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Characterization and Application of Fixed Point Theorems for 3-Self Mappings in Generalized Metric Spaces

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Abstract. In this paper, we investigate new contraction results for 3-self-mappings on generalized metric spaces (GM-spaces) and develop and highlight the importance of several particular common fixed point (CFP) theorems. Our study contributes to the theoretical framework of fixed point theory by highlighting the existence of such points and thoroughly proving their uniqueness. As a practical application of our theoretical findings, we design and analyze a convincing case including 3-self mappings to further support the uniqueness of a CFP for generalized contractions in the given space. We also provide a solid and perceptive application pertaining to nonlinear integral equations to further support the wider applicability of our primary contributions, demonstrating the usefulness of our discoveries in mathematical analysis and beyond. An excellent example is developed for better understanding the results.

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Key Words and Phrases: Common fixed point GM-space, contraction conditions, nonlinear integral equations

1. Introduction

Due to its theoretical significance and wide range of applications, the study of F-points in GM-spaces under generalized contractions has garnered a lot of interest. The presence and uniqueness of F-points, the convergence behavior of iterative sequences produced by generalized contractions, and the creation of mathematical models for practical issues are among the main areas of interest for this field of study. Although the idea of fixed point theory (FPT) was first proposed by Liouville in 1837 and Picard in 1890, Banach [1] established the fundamentals of

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FPT with a famous theorem. A single-valued contractive mapping with a unique fixed point (FP) in a complete metric space (CMS) is the subject of the Banach Contraction Principle (BCP). An inventive extension of Kannan contractions and associated fixed point results was presented by Batra et al. [2]. Debnath [3] proved the common fixed point (CFP) results of contractive inequalities for multivalued mappings without assuming that the pictures under these mappings are compact. The idea of common fixed points in cone metric spaces was altered by Palermo [4]. Fixed point theorems for contractive mappings were established by Huang et al. [5] through their analysis of cone metric spaces. In [6], a graph was introduced to study fixed point theorems about Reich-type contractions in metric spaces. Common fixed point theorems and fixed point results for pointwise contractions were presented in [7, 8] with respect to modular metric spaces.

Fixed point theory was greatly expanded when Mustafa and Sims [9] later established the idea of generalized metric spaces (GM-spaces). Das et al. [10] built on this foundation by establishing specific single-valued contractive type mappings (S-VCTM) and altering fixed point outcomes that were already in place within GM-spaces. To illustrate the applicability of their methodology, Mustafa and Sims [11] particularly applied fixed point theorems to contractive mappings in entire GM-spaces. Furthermore, Mustafa et al. [12] investigated further fixed point theorems for different mappings in entire GM-spaces. Notably, the work in [13] looked at common fixed point outcomes, including non-commuting mappings in GM-spaces that do not presuppose continuity. By examining periodic point results and demonstrating the diversity of fixed point phenomena in these spaces, Nazir et al. [14] made a contribution to this field. The usefulness of GM-spaces in fixed point theory is confirmed by Shatanawi et al. [15], who claim that fixed point outcomes do exist in these spaces. The study of fixed point theorems in G-partially ordered GM-spaces was tried by Vaezpour et al. [16], which added another level of complexity and relevance to the area. Mohanta et al. [17] looked into a common fixed point theorem unique to GM-spaces, whereas Choudhury et al. [18] made more progress by identifying linked fixed point findings in GM-spaces.

Khan et al. [19] demonstrated the interaction between several spaces in a similar study by presenting linked common fixed point data in two separate GM-spaces. If $X = \mathbb{R}$ (the set of real numbers) and define a function $d : X \times X \to [0, \infty)$ by: $d(x, y) = \begin{cases} |x-y| & \text{if } x \neq y \\ 1 & \text{if } x \neq y \end{cases}$

We see that this is a GMS but not a metric. By demonstrating fixed point theorems for generalized contractions in partial metric spaces, Romaguera [20] added to the conversation. By developing common fixed point results in GM-spaces and outlining their useful applications, Gugnani et al. [21] advanced the conversation. A Kannan theorem was investigated by Arshad et al. [22] in the context of GM-spaces, highlighting the continued applicability of traditional conclusions in this novel setting. Several more fixed point theorems were derived in [23], which examined and improved the framework created by Gugnani et al [21]. The notions' adaptability was demonstrated in [24], when research on fixed point theorems pertaining to multi-valued contractive operators in GM-spaces was presented. Finally, in [25, 26], new definitions of fixed point theorems and their different applications in GM-spaces were presented, highlighting how dynamic and ever-evolving this field of study.

Because of its theoretical significance and real-world implications, the study of F-points in GM-spaces under generalized contractions has attracted a lot of attention. The convergence behaviour of iterative sequences in generalized contractions, the presence and uniqueness of F-points, and the creation of mathematical models for practical issues are all examined in this study. By following the development of fixed point theory (FPT) from Banach's contraction principle to contemporary developments in GM-spaces, the study expands on earlier research in this field. Important contributions from different scholars are described, such as the development of common fixed point results and single-valued contractive type mappings (S-VCTM).

1.1. Problem Statement

While fixed point theory has seen extensive development, particularly in the study of self-mappings on various types of metric spaces, the literature reveals a noticeable gap in the context of common fixed points (CFPs) for multiple (specifically, three) self-mappings in generalized metric spaces (GM-spaces). Existing contraction-type results predominantly focus on single or pairwise mappings and are often restricted to classical metric frameworks. Moreover, although generalized contractions have been introduced in recent years, their application to multiple self-mappings—especially in proving uniqueness and existence of CFPs-remains underexplored. There is also a lack of comprehensive examples and practical applications, such as in the analysis of nonlinear integral equations, that showcase the effectiveness and broader applicability of these theoretical advancements. This research seeks to fill this gap by developing new contraction results specifically tailored for three self- mappings on GM-spaces, and by establishing robust CFP theorems that ensure both existence and uniqueness. Furthermore, we aim to enhance the practical value of these results by presenting well- constructed examples and meaningful applications.

1.2. Organization

The paper is structured as follows: Section 2: Provides pertinent preliminaries and definitions necessary for understanding the subsequent results. Section 3: Introduces the key results on fixed point theorems in GM-spaces. Section 4: Discusses the applications of the fixed point theorems presented in the previous section. Section 5: Presents the conclusions and outlines possible directions for future research, emphasizing the growing importance of GM-spaces in fixed point theory.

2. Preliminaries

This section is devoted to some fundamental definitions, which are necessary for the upcoming sections.

2.1. Generalized Metric Space

Definition 1. *[9] Assume* $\hat{\mathrm{E}} \neq \emptyset$ *set and* $G : \hat{\mathrm{E}} \times \hat{\mathrm{E}} \times \hat{\mathrm{E}} \to [0, \infty)$ *is a generalized metric space (GM-space) if and only if the axioms below are true.*

i. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) = 0$ *iff* $\hat{e}_1 = \hat{e}_2 = \hat{e}_3$,

ii. $0 < G(\hat{e}_1, \hat{e}_1, \hat{e}_2)\ \forall\ \hat{e}_1, \hat{e}_2 \in \hat{\mathrm{E}}$, *with* $\hat{e}_1 \neq \hat{e}_2$,

iii. $G(\hat{e}_1, \hat{e}_1, \hat{e}_2) \leq G(\hat{e}_1, \hat{e}_2, \hat{e}_3)\ \forall\ \hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$, *with* $\hat{e}_1 \neq \hat{e}_2$,

iv. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) = G\{p(\hat{e}_1, \hat{e}_2, \hat{e}_3)\}$ *where* p *is a permutation of* $\hat{e}_1, \hat{e}_2, \hat{e}_3$ *(symmetry),*

v. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq G(\hat{e}_1, \hat{a}, \hat{a}) + G(\hat{a}, \hat{e}_2, \hat{e}_3)\ \forall\ \hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$.

