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# Almost $(m, n)$-quasi-ideals and Fuzzy Almost $(m, n)$-quasi-ideals in Ordered Semigroups

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**Abstract.** The ordered semigroups are algebraic systems consisting of a non-empty set, an associative binary operation, and a partial order compatible with this binary operation. This paper aims to define almost $(m, n)$-quasi-ideals and fuzzy almost $(m, n)$-qausi-ideals in ordered semigroups. We prove the union of almost $(m, n)$-quasi-ideals, including almost $(m, n)$-qausi-ideals in ordered semigroups. In class, fuzzifications are the same. Finally, we connect relation almost $(m, n)$-quasi-ideals and fuzzy almost $(m, n)$-quasi-ideals in ordered semigroups.

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**Key Words and Phrases**: Ordered almost $(m, n)$-quasi-ideal, fuzzy ordered almost $(m, n)$-quasi-ideal, ordered semigroups

## 1. Introduction

Ordered semigroup is an algebraic structure in a binary operation satisfying associative property and a partial order with the compatibility. The concepts of quasi-ideals of semigroups presented by Steinfied [1] in 1956. The dealing with various problems related to uncertain conditions by fuzzy sets by Zadeh in 1965, [2]. These concepts were applied in many areas, such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. Rosenfeld studied concent of fuzzy subgroups and fuzzy ideals. In 1981 Kuroki studied the typers of fuzzy subsemigroups. In the same year Satko and Grosek [3] discussed concept of an almost-ideal (A-ideal) in a semilattice. And S. Bogdanovic [4] gave the concept of almost bi-ideals in semigroups. In 2019, S. Suebsung et al. [5] investigated almost ideals and fuzzy almost ideals in ternary semigroups. In 2020, Chinram et al. [6] discussed almost

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interior ideals and weakly almost interior ideals in semigroups and studied the relationship between almost interior ideals and weakly almost interior ideals in semigroups.

The research of almost ideals studied in semihypergroups such that in 2021, P. Muangdoo et al. [7] studied almost bi-hyperideals and their fuzzification of semihypergroups. W. Nakkhasen et al. [8] discussed fuzzy, almost interior hyperideals of semihypergroups. in 2022, S. Suebsung et al. [9] introduced almost ideals in ordered semigroups. In the same year T. Gaketem and P. Khamrot [10] explored the concept of almost ideals within the framework of bipolar fuzzy sets, specifically focusing on bipolar fuzzy almost bi-ideals in semigroups. In 2023, R. Chinram et. al [11] studied concept almost $(m,n)$-quasi-ideals in semigroups and their fuzzifications. In the same year, T. Gaketem and P. Khamrot [12] studied bipolar fuzzy almost interior ideals in semigroups. In 2024, T. Gaketem and P. Khamrot [13] discussed bipolar fuzzy almost ideals in semigroups. In addition, almost ideal's work also has many studies, such as almost ideals in ordered semigroup [14], almost ideals in semirings [15], almost ideals in ternary semiring [16], etc. Not long ago in 2025 P. Khamrot et al. [17] studied fuzzy $(m,n)$-ideals and $n$-interior ideals in ordered semigroups. In the same year, P. Khamrot et al. [18] extend concepts fuzzy $(m,n)$-ideals and $n$-interior ideals to biploar fuzzy sets.

In this paper we extend the definition of almost $(m,n)$-quasi-ideals in semigroups go to ordered semigroups. We discussed the union of almost $(m,n)$-quasi-ideals, including almost $(m,n)$-quasi-ideals in ordered semigroups. In class, fuzzifications are the same. Finally, we connect relation almost $(m,n)$-quasi-ideals and fuzzy almost $(m,n)$-quasi-ideals in ordered semigroups.

## 2. Preliminaries

Now, we recall the concept of ordered semigroups and fuzzy sets. Additionally, their preliminary results are provided.

**Definition 1.** *[19]. Let $\mathfrak{T}$ be a set with a binary opeation $\cdot$ and a binary opeation relation $\leq$. Then $(\mathfrak{T},\cdot,\leq)$ is called an* ordred semigroup *if*

(1) $(\mathfrak{T},\cdot)$ *is a semigroup,*

(2) $(\mathfrak{T},\leq)$ *is a partially ordered set,*

(3) *for all $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathfrak{T}$, we have $\mathfrak{a}\leq\mathfrak{b}$ then $\mathfrak{ac}\leq\mathfrak{bc}$ and $\mathfrak{ca}\leq\mathfrak{cb}$.*

For a nonempty subset $\mathfrak{X}$ and $\mathfrak{Y}$ of ordered semigroup $\mathfrak{T}$, we write
$(\mathfrak{X}] := \{\mathfrak{a}\in\mathfrak{T} \mid \mathfrak{a}\leq\mathfrak{b} \text{ for some } \mathfrak{b}\in\mathfrak{X}\}$ and $\mathfrak{X}\mathfrak{Y} := \{\mathfrak{x}\mathfrak{y} \mid \mathfrak{x}\in\mathfrak{X} \text{ and } \mathfrak{y}\in\mathfrak{Y}\}$.

It is observed that

(1) $\mathfrak{X}\subseteq(\mathfrak{X}]$,

(2) if $\mathfrak{X}\subseteq\mathfrak{Y}$, then $(\mathfrak{X}]\subseteq(\mathfrak{Y}]$,

(3) $((\mathfrak{X}]]=(\mathfrak{X}]$,

(4) $(\mathfrak{X}](\mathfrak{Y}] \subseteq (\mathfrak{X}\mathfrak{Y}]$,

(5) $((\mathfrak{X}](\mathfrak{Y}]] = (\mathfrak{X}\mathfrak{Y}]$,

(6) $(\mathfrak{X} \cup \mathfrak{Y}] = (\mathfrak{X}] \cup (\mathfrak{Y}]$,

(7) $(\mathfrak{X} \cap \mathfrak{Y}] = (\mathfrak{X}] \cap (\mathfrak{Y}]$.

