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Key Characteristics of Quadri-Partitioned Neutrosophic Riemann integrals and Quadri-Partitioned Neutrosophic Soft Topological Spaces

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Abstract. Neutrosophic Set Theory (NST) is an extension of Intuitionistic Fuzzy Set Theory (IFST). While IFST relies on two possibilities for the complete depiction of a set, neutrosophic set theory familiarizes an additional third possibility, thus providing a more delicate representation. Our research builds upon a further extension of neutrosophic set theory, known as quadri-partitioned neutrosophic set theory (QPNST), which brings in a fourth possibility for a more detailed and complete description of sets. In this study, we define the Riemann Integral Theory (RIT) within the framework of QPNST. This opens new doors for probing the properties and characteristics of the Riemann integral in this extended context. One strategic concept that arises in this work is the level cut. In QPNST, the level cut is defined as a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, which represents the different possibilities inherent in the theory. The notion of the Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPNRIT) is explored numerically in this study, and the results are systematically presented in tabular form. This numerical approach sheds light on the integral's properties and facilitates the understanding of its behavior within the QPNST framework. This study explores quadripartitioned neutrosophic soft topological spaces, extending neutrosophic set theory (NST), which incorporates three membership values: true, false, and indeterminacy. The study introduces new concepts such as QPNS semi-open, QPNS pre-open, and QPNS $*_b$ open sets, and builds on these to define QPNS closure, exterior, boundary, and interior. A key development is the definition of a quadripartitioned neutrosophic soft base, which plays a central role in these topological structures. The paper also explores the concept of a quadripartitioned neutrosophic soft sub-base and discusses local bases, as well as the first- and second-countability axioms. The study further examines hereditary properties of these spaces, distinguishing between inherited and non-inherited properties. Key results include that a quadripartitioned neutrosophic soft subspace of a first-countable space is also first-countable, and a second-countable subspace of a second-countable space remains second-countable. It also highlights the relationship between second countability and separability in these spaces, asserting that a second-countable quadripartitioned neutrosophic soft space is separable, though the converse is not always true. This work lays the foundation for further research in neutrosophic soft topologies.

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1. Introduction

Zadeh [1] introduced the concept of fuzzy set theory (FST), where a set is characterized by a membership function. The foundational operations—union, intersection, complement, and convexity—were established. Additionally, the separation theorem for convex fuzzy sets was formulated. Zadeh [2] introduced the concept of linguistic variables and their applications within the context of FST. These variables have found widespread use in various fields, including medicine, law, psychology, economics, and others. Zadeh [3] generalized the ideas presented in [2] and introduced the notion of fuzzy variables. Zadeh [4] applied linguistic variables to approximate reasoning. Zadeh [5] proposed the generalized theory of uncertainty, which enables a broader perspective on uncertainty.

Ye [6] discussed the concept of simplified neutrosophic sets (SNSs), which are a sub-class of neutrosophic sets, and explored several aggregation operators. Furthermore, decision-making techniques were developed. Liu and Luo [7] introduced the notion of multi-attribute group decision-making (MAGDM). Atanassov [8] introduced intuitionistic fuzzy sets (IFS), an extension of fuzzy sets, and described the basic operations associated with this theory, as well as the development of topological operators. Atanassov and Gargov [9] advanced the concept of interval-valued IFSs, utilizing intervals in their formulation. Smarandache [10, 11] explored various techniques, including neutrosophic probability and neutrosophic logic. Ye [12] examined applications of neutrosophic sets in multi-criteria decision-making problems. Techniques based on weighted distance measures and generalized hybrid weighted average operators, employing neutrosophic hesitant sets, were discussed in [13, 14], along with their applications in multiple-attribute decision-making.

Li and Luo [15] introduced the super strong theory, known as soft set theory (SST), and defined its fundamental operations. Maji et al. [16] applied SST to decision-making problems, redefining key operations and illustrating their validity with examples. Molodtsov [17] established a connection between soft sets and fuzzy sets, leading to the development of a hybrid theory known as fuzzy soft set theory (FSST). Wang et al. [18] introduced the concept of HFSS and effectively applied it in multi-criteria decision-making problems. Pei and Miao [19] bridged soft sets with information theory, discussing their practical applications. John [20–22] explored various structures based on SST, offering examples and applications across different areas of mathematics.

Al-Shami et al. [23] proposed a new structure known as Menger space. Al-Shami et al. [24] investigated the structure of infra soft topological spaces (ISTS) with respect to crisp points. Al-Shami et al. [25] discussed weak forms of soft separation axioms and fixed points. Al-Shami et al. [26] defined concepts of connectedness and local connectedness within the context of infra soft topological spaces, presenting several results related to this strong structure. Al-Shami [27] discussed quantum mechanics (QM) in the framework of ISTS. Further results on ISTS and soft topological spaces (STS) were examined in [28, 29]. Ozturk et al. [30] introduced new operators in Neutrosophic Soft Topological Spaces (NSTS), leading to novel approaches to several existing results, with numerous examples provided for clarification. Ahmad et al. [31] discussed Irreversible k-Threshold conversion number for some graph products and neutrosophic graphs. Hatamleh et al. [32] studied complex tangent trigonometric approach applied to q-rung fuzzy set using weighted averaging, geometric operators and its extension. Hatamleh et al. [33] studied different weighted operators such as generalized averaging and generalized geometric based on trigonometric P-rung interval-valued approach and in addition to this some examples were given for clear understanding. Shihadeh et al. [34] discussed algebraic structures towards different intuitionistic fuzzy ideals and its characterization of an ordered ternary semigroup. Hatamleh et al. [35, 36] studied operators via weighted averaging and geometric approach using trigonometric neutrosophic interval-valued set and its extension and characterization of interaction aggregating operators setting interval-valued Pythagorean neutrosophic set. Hatamleh et al. [37] discussed applications of complex interval-valued picture fuzzy soft relations. El-Sheikh and Abd El-Latif [38] discussed decompositions of some types of supra soft sets and soft continuity and cited some excellent examples for clear understanding of the concept. Abd El-Latif [39] discussed soft supra compactness in supra soft topological spaces. Abd El-Latif and Hosny [40] discussed the eye-catching concept of soft separation axioms and provided examples. Abd El-Latif and Hosny discussed some more structures in [41, 42].

1.1. Research Gap

While neutrosophic set theory (NST) extends intuitionistic fuzzy set theory (IFST) by introducing a third possibility for uncertainty representation, the extension to quadri-partitioned neutrosophic set theory (QPNST), which introduces a fourth possibility, remains underexplored. Additionally, the application of QPNST to classical mathematical theories, such as the Riemann Integral, has not been rigorously studied. There is a lack of formal mathematical analysis and numerical exploration of the properties of the Riemann Integral within the QPNST framework, particularly with respect to the behavior of the quadri-partitioned neutrosophic Riemann integral theory (QPNRIT). This research gap hinders a deeper understanding of how higher-order uncertainty models can be integrated into classical mathematical analysis.

1.2. Motivation

The research on neutrosophic Riemann integration and its properties [43], which delves into neutrosophic Riemann set theory, provided a foundational understanding of integrating neutrosophic functions. This exploration highlighted the limitations and potential for further development in the field. As a result, it motivated the development of the theory of quadri-partitioned neutrosophic Riemann integrals, which extends the concept by introducing a fourth partition. This new extension allows for a more comprehensive representation of uncertainty, offering greater flexibility and accuracy in modeling complex systems with multiple layers of uncertainty.

1.3. Novelty

The novelty of neutrosophic set theory (NST) lies in its extension of intuitionistic fuzzy set theory (IFST). While IFST provides two possibilities for a set's complete representation, neutrosophic set theory introduces an additional third possibility, allowing for a more refined and nuanced depiction of sets. Building upon this, our research delves into an even further extension of NST, called quadri-partitioned neutrosophic set theory (QPNST), which incorporates a fourth possibility. This additional possibility enhances the level of detail and completeness in the representation of sets.

In this study, we define the Riemann integral theory (RIT) within the context of QPNST, offering a novel way to explore the properties and characteristics of the Riemann integral in this expanded framework. A key concept that emerges in this work is the level cut, which in the context of QPNST is represented as a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, encapsulating the various possibilities inherent in the theory. We also explore the quadri-partitioned neutrosophic Riemann integral theory (QPNRIT), applying it numerically and presenting the results in tabular form. This numerical exploration allows us to investigate the behavior of the integral in the QPNST framework, providing a deeper understanding of its properties and showcasing the potential of this extended theory for future mathematical and practical applications.