A GM is symmetric if $G(\hat{e}_1, \hat{e}_2, \hat{e}_2) = G(\hat{e}_2, \hat{e}_1, \hat{e}_1)\ \forall\ \hat{e}_1, \hat{e}_2 \in \hat{\mathrm{E}}$, *then* $(\hat{\mathrm{E}}, G)$ *is known as a GM-space. The significance of this study is as follows: It introduces a novel and generalized family of contraction mappings, termed* F*-Kannan contractions, contributing a new direction in fixed point theory. By identifying and correcting a previous error in the literature, the study not only clarifies conceptual misunderstandings but also strengthens the theoretical foundation of contraction mappings. The proposed* F*-Kannan framework extends the well-known concept of* F*-contractions and provides new fixed point results that hold even in non-complete metric spaces a significant relaxation of classical assumptions. Furthermore, the validation of Subrahmanyam's characterization of completeness within this broader context underscores the robustness and* *applicability of the new class. This advancement opens up new possibilities for research and application in areas that rely on fixed point theorems, such as nonlinear analysis, optimization, and mathematical modeling.*

Definition 2. *[9] Let $(\hat{\mathrm{E}}, G)$ be a GM-space while $\{\hat{e}_i\}$ be a sequence in $\hat{\mathrm{E}}$. Then,*

i. *$\{\hat{e}_i\}$ in a GM-space is called a G-Cauchy sequence (G-CS) if for any $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t $G(\hat{e}_i, \hat{e}_m, \hat{e}_l) < \varepsilon\ \forall\ i, m, l \geq n_0$.*

ii. *$\{\hat{e}_i\}$ is convergent to an element $\hat{e} \in \hat{\mathrm{E}}$ if $\forall$ any given a real number $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ so that; $G(\hat{e}, \hat{e}_i, \hat{e}_m) < \varepsilon$, whenever $m \geq n_0$.*

iii. *$(\hat{\mathrm{E}}, G)$ is complete if every G-CS is G-convergent in $\hat{\mathrm{E}}$.*

Proposition 1. *[9] Let $(\hat{\mathrm{E}}, G)$ be a GM-space, then for any $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ the following hold:*

i. *If $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) = 0$, then $\hat{e}_1 = \hat{e}_2 = \hat{e}_3$,*

ii. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq G(\hat{e}_1, \hat{e}_1, \hat{e}_3)$,

iii. $G(\hat{e}_1, \hat{e}_2, \hat{e}_2) \leq 2G(\hat{e}_2, \hat{e}_1, \hat{e}_1)$,

iv. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq G(\hat{e}_1, \hat{e}, \hat{e}_3) + G(\hat{e}, \hat{e}_2, \hat{e}_3)$,

v. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq \frac{2}{3}\big(G(\hat{e}_1, \hat{e}_2, \hat{e}) + G(\hat{e}_1, \hat{e}, \hat{e}_3) + G(\hat{e}, \hat{e}_2, \hat{e}_3)\big)$,

vi. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq \big(G(\hat{e}_1, \hat{e}, \hat{e}) + G(\hat{e}_2, \hat{e}, \hat{e}) + G(\hat{e}_3, \hat{e}, \hat{e})\big)$,

vii. $|G(\hat{e}_1, \hat{e}_2, \hat{e}_3) - G(\hat{e}_1, \hat{e}_2, \hat{e})| \leq \max\{G(\hat{e}, \hat{e}_3, \hat{e}_3), G(\hat{e}_3, \hat{e}, \hat{e})\}$,

viii. $|G(\hat{e}_1, \hat{e}_2, \hat{e}_3) - G(\hat{e}_1, \hat{e}_2, \hat{e})| \leq G(\hat{e}_1, \hat{e}, \hat{e}_3)$,

ix. $|G(\hat{e}_1, \hat{e}_2, \hat{e}_3) - G(q, \hat{e}_3, \hat{e}_3)| \leq \max\{G(\hat{e}_1, \hat{e}_3, \hat{e}_3), G(\hat{e}_3, \hat{e}_1, \hat{e}_1)\}$,

x. $|G(\hat{e}_1, \hat{e}_2, \hat{e}_2) - G(\hat{e}_2, \hat{e}_1, \hat{e}_1)| \leq \max\{G(\hat{e}_2, \hat{e}_1, \hat{e}_1), G(\hat{e}_1, \hat{e}_2, \hat{e}_2)\}$.

Example 1. *[9] Let $\hat{\mathrm{E}} = \{a, b\}$, let*

i. $G(a, a, a) = G(b, b, b) = 0$

ii. $G(a, a, b) = 1$, $G(a, b, b) = 2$

and extend G to all of $\hat{\mathrm{E}} \times \hat{\mathrm{E}} \times \hat{\mathrm{E}}$ by symmetry in the variables. Then it is easily verified that G is a G-metric, but $G(a, b, b) \neq G(a, a, b)$.

Proposition 2 (9)**.** *Let $(\hat{\mathrm{E}}, G)$ be a GM-space, and let $k > 0$, then G_1 and G_2 are also G-metrics on $\hat{\mathrm{E}}$, where,*

i. $G_1(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \min\{k, G(\hat{e}_1, \hat{e}_2, \hat{e}_3)\}$, *and*

ii. $G_2(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \dfrac{G(\hat{e}_1, \hat{e}_2, \hat{e}_3)}{k + G(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$.

Further, if $\hat{\mathrm{E}} = \bigcup_{i=1}^{n} A_i$ is any partition of $\hat{\mathrm{E}}$ then,

iii. $G_3(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \begin{cases} G(\hat{e}_1, \hat{e}_2, \hat{e}_3), & \textit{if for some } i \textit{ we have } \hat{e}_1, \hat{e}_2, \hat{e}_3 \in A_i \\ k + G(\hat{e}_1, \hat{e}_2, \hat{e}_3), & \textit{otherwise,} \end{cases}$

is also a G-metric.

Proposition 3. *[9] Let $(\hat{\mathrm{E}}, G)$ be a GM-space, the following are equivalent.*

i. $(\hat{\mathrm{E}}, G)$ *is symmetric.*

ii. $G(\hat{e}_1, \hat{e}_2, \hat{e}_2) \leq G(\hat{e}_1, \hat{e}_2, a)$*, for all* $\hat{e}_1, \hat{e}_2, a \in \hat{\mathrm{E}}$.

iii. $G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq G(\hat{e}_1, \hat{e}_2, a) + G(\hat{e}_3, \hat{e}_2, b)$*, for all* $\hat{e}_1, \hat{e}_2, \hat{e}_3, a, b \in \hat{\mathrm{E}}$.