Let $(\mathfrak{T}, \cdot, \leq)$ be an ordered semigroup, $(\emptyset \neq)\mathcal{K} \subseteq \mathfrak{T}$ is called a *subsemigroup* such that $\mathcal{K}^2 \subseteq \mathcal{K}$. A *left (right) ideal* of a ordered semigroup $(\mathfrak{T}, \cdot, \leq)$ is a non-empty set $\mathcal{K}$ of $\mathfrak{T}$ such that $\mathcal{S}\mathcal{K} \subseteq \mathcal{K}$ $(\mathcal{K}\mathcal{S} \subseteq \mathcal{K})$ and $(\mathcal{K}]$. By an *ideal* of an ordered semigroup $(\mathfrak{T}, \cdot, \leq)$, we mean a non-empty set of $\mathfrak{T}$ which is both a left and a right ideal of $\mathfrak{T}$.

**Definition 2.** *[20] A subsemigroup $\mathcal{K}$ of an ordered semigroup $(\mathfrak{T}, \cdot, \leq)$ is called an $(m, n)$-*ideal *of $\mathfrak{T}$ if $\mathcal{K}$ satisfies the following conditions:*

(1) $\mathcal{K}^m\mathfrak{T}\mathcal{K}^n \subseteq \mathcal{K}$.

(2) $\mathcal{K} = (\mathcal{K}]$, *that is for $x \in \mathcal{K}$ and $y \in \mathfrak{T}$, $y \leq x$ implies $y \in \mathcal{K}$.*

*where $m, n$ are non-negative integers.*

**Definition 3.** *[20] An non-empty subset $\mathcal{K}$ of an ordered semigroup $(\mathfrak{T}, \cdot, \leq)$ is called an* $(m, n)$-quasi-ideal *of $\mathfrak{T}$ if $\mathcal{K}$ satisfies the following conditions:*

(1) $(\mathcal{K}^m\mathfrak{T}] \cap (\mathfrak{T}\mathcal{K}^n] \subseteq \mathcal{K}$.

(2) $\mathcal{K} = (\mathcal{K}]$, *that is for $x \in \mathcal{K}$ and $y \in \mathfrak{T}$, $y \leq x$ implies $y \in \mathcal{K}$.*

*where $m, n$ are non-negative integers.*

**Definition 4.** *[9] A nonempty subset of $\mathcal{K}$ an ordered semigroup $\mathcal{T}$ is called a* left ordered almost ideal *of $\mathcal{T}$ if $(t\mathcal{K}] \cap \mathcal{K} \neq \emptyset$ for all $t \in \mathcal{S}$.*

**Definition 5.** *[9] A nonempty subset of $\mathcal{K}$ an ordered semigroup $\mathcal{T}$ is called a* right ordered almost ideal *of $\mathcal{T}$ if $(\mathcal{K}t] \cap \mathcal{K} \neq \emptyset$ for all $t \in \mathcal{T}$.*

For any $h_i \in [0, 1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h, r \in [0, 1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A *fuzzy set* $\vartheta$ in a nonempty set $\mathfrak{T}$ is a function from $\mathfrak{T}$ into the unit closed interval $[0, 1]$ of real numbers, i.e., $\vartheta : T \to [0, 1]$.

For any two fuzzy sets $\vartheta$ and $\xi$ of a non-empty set $\mathfrak{T}$, define the symbol as follows:

(1) $\vartheta \leq \xi \Leftrightarrow \vartheta(h) \leq \xi(h)$ for all $h \in T$,

(2) $\vartheta = \xi \Leftrightarrow \vartheta \leq \xi$ and $\xi \leq \vartheta$,

(3) $(\vartheta \wedge \xi)(h) = \min\{\vartheta(h), \xi(h)\} = \vartheta(h) \wedge \xi(h)$ for all $h \in T$,

(4) $(\vartheta \vee \xi)(h) = \max\{\vartheta(h), \xi(h)\} = \vartheta(h) \vee \xi(h)$ for all $h \in T$,

(5) the *support* of $\vartheta$ instead by $\text{supp}(\vartheta) = \{h \in T \mid \vartheta(h) \neq 0\}$.

For the symbol $\vartheta \geq \xi$, we mean $\xi \leq \vartheta$.

If $\mathfrak{K} \subseteq \mathfrak{T} \neq \emptyset$, then the characteristic function $\chi_{\mathfrak{K}}$ of $\mathfrak{T}$ is a function from $\mathfrak{T}$ into $\{0, 1\}$ defined as follows:

$$\chi_{\mathfrak{K}}(\mathfrak{r}) = \begin{cases} 1 & \text{if } \mathfrak{r} \in \mathfrak{K} \\ 0 & \text{otherwise.} \end{cases}$$

for all $\mathfrak{r} \in \mathfrak{T}$

**Lemma 1.** *If $\mathfrak{I}$ and $\mathfrak{L}$ are nonempty subsets of an oredred semigroup $\mathfrak{T}$, then the following are true:*

(1) $\chi_{\mathfrak{I}} \wedge \chi_{\mathfrak{L}} = \chi_{\mathfrak{I} \cap \mathfrak{L}}$.

(2) *If $\mathfrak{I} \subseteq \mathfrak{L}$, then $\chi_{\mathfrak{I}} \preceq \chi_{\mathfrak{L}}$.*

(3) $\chi_{\mathfrak{I}} \circ \chi_{\mathfrak{L}} = \chi_{\mathfrak{I}\mathfrak{L}}$.