1.4. Importance of The Study

A significant development in our study is the application of the Riemann Integral Theory (RIT) within the context of QPNST. This extension paves the way for exploring the properties and characteristics of the Riemann integral in a richer and more nuanced mathematical setting.

The concept of the level cut, defined as a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, is critical in this work as it captures the different possibilities within the QPNST framework, offering a unique perspective for analyzing integral properties.

Through the introduction of the quadri-partitioned neutrosophic Riemann integral theory (QPNRIT), our study provides numerical insights into the behavior of integrals within this extended framework. The results are presented systematically in tabular form, enhancing the understanding of the integrals properties and illustrating how it behaves under different conditions of uncertainty. This numerical approach not only enriches the theoretical foundations of QPNST but also holds promise for practical applications in fields such as engineering, decision-making, and data analysis, where complex uncertainties and ambiguities need to be addressed.

1.5. Literature Review

Ozturk et al. [44] explored soft continuous mappings. Gunduz et al. [45] focused on critical structures within STS, particularly separation axioms. Ozturk [46] expanded the analysis of additional structures within STS. Mehmood et al. [47] made significant contributions to Neutrosophic Soft Bounded Topological Spaces (NSBTS) and discussed a comprehensive set of results related to crisp points. Mehmood et al. [48, 49] presented results on Neutrosophic Soft Open Sets (NSOSs). Mehmood et al.[50] introduced the concept of Neutrosophic Soft Quasi-Sets (NSQS), with respect to neutrosophic soft points (NSP). Mehmood et al. [51] provided an in-depth analysis of NSQS, specifically focusing on soft p-open sets. Kim et al. [52] addressed several differential problems and illustrated their solutions with examples. Moi et al. [53] investigated second-order problems within the context of NST, enhancing applicability through well-chosen examples. Shami et al. [54] introduced supra-soft topologically ordered spaces as an extension of soft topologically ordered spaces. They discussed key notions like monotone interior and closure operators, formulated supra-soft separation axioms, and explored the relationships between these concepts and their parametric supra topologies. They also characterized supra p-soft Ti-ordered spaces, supra p-soft regularly ordered spaces, and supra-soft normally ordered spaces. T. M. Al-Shami and Shafei [55] discussed two types of separation axioms in supra-soft topological spaces and provided the best examples for better understanding of the results.

2. Preliminaries

Definition 1. *[43] Let U be the universal set. Then, a single-valued neutrosophic set (SVNS) N over the set U is a neutrosophic set over U, but the truth, indeterminacy, and falsity membership functions are defined as*

$$T_N : U \to [0,1], \quad I_N : U \to [0,1], \quad F_N : U \to [0,1],$$

respectively.

Definition 2. *[43] A neutrosophic set N over the universal set of real numbers R is said to be a neutrosophic number if it satisfies the following conditions:*

1. N is normal, i.e., there exists $\mathrm{X}_0 \in \mathrm{R}$ such that:

$$T_N(\mathrm{X}_0) = 1, \quad I_N(\mathrm{X}_0) = 0, \quad F_N(\mathrm{X}_0) = 0.$$

2. N is convex for the truth membership function $T_N(\mathrm{X})$, i.e.,

$$T_N(\mu\mathrm{X}_1 + (1-\mu)\mathrm{X}_2) \geq \min\left(T_N(\mathrm{X}_1), T_N(\mathrm{X}_2)\right), \quad \forall \mathrm{X}_1, \mathrm{X}_2 \in \mathrm{R}, \quad \mu \in [0,1].$$

3. N is concave for the indeterminacy and falsity membership functions, $I_N(\mathrm{X})$ and $F_N(\mathrm{X})$, respectively:

$$I_N(\mu\mathrm{X}_1 + (1-\mu)\mathrm{X}_2) \geq \max\left(I_N(\mathrm{X}_1), I_N(\mathrm{X}_2)\right),$$

$$F_N(\mu X_1 + (1-\mu)X_2) \geq \max\big(F_N(X_1), F_N(X_2)\big),$$

for all $X_1, X_2 \in R$ *and* $\mu \in [0,1]$.

Definition 3. *[43] A truth, indeterminacy, and falsity membership function describes an interval neutrosophic set N over the universal set U and is given as $T_N(X)$, $I_N(X)$, and $F_N(X)$, respectively. For all $X \in U$, we have:*

$$T_N(X) = [\inf T_N(X), \sup T_N(X)],$$

$$I_N(X) = [\inf I_N(X), \sup I_N(X)],$$

$$F_N(X) = [\inf F_N(X), \sup F_N(X)] \subseteq [0,1], \quad \forall X \in U.$$

Here, we focus on the sub-unitary range $[0,1]$. *Let*

$$\tilde{n} = \langle [T^L_{\tilde{n}}, T^U_{\tilde{n}}], [I^L_{\tilde{n}}, I^U_{\tilde{n}}], [F^L_{\tilde{n}}, F^U_{\tilde{n}}] \rangle$$

indicate a neutrosophic interval number (INN), where $T^L_{\tilde{n}}, T^U_{\tilde{n}}, I^L_{\tilde{n}}, I^U_{\tilde{n}}, F^L_{\tilde{n}}$, and $F^U_{\tilde{n}}$ denote:

$$\inf T_{\tilde{N}}(X), \quad \sup T_{\tilde{N}}(X), \quad \inf I_{\tilde{N}}(X), \quad \sup I_{\tilde{N}}(X), \quad \inf F_{\tilde{N}}(X), \quad \sup F_{\tilde{N}}(X),$$

respectively.

3. Single Valued Quadri-partitioned Neutrosophic Set

This section introduces the Single Valued Quadri-Partitioned Neutrosophic Set (SVQNS), characterized by membership functions representing absolute truth, relative truth, absolute falsehood, and relative falsehood. We define the inclusion of one SVQNS within another based on membership values. The union and intersections are also defined in this study. The Quadri Single Valued Neutrosophic Number (QSVNN) is also proposed, a unique neutrosophic set on the real number line R, incorporating truth, indeterminacy, hesitation, and falsity membership functions. Additionally, we define the cut of a neutrosophic set, providing a framework for analyzing subsets.

Definition 4. *A Single Valued Quadri-Partitioned Neutrosophic Set (SVQNS) A on the universe of discourse X is characterized by the following membership functions:*

- *Absolute true membership function: $T_A(x)$,*
- *Relative true membership function: $ReT_A(x)$,*
- *Absolute false membership function: $F_A(x)$,*
- *Relative false membership function: $ReF_A(x)$.*

These functions are subsets of $]0,1[$, *i.e.,*

$$T_A(x) : X \to]0,1[, \quad ReT_A(x) : X \to]0,1[, \quad F_A(x) : X \to]0,1[, \quad ReF_A(x) : X \to]0,1[$$

with the condition:

$$0 \leq \sup T_A(x) + \sup ReT_A(x) + \sup F_A(x) + \sup ReF_A(x) \leq 4.$$

Thus, the SVQNS can be represented as:

$$A = \{\langle x, T_A(x), ReT_A(x), ReF_A(x), F_A(x)\rangle : x \in X\}.$$

Definition 5. *A single valued quadri-partitioned neutrosophic set A is contained in another single valued quadri-partitioned neutrosophic set B if the following conditions hold for each $\mathrm{X} \in X$:*

$$T_A(\mathrm{X}) \leq T_B(\mathrm{X}), \quad ReT_A(\mathrm{X}) \leq ReT_B(\mathrm{X}), \quad F_A(\mathrm{X}) \geq F_B(\mathrm{X}), \quad ReF_A(\mathrm{X}) \geq ReF_B(\mathrm{X}).$$

Equality of two SVQNSs is defined as:

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

The complement of A, denoted as A^c, is given by:

$$A^c = \{\langle \mathrm{X}, F_A(\mathrm{X}), ReF_A(\mathrm{X}), T_A(\mathrm{X}), ReT_A(\mathrm{X})\rangle : \mathrm{X} \in X\}.$$

Definition 6. *Let A and B be two single valued quadri-partitioned neutrosophic sets on the universe of discourse X, then:*

