ii. $i_a(A) \subseteq i_{\alpha ad} \subseteq i_{Pad} \subseteq i_{\gamma ad} \subseteq i_{\beta ad} \subseteq A \subseteq \Gamma_{\beta ad} \subseteq \Gamma_{\gamma ad}(A) \subseteq \Gamma_{Pad}(A) \subseteq \Gamma_{\alpha ad} \subseteq \Gamma_d(A)$,

proof: *We will prove the part (i) and we can prove (ii) in the same way.*

i. *Since* $i_d(A) = \{x \in X : R(x) \subseteq A\} \subseteq \cup\{G_1 \in F\alpha_{ad}(X) : G_1 \subseteq A\} \subseteq \cup\{G_1 \in FS_{ad}(X) : G_1 \subseteq A\} \subseteq \cup\{G_1 \in F\gamma_{ad}(X) : G_1 \subseteq A\} \subseteq \cup\{G_1 \in F\beta_{ad}(X) : G_1 \subseteq A\} \subseteq A \subseteq \cap\{G_2 \in F\beta^c_{ad}(X) : A \subseteq H\} \subseteq \cap\{G_2 \in F\gamma^c_{ad}(X) : A \subseteq G_2\} \subseteq \cap\{G_2 \in FS^c_{ad}(X) : A \subseteq G_2\} \subseteq \cap\{G_2 \in F\alpha^c_{ad}(X) : A \subseteq G_2\} \subseteq \{x \in X : R(x) \cap A \neq \emptyset\}$. *Therefore,* $i_d(A) \subseteq i_{\alpha ad} \subseteq i_{Sad} \subseteq i_{\gamma ad} \subseteq i_{\beta ad} \subseteq A \subseteq \Gamma_{\beta ad} \subseteq \Gamma_{\gamma ad}(A) \subseteq \Gamma_{Sad}(A) \subseteq \Gamma_{\alpha ad} \subseteq \Gamma_d(A)$. □

Example 3.2. *Suppose that* $A_1 = \{a, c\}, A_2 = \{c, d\}$. *Then by using Example 3.1,* $i_{\alpha ad}(A_1) = \{a\}$, $i_{pad}(A) = \{a, c\}$, $\Gamma_{pad}(A_2) = \{c, d\}$, $\Gamma_{\alpha ad}(A_2) = \{b, c, d\}$. *Therefore,* $i_{\alpha ad}(A_1) \subseteq i_{pad}(A_1)$, $\Gamma_{pad}(A_2) \subseteq \Gamma_{\alpha ad}(A_2)$.

Proposition 3.3. *Assume that* (X, R) *is a pretopological approximation structure and* $A_1, A_2 \subseteq X$. *Hence,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *the following axioms are satisfied:*

i. $i_{\mu ad}(X) = \Gamma_{\mu ad}(X) = X$,

ii. $i_{\mu ad}(\emptyset) = \Gamma_{\mu ad}(\emptyset) = \emptyset$,

iii. *If* $A_1 \subseteq A_2$, *then* $i_{\mu ad}(A_1) \subseteq i_{\mu ad}(A_2)$, $\Gamma_{\mu ad}(A_1) \subseteq \Gamma_{\mu ad}(A_2)$,

iv. $i_{\mu ad}(A_1) \cup i_{\mu ad}(A_2) \subseteq i_{\mu ad}(A_1 \cup A_2)$,

v. $\Gamma_{\mu ad}(A_1 \cap A_2) \subseteq \Gamma_{\mu ad}(A_1) \cap \Gamma_{\mu ad}(A_2)$,

vi. $\Gamma_{\mu ad}(A_1) \cup \Gamma_{\mu ad}(A_2) \subseteq \Gamma_{\mu ad}(A_1 \cup A_2)$,

vii. $i_{\mu ad}(A_1 \cap A_2) \subseteq i_{\mu ad}(A_1) \cap i_{\mu ad}(A_2)$,

viii. $i_{\mu ad}(A_1^c) = (\Gamma_{\mu ad}(A_1))^c$,

ix. $\Gamma_{\mu ad}(A_1^c) = (i_{\mu ad}(A_1))^c$.

proof *Since* $\mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$, *then by the properties of* $i_{\mu ad}$ *and* $\Gamma_{\mu ad}$, *the proof is complete.*

The converse of axioms iv and v in Proposition 3.3 do not hold in general. This will be shown in the next example,so take $\mu ad = S_{ad}$.