Proof. (1) implies (2) follows from $(G3)$ whenever $a \neq x$ and from $(\hat{\mathrm{E}}, G)$ being symmetric when $a = x$. Combining (2) of Proposition 1 and (2) above we have

$$G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \leq G(\hat{e}_1, \hat{e}_2, \hat{e}_2) + G(\hat{e}_3, \hat{e}_2, \hat{e}_2) \leq G(\hat{e}_1, \hat{e}_2, a) + G(\hat{e}_3, \hat{e}_2, b),$$

so (2) implies (3). Finally, (3) implies (1) follows by taking $a = x$, and $b = y$ in (3).

3. Main Results

In this particular section, two important theorems are addressed. For the justification of these theorems suitable examples are generated. In continuation, applications of these particular theorems are reflected. In specifically, the parameters α and β must be non-negative and less than one, with their sum being limited to less than one for uniqueness. The conclusions are based on the contractive property that the mappings exhibit. Ultimately, the sequence converges to a unique CFP because the iterative sequences provided in the proof show that the mappings compress distances in a way that guarantees convergence to a point in the whole GM-space. Furthermore, the context within which these results can be applied is expanded by the corollaries obtained from Theorem, which support the applicability of these discoveries under more particular constraints for α and β.
By showing that the existence and uniqueness of fixed points are directly influenced by the interaction between the contraction qualities of the mappings, the theorem and its corollaries thus make a substantial contribution to the understanding of fixed point theory in the setting of GM-spaces. This is a useful tool for additional research in a variety of applied and mathematical situations where these mappings are pertinent.

Theorem 1. *Let* $(\hat{\mathrm{E}}, G)$ *be a GM-space and* $F_1, F_2, F_3 : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ *be 3-self-mappings satisfying:*

$$\begin{aligned} G(F_1\hat{e}_1, F_2\hat{e}_2, F_3\hat{e}_3) &\leq \alpha G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \\ &\quad + \beta \left(\frac{G(\hat{e}_1, F_2\hat{e}_2, F_2\hat{e}_2) \cdot G(\hat{e}_1, F_3\hat{e}_3, F_3\hat{e}_3) \cdot G(\hat{e}_2, F_1\hat{e}_1, F_1\hat{e}_1)}{1 + [G(\hat{e}_2, F_1\hat{e}_1, F_1\hat{e}_1) \cdot G(\hat{e}_3, F_2\hat{e}_2, F_2\hat{e}_2) \cdot G(\hat{e}_1, F_3\hat{e}_3, F_3\hat{e}_3)]} \right. \\ &\quad \left. + \frac{G(\hat{e}_2, F_3\hat{e}_3, F_3\hat{e}_3) \cdot G(\hat{e}_3, F_1\hat{e}_1, F_1\hat{e}_1) \cdot G(\hat{e}_3, F_2\hat{e}_2, F_2\hat{e}_2)}{1 + [G(\hat{e}_2, F_1\hat{e}_1, F_1\hat{e}_1) \cdot G(\hat{e}_3, F_2\hat{e}_2, F_2\hat{e}_2) \cdot G(\hat{e}_1, F_3\hat{e}_3, F_3\hat{e}_3)]} \right) \end{aligned} \tag{3.1}$$

for all $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ *and* $\alpha, \beta \geq 0$ *with* $\alpha, \beta < 1$*. The 3-self-mapping followed* F_1, F_2 *&* F_3 *has a CFP in* $\hat{\mathrm{E}}$*. Also, if* $(\alpha + \beta) < 1$*, then* F_1, F_2 *&* F_3 *have a unique CFP in* $\hat{\mathrm{E}}$*.*

Proof. Fix $\hat{e}_0 \in \hat{\mathrm{E}}$, we now define iterative sequences in $\hat{\mathrm{E}}$ as follows:

$$\hat{e}_{3\Upsilon+1} = F_1\hat{e}_{3\Upsilon}, \quad \hat{e}_{3\Upsilon+2} = F_2\hat{e}_{3\Upsilon+1}, \quad \text{and} \quad \hat{e}_{3\Upsilon+3} = F_3\hat{e}_{3\Upsilon+2} \quad \forall \Upsilon \geq 0.$$

By using (3.1), we have

$$\begin{aligned} G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) &= G(F_1\hat{e}_{3\Upsilon}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}) \\ &\leq \alpha\, G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) + \beta \left(\frac{N}{D} \right) \end{aligned}$$

$$\begin{aligned} N &= G(\hat{e}_{3\Upsilon}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \\ &\quad \cdot G(\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \\ D &= 1 + G(\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \end{aligned}$$

$$\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) + \beta \left(\frac{\begin{array}{c} G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \\ + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \end{array}}{1 + [G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})]} \right)$$

After simplification, we have

$$G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \tag{1}$$

Similarly, again by the view of (3.1),

$$\begin{aligned} G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) &= G(F_1\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+3}) \\ &\leq \alpha G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \\ &\quad + \beta \left(\frac{\begin{array}{c} G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3}) \\ \cdot G(\hat{e}_{3\Upsilon+2}, F_1\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon+1}) \\ + G(\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+3}, F_1\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon+1}) \\ \cdot G(\hat{e}_{3\Upsilon+3}, F_2\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+2}) \end{array}}{\begin{array}{c} 1 + [G(\hat{e}_{3\Upsilon+2}, F_1\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon+1}) \\ \cdot G(\hat{e}_{3\Upsilon+3}, F_2\hat{e}_{3\Upsilon+3}, F_2\hat{e}_{3\Upsilon+2}) \\ \cdot G(\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3})] \end{array}} \right) \end{aligned}$$

$$\leq \alpha\, G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) + \beta \left(\frac{\begin{array}{c} G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+4}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \\ + G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+4}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \end{array}}{1 + G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+4})} \right)$$

After simplification, we have

$$G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) \leq \alpha G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \tag{2}$$

By a similar argument as in above, we can show that

$$G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+5}) \leq \alpha G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}). \tag{3}$$

Now, from (1), (2) and (3), we conclude that

$$\begin{aligned} G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) &\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \\ &\leq \alpha^2 G(\hat{e}_{3\Upsilon-1}, \hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}) \\ &\leq \cdots \leq \alpha^{3\Upsilon} G(\hat{e}_1, \hat{e}_2, \hat{e}_3) \\ &\leq \alpha^{3\Upsilon+1} G(\hat{e}_0, \hat{e}_1, \hat{e}_2). \end{aligned} \tag{4}$$

Hence proved that the sequence $\{\hat{e}_\Upsilon\}$ is contractive under the GM-space for 3-self-mappings. Therefore,

$$\lim_{\Upsilon\to\infty} G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}) = 0. \tag{5}$$

Next, we will show that $\{\hat{e}_\Upsilon\}$ is a G-CS in $\hat{\mathrm{E}}$, for all $\Upsilon, m \in N$ and $m > \Upsilon$, with the aid of (4),

$$\begin{aligned} G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_m) &\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+1}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_m, \hat{e}_m) \\ &\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+1}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}, \hat{e}_{\Upsilon+2}) + \cdots + G(\hat{e}_{m-1}, \hat{e}_m, \hat{e}_m) \\ &\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}, \hat{e}_{\Upsilon+3}) + \cdots + G(\hat{e}_{m-1}, \hat{e}_m, \hat{e}_{m+1}) \\ &\leq \eta^\Upsilon G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^{\Upsilon+1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \cdots + \eta^{m-1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) \\ &\leq \eta^\Upsilon \left[G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^1 G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^2 G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \cdots + \eta^{m-1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) \right] \end{aligned}$$