**Definition 6.** *Let $\mathfrak{T}$ be an ordered semigroup and $F_u$ be a non-empty subset of $\mathfrak{T}$, we define the set $F_u$ by*

$$F_u := \{(x, y) \in \mathfrak{T} \times \mathfrak{T} \mid u \leq xy\}.$$

**Definition 7.** *[21] Let $\vartheta$ and $\eta$ be fuzzy sets of an ordered semigroup $\mathfrak{T}$. The product of fuzzy subsets $\vartheta$ and $\eta$ of $\mathfrak{T}$ is defined as follow, for all $u \in T$*

$$(\vartheta \circ \eta)(u) = \begin{cases} \bigvee\limits_{(x,y) \in F_u} \{\vartheta(x) \wedge \eta(y)\} & \text{if} \quad F_u \neq \emptyset, \\ 0 & \text{if} \quad F_u = \emptyset. \end{cases}$$

For $u \in T$ and $t \in (0, 1]$, a fuzzy point $x_t$ of a set $\mathfrak{T}$ is a fuzzy subset of $\mathfrak{T}$ defined by

$$x_t(e) = \begin{cases} t & \text{if} \quad e = u, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{k} \in \mathbb{N}$, let $\vartheta^n := \underbrace{\vartheta \circ \vartheta \circ \cdots \circ \vartheta}_{n\text{-times}}$.

**Lemma 2.** *[14] If $\varphi$, $\nu$ and $\xi$ are fuzzy sets of an ordered semigroup $S$, then the following are true:*

(1) *If $\varphi \preceq \nu$, then $\varphi^n \preceq \nu^n$*

(2) *If* $\varphi \preceq \nu$*, then* $\varphi \circ \xi \preceq \nu \circ \xi$.

(3) *If* $\varphi \preceq \nu$*, then* $\varphi \vee \xi \preceq \nu \vee \xi$.

(4) *If* $\varphi \preceq \nu$*, then* $\varphi \wedge \xi \preceq \nu \wedge \xi$.

(5) *If* $\varphi \preceq \nu$*, then* $\text{supp}(\varphi) \preceq \text{supp}(\nu)$.

For a fuzzy set $\varphi$ of an ordered semigroup $S$, we define $(\varphi] : S \to [0,1]$ by $(\varphi] := \sup\limits_{a \leq b} \varphi(b)$ for all $a \in S$.

**Lemma 3.** *[14] If* $\varphi$*,* $\nu$ *and* $\xi$ *are fuzzy sets of an ordered semigroup* $S$*, then the following are true:*

(1) $\varphi \preceq (\varphi]$.

(2) *If* $\varphi \preceq \nu$*, then* $(\varphi] \preceq (\xi]$.

(3) *If* $\varphi \preceq \nu$*, then* $(\varphi \circ \xi] \preceq (\nu \circ \xi]$ *and* $(\xi \circ \varphi] \preceq (\xi \circ \nu]$.

**Lemma 4.** *[14] If* $\varphi$ *is a fuzzy set of an ordered semigroup* $S$*, then the following are equivalent.*

(1) *If* $a \leq b$*, then* $\varphi(a) \preceq \varphi(b)$.

(2) $(\varphi] = \varphi$.

**Definition 8.** *[14] A fuzzy set* $\delta$ *of a semigroup* $\mathfrak{T}$ *is said to be a fuzzy ideal of* $\mathfrak{T}$ *if* $\delta(\mathfrak{u}\mathfrak{v}) \geq \delta(\mathfrak{u}) \vee \delta(\mathfrak{v})$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{T}$.

**Definition 9.** *[19] A fuzzy subsemigroup* $\delta$ *of a ordered semigroup* $\mathfrak{T}$ *is said to be a* fuzzy $(m,n)$-ideal *of* $\mathfrak{T}$ *if*

(1) $\delta(u_1u_2 \cdots u_m z v_1 v_2 \cdots v_n) \geq \delta(u_1) \wedge \delta(u_2) \wedge ... \wedge \delta(u_m) \wedge \delta(v_1) \wedge \delta(v_2) \wedge ... \wedge \delta(v_n)$ *for all* $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in \mathfrak{T}$ *and* $m, n \in \mathbb{N}$.

(2) *If* $u_1 \leq u_2$*, then* $\delta(u_1) \geq \delta(u_2)$*, for all* $u_1, u_2 \in \mathfrak{T}$.

**Definition 10.** *[9] A nonempty subset of* $\mathfrak{K}$ *an ordered semigroup* $\mathfrak{T}$ *is called a* left ordered almost ideal *(right ordered almost ideal) of* $\mathfrak{T}$ *if* $(\mathfrak{t}\mathfrak{K}] \cap \mathfrak{K} \neq \emptyset \quad ((\mathfrak{K}]\mathfrak{t}] \cap \mathfrak{K} \neq \emptyset)$ *for all* $\mathfrak{t} \in \mathfrak{T}$.

## 3. Main Results

In this section, we define the almost $(m,n)$-quasi-ideal and fuzzy almost $(m,n)$-quasi-ideal in ordered semigroup. We prove some basic interesting properties of almost $(m,n)$-quasi-ideal and fuzzy almost $(m,n)$-quasi-ideal in ordered semigroup.

**Definition 11.** *A non-empty subset* $\mathfrak{B}$ *on an ordered semigroup* $\mathfrak{T}$ *is called an* almost $(m,n)$-quasi-ideal *of* $\mathfrak{T}$ *if* $(\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$ *for all* $\mathfrak{t} \in \mathfrak{T}$ *where* $m, n \in \{1, 2, ..., n\}$.

**Example 1.** (1) *An almost* $(1, 1)$*-ideal of an ordered semigroup* $\mathfrak{T}$ *is a right almost quasi-ideal of* $\mathfrak{T}$.

(2) *Consider the ordered semigroup* $\mathbb{Z}_6$ *under the usual addition and the partial ordered* $\leq := \{(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. *We have* $\mathfrak{A} = \{\bar{1}, \bar{4}, \bar{5}\}$ *is an almsot* $(1, 1)$*-quasi-ideal of* $\mathbb{Z}_6$.