(i) The union of A and B is defined as:

$$A \cup B = \Big\{\mathrm{X}, \Big[\max(T_A(\mathrm{X}), T_B(\mathrm{X})), \max(ReT_A(\mathrm{X}), ReT_B(\mathrm{X})), \min(ReF_A(\mathrm{X}), ReF_B(\mathrm{X})), \min(F_A(\mathrm{X}), F_B(\mathrm{X}))\Big] : \mathrm{X} \in X\Big\}.$$

(ii) The intersection of A and B is defined as:

$$A \cap B = \Big\{\mathrm{X}, \Big[\min(T_A(\mathrm{X}), T_B(\mathrm{X})), \min(ReT_A(\mathrm{X}), ReT_B(\mathrm{X})), \max(ReF_A(\mathrm{X}), ReF_B(\mathrm{X})), \max(F_A(\mathrm{X}), F_B(\mathrm{X}))\Big] : \mathrm{X} \in X\Big\}.$$

Definition 7. *A Quadri-Single-Valued-Neutrosophic-Number (Q-SVNN) is denoted as:*

$$N = \langle (p, q, r, s); \rho_N, \upsilon_N, \kappa_N, \tau_N\rangle$$

where N is a unique neutrosophic set on $\mathbb{R}$, with its truth-membership function $T_N(\mathrm{X})$ defined as:

$$T_N(\mathrm{X}) = \begin{cases} \frac{\mathrm{X}-p}{q-p}\rho_N, & \text{for } p \preceq \mathrm{X} \preceq q, \\ \frac{r-\mathrm{X}}{r-q}\rho_N, & \text{for } q \preceq \mathrm{X} \preceq r, \\ 0, & \text{otherwise.} \end{cases}$$

The definitions of the indeterminacy-membership, hesitation, and falsity-membership functions follow a similar structure.

$$I_N(\mathrm{X}) = \begin{cases} \frac{q-\mathrm{X}+\upsilon_N(\mathrm{X}-p)}{q-p}, & \text{for } p \preceq \mathrm{X} \preceq q, \\ \frac{q-\mathrm{X}+\upsilon_N(r-\mathrm{X})}{r-q}, & \text{for } q \preceq \mathrm{X} \preceq r, \\ 0, & \text{otherwise.} \end{cases}$$

$$H_N(\mathrm{X}) = \begin{cases} \frac{q-\mathrm{X}+\kappa_N(\mathrm{X}-p)}{q-p}, & \text{for } p \preceq \mathrm{X} \preceq q, \\ \frac{q-\mathrm{X}+\kappa_N(r-\mathrm{X})}{r-q}, & \text{for } q \preceq \mathrm{X} \preceq r, \\ 0, & \text{otherwise.} \end{cases}$$

$$F_N(\mathrm{X}) = \begin{cases} \frac{q-\mathrm{X}+\tau_N(\mathrm{X}-p)}{q-p}, & \textit{for } p \preceq \mathrm{X} \preceq q, \\ \frac{q-\mathrm{X}+\tau_N(r-\mathrm{X})}{r-q}, & \textit{for } q \preceq \mathrm{X} \preceq r, \\ 0, & \textit{otherwise.} \end{cases}$$

Respectively, these represent the indeterminacy-membership function $I_N(\mathrm{X})$, hesitation-membership function $H_N(\mathrm{X})$, and falsity-membership function $F_N(\mathrm{X})$.

Definition 8. *A Quadri-Single-Valued-Neutrosophic Number (Q-SVN number) is defined as:*

$$N = \langle (p, q, r, s); \rho_N, \upsilon_N, \kappa_N, \tau_N \rangle$$

where N represents a unique neutrosophic set on $\mathbb{R}$. The corresponding membership functions for truth, indeterminacy, hesitation, and falsity are defined as follows.

$$N(\mathrm{X}) = \begin{cases} \frac{\mathrm{X}-p}{q-p}\rho_N, & \textit{for } p \preceq \mathrm{X} \preceq q, \\ \frac{s-\mathrm{X}}{s-q}\rho_N, & \textit{for } q \preceq \mathrm{X} \preceq r, \\ \tau_N, & \textit{for } r \preceq \mathrm{X} \preceq s, \\ 0, & \textit{otherwise.} \end{cases}$$

$$T_N(\mathrm{X}) = \begin{cases} \frac{q-\mathrm{X}+\upsilon_N(\mathrm{X}-p)}{q-p}, & \textit{for } p \preceq \mathrm{X} \preceq q, \\ \frac{\mathrm{X}-r+\upsilon_N(s-\mathrm{X})}{s-r}, & \textit{for } q \preceq \mathrm{X} \preceq r, \\ \tau_N, & \textit{for } r \preceq \mathrm{X} \preceq s, \\ 0, & \textit{otherwise.} \end{cases}$$

$$F_N(\mathrm{X}) = \begin{cases} \frac{q-\mathrm{X}+\kappa_N(\mathrm{X}-p)}{q-p}, & \textit{for } p \preceq \mathrm{X} \preceq q, \\ \frac{\mathrm{X}-r+\kappa_N(s-\mathrm{X})}{s-r}, & \textit{for } q \preceq \mathrm{X} \preceq r, \\ \tau_N, & \textit{for } r \preceq \mathrm{X} \preceq s, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 9. *The $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$-cut of a neutrosophic set N is denoted by $N(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, where $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) \in [0, 1]$. Then,*

$$N(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) = \Big\{ \langle T_N(\mathrm{X}), ReT_N(\mathrm{X}), ReF_N(\mathrm{X}), F_N(\mathrm{X}) \rangle : \mathrm{X} \in U,\, T_N(\mathrm{X}) \succeq \mathfrak{i},\, ReT_N(\mathrm{X}) \preceq \mathfrak{j},\, ReF_N(\mathrm{X}) \preceq \mathfrak{k},\, F_N(\mathrm{X}) \preceq$$

4. Characterization of Few Results in Quadri-partitioned Neutrosophic Space

In this section, some properties of single valued quadri-partitioned neutrosophic numbers and operations of single valued quadri-partitioned neutrosophic numbers are presented. In addition to this few results are also addressed, which are necessary for the up-coming sections.

Definition 10. *(i) A neutrosophic number $\tilde{n}$ is called a closed single valued quadri-partitioned neutrosophic number if $\tilde{n}$ is a quadri-partitioned neutrosophic number and the truth membership function and relative truth membership functions are upper semi-continuous, while the false and relative false membership functions are lower semi-continuous.*

(ii) A neutrosophic number $\tilde{n}$ is called a bounded single valued quadri-partitioned neutrosophic number if $\tilde{n}$ is a quadri-partitioned neutrosophic number and the truth membership, relative truth membership, relative false membership, and false membership functions have compact support.

(iii) *Let $\tilde{m}$ and $\tilde{n}$ be two single valued quadri-partitioned neutrosophic numbers. Then, $\tilde{m}$ and $\tilde{n}$ are said to be equal, denoted as $\tilde{m} = \tilde{n}$, if and only if $\tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ and $\tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$. Here, $\tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ and $\tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ denote the $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$-cut of $\tilde{m}$ and $\tilde{n}$, respectively.*

$$\tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}.$$

Proposition 1. *If $\tilde{n}$ is a closed single-valued quadri-partitioned neutrosophic number, then the $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$-cut of $\tilde{n}$, denoted as*

$$\tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \left\langle [\tilde{n}_{\mathfrak{i}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{i}}^{\mathcal{U}}], [\tilde{n}_{\mathfrak{j}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{j}}^{\mathcal{U}}], [\tilde{n}_{\mathfrak{k}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{k}}^{\mathcal{U}}], [\tilde{n}_{\mathfrak{l}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{l}}^{\mathcal{U}}] \right\rangle$$

is a closed interval Quadri-Neutrosophic Single Number (QNSN) or Interval Quadri-Neutrosophic Number (IQNN), where $[\tilde{n}_{\mathfrak{k}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{k}}^{\mathcal{U}}]$ and $[\tilde{n}_{\mathfrak{l}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{l}}^{\mathcal{U}}]$ are all closed intervals. Here,

$$\tilde{n}_{\mathfrak{i}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{i}}^{\mathcal{U}}, \tilde{n}_{\mathfrak{j}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{j}}^{\mathcal{U}}, \tilde{n}_{\mathfrak{k}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{k}}^{\mathcal{U}}, \tilde{n}_{\mathfrak{l}}^{\mathcal{L}}, \tilde{n}_{\mathfrak{l}}^{\mathcal{U}}$$

denote $\inf T_{\tilde{N}}(\mathrm{X})$, $\sup T_{\tilde{N}}(\mathrm{X})$, $\inf I_{\tilde{N}}(\mathrm{X})$, $\sup I_{\tilde{N}}(\mathrm{X})$, $\inf F_{\tilde{N}}(\mathrm{X})$, $\sup F_{\tilde{N}}(\mathrm{X})$, $\inf H_{\tilde{N}}(\mathrm{X})$, *and* $\sup H_{\tilde{N}}(\mathrm{X})$, *respectively.*