Example 3.3. *If* $A_1 = \{d\}$, $A_2 = \{a, c\}$ *and by using Example 3.1. Then,* $i_{Sad}(A_1) = \{d\}$, $i_{Sad}(A_2) = \{a\}$, $i_{Sad}(A_1 \cup A_2) = \{a, c, d\}$. *Therefore* $i_{Sad}(A_1) \cup i_{Sad}(A_2) \neq i_{Sad}(A_1 \cup A_2)$. *Also,* $\Gamma_{Sad}(A_1) = \{b, c, d\}$, $\Gamma_{Sad}(A_2) = \{a, c\}$,$\Gamma_{Sad}(A_1 \cap A_2) = \emptyset$, *Hence* $\Gamma_{\mu ad}(A_1 \cap A_2) \neq \Gamma_{\mu ad}(A_1) \cap \Gamma_{\mu ad}(A_2)$

Proposition 3.4. *If* $A \subseteq X$ *and* (X, R) *is a pretopological approximation space. Then,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *the following axioms do not satisfied:*

i. $i_{\mu ad}(i_{\mu ad}(A)) = i_{\mu ad}(A) \neq \Gamma_{\mu ad}(i_{\mu ad}(A))$, *ii.* $\Gamma_{\mu ad}(\Gamma_{\mu ad}(A)) = \Gamma_{\mu ad}(A) \neq i_{\mu ad}(\Gamma_{\mu ad}(A))$,

The following is an counter example for the above proposition by taking $\mu_{ad} = S_{ad}, P_{ad}$.

Example 3.4. *Let* $A_1 = \{b, c, d\}$ *and by using Example 3.1.,* $i_{Sad}(A_1) = \{b, d\}$, $i_{Sad}(i_Sad(A_1)) = \{b, d\}$, *then* $\Gamma_{Sad}(i_{Sad}(A_1)) = i_{Sad}\{b, d\} = \{b, c, d\} \neq i_{Sad}(i_{Sad}(A_1))$. *Let* $A_2 = \{c\}$, $\Gamma_{Pad}(A_2) = \{c\}, \Gamma_{Pad}(\Gamma_{Pad}(A_2)) = \emptyset$, *then* $i_{Pad}(\Gamma_{Pad}(A_2)) = \Gamma_{\mu ad}(\Gamma_{\mu ad}(A_2)) = \emptyset \neq \Gamma_{\mu ad}(A_2)$.

4. Pretopological generalizations of rough theory sets concepts

Through this section, some generalizations of rough theory concepts based on pretopological space are introduced by using $i_{\mu ad}$ and $\Gamma_{\mu ad}$ approximation operators.

Definition 4.1. *Assume that* (X, R) *is a pretopological approximation structure and* $A \subseteq X$. *Hence,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *we define the following:*

i. *A is totally pretopoligical* μ_{ad}*-definable (*μ_{ad}*-exact) set if* $i_{\mu ad}(A) = \Gamma_{\mu ad}(A) = A$,

ii. *A is internally pretopoligical* μ_{ad}*-definable set if* $i_{\mu ad}(A) = A$ *and* $\Gamma_{\mu ad}(A) \neq A$,

iii. *A is externally pretopoligical* μ_{ad}*-definable set if* $i_{\mu ad}(A) \neq A$ *and* $\Gamma_{\mu ad}(A) = A$,

iv. *A is topologically* μ_{ad}*-indefinable (*μ_{ad}*-rough) set if* $i_{\mu ad}(A) \neq A$ *and* $\Gamma_{\mu ad}(A) \neq A$.