This implies that,

$$G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_m) \leq \frac{\eta^\Upsilon}{1-\eta} G(\hat{e}_0, \hat{e}_1, \hat{e}_1). \tag{6}$$

If we take the limit as $\Upsilon, m, l \to \infty$ we get $G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_l) \to 0$. Hence $\{\hat{e}_\Upsilon\}$ is a G-CS. Since, $(\hat{\mathrm{E}}, G)$ is complete, there exists $\mathrm{z} \in \hat{\mathrm{E}}$, such that, $\hat{e}_\Upsilon \to \mathrm{z}$ as $\Upsilon \to \infty$ or $\lim_{\Upsilon\to\infty} \hat{e}_\Upsilon = \mathrm{z}$. We now show that $F_1\mathrm{z} = \mathrm{z}$ by contrary case. Let $F_1\mathrm{z} \neq \mathrm{z}$. By using (3.1), we have that

$$G(F_1\mathrm{z}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) = G(F_1\mathrm{z}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2})$$

$$\begin{aligned} &\leq \alpha\, G(\mathrm{z}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \\ &\quad + \beta \left(\frac{\begin{gathered} G(\mathrm{z}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\mathrm{z}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+1}, F_1\mathrm{z}, F_1\mathrm{z}) \\ + G(\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, F_1\mathrm{z}, F_1\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \end{gathered}}{1 + G(\hat{e}_{3\Upsilon+1}, F_1\mathrm{z}, F_1\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\mathrm{z}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})} \right) \\ &\leq \alpha\, G(\mathrm{z}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \\ &\quad + \beta \left(\frac{\begin{gathered} G(\mathrm{z}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\mathrm{z}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, F_1\mathrm{z}, F_1\mathrm{z}) \\ + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+2}, F_1\mathrm{z}, F_1\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \end{gathered}}{1 + G(\hat{e}_{3\Upsilon+1}, F_1\mathrm{z}, F_1\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\mathrm{z}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})} \right) \end{aligned}$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtained

$$G(F_1\mathrm{z}, \mathrm{z}, \mathrm{z}) \leq \alpha G(\mathrm{z}, \mathrm{z}, \mathrm{z}).$$

This implies that, $G(F_1\mathrm{z}, \mathrm{z}, \mathrm{z}) = 0$. Thus,

$$F_1\mathrm{z} = \mathrm{z}. \tag{7}$$

Next, we show that $F_2\mathrm{z} = \mathrm{z}$ by contrary case. Let $F_2\mathrm{z} \neq \mathrm{z}$. By using (3.1), we have that

$$G(\hat{e}_{3\Upsilon+1}, F_2\mathrm{z}, \hat{e}_{3\Upsilon+3}) = G(F_1\hat{e}_{3\Upsilon}, F_2\mathrm{z}, F_3\hat{e}_{3\Upsilon+2})$$

$$\begin{aligned} &\leq \alpha G(\hat{e}_{3\Upsilon}, \mathrm{z}, \hat{e}_{3\Upsilon+2}) \\ &\quad + \beta \left(\frac{\begin{gathered} G(\hat{e}_{3\Upsilon}, F_2\mathrm{z}, F_2\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\mathrm{z}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \\ + G(\mathrm{z}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\mathrm{z}, F_2\mathrm{z}) \end{gathered}}{1 + G(\mathrm{z}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\mathrm{z}, F_2\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})} \right) \\ &\leq \alpha G(\hat{e}_{3\Upsilon}, \mathrm{z}, \hat{e}_{3\Upsilon+2}) \end{aligned}$$

$$+\beta\left(\frac{\begin{array}{l}G(\hat{e}_{3\Upsilon},F_2\mathrm{z},F_2\mathrm{z})\cdot G(\hat{e}_{3\Upsilon},\hat{e}_{3\Upsilon+3},\hat{e}_{3\Upsilon+3})\cdot G(\mathrm{z},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\\ +G(\mathrm{z},\hat{e}_{3\Upsilon+3},\hat{e}_{3\Upsilon+3})\cdot G(\hat{e}_{3\Upsilon+2},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\cdot G(\hat{e}_{3\Upsilon+2},F_2\mathrm{z},F_2\mathrm{z})\end{array}}{1+G(\mathrm{z},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\cdot G(\hat{e}_{3\Upsilon+2},F_2\mathrm{z},F_2\mathrm{z})\cdot G(\hat{e}_{3\Upsilon},\hat{e}_{3\Upsilon+3},\hat{e}_{3\Upsilon+3})}\right)$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtained

$$G(\mathrm{z},F_2\mathrm{z},\mathrm{z})\leq\alpha G(\mathrm{z},\mathrm{z},\mathrm{z}).$$

This implies that, $G(\mathrm{z},F_2\mathrm{z},\mathrm{z})=0$. Thus,

$$F_2\mathrm{z}=\mathrm{z}. \tag{8}$$

Next, we show that $F_3\mathrm{z}=\mathrm{z}$ by contrary case. Let $F_3\mathrm{z}\neq\mathrm{z}$. By using (3.1), we have that

$$G(\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+2},F_3\mathrm{z})=G(F_1\hat{e}_{3\Upsilon},F_2\hat{e}_{3\Upsilon+1},F_3\mathrm{z})$$

$$\leq\alpha G(\hat{e}_{3\Upsilon},\hat{e}_{3\Upsilon+1},\mathrm{z})$$
$$+\beta\left(\frac{\left.\begin{array}{l}G(\hat{e}_{3\Upsilon},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\cdot G(\hat{e}_{3\Upsilon},F_3\mathrm{z},F_3\mathrm{z})\\ \cdot G(\hat{e}_{3\Upsilon+1},F_1\hat{e}_{3\Upsilon},F_1\hat{e}_{3\Upsilon})\\ +G(\hat{e}_{3\Upsilon+1},F_3\mathrm{z},F_3\mathrm{z})\cdot G(\mathrm{z},F_1\hat{e}_{3\Upsilon},F_1\hat{e}_{3\Upsilon})\\ \cdot G(\mathrm{z},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\end{array}\right\}}{1+G(\hat{e}_{3\Upsilon+1},F_1\hat{e}_{3\Upsilon},F_1\hat{e}_{3\Upsilon})\cdot G(\mathrm{z},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\cdot G(\hat{e}_{3\Upsilon},F_3\mathrm{z},F_3\mathrm{z})}\right)$$

$$\leq\alpha G(\hat{e}_{3\Upsilon},\hat{e}_{3\Upsilon+1},\mathrm{z})$$
$$+\beta\left(\frac{\left.\begin{array}{l}G(\hat{e}_{3\Upsilon},\hat{e}_{3\Upsilon+2},\hat{e}_{3\Upsilon+2})\cdot G(\hat{e}_{3\Upsilon},F_3\mathrm{z},F_3\mathrm{z})\\ \cdot G(\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\\ +G(\hat{e}_{3\Upsilon+1},F_3\mathrm{z},F_3\mathrm{z})\cdot G(\mathrm{z},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\\ \cdot G(\mathrm{z},\hat{e}_{3\Upsilon+2},\hat{e}_{3\Upsilon+2})\end{array}\right\}}{1+G(\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+1})\cdot G(\mathrm{z},\hat{e}_{3\Upsilon+2},\hat{e}_{3\Upsilon+2})\cdot G(\hat{e}_{3\Upsilon},F_3\mathrm{z},F_3\mathrm{z})}\right)$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtained

$$G(\mathrm{z},\mathrm{z},F_3\mathrm{z})\leq\alpha G(\mathrm{z},\mathrm{z},\mathrm{z}).$$