Proof. If $T_{\tilde{N}}(\mathrm{X})$ is upper semi-continuous, and $I_{\tilde{N}}(\mathrm{X})$ and $F_{\tilde{N}}(\mathrm{X})$ are lower semi-continuous, then the $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$-level set of $\tilde{n}$, i.e.,

$$\tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \left\{ T_{\tilde{N}}(\mathrm{X}) \succeq \mathfrak{i},\ I_{\tilde{N}}(\mathrm{X}) \preceq \mathfrak{j},\ F_{\tilde{N}}(\mathrm{X}) \preceq \mathfrak{k},\ H_{\tilde{N}}(\mathrm{X}) \preceq \mathfrak{l},\ \mathrm{X} \in \mathbb{R} \right\}$$

is a closed set. Then, from Definition 7, $\tilde{n}$ is an interval Quadri-Neutrosophic Single Number (QNSN), and intervals are closed intervals.

Proposition 2. *Let $\tilde{m}$ and $\tilde{n}$ be two single-valued quadri-partitioned neutrosophic numbers, then:*

(i) $(\tilde{m} \odot \tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} \odot \tilde{n}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$

(ii) $(\lambda\tilde{m})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \lambda\tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$, *where $\lambda \neq 0$ is any real number.*

Proof.

1 Since

$$\tilde{m} \odot \tilde{n} = \Big\{ Z, \max_{\mathrm{X}} Z = (\mathrm{X}) \circledast y \left\{ \min(T_{\tilde{m}}(\mathrm{X}), T_{\tilde{n}}(y)) \right\},$$
$$\min Z = (\mathrm{X}) \circledast y \left\{ \max(I_{\tilde{m}}(\mathrm{X}), I_{\tilde{n}}(y)) \right\},$$
$$\min Z = (\mathrm{X}) \circledast y \left\{ \max(F_{\tilde{m}}(\mathrm{X}), F_{\tilde{n}}(y)) \right\}, \min Z = (\mathrm{X}) \circledast y \left\{ \max(H_{\tilde{m}}(\mathrm{X}), H_{\tilde{n}}(y)) \right\} \Big\}.$$

And

$$(\tilde{m} \odot \tilde{n})(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = \left\{ Z : T_{\tilde{m}\odot\tilde{n}}(z) \succeq \mathfrak{i}, I_{\tilde{m}\odot\tilde{n}}(z) \succeq \mathfrak{j}, F_{\tilde{m}\odot\tilde{n}}(z) \succeq \mathfrak{k}, H_{\tilde{m}\odot\tilde{n}}(z) \succeq \mathfrak{l} \right\}.$$

Let $Z \in (\tilde{m} \odot \tilde{n})(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$, then there exists at least one $\mathrm{X} \in \tilde{m}$ and $y \in \tilde{n}$ such that $Z = \tilde{m} \odot \tilde{n}$. Then,

$$T_{\tilde{m}\odot\tilde{n}}(z) = \max \left\{ \min(T_{\tilde{m}}(\mathrm{X}), T_{\tilde{n}}(y)) \right\} \succeq \mathfrak{i}.$$
$$T_{\tilde{m}}(\mathrm{X}) \succeq \mathfrak{i} \quad \text{and} \quad T_{\tilde{n}}(y) \succeq \mathfrak{i}$$

$$I_{\tilde{m}\odot\tilde{n}}(z) = \min \left\{ \max(I_{\tilde{m}}(\mathrm{X}), I_{\tilde{n}}(y)) \right\} \preceq \mathfrak{j}$$

$$I_{\tilde{m}}(\mathrm{X}) \preceq \mathfrak{j} \quad \text{and} \quad I_{\tilde{n}}(y) \preceq \mathfrak{j}$$

$$F_{\tilde{m}\odot\tilde{n}}(z) = \min\{\max(F_{\tilde{m}}(\mathrm{X}), F_{\tilde{n}}(y))\} \preceq \mathfrak{k}$$

$$F_{\tilde{m}}(\mathrm{X}) \preceq \mathfrak{k} \quad \text{and} \quad F_{\tilde{n}}(y) \preceq \mathfrak{k}$$

$$H_{\tilde{m}\odot\tilde{n}}(z) = \min\{\max(H_{\tilde{m}}(\mathrm{X}), H_{\tilde{n}}(y))\} \preceq \mathfrak{l}$$

$$H_{\tilde{m}}(\mathrm{X}) \preceq \mathfrak{l} \quad \text{and} \quad H_{\tilde{n}}(y) \preceq \mathfrak{l}$$

This implies $\mathrm{X} \in \tilde{m}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ and $y \in \tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$. Therefore,

$$z = \mathrm{X} \odot y \in \tilde{m}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) \odot \tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$$

Again, let $z^* \in \tilde{m}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) \odot \tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$. Then there exists at least one $\mathrm{X}^* \in \tilde{m}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ and $y^* \in \tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ such that

$$z^* = \mathrm{X}^* \odot y^*$$

Then, we have

$$T_{\tilde{m}}(\mathrm{X}^*) \succeq \mathfrak{i} \quad \text{and} \quad T_{\tilde{n}}(y^*) \succeq \mathfrak{i}$$

$$I_{\tilde{m}}(\mathrm{X}^*) \preceq \mathfrak{j} \quad \text{and} \quad I_{\tilde{n}}(y^*) \preceq \mathfrak{j}$$

$$F_{\tilde{m}}(\mathrm{X}^*) \preceq \mathfrak{k} \quad \text{and} \quad F_{\tilde{n}}(y^*) \preceq \mathfrak{k}$$

$$H_{\tilde{m}}(\mathrm{X}^*) \preceq \mathfrak{l} \quad \text{and} \quad H_{\tilde{n}}(y^*) \preceq \mathfrak{l}$$

This implies that

$$\min(T_{\tilde{m}}(\mathrm{X}^*), T_{\tilde{n}}(y^*)) \succeq \mathfrak{i}, \quad \max(I_{\tilde{m}}(\mathrm{X}^*), I_{\tilde{n}}(y^*)) \preceq \mathfrak{j},$$

$$\max(F_{\tilde{m}}(\mathrm{X}^*), F_{\tilde{n}}(y^*)) \preceq \mathfrak{k}, \quad \max(H_{\tilde{m}}(\mathrm{X}^*), H_{\tilde{n}}(y^*)) \preceq \mathfrak{l}$$

Then, we have

$$\max_{\mathrm{X}} z^* = \mathrm{X}^* \odot y^* \{\min(T_{\tilde{m}}(\mathrm{X}^*), T_{\tilde{n}}(y^*))\} \succeq \mathfrak{i}$$

$$\Rightarrow T_{\tilde{m}\odot\tilde{n}}(z^*) = T_{\tilde{m}\odot\tilde{n}}(\mathrm{X}^* \odot y^*) \succeq \mathfrak{i}$$

Similarly,

$$I_{\tilde{m}\odot\tilde{n}}(z^*) = I_{\tilde{m}\odot\tilde{n}}(\mathrm{X}^* \odot y^*) \preceq \mathfrak{j}$$

$$F_{\tilde{m}\odot\tilde{n}}(z^*) = F_{\tilde{m}\odot\tilde{n}}(\mathrm{X}^* \odot y^*) \preceq \mathfrak{k}$$

$$H_{\tilde{m}\odot\tilde{n}}(z^*) = H_{\tilde{m}\odot\tilde{n}}(\mathrm{X}^* \odot y^*) \preceq \mathfrak{l}$$

Therefore,

$$z^* \in (\tilde{m} \odot \tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}.$$

2 Since

$$\tilde{m} = \left\{ Z \mid \max_{\zeta} z = \lambda(\zeta) T_{\tilde{m}}(\zeta), \min_{\zeta} z = \lambda(\zeta) I_{\tilde{m}}(\zeta), \min_{\zeta} z = \lambda(\zeta) F_{\tilde{m}}(\zeta), \min_{\zeta} z = \lambda(\zeta) H_{\tilde{m}}(\zeta) \right\}$$