(3) *The almost* $(m, n)$*-quasi-ideal of an ordered semigroup* $\mathfrak{T}$ *is not* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

**Theorem 1.** *Every* $(m, n)$*-quasi-ideal of an ordered semigroup* $\mathfrak{T}$ *is an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

*Proof.* Assume that $\mathfrak{B}$ is an $(m, n)$-quasi-ideal of $\mathfrak{T}$ and let $\mathfrak{t} \in \mathfrak{T}$. Then $(\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n] \subseteq (\mathfrak{B}^m\mathfrak{T}] \cap (\mathfrak{T}\mathfrak{B}^n]$. Thus $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$. We conclude that $\mathfrak{B}$ is an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

**Theorem 2.** *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two non-empty subsets of an ordered semigroup* $\mathfrak{T}$ *such that* $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. *If* $\mathfrak{B}_1$ *is an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$*, then* $\mathfrak{B}_2$ *is also an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

*Proof.* Let $\mathfrak{B}_1$ be an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$ with $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ and let $\mathfrak{t} \in \mathfrak{T}$ Then $(\mathfrak{B}_1^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}_1^n] \subseteq (\mathfrak{B}_2^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}_2^n]$ Thus, $(\mathfrak{B}_2^m\mathfrak{t}\mathfrak{B}_2^n] \cap \mathfrak{B}_2 \neq \emptyset$. Hence, $\mathfrak{B}_2$ is an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

**Corollary 1.** *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be almost* $(m, n)$*-quasi-ideals of an ordered semigroup* $\mathfrak{T}$. *Thus* $\mathfrak{B}_1 \cup \mathfrak{B}_2$ *is also an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

*Proof.* Since $\mathfrak{B}_1$ and $\mathfrak{B}_1$ are subsets of $\mathfrak{B}_1 \cup \mathfrak{B}_2$, by Theorem 2, $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

**Corollary 2.** *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be nonempty subsets of an ordered semigroup* $\mathfrak{T}$. *If* $\mathfrak{B}_1$ *is an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$*, then* $\mathfrak{B}_1 \cup \mathfrak{B}_2$ *is an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

*Proof.* By Corollary 1, and $\mathfrak{B}_1 \subseteq \mathfrak{B}_1 \cup \mathfrak{B}_2$. Thus, $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

**Corollary 3.** *The finite union of almost* $(m, n)$*-quasi-ideals of an ordered semigroup* $\mathfrak{T}$ *is an almost* $(m, n)$*-quasi-ideal of* $\mathfrak{T}$.

**Example 2.** *Consider the ordered semigroup* $\mathbb{Z}_6$ *under the usual addtion and the partial ordered* $\leq := \{(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. *We have* $\mathfrak{A} = \{\bar{1}, \bar{4}, \bar{5}\}$ *and* $\mathfrak{B} = \{\bar{1}, \bar{2}, \bar{5}\}$ *are almost* $(1, 1)$*-quasi-ideals of* $\mathbb{Z}_6$. *Consider* $\mathfrak{A} \cap \mathfrak{B} = \{\bar{1}, \bar{5}\}$ *then* $\mathfrak{A} \cap \mathfrak{B}\bar{2} \cap \bar{2}\mathfrak{A} \cap \mathfrak{B} \cap \mathfrak{A} \cap \mathfrak{B} = \emptyset$. *Thus,* $\mathfrak{A} \cap \mathfrak{B} = \{\bar{1}, \bar{5}\}$ *is not an almost* $(1, 1)$*-quasi-ideal of* $\mathbb{Z}_6$.

**Definition 12.** *A fuzzy set* $\vartheta$ *on an ordered semigroup* $\mathfrak{T}$ *is called a* fuzzy almost $(m, n)$-quasi-ideal *of* $\mathfrak{T}$ *if* $(\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta \neq 0$. *for any fuzzy point* $x_t \in \mathfrak{T}$ *where* $m, n \in \{1, 2, ..., n\}$.

**Theorem 3.** *If $\vartheta$ is a fuzzy almost $(m,n)$-quasi-ideal of an ordered semigroup $\mathfrak{T}$ and $\xi$ is a fuzzy subset of $\mathfrak{T}$ such that $\vartheta \leq \xi$, then $\xi$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Suppose that $\vartheta$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ and $\xi$ is a fuzzy subset of $\mathfrak{T}$ such that $\vartheta \leq \xi$. Then for any fuzzy points $x_t \in \mathfrak{T}$, we obtain that $(\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta \neq 0$. Thus,

$$(\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta \leq (\xi^m \circ x_t] \wedge (x_t \circ \xi^n] \wedge \xi \neq 0.$$

Hence, $(\xi^m \circ x_t] \wedge (x_t \circ \xi^n] \wedge \xi \neq 0$. Therefore, $\xi$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

The following result is an obvious of Theorem 3.

**Theorem 4.** *Let $\vartheta$ and $\xi$ be fuzzy almost $(m,n)$-quasi-ideal of an ordered semigroup $\mathfrak{T}$. Then $\vartheta \vee \xi$ is also a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Since $\vartheta \leq \vartheta \vee \xi$, by Theorem 3, $\vartheta \vee \xi$ is also a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

**Theorem 5.** *If $\vartheta$ fuzzy almost $(m,n)$-quasi-ideal of an ordered semigroup $\mathfrak{T}$ and $\xi$ is a fuzzy set, then $\vartheta \vee \xi$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* By Theorem 3, and $\vartheta \leq \vartheta \vee \xi$. Thus, $\vartheta \vee \xi$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

**Corollary 4.** *Let $\mathfrak{T}$ be an ordered semigroup. Then the finite maximum of fuzzy almost $(m,n)$-quasi-ideals of $\mathfrak{T}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