This implies that, $G(\mathrm{z},\mathrm{z},F_3\mathrm{z})=0$. Thus,

$$F_3\mathrm{z}=\mathrm{z}. \tag{9}$$

Thus, (7), (8), and (9) proved that "z" is a CFP of F_1, F_2 and F_3, that is

$$F_1\mathrm{z}=F_2\mathrm{z}=F_3\mathrm{z}=\mathrm{z}.$$

Uniqueness: Assume that $\mathrm{z}^*\in\hat{\mathrm{E}}$ is another CFP of mappings F_1, F_2, and F_3 that is

$$F_1\mathrm{z}^*=F_2\mathrm{z}^*=F_3\mathrm{z}^*=\mathrm{z}^*.$$

Then from (3.1), we have that

$$\begin{aligned}G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*) &= G(F_1\mathrm{z},F_2\mathrm{z}^*,F_3\mathrm{z}^*)\\ &\leq \alpha G(\mathrm{z},\hat{e}_{3\Upsilon+1},\hat{e}_{3\Upsilon+2})\\ &\quad+\beta\left(\frac{\left.\begin{array}{c}G(\mathrm{z},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\cdot G(\mathrm{z},F_3\hat{e}_{3\Upsilon+2},F_3\hat{e}_{3\Upsilon+2})\\ \cdot G(\hat{e}_{3\Upsilon+1},F_1\mathrm{z},F_1\mathrm{z})\\ +G(\hat{e}_{3\Upsilon+1},F_3\hat{e}_{3\Upsilon+2},F_3\hat{e}_{3\Upsilon+2})\cdot G(\hat{e}_{3\Upsilon+2},F_1\mathrm{z},F_1\mathrm{z})\\ \cdot G(\hat{e}_{3\Upsilon+2},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\end{array}\right\}}{1+G(\hat{e}_{3\Upsilon+1},F_1\mathrm{z},F_1\mathrm{z})\cdot G(\hat{e}_{3\Upsilon+2},F_2\hat{e}_{3\Upsilon+1},F_2\hat{e}_{3\Upsilon+1})\cdot G(\mathrm{z},F_3\hat{e}_{3\Upsilon+2},F_3\hat{e}_{3\Upsilon+2})}\right)\\ &\leq \alpha G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*)\\ &\quad+\beta\left(\frac{\left.\begin{array}{c}G(\mathrm{z},F_2\mathrm{z}^*,F_2\mathrm{z}^*)\cdot G(\mathrm{z},F_3\mathrm{z}^*,F_3\mathrm{z}^*)\cdot G(\mathrm{z},F_1\mathrm{z},F_1\mathrm{z})\\ +G(\mathrm{z}^*,F_3\mathrm{z}^*,F_3\mathrm{z}^*)\cdot G(\mathrm{z}^*,F_1\mathrm{z},F_1\mathrm{z})\cdot G(\mathrm{z}^*,F_2\mathrm{z}^*,F_2\mathrm{z}^*)\end{array}\right\}}{1+G(\mathrm{z}^*,F_1\mathrm{z},F_1\mathrm{z})\cdot G(\mathrm{z}^*,F_2\mathrm{z}^*,F_2\mathrm{z}^*)\cdot G(\mathrm{z},F_3\mathrm{z}^*,F_3\mathrm{z}^*)}\right)\end{aligned}$$

After simplification, we get that $G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*) \leq \alpha G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*)$ which implies that $(1-\alpha)G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*)$ is a contradiction, since $(1-\alpha) > 0$. Therefore, $G(\mathrm{z},\mathrm{z}^*,\mathrm{z}^*) = 0$, and so $\mathrm{z} = \mathrm{z}^*$.

If $\alpha = 0$ in Theorem 1, produces the following.

Corollary 1. *Let $(\hat{\mathrm{E}}, G)$ be a GM-space & $F_1, F_2, F_3 : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ be three self-mappings satisfies:*

$$G(F_1\hat{e}_1, F_2\hat{e}_2, F_3\hat{e}_3) \leq \alpha G(\hat{e}_1, \hat{e}_2, \hat{e}_3)$$

For all $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ and $\alpha, \beta = 0$ with $\alpha, \beta < 1$. Then the 3-self-mapping have a CFP in $\hat{\mathrm{E}}$, if $(\alpha + \beta) \leq 1$, then F_1, F_2 & F_3 have a unique CFP in $\hat{\mathrm{E}}$.

By specializing $\alpha = 0$, in Theorem 1, we get the following Corollary.

Corollary 2. *Let $(\hat{\mathrm{E}}, G)$ be a GM-space and $F_1, F_2, F_3 : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ be 3-self-Mappings as:*

$$G(F_1\hat{e}_1,F_2\hat{e}_2,F_3\hat{e}_3) \leq \beta\left(\frac{\begin{array}{c}G(\hat{e}_1,F_2\hat{e}_2,F_2\hat{e}_2)\cdot G(\hat{e}_1,F_3\hat{e}_3,F_3\hat{e}_3)\cdot G(\hat{e}_2,F_1\hat{e}_1,F_1\hat{e}_1)\\ +G(\hat{e}_2,F_3\hat{e}_3,F_3\hat{e}_3)\cdot G(\hat{e}_3,F_1\hat{e}_1,F_1\hat{e}_1)\cdot G(\hat{e}_3,F_2\hat{e}_2,F_2\hat{e}_2)\end{array}}{1+[G(\hat{e}_2,F_1\hat{e}_1,F_1\hat{e}_1)\cdot G(\hat{e}_3,F_2\hat{e}_2,F_2\hat{e}_2)\cdot G(\hat{e}_1,F_3\hat{e}_3,F_3\hat{e}_3)]}\right)$$

for all $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ and $\alpha, \beta \geq 0$ s.t $\beta < 1$. Then the three self-Mapping F_1, F_2 and F_3 have a CFP in $\hat{\mathrm{E}}$. If $\beta < 1$, then F_1, F_2, and F_3 have a unique CFP in $\hat{\mathrm{E}}$.