Example 4.1. *By using Example 3.1, the set* $A = \{a\}$ *is topologically* α_{ad}*-indefinable (*α_{ad}*-rough) set, the set* $B = \{a, c\}$ *is totally pretopoligical* P_{ad}*-definable (*P_{ad}*-exact) set.*

Definition 4.2. *Suppose that* (X, R) *is a pretopological approximation structure and* $A \subseteq X$. *Then, we define the accuracy measure of any set A as:*
$acc_{\mu_{ad}}(A) = \frac{|i_{\mu ad}(A)|}{|\Gamma_{\mu ad}(A)|}$, $\Gamma_{\mu ad}(A) \neq \emptyset$, *where* $\mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *and where* $\mid A \mid$ *is cardinality of A.*

By the accuracy measure, we can determine the exactness of any subset $A \subseteq X$. The relationship among the four types of accuracy measure is given as the following:

i. $0 \leq acc_d(A) \leq acc_{\alpha ad} \leq acc_{Sad} \leq acc_{\gamma ad} \leq acc_{\beta ad} \leq 1$,

ii. $0 \leq acc_d(A) \leq acc_{\alpha ad} \leq acc_{Pad} \leq acc_{\gamma ad} \leq acc_{\beta ad} \leq 1$.

Hence, the best the accuracy for approximation is $\mu_{ad} = \beta_{ad}$ as in the next example:

Example 4.2. *Continued from Example 3.1., the compression between some types of accuracy measure are listed in Table 1*

Definition 4.3. *Suppose that* (X, R) *is a pretopological approximation structure and* $A_1, A_2 \subseteq X$. *Then,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$ *the following are defined:*

i. $A_1 \tilde{\subseteq}_{\mu ad} A_2$ *if* $i_{\mu ad}(A_1) \subseteq i_{\mu ad}(A_2)$,

Table 1: the compression between some types of accuracy measure

$SetA$	acc_{Sad}	$acc_{\beta ad}$
$\{a,b\}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{a,c\}$	$\frac{1}{2}$	1
$\{b,d\}$	$\frac{2}{3}$	1
$\{a,b,c\}$	$\frac{1}{3}$	1

ii. $A_1 \tilde{\subseteq}^{\mu ad} A_2$ *if* $\Gamma_{\mu ad}(A_1) \subseteq \Gamma_{\mu ad}(A_2)$.

Example 4.3. *By using Example 3.1, consider* $A_1 = \{d\}, A_2 = \{a,c\}, A_3 = \{b,d\}$ *and* $A_4 = \{c,d\}$. *Hence, we have:* $A_1 \tilde{\subseteq}^{Sad} A_2$ *and* $A_3 \tilde{\subseteq}_{\beta ad} A_4$.

Proposition 4.1. *Assume that* (X,R) *be a pretopological approximation structure and* $A \subseteq X$. *Then,* $\forall \mu_{ad} \in \{S_{ad}, P_{ad}, \beta_{ad}, \alpha_{ad}, \gamma_{ad}\}$, $x \in X$ *the following are defined:*

i. *If* $x \tilde{\in}_{\mu ad} A$, *hence* $x \in A$,

ii. *If* $x \tilde{\notin}_{\mu ad} A$, *hence* $x \notin A$.

proof *It's clear from the above definition.* □

The next example shows that the converse of Proposition 4.1 doesn't hold in general: The converse of the Proposition 4.4 is not true as in the following example:

Example 4.4. *Continued from Example 3.1, let* $A_1 = \{a, b.c\}$ *and* $A_2 = \{a,d\}$, *then we get* $b \in A_1$, *but* $b \tilde{\notin}_{\mu ad} A_1$. *Also,* $b \notin A_2$, *but* $b \tilde{\in}_{Sad} A_2$ *and* $b \tilde{\in}_{\gamma ad} A_2$.

5. Conclusion

This paper used the pretopological concepts to generate rough approximation space. Different types of lower and upper approximation are generated based on pretopological space. We have got the best the accuracy for approximation using our approach. Our approach will be useful in knowledge discovery. In the future work, we will study more applications of these tools based on generalizations of pretopological concepts. Moreover, we will study the connection between pretopological spaces and soft set theory.

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