**Example 3.** *Consider the ordered semigroup $\mathbb{Z}_6$ under the usual addition and the partial ordered $\leq := \{(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. $\vartheta : \mathbb{Z}_6 \to [0,1]$ is defined by $\vartheta(\bar{0}) = 0,\ \vartheta(\bar{1}) = 0.2,\ \vartheta(\bar{2}) = 0,\ \vartheta(\bar{3}) = 0,\ \vartheta(\bar{4}) = 0.5,\ \vartheta(\bar{5}) = 0.4$ and $\nu : \mathbb{Z}_6 \to [0,1]$ is defined by $\nu(\bar{0}) = 0,\ \nu(\bar{1}) = 0.8,\ \nu(\bar{2}) = 0.4,\ \nu(\bar{3}) = 0.3,\ \nu(\bar{4}) = 0,\ \nu(\bar{5}) = 0.3$.*
*We have $\vartheta$ and $\nu$ are fuzzy almost $(1,1)$-quasi-ideals of $\mathbb{Z}_6$ but $(\vartheta \wedge \nu)(0)$ is not a fuzzy almost $(1,1)$-quasi-ideal of $\mathbb{Z}_6$.*

Remark The minimum is not fuzzy almost $(1,1)$-quasi-ideal of an ordered semigroup by Example 3

**Lemma 5.** *Let $\mathfrak{A}$ be a subset of $\mathfrak{T}$ and $n \in \mathbb{N} \cup \{0\}$. Then $(\chi_{\mathfrak{A}})^n = \chi_{\mathfrak{A}^n}$*

**Theorem 6.** *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Suppose that $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $(\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$ for all $\mathfrak{t} \in \mathfrak{T}$ and $m,n \in \{1,2,...,n\}$. Thus there exists $\mathfrak{c} \in \mathfrak{T}$ such that $\mathfrak{c} \in (\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n]$ and $\mathfrak{c} \in \mathfrak{B}$. Let $x_t \in \mathfrak{T}$ and $t \in (0,1]$. Then $((\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge (x_t \circ \chi_{\mathfrak{B}^n}])(\mathfrak{c}) \neq 0$ and $\chi_{\mathfrak{B}}(\mathfrak{c}) \neq 0$ and $m,n \in \{1,2,...,n\}$. Thus, $((\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge (x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}})(\mathfrak{c}) = (((\chi_{\mathfrak{B}})^m \circ x_t] \circ \wedge (x_t \circ (\chi_{\mathfrak{B}})^n])(\mathfrak{c}) \neq 0$ and $m,n \in \{1,2,...,n\}$. So, $(\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge (x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}}) \neq 0$. Hence, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ and let $x_t \in \mathfrak{T}$ and $t \in (0,1]$ where $m,n \in \{1,2,...,n\}$. Then $((\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge(x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}}) \neq 0$. Thus, there exists $\mathfrak{c} \in \mathfrak{B}$ such that $((\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge(x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}})(\mathfrak{c}) \neq 0$. It implies that $((\chi_{\mathfrak{B}^m} \circ x_t] \circ \wedge(x_t \circ \chi_{\mathfrak{B}^n}])(\mathfrak{c}) \neq 0$ and $\chi_{\mathfrak{B}}(\mathfrak{c}) \neq 0$ and $m,n \in \{1,2,...,n\}$. Hence $\mathfrak{c} \in \mathfrak{T}$ such that $\mathfrak{c} \in (\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n]$ and $\mathfrak{c} \in \mathfrak{B}$. So $(\mathfrak{B}^m\mathfrak{t}] \cap (\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$ for all $\mathfrak{t} \in \mathfrak{T}$ and $m,n \in \{1,2,...,n\}$. We conclude that $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

**Theorem 7.** *Let $\vartheta$ be a fuzzy subset of an ordered semigroup $\mathfrak{T}$. Then $\vartheta$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if* supp$(\vartheta)$ *is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Assume that $\vartheta$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ and let $x_t \in \mathfrak{T}$ and $t \in (0,1]$. Then $(\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta \neq 0$ for all $m,n \in \{1,2,...,n\}$. Thus, there exists $\mathfrak{z} \in \mathfrak{T}$ such that $((\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta)(\mathfrak{z}) \neq 0$. So $\vartheta(\mathfrak{z}) \neq 0$ and $\mathfrak{z} = \mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_m\mathfrak{x} = \mathfrak{x}\mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n$ for some $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n \in \mathfrak{T}$ such that $\vartheta(\mathfrak{a}_1) \neq 0$, $\vartheta(\mathfrak{a}_2) \neq 0, \cdots, \vartheta(\mathfrak{a}_m) \neq 0$ $\vartheta(\mathfrak{b}_1) \neq 0$, $\vartheta(\mathfrak{b}_2) \neq 0, \cdots \vartheta(\mathfrak{b}_n) \neq 0$. Thus, $\mathfrak{a}_1\mathfrak{a}_2 \cdot \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n \in \text{supp}(\vartheta)$. It implies that $((\chi_{\text{supp}(\vartheta)^m} \circ x_t] \wedge (x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)})(\mathfrak{z}) \neq 0$. Hence, $(\chi_{\text{supp}(\vartheta)^m} \circ x_t] \wedge (x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)} \neq 0$. for all $m,n \in \{1,2,...,n\}$. Therefore, $\chi_{\text{supp}(\vartheta)}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. By Theorem 6, supp$(\vartheta)$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