Let

$$Z\lambda\tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \{Z \mid T_{\tilde{m}}(\zeta) \succeq \mathfrak{i}, I_{\tilde{m}}(\zeta) \preceq \mathfrak{j}, F_{\tilde{m}}(\zeta) \preceq \mathfrak{k}, H_{\tilde{m}}(\zeta) \preceq \mathfrak{l}\}$$

Let $Z \in (\lambda m)$, then there exists $\zeta \in \tilde{m}$ such that

$$z = \lambda(\zeta)$$

which implies

$$T_{\tilde{m}}(\zeta) \succeq \mathfrak{i}, \quad I_{\tilde{m}}(\zeta) \preceq \mathfrak{j}, \quad F_{\tilde{m}}(\zeta) \preceq \mathfrak{k}, \quad H_{\tilde{m}}(\zeta) \preceq \mathfrak{l}$$

Thus,

$$\zeta \in \tilde{m}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} \Rightarrow z = \lambda(\zeta) \in (\lambda m)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

Again, let $Z^* \in (\lambda m)$, then there exists $\zeta^* \in \tilde{m}$ such that

$$Z^* = \lambda(\zeta^*)$$

which implies

$$T_{\tilde{m}}(\zeta^*) \succeq \mathfrak{i}, \quad I_{\tilde{m}}(\zeta^*) \preceq \mathfrak{j}, \quad F_{\tilde{m}}(\zeta^*) \preceq \mathfrak{k}, \quad H_{\tilde{m}}(\zeta^*) \preceq \mathfrak{l}$$

Then we have

$$Z^* = \lambda(\zeta)^* \quad T_{\tilde{m}}(\zeta^*) \succeq \mathfrak{i} \Rightarrow T(\lambda m)_{\tilde{m}}(\lambda(\zeta)^*) = T(\lambda m)_{\tilde{m}}(z^*) \succeq \mathfrak{i}$$

$$Z^* = \lambda(\zeta)^* \quad I_{\tilde{m}}(\zeta^*) \preceq \mathfrak{j} \Rightarrow I(\lambda m)_{\tilde{m}}(\lambda(\zeta)^*) = I(\lambda m)_{\tilde{m}}(z^*) \preceq \mathfrak{j}$$

$$Z^* = \lambda(\zeta)^* \quad F_{\tilde{m}}(\zeta^*) \preceq \mathfrak{k} \Rightarrow F(\lambda m)_{\tilde{m}}(\lambda(\zeta)^*) = F(\lambda m)_{\tilde{m}}(z^*) \preceq \mathfrak{k}$$

$$Z^* = \lambda(\zeta)^* \quad H_{\tilde{m}}(\zeta^*) \preceq \mathfrak{l} \Rightarrow H(\lambda m)_{\tilde{m}}(\lambda(\zeta)^*) = H(\lambda m)_{\tilde{m}}(z^*) \preceq \mathfrak{l}$$

Therefore,

$$Z^* \in (\lambda m)_{\tilde{m}(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

Proposition 3. *Let $\tilde{m}$ and $\tilde{n}$ be two closed quadri-partitioned neutrosophic numbers. Then, $\tilde{m}+\tilde{n}$, $\tilde{m}-\tilde{n}$, $\tilde{m}\times\tilde{n}$, and $\lambda\tilde{m}$ are also closed quadri-partitioned neutrosophic numbers, where $0 \neq \lambda \in \mathbb{R}$ and λ is any real number.*

Proof. Since $T_{\tilde{m}\odot\tilde{n}}$ and $T(\lambda\tilde{m})$ are upper semi-continuous, and $I_{\tilde{m}\odot\tilde{n}}$ and $I(\lambda\tilde{m})$ are lower semi-continuous, it follows that $(\tilde{m}\odot\tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ and $(\lambda\tilde{m})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$, along with $F_{\tilde{m}\odot\tilde{n}}$, $F(\lambda\tilde{m})$, $H_{\tilde{m}\odot\tilde{n}}$, and $H(\lambda\tilde{m})$, are closed sets for all $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$.

Proposition 4.

Let $\tilde{m},\tilde{n}$ be closed quadri-partitioned neutrosophic numbers, then

$$(\tilde{m}+\tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \left\langle \left[m_{\mathfrak{i}}^{\mathcal{L}}+n_{\mathfrak{i}}^{\mathcal{L}}, m_{\mathfrak{i}}^{\mathcal{U}}+n_{\mathfrak{i}}^{\mathcal{U}}\right], \left[m_{\mathfrak{j}}^{\mathcal{L}}+n_{\mathfrak{j}}^{\mathcal{L}}, m_{\mathfrak{j}}^{\mathcal{U}}+n_{\mathfrak{j}}^{\mathcal{U}}\right], \left[m_{\mathfrak{k}}^{\mathcal{L}}+n_{\mathfrak{k}}^{\mathcal{L}}, m_{\mathfrak{k}}^{\mathcal{U}}+n_{\mathfrak{k}}^{\mathcal{U}}\right], \left[m_{\mathfrak{l}}^{\mathcal{L}}+n_{\mathfrak{l}}^{\mathcal{L}}, m_{\mathfrak{l}}^{\mathcal{U}}+n_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle$$

$$(\tilde{m}-\tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \left\langle \left[m_{\mathfrak{i}}^{\mathcal{L}}-n_{\mathfrak{i}}^{\mathcal{L}}, m_{\mathfrak{i}}^{\mathcal{U}}-n_{\mathfrak{i}}^{\mathcal{U}}\right], \left[m_{\mathfrak{j}}^{\mathcal{L}}-n_{\mathfrak{j}}^{\mathcal{L}}, m_{\mathfrak{j}}^{\mathcal{U}}-n_{\mathfrak{j}}^{\mathcal{U}}\right], \left[m_{\mathfrak{k}}^{\mathcal{L}}-n_{\mathfrak{k}}^{\mathcal{L}}, m_{\mathfrak{k}}^{\mathcal{U}}-n_{\mathfrak{k}}^{\mathcal{U}}\right], \left[m_{\mathfrak{l}}^{\mathcal{L}}-n_{\mathfrak{l}}^{\mathcal{L}}, m_{\mathfrak{l}}^{\mathcal{U}}-n_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle$$

$$(\lambda\tilde{m})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \begin{cases} \left\langle \left[\lambda m_{\mathfrak{i}}^{\mathcal{L}}, \lambda m_{\mathfrak{i}}^{\mathcal{U}}\right], \left[\lambda m_{\mathfrak{j}}^{\mathcal{L}}, \lambda m_{\mathfrak{j}}^{\mathcal{U}}\right], \left[\lambda m_{\mathfrak{k}}^{\mathcal{L}}, \lambda m_{\mathfrak{k}}^{\mathcal{U}}\right], \left[\lambda m_{\mathfrak{l}}^{\mathcal{L}}, \lambda m_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle & \text{for } \lambda > 0 \\ \left\langle \left[\lambda m_{\mathfrak{i}}^{\mathcal{U}}, \lambda m_{\mathfrak{i}}^{\mathcal{L}}\right], \left[\lambda m_{\mathfrak{j}}^{\mathcal{U}}, \lambda m_{\mathfrak{j}}^{\mathcal{L}}\right], \left[\lambda m_{\mathfrak{k}}^{\mathcal{U}}, \lambda m_{\mathfrak{k}}^{\mathcal{L}}\right], \left[\lambda m_{\mathfrak{l}}^{\mathcal{U}}, \lambda m_{\mathfrak{l}}^{\mathcal{L}}\right] \right\rangle & \text{for } \lambda < 0 \end{cases}$$

$$\text{Since } (\tilde{m}+\tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = (\tilde{m})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} + (\tilde{n})_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

$$= \left\langle \left[m_{\mathfrak{i}}^{\mathcal{L}}, m_{\mathfrak{i}}^{\mathcal{U}}\right], \left[m_{\mathfrak{j}}^{\mathcal{L}}, m_{\mathfrak{j}}^{\mathcal{U}}\right], \left[m_{\mathfrak{k}}^{\mathcal{L}}, m_{\mathfrak{k}}^{\mathcal{U}}\right], \left[m_{\mathfrak{l}}^{\mathcal{L}}, m_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle + \left\langle \left[n_{\mathfrak{i}}^{\mathcal{L}}, n_{\mathfrak{i}}^{\mathcal{U}}\right], \left[n_{\mathfrak{j}}^{\mathcal{L}}, n_{\mathfrak{j}}^{\mathcal{U}}\right], \left[n_{\mathfrak{k}}^{\mathcal{L}}, n_{\mathfrak{k}}^{\mathcal{U}}\right], \left[n_{\mathfrak{l}}^{\mathcal{L}}, n_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle$$