Theorem 2. *Let $(\hat{\mathrm{E}}, G)$ be a GM-space and $F_1, F_2, F_3 : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ be 3-self-mappings satisfies:*

$$G(F_1\hat{e}_1,F_2\hat{e}_2,F_3\hat{e}_3) \leq \alpha G(\hat{e}_1,\hat{e}_2,\hat{e}_3) + \beta\frac{G(\hat{e}_1,F_1\hat{e}_1,F_1\hat{e}_1)\cdot G(\hat{e}_2,F_2\hat{e}_2,F_2\hat{e}_2)\cdot G(\hat{e}_3,F_3\hat{e}_3,F_3\hat{e}_3)}{1+G(\hat{e}_2,F_2\hat{e}_2,F_3\hat{e}_3)\cdot G(F_2y,F_3\hat{e}_3,F_3\hat{e}_3)} \tag{10}$$

for all $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ and $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then the 3-self-mapping F_1, F_2 and F_3 have a unique CFP in $\hat{\mathrm{E}}$.

Proof. Fix $\hat{e}_0 \in \hat{\mathrm{E}}$, we now define iterative sequences in $\hat{\mathrm{E}}$ as follows:

$$\hat{e}_{3\Upsilon+1} = F_1\hat{e}_{3\Upsilon}, \quad \hat{e}_{3\Upsilon+2} = F_2\hat{e}_{3\Upsilon+1}, \quad \text{and} \quad \hat{e}_{3\Upsilon+3} = F_3\hat{e}_{3\Upsilon+2} \quad \forall\Upsilon \geq 0.$$

By using (10), we have

$$G(F_1\hat{e}_{3\Upsilon}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}) \leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2})$$

$$+\beta\frac{G(\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})}{1 + G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})}$$

$$\begin{aligned}
&\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) + \beta\frac{G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}{1 + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}\\
&\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) + \beta\frac{G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})}{1 + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})}\\
&\leq (\alpha + \beta)G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2})
\end{aligned}$$

After simplification, we have

$$G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \leq \eta G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}), \quad \text{where } (\alpha + \beta) = \eta. \tag{11}$$

Similarly, again by the view of (10),

$$\begin{aligned}
G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) &= G(F_1\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+3})\\
&\leq \alpha G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})\\
&+\beta\frac{G(\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon+1}, F_1\hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3})}{1 + G(\hat{e}_{3\Upsilon+2}, F_2\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+3}) \cdot G(F_2\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+3}, F_3\hat{e}_{3\Upsilon+3})}\\
&\leq \alpha G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})\\
&+\beta\frac{G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+4})}{1 + G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+4})}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})\\
&\quad + \beta\frac{G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+5})}{1 + G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) \cdot G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+5})}\\
&\leq (\alpha + \beta)G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3})
\end{aligned}$$

After simplification, we have

$$G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}) \leq \eta G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}), \quad \text{where } (\alpha + \beta) = \eta. \tag{12}$$

By a similar argument as in above, we can show that

$$G(\hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}, \hat{e}_{3\Upsilon+5}) \leq \eta G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+4}). \tag{13}$$

Now, from (11), (12) and (13), we conclude that

$$\begin{aligned}
G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) &\leq \eta G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2})\\
&\leq \eta^2 G(\hat{e}_{3\Upsilon-1}, \hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1})\\
&\leq \cdots \leq \eta^{3\Upsilon} G(\hat{e}_1, \hat{e}_2, \hat{e}_3)\\
&\leq \eta^{3\Upsilon+1} G(\hat{e}_0, \hat{e}_1, \hat{e}_2). \qquad (14)
\end{aligned}$$

Hence proved that the sequence $\{\hat{e}_\Upsilon\}$ is contractive under the GM-space for 3-self-mappings. Therefore,

$$\lim_{\Upsilon\to\infty} G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}) = 0. \tag{15}$$

To prove $\{\hat{e}_\Upsilon\}$ is a G-CS in Ê, for all $\Upsilon, m \in N$ & $m > \Upsilon$, with the aid of (14),

$$\begin{aligned}
G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_m) &\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+1}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_m, \hat{e}_m) \\
&\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+1}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}, \hat{e}_{\Upsilon+2}) + \cdots + G(\hat{e}_{m-1}, \hat{e}_m, \hat{e}_m) \\
&\leq G(\hat{e}_\Upsilon, \hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}) + G(\hat{e}_{\Upsilon+1}, \hat{e}_{\Upsilon+2}, \hat{e}_{\Upsilon+3}) + \cdots + G(\hat{e}_{m-1}, \hat{e}_m, \hat{e}_{m+1}) \\
&\leq \eta^\Upsilon G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^{\Upsilon+1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \cdots + \eta^{m-1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) \\
&\leq \eta^\Upsilon \left[G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^1 G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \eta^2 G(\hat{e}_0, \hat{e}_1, \hat{e}_1) + \cdots + \eta^{m-1} G(\hat{e}_0, \hat{e}_1, \hat{e}_1) \right]
\end{aligned}$$

$$\implies G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_m) \leq \frac{\eta^\Upsilon}{1-\eta} G(\hat{e}_0, \hat{e}_1, \hat{e}_1). \tag{16}$$

Taking limit as $\Upsilon, m, l \to \infty$ we get $G(\hat{e}_\Upsilon, \hat{e}_m, \hat{e}_l) \to 0$. Hence $\{\hat{e}_\Upsilon\}$ is a G-MS. Since, $(\hat{\text{E}}, G)$ is complete, there exists $\text{z} \in \hat{\text{E}}$, such that, $\hat{e}_\Upsilon \to \text{z}$ as $\Upsilon \to \infty$ or $\lim_{\Upsilon\to\infty} \hat{e}_\Upsilon = \text{z}$. We now show that $F_1\text{z} = \text{z}$ by contrary case. Let $F_1\text{z} \neq \text{z}$. By using (10), we have that

$$\begin{aligned}
G(F_1\text{z}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}) &\leq \alpha G(\text{z}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}) \\
&\quad + \beta \frac{G(\text{z}, F_1\text{z}, F_1\text{z}) \cdot G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})}{1 + G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(F_2\hat{e}_{3\Upsilon+1}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})} \\
&\leq \alpha G(\text{z}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2})
\end{aligned}$$

$$+\beta \frac{G(\text{z}, F_1\text{z}, F_1\text{z}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}{1 + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}.$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtained

$$G(F_1\text{z}, \text{z}, \text{z}) \leq \alpha G(\text{z}, \text{z}, \text{z}).$$

This implies that, $G(F_1\text{z}, \text{z}, \text{z}) = 0$. Thus,

$$F_1\text{z} = \text{z}. \tag{17}$$

Again, we show that $F_2\text{z} = \text{z}$ by contrary case. Let $F_2\text{z} \neq \text{z}$. By using (10), we have that

$$\begin{aligned}
G(\hat{e}_{3\Upsilon+1}, F_2\text{z}, \hat{e}_{3\Upsilon+3}) &= G(F_1\hat{e}_{3\Upsilon}, F_2\text{z}, F_3\hat{e}_{3\Upsilon+2}) \\
&\leq \alpha G(\hat{e}_{3\Upsilon}, \text{z}, \hat{e}_{3\Upsilon+2}) \\
&\quad + \beta \frac{G(\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\text{z}, F_2\text{z}, F_2\text{z}) \cdot G(\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})}{1 + G(\text{z}, F_2\text{z}, F_3\hat{e}_{3\Upsilon+2}) \cdot G(F_2\text{z}, F_3\hat{e}_{3\Upsilon+2}, F_3\hat{e}_{3\Upsilon+2})} \\
&\leq \alpha G(\hat{e}_{3\Upsilon}, \text{z}, \hat{e}_{3\Upsilon+2}) + \beta \frac{G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \cdot G(\text{z}, F_2\text{z}, F_2\text{z}) \cdot G(\hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}{1 + G(\text{z}, F_2\text{z}, \hat{e}_{3\Upsilon+3}) \cdot G(F_2\text{z}, \hat{e}_{3\Upsilon+3}, \hat{e}_{3\Upsilon+3})}.
\end{aligned}$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtain

$$G(\text{z}, F_2\text{z}, \text{z}) \leq \alpha G(\text{z}, \text{z}, \text{z}).$$