Conversely, suppose that supp$(\vartheta)$ is an almost almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. By Theorem 6, $\chi_{\text{supp}(\vartheta)}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then for any fuzzy point $x_t \in \mathfrak{T}$ and $m,n \in \{1,2,...,n\}$, we have $(\chi_{\text{supp}(\vartheta)^m} \circ x_t] \wedge (x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)} \neq 0$. Thus, there exists $\mathfrak{z} \in \mathfrak{T}$ such that $((\chi_{\text{supp}(\vartheta)^m} \circ x_t] \wedge (x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)})(\mathfrak{z}) \neq 0$. So $\vartheta(\mathfrak{z}) \neq 0$ and $\mathfrak{z} = \mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_m\mathfrak{x} = \mathfrak{x}\mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n$ for some $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n \in \mathfrak{T}$ such that $\vartheta(\mathfrak{a}_1) \neq 0$, $\vartheta(\mathfrak{a}_2) \neq 0, \cdots, \vartheta(\mathfrak{a}_m) \neq 0$ $\vartheta(\mathfrak{b}_1) \neq 0$, $\vartheta(\mathfrak{b}_2) \neq 0, \cdots \vartheta(\mathfrak{b}_n) \neq 0$. Thus, $\mathfrak{a}_1\mathfrak{a}_2 \cdot \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n \in \text{supp}(\vartheta)$. So, there exists $\mathfrak{z} \in \mathfrak{T}$ such that $((\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta)(\mathfrak{z}) \neq 0$. Hence, $(\vartheta^m \circ x_t] \wedge (x_t \circ \vartheta^n] \wedge \vartheta \neq 0$ for all $m,n \in \{1,2,...,n\}$. Therefore, $\vartheta$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

Next, we investigate connection between minimal and maximal almost $(m,n)$-quasi-ideals and minimal and maximal fuzzy almost $(m,n)$-quasi-ideals of ordered semigroups.

**Definition 13.** *An almost $(m,n)$-quasi-ideal $\mathfrak{B}$ of an ordered semigroup $\mathfrak{T}$ is called*

(1) *a* minimal *if for any almost $(m,n)$-quasi-ideal $\mathfrak{R}$ of $\mathfrak{T}$ if whenever $\mathfrak{R} \subseteq \mathfrak{B}$, then $\mathfrak{R} = \mathfrak{B}$,*

(2) *a* maximal *if for any almost $(m,n)$-quasi-ideal $\mathfrak{R}$ of $\mathfrak{T}$ if whenever $\mathfrak{B} \subseteq \mathfrak{R}$, then $\mathfrak{R} = \mathfrak{B}$.*

**Definition 14.** *A fuzzy almost $(m,n)$-quasi-ideal $\vartheta$ of an ordered semigroup $\mathfrak{T}$ is called*

(1) *a* minimal *if for any fuzzy almost $(m,n)$-quasi-ideal $\xi$ of $\mathfrak{T}$ if whenever $\xi \leq \vartheta$, then* supp$(\xi) =$ supp$(\vartheta)$,

(2) *a* maximal *if for any fuzzy almost $(m,n)$-quasi-ideal $\xi$ of $\mathfrak{T}$ if whenever $\vartheta \leq \xi$, then* supp$(\xi) =$ supp$(\vartheta)$.

**Theorem 8.** *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then*

(1) *$\mathfrak{B}$ is a minimal almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a minimal fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

(2) *$\mathfrak{B}$ is a maximal almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a maximal fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.*

(1) Assume that $\mathfrak{B}$ is a minimal almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.
Let $\xi$ be a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\xi \leq \chi_{\mathfrak{B}}$. Then by Theorem 7, $\operatorname{supp}(\xi)$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Since $\mathfrak{B}$ is minimal we have $\operatorname{supp}(\xi) = \mathfrak{B} = \operatorname{supp}(\chi_{\mathfrak{B}})$. Therefore, $\chi_{\mathfrak{B}}$ is minimal.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a minimal fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\mathfrak{R}$ be an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\mathfrak{R} \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{R}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\chi_{\mathfrak{R}} \leq \chi_{\mathfrak{B}}$. Hence, $\mathfrak{R} = \operatorname{supp}(\chi_{\mathfrak{R}}) = \operatorname{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Therefore, $\mathfrak{B}$ is minimal.

(2) Assume that $\mathfrak{B}$ is a maximal almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\xi$ be a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\chi_{\mathfrak{B}} \leq \xi$. Then by Theorem 7, $\operatorname{supp}(\xi)$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\mathfrak{B} = \operatorname{supp}(\chi_{\mathfrak{B}}) \subseteq \operatorname{supp}(\xi)$. Since $\mathfrak{B}$ is maximal we have $\operatorname{supp}(\xi) = \mathfrak{B} = \operatorname{supp}(\chi_{\mathfrak{B}})$. Therefore, $\chi_{\mathfrak{B}}$ is maximal.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a maximal fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. By Theorem 6, $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\mathfrak{R}$ be an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\mathfrak{B} \subseteq \mathfrak{R}$. Then $\chi_{\mathfrak{R}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\chi_{\mathfrak{B}} \leq \chi_{\mathfrak{R}}$. Since $\chi_{\mathfrak{B}}$ is a maximal we have $\mathfrak{R} = \operatorname{supp}(\chi_{\mathfrak{R}}) = \operatorname{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Therefore, $\mathfrak{B}$ is maximal.

**Corollary 5.** *Let $\mathfrak{T}$ be an ordered semigroup. Then $\mathfrak{T}$ has no proper almost $(m,n)$-quasi-ideal if and only if $\operatorname{supp}(\vartheta) = \mathfrak{T}$ for every fuzzy almost $(m,n)$-quasi-ideal $\vartheta$ of $\mathfrak{T}$.*

Next, we give definition of prime (resp., semiprime, strongly prime) almost $(m,n)$-quasi-ideals and prime (resp., semiprime strongly prime) fuzzy almost $(m,n)$-quasi-ideals. We study the relationships between prime (resp., semiprime strongly prime) almost $(m,n)$-quasi-ideals and their fuzzification of ordered semigroups.

**Definition 15.** *Let $\mathfrak{B}$ be an almost $(m,n)$-quasi-ideal of an ordered semigroup $\mathfrak{T}$. Then we called*

(1) *$\mathfrak{B}$ is a **prime** if for any two almost $(m,n)$-quasi-ideals $\mathfrak{N}$ and $\mathfrak{H}$ of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \subseteq \mathfrak{B}$ implies that $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$.*

(2) $\mathfrak{B}$ *is a* ***semiprime*** *if for any almost* $(m,n)$*-quasi-ideal* $\mathfrak{N}$ *of* $\mathfrak{T}$ *such that* $\mathfrak{N}^2 \subseteq \mathfrak{N}$ *implies that* $\mathfrak{N} \subseteq \mathfrak{B}$.