$$= \left\langle \left[m_{\mathfrak{i}}^{\mathcal{L}}+n_{\mathfrak{i}}^{\mathcal{L}}, m_{\mathfrak{i}}^{\mathcal{U}}+n_{\mathfrak{i}}^{\mathcal{U}}\right], \left[m_{\mathfrak{j}}^{\mathcal{L}}+n_{\mathfrak{j}}^{\mathcal{L}}, m_{\mathfrak{j}}^{\mathcal{U}}+n_{\mathfrak{j}}^{\mathcal{U}}\right], \left[m_{\mathfrak{k}}^{\mathcal{L}}+n_{\mathfrak{k}}^{\mathcal{L}}, m_{\mathfrak{k}}^{\mathcal{U}}+n_{\mathfrak{k}}^{\mathcal{U}}\right], \left[m_{\mathfrak{l}}^{\mathcal{L}}+n_{\mathfrak{l}}^{\mathcal{L}}, m_{\mathfrak{l}}^{\mathcal{U}}+n_{\mathfrak{l}}^{\mathcal{U}}\right] \right\rangle$$

Let me know if you need further modifications!

Proof. Trivial.

Theorem 1.

Let N be a quadri-partitioned neutrosophic set. Then,

$$(N)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \{(\mathrm{X}) : TN(\mathrm{X}) \succeq \mathfrak{i},\ IN(\mathrm{X}) \preceq \mathfrak{j},\ FN(\mathrm{X}) \preceq \mathfrak{k},\ HN(\mathrm{X}) \preceq \mathfrak{l}\}$$

$$\textbf{1.} \quad N(\mathfrak{i}_2,\mathfrak{j}_2,\mathfrak{k}_2,\mathfrak{l}_2) \subseteq N(\mathfrak{i}_1,\mathfrak{j}_1,\mathfrak{k}_1,\mathfrak{l}_1)$$

where $\mathfrak{i}_1 \prec \mathfrak{i}_2, \quad \mathfrak{j}_1 \succ \mathfrak{j}_2, \quad \mathfrak{k}_1 \succ \mathfrak{k}_2, \quad \mathfrak{l}_1 \succ \mathfrak{l}_2.$

$$\textbf{2.} \quad \bigcap_{n=1}^{\infty} N(\mathfrak{i}_n,\mathfrak{j}_n,\mathfrak{k}_n,\mathfrak{l}_n) = (N)(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$$

where $\mathfrak{i}_1 \prec \mathfrak{i}_2, \quad \mathfrak{j}_1 \succ \mathfrak{j}_2, \quad \mathfrak{k}_1 \succ \mathfrak{k}_2, \quad \mathfrak{l}_1 \succ \mathfrak{l}_2.$

Proof. **1.** Let (X) $\in N(\mathfrak{i}_2, \mathfrak{j}_2, \mathfrak{k}_2, \mathfrak{l}_2)$, then we have:

$$TN(\mathrm{X}) \succeq \mathfrak{i}_2, \quad IN(\mathrm{X}) \preceq \mathfrak{j}_2, \quad FN(\mathrm{X}) \preceq \mathfrak{k}_2, \quad HN(\mathrm{X}) \preceq \mathfrak{l}_2.$$

Now,

$$TN(\mathrm{X}) \succeq \mathfrak{i}_2 \succ \mathfrak{i}_1, \quad IN(\mathrm{X}) \preceq \mathfrak{j}_2 \prec \mathfrak{j}_1, \quad FN(\mathrm{X}) \preceq \mathfrak{k}_2 \prec \mathfrak{k}_1, \quad HN(\mathrm{X}) \preceq \mathfrak{l}_2 \prec \mathfrak{l}_1$$

this implies that

$$TN(\mathrm{X}) \succeq \mathfrak{i}_1, \quad IN(\mathrm{X}) \preceq \mathfrak{j}_1, \quad FN(\mathrm{X}) \preceq \mathfrak{k}_1, \quad HN(\mathrm{X}) \preceq \mathfrak{l}_1$$

Therefore, $\mathrm{X} \in N(\mathfrak{i}_1, \mathfrak{j}_1, \mathfrak{k}_1, \mathfrak{l}_1) \Rightarrow N(\mathfrak{i}_2, \mathfrak{j}_2, \mathfrak{k}_2, \mathfrak{l}_2) \subseteq N(\mathfrak{i}_1, \mathfrak{j}_1, \mathfrak{k}_1, \mathfrak{l}_1)$.

2. Let $\mathrm{X} \in \bigcap_{n=1}^{\infty} N(\mathfrak{i}_n, \mathfrak{j}_n, \mathfrak{k}_n, \mathfrak{l}_n) \Rightarrow \mathrm{X} \in N(\mathfrak{i}_n, \mathfrak{j}_n, \mathfrak{k}_n, \mathfrak{l}_n)$.

Since

$$\lim_{n\to\infty} \mathfrak{i}_n = \mathfrak{i}, \quad \lim_{n\to\infty} \mathfrak{j}_n = \mathfrak{j}, \quad \lim_{n\to\infty} \mathfrak{k}_n = \mathfrak{k}, \quad \lim_{n\to\infty} \mathfrak{l}_n = \mathfrak{l}$$

then

$$TN(\mathrm{X}) \succeq \lim_{n\to\infty} \mathfrak{i}_n = \mathfrak{i}, \quad IN(\mathrm{X}) \preceq \lim_{n\to\infty} \mathfrak{j}_n = \mathfrak{j}, \quad FN(\mathrm{X}) \preceq \lim_{n\to\infty} \mathfrak{k}_n = \mathfrak{k}, \quad HN(\mathrm{X}) \preceq \lim_{n\to\infty} \mathfrak{l}_n = \mathfrak{l}$$

Thus,

$$TN(\mathrm{X}) \succeq \mathfrak{i}, \quad IN(\mathrm{X}) \preceq \mathfrak{j}, \quad FN(\mathrm{X}) \preceq \mathfrak{k}, \quad HN(\mathrm{X}) \preceq \mathfrak{l}$$

Therefore, $\mathrm{X} \in (N)(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) \Rightarrow N(\mathfrak{i}_1, \mathfrak{j}_1, \mathfrak{k}_1, \mathfrak{l}_1) \subseteq (N)(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$.

Again, let $\mathrm{X} \in (N)(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, then

$$TN(\mathrm{X}) \succeq \mathfrak{i}, \quad IN(\mathrm{X}) \preceq \mathfrak{j}, \quad FN(\mathrm{X}) \preceq \mathfrak{k}, \quad HN(\mathrm{X}) \preceq \mathfrak{l}$$

Since $\mathfrak{i}_n \uparrow \mathfrak{i}, \quad \mathfrak{j}_n \downarrow \mathfrak{j}, \quad \mathfrak{k}_n \downarrow \mathfrak{k}, \quad \mathfrak{l}_n \downarrow \mathfrak{l}$, we obtain:

$$TN(\mathrm{X}) \succeq \mathfrak{i} \succeq \mathfrak{i}_n, \quad IN(\mathrm{X}) \preceq \mathfrak{j} \preceq \mathfrak{j}_n, \quad FN(\mathrm{X}) \preceq \mathfrak{k} \preceq \mathfrak{k}_n, \quad HN(\mathrm{X}) \preceq \mathfrak{l} \preceq \mathfrak{l}_n$$

for all n. This implies that

$$\mathrm{X} \in \bigcap_{n=1}^{\infty} N(\mathfrak{i}_n, \mathfrak{j}_n, \mathfrak{k}_n, \mathfrak{l}_n) \Rightarrow (N)(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) \subseteq N(\mathfrak{i}_1, \mathfrak{j}_1, \mathfrak{k}_1, \mathfrak{l}_1)$$

Thus, we conclude:

$$\bigcap_{n=1}^{\infty} N(\mathfrak{i}_n, \mathfrak{j}_n, \mathfrak{k}_n, \mathfrak{l}_n) = (N)(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}).$$