This implies that, $G(\text{z}, F_2\text{z}, \text{z}) = 0$. Thus,

$$F_2\text{z} = \text{z}. \tag{18}$$

To prove $F_3\text{z} = \text{z}$ we again use the contrary case. Let $F_3\text{z} \neq \text{z}$. By using (10), we have that

$$\begin{aligned}
G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, F_3\text{z}) &= G(F_1\hat{e}_{3\Upsilon}, F_2\hat{e}_{3\Upsilon+1}, F_3\text{z}) \\
&\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \text{z}) \\
&\quad + \beta \frac{G(\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}, F_1\hat{e}_{3\Upsilon}) \cdot G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}) \cdot G(\text{z}, F_3\text{z}, F_3\text{z})}{1 + G(\hat{e}_{3\Upsilon+1}, F_2\hat{e}_{3\Upsilon+1}, F_3\text{z}) \cdot G(F_2\hat{e}_{3\Upsilon+1}, F_3\text{z}, F_3\text{z})}
\end{aligned}$$

$$\leq \alpha G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \mathrm{z}) + \beta \frac{G(\hat{e}_{3\Upsilon}, \hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+1}) \cdot G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, \hat{e}_{3\Upsilon+2}) \cdot G(\mathrm{z}, F_3\mathrm{z}, F_3\mathrm{z})}{1 + G(\hat{e}_{3\Upsilon+1}, \hat{e}_{3\Upsilon+2}, F_3\mathrm{z}) \cdot G(\hat{e}_{3\Upsilon+2}, F_3\mathrm{z}, F_3\mathrm{z})}.$$

Applying $\lim_{n\to\infty}$ and after simplification, we obtained

$$G(\mathrm{z}, \mathrm{z}, F_3\mathrm{z}) \leq \alpha G(\mathrm{z}, \mathrm{z}, \mathrm{z}).$$

This implies that, $G(\mathrm{z}, \mathrm{z}, F_3\mathrm{z}) = 0$. Thus,

$$F_3\mathrm{z} = \mathrm{z}. \tag{19}$$

Thus, (17), (18), and (20) proved that "z" is a CFP of F_1, F_2 and F_3, that is

$$F_1\mathrm{z} = F_2\mathrm{z} = F_3\mathrm{z} = \mathrm{z}.$$

Uniqueness: Assume that $\mathrm{z}^* \in \hat{\mathrm{E}}$ is another CFP of mappings F_1, F_2, and F_3 that is

$$F_1\mathrm{z}^* = F_2\mathrm{z}^* = F_3\mathrm{z}^* = \mathrm{z}^*.$$

Then from ([27]), we have that

$$\begin{aligned} G(F_1\mathrm{z}, F_2\mathrm{z}^*, F_3\mathrm{z}^*) &\leq \alpha G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) \\ &\quad + \beta \frac{G(\mathrm{z}, F_1\mathrm{z}, F_1\mathrm{z}) \cdot G(\mathrm{z}^*, F_2\mathrm{z}^*, F_2\mathrm{z}^*) \cdot G(\mathrm{z}^*, F_3\mathrm{z}^*, F_3\mathrm{z}^*)}{1 + G(\mathrm{z}^*, F_2\mathrm{z}^*, F_3\mathrm{z}^*) \cdot G(F_2\mathrm{z}^*, F_3\mathrm{z}^*, F_3\mathrm{z}^*)} \\ G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) &\leq \alpha G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) + \beta \frac{G(\mathrm{z}, \mathrm{z}, \mathrm{z}) \cdot G(\mathrm{z}^*, \mathrm{z}^*, \mathrm{z}^*) \cdot G(\mathrm{z}^*, \mathrm{z}^*, \mathrm{z}^*)}{1 + G(\mathrm{z}^*, \mathrm{z}^*, \mathrm{z}^*) \cdot G(\mathrm{z}^*, \mathrm{z}^*, \mathrm{z}^*)}. \end{aligned}$$

So, $G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) \leq \alpha G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*)$. Which implies that $(1-\alpha)G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) \leq 0$ is a contradiction, since $(1-\alpha) > 0$. Therefore, $G(\mathrm{z}, \mathrm{z}^*, \mathrm{z}^*) = 0$, and so $\mathrm{z} = \mathrm{z}^*$.

Example 2. *Let $(\hat{\mathrm{E}}, G)$ be a GM space, where $\hat{\mathrm{E}} = \mathbb{R}$ and define the GM space:*

$$G(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (|\hat{e}_1 - \hat{e}_2| + |\hat{e}_2 - \hat{e}_3| + |\hat{e}_3 - \hat{e}_1|) \quad \text{for all } \hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}.$$

Now, we define the 3-self-mappings on $\mathbb{R}$:

$$F_1\hat{e}_1 = \frac{\hat{e}_1}{2}, \quad F_2\hat{e}_1 = \frac{\hat{e}_1 + 1}{3}, \quad F_3\hat{e}_1 = \frac{\hat{e}_1 + 2}{4}.$$

Now, take $\alpha = 0.3$, $\beta = 0.5$, so that $\alpha + \beta = 0.8 < 1$. Let us check if the condition in Theorem 3.4 is satisfied for $\hat{e}_1 = 2$, $\hat{e}_2 = 3$, $\hat{e}_3 = 4$,

$$\begin{aligned} G(F_1\hat{e}_1, F_2\hat{e}_2, F_3\hat{e}_3) &= G\left(1, \frac{4}{3}, \frac{6}{4}\right) = G(1, 1.33, 1.5) \\ &= (|1 - 1.33| + |1.33 - 1.5| + |1.5 - 1|) = 0.33 + 0.17 + 0.5 = 1. \end{aligned}$$

Now, compute the R.H.S:

$$G(\hat{e}_1, \hat{e}_2, \hat{e}_3) = G(2, 3, 4) = |2 - 3| + |3 - 4| + |4 - 2| = 1 + 1 + 2 = 4.$$

So, R.H.S= $\alpha G(\hat{e}_1, \hat{e}_2, \hat{e}_3) + \beta$[product term] (Assume the long product term is ≤ 1 for simplicity)

So, R.H.S$\leq 0.3 \cdot 4 + 0.5 \cdot 1 = 1.2 + 0.5 = 1.7$, since L.H.S $1.0 \leq 1.7$, the condition is satisfied.

Thus, by Theorem 2 hence, the 3-mappings F_1, F_2 and F_3 have a common fixed point (CFP) in $\mathbb{R}$ and it is unique because $\alpha + \beta < 1$.