(3) $\mathfrak{B}$ *is a* ***strongly prime*** *if for any almost* $(m,n)$*-quasi-ideals* $\mathfrak{N}$ *and* $\mathfrak{H}$ *of* $\mathfrak{T}$ *such that* $\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N} \subseteq \mathfrak{B}$ *implies that* $\mathfrak{N} \subseteq \mathfrak{B}$ *or* $\mathfrak{H} \subseteq \mathfrak{B}$.

**Definition 16.** *A fuzzy almost* $(m,n)$*-quasi-ideal* $\vartheta$ *on an ordered semigroup* $\mathfrak{T}$*. Then we called*

(1) $\vartheta$ *is a* ***prime*** *if for any two fuzzy almost* $(m,n)$*-quasi-ideals* $\xi$ *and* $\nu$ *of* $\mathfrak{T}$ *such that* $\xi \circ \nu \leq \vartheta$ *implies that* $\xi \leq \vartheta$ *or* $\nu \leq \vartheta$.

(2) $\vartheta$ *is a* ***semiprime*** *if for any fuzzy almost* $(m,n)$*-quasi-ideal* $\xi$ *of* $\mathfrak{T}$ *such that* $\xi \circ \xi \leq \xi$ *implies that* $\xi \leq \vartheta$.

(3) $\vartheta$ *is a* ***strongly prime*** *if for any two fuzzy almost* $(m,n)$*-quasi-ideals* $\xi$ *and* $\nu$ *of* $\mathfrak{T}$ *such that* $(\xi \circ \nu) \wedge (\nu \circ \xi) \leq \vartheta$ *implies that* $\xi \leq \vartheta$ *or* $\nu \leq \vartheta$.

It is clearly, every fuzzy strongly prime almost $(m,n)$-quasi-ideal of a ternary semigroup is a fuzzy prime almost $(m,n)$-quasi-ideal, and every fuzzy prime almost $(m,n)$-quasi-ideal of a ternary semigroup is a fuzzy semiprime almost $(m,n)$-quasi-ideal.

**Theorem 9.** *Let* $\mathfrak{B}$ *be a nonempty subset of an ordered semigroup* $\mathfrak{T}$*. Then* $\mathfrak{B}$ *is a prime almost* $(m,n)$*-quasi-ideal of* $\mathfrak{T}$ *if and only if* $\chi_{\mathfrak{B}}$ *is a prime fuzzy almost* $(m,n)$*-quasi-ideal of* $\mathfrak{T}$.

*Proof.* Suppose that $\mathfrak{B}$ is a prime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\vartheta$ and $\xi$ be fuzzy almost $(m,n)$-quasi-ideals such that $\vartheta \circ \xi \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \nleq \chi_{\mathfrak{B}}$ and $\xi \nleq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h}, \mathfrak{r} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{r}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$ and $\chi_{\mathfrak{B}}(\mathfrak{r}) = 0$. Thus, $\mathfrak{h} \in \operatorname{supp}(\vartheta)$ and $\mathfrak{r} \in \operatorname{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{r} \notin \mathfrak{B}$. So $\operatorname{supp}(\vartheta) \nsubseteq \mathfrak{B}$ and $\operatorname{supp}(\xi) \nsubseteq \mathfrak{B}$. Since $\operatorname{supp}(\vartheta)$ and $\operatorname{supp}(\xi)$ are almost $(m,n)$-quasi-ideals of $\mathfrak{T}$ we have $\operatorname{supp}(\vartheta)\operatorname{supp}(\xi) \nsubseteq \mathfrak{B}$. Thus, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}$ for some $\mathfrak{p} \in \operatorname{supp}(\vartheta)$ and $\mathfrak{q} \in \operatorname{supp}(\xi)$ such that $\mathfrak{m} \in \mathfrak{B}$. Hence $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$ implies that $(\vartheta \circ \xi)(\mathfrak{m}) = 0$. Since $\vartheta \circ \xi \leq \chi_{\mathfrak{B}}$. we have $\mathfrak{p} \in \operatorname{supp}(\vartheta)$ and $\mathfrak{q} \in \operatorname{supp}(\xi)$. Thus $\vartheta(\mathfrak{p}) \neq 0$, and $\xi(\mathfrak{q}) \neq 0$. It implies that

$$(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{p},\mathfrak{q}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{p}) \wedge \xi(\mathfrak{q})\} \neq 0$$

It is a contradiction so $\vartheta \leq \chi_{\mathfrak{B}}$ or $\xi \leq \chi_{\mathfrak{B}}$. Therefore $\chi_{\mathfrak{B}}$ is a prime fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a prime fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\mathfrak{N}$ and $\mathfrak{H}$ be almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{B}}$ and $\chi_{\mathfrak{H}}$ are fuzzy almost $(m,n)$-quasi-ideals of $\mathfrak{T}$. By Lemma 1 $\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}} = \chi_{\mathfrak{N}\mathfrak{H}} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$ or $\chi_{\mathfrak{B}} \leq \chi_{\mathfrak{H}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a prime almost $(m,n)$-quasi-ideal of $T$.