Proposition 5. *Let*

$$N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \langle [l_\mathfrak{i}, U_\mathfrak{i}], [l_\mathfrak{j}, U_\mathfrak{j}], [l_\mathfrak{k}, U_\mathfrak{k}], [l_\mathfrak{l}, U_\mathfrak{l}] \rangle,$$

where $0 \preceq \mathfrak{i} \preceq 1$, $0 \preceq \mathfrak{j} \preceq 1$, $0 \preceq \mathfrak{k} \preceq 1$, $0 \preceq \mathfrak{l} \preceq 1$ *be a domain of interval quadri-neutrosophic sets (QNS), and each interval* $[l_\mathfrak{i}, U_\mathfrak{i}], [l_\mathfrak{j}, U_\mathfrak{j}], [l_\mathfrak{k}, U_\mathfrak{k}], [l_\mathfrak{l}, U_\mathfrak{l}]$ *is closed.*

Suppose $N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ *is decreasing with respect to* $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ *and* $\tilde{n}$ *is a closed QNSN. Then* $\{N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}\}$ *can induce* $\tilde{n}$*, and*

$$\tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}) = N(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$$

where $\tilde{n}(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ *and* $N(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ *denote the* $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$*-cut of* $\tilde{n}$ *and* N*, respectively.*

Proof. Let $\tilde{n}$ be a single-valued quadri-partitioned neutrosophic number, and suppose $N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ is decreasing with respect to $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$. From Theorem 1, we have

$$\bigcap_{n=1}^{\infty} N(\mathfrak{i}_n,\mathfrak{j}_n,\mathfrak{k}_n,\mathfrak{l}_n) = N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

for $\mathfrak{i}_n \uparrow \mathfrak{i}, \mathfrak{j}_n \downarrow \mathfrak{j}, \mathfrak{k}_n \downarrow \mathfrak{k}, \mathfrak{l}_n \downarrow \mathfrak{l}$.

Now, from Definitions 2 and 3, it follows that $\{N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}\}$ can induce $\tilde{n}$. From Definition 4, we have

$$\{\mathrm{X} : T_{\tilde{n}}(\mathrm{X}) \succeq \mathfrak{i}, \quad I_{\tilde{n}}(\mathrm{X}) \preceq \mathfrak{j}, \quad F_{\tilde{n}}(\mathrm{X}) \preceq \mathfrak{k}, \quad H_{\tilde{n}}(\mathrm{X}) \preceq \mathfrak{l}\}$$
$$= N_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \langle [l_{\mathfrak{i}}, U_{\mathfrak{i}}], [l_{\mathfrak{j}}, U_{\mathfrak{j}}], [l_{\mathfrak{k}}, U_{\mathfrak{k}}], [l_{\mathfrak{l}}, U_{\mathfrak{l}}]\rangle$$

is a closed single-valued quadri-partitioned neutrosophic number.

Proposition 6. *If $\tilde{n}$ is a closed single-valued quadri-partitioned neutrosophic number, then*

$$\tilde{n} \textit{ and } \tilde{n}$$

$$U \downarrow \tilde{n}_{\mathfrak{i}}^U \quad \textit{for } \mathfrak{i}_n \uparrow \mathfrak{i}, \quad \tilde{n}_{\mathfrak{j}_n}^L \uparrow \tilde{n}_{\mathfrak{j}}^L, \quad \tilde{n}_{\mathfrak{j}_n}^U \downarrow \tilde{n}_{\mathfrak{j}}^U \quad \textit{for } \mathfrak{j}_n \downarrow \mathfrak{j},$$

$$\tilde{n}_{\mathfrak{k}_n}^L \uparrow \tilde{n}_{\mathfrak{k}}^L, \quad \tilde{n}_{\mathfrak{k}_n}^U \downarrow \tilde{n}_{\mathfrak{k}}^U \quad \textit{for } \mathfrak{k}_n \downarrow \mathfrak{k},$$

$$\tilde{n}_{\mathfrak{l}_n}^L \uparrow \tilde{n}_{\mathfrak{l}}^L, \quad \tilde{n}_{\mathfrak{l}_n}^U \downarrow \tilde{n}_{\mathfrak{l}}^U \quad \textit{for } \mathfrak{l}_n \downarrow \mathfrak{l}.$$

Proof. Since $(N)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \tilde{n}(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = \langle [\tilde{n}_{\mathfrak{i}}^L, \tilde{n}_{\mathfrak{i}}^U], [\tilde{n}_{\mathfrak{j}}^L, \tilde{n}_{\mathfrak{j}}^U], [\tilde{n}_{\mathfrak{k}}^L, \tilde{n}_{\mathfrak{k}}^U], [\tilde{n}_{\mathfrak{l}}^L, \tilde{n}_{\mathfrak{l}}^U]\rangle$, then $(N)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$ is decreasing with respect to $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$. We have:

$$\lim \tilde{n}_{an}^L \preceq \tilde{n}_{\mathfrak{i}}^L, \quad \lim \tilde{n}_{an}^U \succeq \tilde{n}_{\mathfrak{i}}^U, \quad \lim \tilde{n}_{\mathfrak{j}n}^L \preceq \tilde{n}_{\mathfrak{j}}^L, \quad \lim \tilde{n}_{\mathfrak{j}n}^U \succeq \tilde{n}_{\mathfrak{j}}^U,$$

$$\lim \tilde{n}_{\mathfrak{k}n}^L \preceq \tilde{n}_{\mathfrak{k}}^L, \quad \lim \tilde{n}_{\mathfrak{k}n}^U \succeq \tilde{n}_{\mathfrak{k}}^U, \quad \lim \tilde{n}_{\mathfrak{l}n}^L \preceq \tilde{n}_{\mathfrak{l}}^L, \quad \lim \tilde{n}_{\mathfrak{l}n}^U \succeq \tilde{n}_{\mathfrak{l}}^U.$$

This implies that:

$$\langle [\lim \tilde{n}_{an}^L, \lim \tilde{n}_{an}^U], [\lim \tilde{n}_{\mathfrak{j}n}^L, \lim \tilde{n}_{\mathfrak{j}n}^U], [\lim \tilde{n}_{\mathfrak{k}n}^L, \lim \tilde{n}_{\mathfrak{k}n}^U], [\lim \tilde{n}_{\mathfrak{l}n}^L, \lim \tilde{n}_{\mathfrak{l}n}^U]\rangle \subseteq \langle [\tilde{n}_{\mathfrak{i}}^L, \tilde{n}_{\mathfrak{i}}^U], [\tilde{n}_{\mathfrak{j}}^L, \tilde{n}_{\mathfrak{j}}^U], [\tilde{n}_{\mathfrak{k}}^L, \tilde{n}_{\mathfrak{k}}^U], [\tilde{n}_{\mathfrak{l}}^L, \tilde{n}_{\mathfrak{l}}^U]\rangle.$$

Thus, we conclude:

$$\lim \tilde{n}_{an}^L \succeq \tilde{n}_{\mathfrak{i}}^L, \quad \lim \tilde{n}_{an}^U \preceq \tilde{n}_{\mathfrak{i}}^U, \quad \lim \tilde{n}_{\mathfrak{j}n}^L \succeq \tilde{n}_{\mathfrak{j}}^L, \quad \lim \tilde{n}_{\mathfrak{j}n}^U \preceq \tilde{n}_{\mathfrak{j}}^U,$$

$$\lim \tilde{n}_{\mathfrak{k}n}^L \succeq \tilde{n}_{\mathfrak{k}}^L, \quad \lim \tilde{n}_{\mathfrak{k}n}^U \preceq \tilde{n}_{\mathfrak{k}}^U, \quad \lim \tilde{n}_{\mathfrak{l}n}^L \succeq \tilde{n}_{\mathfrak{l}}^L, \quad \lim \tilde{n}_{\mathfrak{l}n}^U \preceq \tilde{n}_{\mathfrak{l}}^U.$$

5. Characterization of Riemann Integration in terms of Quadri-partitioned Neutrosophic Structure

This section is devoted to Riemann integral theory based on quadri-partitioned neutrosophic sets, and all the fundamentals are defined according to this new theory. Some new effects are given, and this whole scenario has been established by introducing interesting examples.

Definition 11. *Let $\mathcal{Q}$ be the set of all Quadri-Partitioned Neutrosophic Numbers (QNSN), $\mathcal{Q}_{cl}$ be the set of all closed (QNSN), and $\mathcal{Q}_b$ signify the set of all bounded (QNSN).*

(i) $\tilde{f}_{\mathcal{Q}}(x)$ *is a Q-NSVF if* $\tilde{f}_{\mathcal{Q}} : X \to \mathcal{Q}$.