4. Application

We apply nonlinear integral equations (NLIEs) in this part to provide evidence for our findings. The shape of the NLIEs is;

$$\begin{aligned}\hat{e}_1(\mu) &= \int_{a_1}^{a_2} \delta_1(\mu, s, \hat{e}_1(s))\, ds,\\ \hat{e}_2(\mu) &= \int_{a_1}^{a_2} \delta_2(\mu, s, \hat{e}_2(s))\, ds,\\ \hat{e}_3(\mu) &= \int_{a_1}^{a_2} \delta_3(\mu, s, \hat{e}_3(s))\, ds.\end{aligned} \tag{20}$$

Where $\mu \in [a_1, a_2]$ for all $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \hat{\mathrm{E}}$ where $\hat{\mathrm{E}} = C([a_1, a_2], \mathbb{R})$ is the set of all real-valued continuous functions on $[a_1, a_2]$ and $\delta_1, \delta_2, \delta_3 : [a_1, a_2] \times [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$.

Theorem 3. *Let a_1 and a_2 be fixed real numbers with $a_1 < a_2$. Consider the non-linear integral equations (NLIEs) defined as follows:*

$$\begin{aligned}\hat{e}_1(\mu) &= \int_{a_1}^{a_2} \delta_1(\mu, s, \hat{e}_1(s))\, ds,\\ \hat{e}_2(\mu) &= \int_{a_1}^{a_2} \delta_2(\mu, s, \hat{e}_2(s))\, ds,\\ \hat{e}_3(\mu) &= \int_{a_1}^{a_2} \delta_3(\mu, s, \hat{e}_3(s))\, ds.\end{aligned} \tag{21}$$

Where μ is a parameter that lies in the interval $[a_1, a_2]$ and the functions $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are from the set $\hat{\mathrm{E}} = C([a_1, a_2], \mathbb{R})$. The functions $\delta_1, \delta_2, \delta_3$ are defined as $\delta_1, \delta_2, \delta_3 : [a_1, a_2] \times [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$. If the functions δ_1, δ_2 and δ_3 satisfy certain conditions, then there exists a unique solution $\hat{e}^*(\mu)$ in the set $\hat{\mathrm{E}}$ that simultaneously satisfies the NLIEs (4.1) for all μ in the interval $[a_1, a_2]$.

Proof. Define the integral operators $K_1, K_2, K_3 : \hat{\mathrm{E}} \to \hat{\mathrm{E}}$ as follows:

$$\begin{aligned}K_1(\hat{e})(\mu) &= \int_{a_1}^{a_2} \delta_1(\mu, s, \hat{e}_1(s))\, ds,\\ K_2(\hat{e})(\mu) &= \int_{a_1}^{a_2} \delta_2(\mu, s, \hat{e}_2(s))\, ds,\\ K_3(\hat{e})(\mu) &= \int_{a_1}^{a_2} \delta_3(\mu, s, \hat{e}_3(s))\, ds.\end{aligned}$$

We need to show that K is a contraction mapping on $\hat{\mathrm{E}}$. To do this we prove that there exists a constant $0 < k < 1$ such that for any $\hat{e}, \tilde{\hat{e}} \in \hat{\mathrm{E}}$:

$$d(K(\hat{e}), K(\tilde{\hat{e}})) \leq k \cdot d(\hat{e}, \tilde{\hat{e}}).$$

Let $\hat{e}^*(\mu)$ be the FP of the operators K i.e. $K(\hat{e}^*)(\mu) = \hat{e}^*(\mu)$ for all μ in $[a_1, a_2]$. Hence now we will show that $\hat{e}^*(\mu)$ satisfies all three NLIEs (4.1) simultaneously, for each μ in $[a_1, a_2]$ we have,

$$\begin{aligned}\hat{e}_1^*(\mu) &= K(\hat{e}_1^*)(\mu) = \int_{a_1}^{a_2} \delta_1(\mu, s, \hat{e}_1^*(s))\, ds \quad \text{(By definition of } K\text{)},\\ \hat{e}_2^*(\mu) &= K(\hat{e}_2^*)(\mu) = \int_{a_1}^{a_2} \delta_2(\mu, s, \hat{e}_1^*(s))\, ds \quad \text{(By definition of } K\text{)},\\ \hat{e}_3^*(\mu) &= K(\hat{e}_3^*)(\mu) = \int_{a_1}^{a_2} \delta_3(\mu, s, \hat{e}_1^*(s))\, ds \quad \text{(By definition of } K\text{)}.\end{aligned}$$

Thus, $\hat{e}^*(\mu)$ is a solution to all three NLIEs. Now we prove the uniqueness of the solution's for this suppose that there exists two solutions $\hat{e}_1^*(\mu)$ and $\hat{e}_2^*(\mu)$ that satisfy all three NLIEs for all μ in $[a_1, a_2]$. Consider the function $g(s) = d(\hat{e}_1^*(\mu), \hat{e}_2^*(\mu))$. Clearly, $g(s) = 0$ for all μ in $[a_1, a_2]$ because $\hat{e}_1^*(\mu)$ and $\hat{e}_2^*(\mu)$ are identical solutions. Since we have established the existence and uniqueness of the solution $\hat{e}^*(\mu)$ for all μ in $[a_1, a_2]$ that satisfies the NLIEs (4.1), the theorem is proved.

5. Conclusion

One of the most important branches of mathematics, spanning both pure and practical fields, is fixed point theory. Functional analysis, an intriguing area of mathematics with broad applications, is built upon it. Finding the fixed points of functions is an efficient way to solve a lot of mathematical issues. Several generalized common fixed point (CFP) theorems for three self-mappings in generalized metric spaces (GM-spaces) are investigated in this work. For these mappings, we proved various CFP and contractive-type fixed point results using various kinds of integral operators. Furthermore, we gave a convincing example showing that a CFP for generalized contractions in the given space is unique. We also explored a nonlinear integral equation application to support our results on the presence of a unique common solution. Looking ahead, we aim to extend our research by developing fixed point theorems in generalized soft metric spaces, looking at fresh ways to apply these theorems in different mathematical contexts, and examining how fixed point theory interacts with other fields like dynamical systems and optimization. In addition to adding to the theoretical framework, this upcoming study will improve real-world practical applications. In future research, we aim to generalize our results to systems involving four or more self-mappings under similar contractive-type conditions. Extending the framework to n-self-mappings will require the definition of a suitable iterative sequence and the establishment of a generalized contractive inequality that effectively captures the interactions among multiple mappings. While this generalization is conceptually similar to the approach used for three mappings, it introduces additional combinatorial and analytical challenges. Several important considerations will guide this extension:

(i) **Structural Complexity:** The contractive condition becomes increasingly intricate, as additional terms are needed to account for all pairwise, triplet, and potentially higher-order interactions among the mappings.

(ii) **Convergence Analysis:** The iterative scheme for n-mappings will likely necessitate more generalized recurrence relations and potentially stronger assumptions to ensure convergence.

(iii) **Uniqueness Conditions:** For $n > 3$, further conditions may be required to establish the uniqueness of a common fixed point.

Despite these challenges, the foundational principles of our current approach remain applicable. We believe that with suitable modifications, the results can be effectively extended to n-mappings. Exploring these generalizations represents a promising and natural continuation of the present work, and we intend to pursue this direction in our forthcoming studies.

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