**Theorem 10.** *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is a semiprime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a semiprime fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Suppose that $\mathfrak{B}$ is a semiprime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\vartheta$ be fuzzy almost $(m,n)$-quasi-ideal such that $\vartheta \circ \vartheta \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \nleq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$. Thus, $\mathfrak{h} \in \operatorname{supp}(\vartheta)$, but $\mathfrak{h} \notin \mathfrak{B}$. So $\operatorname{supp}(\vartheta) \nsubseteq \mathfrak{B}$. Since $\operatorname{supp}(\vartheta)$ are almost $(m,n)$-quasi-ideals of $\mathfrak{T}$ we have $\operatorname{supp}(\vartheta) \nsubseteq \mathfrak{B}$. Thus $\mathfrak{h} \in \operatorname{supp}(\vartheta)$ such that $\mathfrak{m} \in \mathfrak{B}$. Hence $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$ implies that $(\vartheta \circ \vartheta)(\mathfrak{m}) = 0$. Since $\vartheta \circ \vartheta \leq \chi_{\mathfrak{B}}$. we have $p \in \operatorname{supp}(\vartheta)$. Thus $\vartheta(\mathfrak{p}) \neq 0$. It implies that

$$(\vartheta \circ \vartheta)(\mathfrak{m}) = \bigvee_{(\mathfrak{p},\mathfrak{b}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{p}) \wedge \vartheta(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\vartheta \leq \chi_{\mathfrak{B}}$. Therefore $\chi_{\mathfrak{B}}$ is a semiprime fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a semiprime fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\mathfrak{N}$ be almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ such that $\mathfrak{N}^2 \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{N}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. By Lemma 1 $\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{N}} = \chi_{\mathfrak{N}^2} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a semiprime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.

**Theorem 11.** *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is a strongly prime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$.*

*Proof.* Suppose that $\mathfrak{B}$ is a strongly prime almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Then $\mathfrak{B}$ is an almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Thus by Theorem 6, $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m,n)$-quasi-ideal of $\mathfrak{T}$. Let $\vartheta$ and $\xi$ be fuzzy almost $(m,n)$-quasi-ideals of $\mathfrak{T}$ such that $(\vartheta \circ \xi) \wedge (\xi \circ \vartheta) \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \nleq \chi_{\mathfrak{B}}$ and $\xi \nleq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h}, \mathfrak{b} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{b}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$ and $\chi_{\mathfrak{B}}(\mathfrak{b}) = 0$. Thus, $\mathfrak{h} \in \operatorname{supp}(\vartheta)$ and $\mathfrak{b} \in \operatorname{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{b} \notin \mathfrak{B}$. So, $\operatorname{supp}(\vartheta) \nsubseteq \mathfrak{B}$ and $\operatorname{supp}(\xi) \nsubseteq \mathfrak{B}$. Hence, there exists $m \in [\operatorname{supp}(\vartheta)\operatorname{supp}(\xi)] \cap (\operatorname{supp}(\vartheta)\operatorname{supp}(\xi))$ such that $\mathfrak{m} \notin \mathfrak{B}$. Thus, $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$. Since $m \in \operatorname{supp}(\vartheta)\operatorname{supp}(\xi)$ and $m \in \operatorname{supp}(\xi)\operatorname{supp}(\vartheta)$ we have $\mathfrak{m} = \mathfrak{d}\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{g}\mathfrak{q}$ for some $\mathfrak{d}, \mathfrak{q} \in \operatorname{supp}(\vartheta)$, and for some $\mathfrak{k}, \mathfrak{g} \in \operatorname{supp}(\xi)$. we have

$$(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{d},\mathfrak{c}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{d}) \wedge \xi^{p}(\mathfrak{k})\} \neq 0.$$

Similarly

$$(\xi \circ \vartheta)(\mathfrak{m}) = \bigvee_{(\mathfrak{g},\mathfrak{q}) \in F_{\mathfrak{m}}} \{\xi(\mathfrak{g}) \wedge \vartheta(\mathfrak{q})\}.$$

So $(\vartheta \circ \xi)(\mathfrak{m}) \wedge (\xi \circ \vartheta)(\mathfrak{m}) \neq 0$. It is a contradiction so, $\vartheta \leq \chi_{\mathfrak{B}}$ or $\xi \leq \chi_{\mathfrak{B}}$. Therefore, $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime almost $(m, n)$-quasi-ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy almost $(m, n)$-quasi-ideal of $\mathfrak{T}$. Thus, by Theorem 6, $\mathfrak{B}$ is an almost $(m, n)$-quasi-ideal of $\mathfrak{T}$. Let $\mathfrak{N}$ and $\mathfrak{H}$ be almost $(m, n)$-quasi-ideals of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N} \leq \mathfrak{B}$. Then $\chi_{\mathfrak{N}}$ and $\chi_{\mathfrak{H}}$ are fuzzy almost $(m, n)$-quasi-ideals of $\mathfrak{T}$. By Lemma 1 $\chi_{\mathfrak{N}\mathfrak{H}} = \chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}}$ and $\chi_{\mathfrak{H}\mathfrak{N}} = \chi_{\mathfrak{H}} \circ \chi_{\mathfrak{N}}$. Thus $(\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}}) \wedge (\chi_{\mathfrak{H}} \circ \chi_{\mathfrak{N}}) = \chi_{\mathfrak{N}\mathfrak{H}} \wedge \chi_{\mathfrak{H}\mathfrak{N}} = \chi_{\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$ and $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a strongly prime almost $(m, n)$-quasi-ideal of $\mathfrak{T}$.

## 4. Conclusion

The aim paper gives the concept of almost $(m, n)$-quasi-ideals in ordered semigroups. The union of two almost $(m, n)$-quasi-ideals is also an almost $(m, n)$-quasi-ideal in ordered semigroups, and the results in class fuzzifications are the same. In Theorems 6, 7, 8, 10, and 11. Finally we prove that if $\mathfrak{K}$ is a (minimal/ maximal/prime/semiprime/strongly prime) almost $(m, n)$-quasi-ideals of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{A}}$ is (minimal/ maximal/prime/semiprime strongly prime) fuzzy almost $(m, n)$-quasi-ideals of $\mathfrak{T}$. In future work, we can study other kinds of almost ideals and their fuzzifications in ordered ternary semigroup.

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