(ii) $\tilde{f}_{\mathcal{Q}}(x)$ *is a closed Q-NSVF if* $\tilde{f}_{\mathcal{Q}} : X \to \mathcal{Q}_{cl}$.

(iii) $\tilde{f}_{\mathcal{Q}}(x)$ *is a bounded Q-NSVF if* $\tilde{f}_{\mathcal{Q}} : X \to \mathcal{Q}_b$.

Definition 12. *Let $\tilde{f}_{\mathcal{Q}}(\mathrm{X})$ be a closed bounded Q-NSVF on a closed interval $[a_1, b_1]$. Let*

$$\tilde{f}^{\mathcal{L}}_{\mathcal{Q}i}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}i}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}j}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}j}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}k}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}k}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}l}(\mathrm{X}), \quad \text{and} \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}l}(\mathrm{X})$$

are all quadri-neutrosophic Riemann integrable (QNRI) on $[a_1, b_1]$. For all $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, let

$$\mathcal{I}_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \begin{bmatrix} \left[\int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}i}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}i}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}j}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}j}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}k}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}k}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}l}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}l}(\mathrm{X})d\mathrm{X}\right] \end{bmatrix}$$

Here, $\left[\tilde{f}^{\mathcal{L}}_{\mathcal{Q}i}(\mathrm{X}), \tilde{f}^{\mathcal{U}}_{\mathcal{Q}i}(\mathrm{X})\right]$, $\left[\tilde{f}^{\mathcal{L}}_{\mathcal{Q}j}(\mathrm{X}), \tilde{f}^{\mathcal{U}}_{\mathcal{Q}j}(\mathrm{X})\right]$, $\left[\tilde{f}^{\mathcal{L}}_{\mathcal{Q}k}(\mathrm{X}), \tilde{f}^{\mathcal{U}}_{\mathcal{Q}k}(\mathrm{X})\right]$, and $\left[\tilde{f}^{\mathcal{L}}_{\mathcal{Q}l}(\mathrm{X}), \tilde{f}^{\mathcal{U}}_{\mathcal{Q}l}(\mathrm{X})\right]$ denote the $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$-cut of $\tilde{f}_{\mathcal{Q}}(\mathrm{X})$, respectively.

Proposition 7. *[56] If $g(x)$ is defined on $[a_1, b_1]$ and is a bounded function over $[a_1, b_1]$, then $g(x)$ is also Lebesgue integrable over $[a_1, b_1]$.*

Proposition 8. *[56] If $g(x)$ is a bounded function defined on $[a_1, b_1]$, then $g(x)$ is Riemann integrable on $[a_1, b_1]$ if and only if $g(x)$ is continuous on the closed bounded interval $[a_1, b_1]$.*

Proposition 9. *[56] If $g(x)$ is Riemann integrable on $[a_1, b_1]$ and $\mu \in \mathbb{R}$, then $\mu g(\mathrm{X})$ is also Riemann integrable on $[a_1, b_1]$ and*

$$\int_{a_1}^{b_1} \mu g(\mathrm{X})\, dx = \mu \int_{a_1}^{b_1} g(\mathrm{X})\, dx.$$

Theorem 2. *Let $\tilde{\mathfrak{f}}_{\mathcal{Q}}(x)$ be a closed bounded quadri-neutrosophic valued function on the closed quadri-neutrosophic bounded interval $[a_1, b_1]$. If $\tilde{\mathfrak{f}}_{\mathcal{Q}}(x) \in \mathcal{Q}_{RI}$ on the closed quadri-neutrosophic bounded interval $[a_1, b_1]$, then the quadri-neutrosophic Riemann integral*

$$\int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(x)dx$$

is a closed quadri-neutrosophic single number (QNSN). The $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$-cut of

$$\int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(x)dx$$

is given by:

$$\left(\int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(x)dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \begin{bmatrix} \left[\int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})d\mathrm{X}\right], \\ \left[\int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})d\mathrm{X}, \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})d\mathrm{X}\right] \end{bmatrix}$$

where $[\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X}), \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})]$, $[\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X}), \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})]$, $[\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X}), \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})]$, *and* $[\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X}), \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})]$ *denote the* $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$-*cut of* $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$, *respectively.*

Proof. Let

$$\mathfrak{Q}(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = \begin{bmatrix} \int \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}} d(\mathrm{X}), & \int \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}} d(\mathrm{X}) \\ \int \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}} d(\mathrm{X}), & \int \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}} d(\mathrm{X}) \\ \int \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}} d(\mathrm{X}), & \int \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}} d(\mathrm{X}) \\ \int \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}} d(\mathrm{X}), & \int \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}} d(\mathrm{X}) \end{bmatrix}$$

For $\mathfrak{i}_2 \succ \mathfrak{i}_1$, $\mathfrak{j}_2 \prec \mathfrak{j}_1$, $\mathfrak{k}_2 \prec \mathfrak{k}_1$, $\mathfrak{l}_2 \prec \mathfrak{l}_1$, we have:

$$\begin{aligned} &\tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}_1} \preceq \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}_1} \succeq \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{i}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}_1} \succeq \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}_1} \preceq \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{j}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}_1} \succeq \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}_1} \preceq \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{k}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}_1} \succeq \tilde{\mathfrak{f}}^{\mathcal{L}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}_2}, \\ &\tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}_1} \preceq \tilde{\mathfrak{f}}^{\mathcal{U}(\mathrm{X})}_{\mathcal{Q}\mathfrak{l}_2}. \end{aligned}$$

Then,

$$\mathcal{Q}(\mathfrak{i}_2,\mathfrak{j}_2,\mathfrak{k}_2,\mathfrak{l}_2) \subseteq \mathcal{Q}(\mathfrak{i}_1,\mathfrak{j}_1,\mathfrak{k}_1,\mathfrak{l}_1)$$

for $0 \preceq \mathfrak{i}_1^n \preceq 1$, we get:

$$\tilde{\mathfrak{f}}_{\mathcal{Q}_0}\mathcal{L}(\mathrm{X}) \preceq \tilde{\mathfrak{f}}_{\mathcal{Q}_{\mathfrak{i}_1^n}}(\mathrm{X}) \preceq \tilde{\mathfrak{f}}_{\mathcal{Q}_1}\mathcal{L}(\mathrm{X}).$$

Thus,

$$|\tilde{\mathfrak{f}}_{\mathcal{Q}_{\mathfrak{i}_1^n}}(\mathrm{X})| \preceq \max\{|\tilde{\mathfrak{f}}_{\mathcal{Q}_0}\mathcal{L}(\mathrm{X})|, |\tilde{\mathfrak{f}}_{\mathcal{Q}_1}\mathcal{L}(\mathrm{X})|\} = h(\mathrm{X}).$$

Since $\tilde{\mathfrak{f}}_{\mathcal{Q}_0}\mathcal{L}(\mathrm{X})$ and $\tilde{\mathfrak{f}}_{\mathcal{Q}_1}\mathcal{L}(\mathrm{X})$ are qudri-neutrosophic Riemann integrable on $[a_1, b_1]$, $|\tilde{\mathfrak{f}}_{\mathcal{Q}_0}\mathcal{L}(\mathrm{X})|$, $|\tilde{\mathfrak{f}}_{\mathcal{Q}_1}\mathcal{L}(\mathrm{X})|$, and $h(\mathrm{X})$ are also qudri-neutrosophic Riemann integrable on $[a_1, b_1]$.

From Proposition 9, it follows that $h(\mathrm{X})$ is qudri-neutrosophic Lebesgue integrable on $[a_1, b_1]$. Now we should apply the quadri-neutrosophic Lebesgue dominated convergence theorem. For $\mathfrak{i}_1^n \uparrow \mathfrak{i}_1$, we have:

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X})d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}_1}(\mathrm{X})d(\mathrm{X}).$$

Since $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$ is a closed QNVF, we have

$$\lim_{n\to\infty} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}) = \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}),$$

by Definition 9. Again, for $0 \preceq \mathfrak{i}_1^n \preceq 1$, we have:

$$\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}1}(\mathrm{X}) \preceq \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}) \preceq \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}0}(\mathrm{X}).$$

Then,

$$|\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X})| \preceq \max\{|\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}1}(\mathrm{X})|, |\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}0}(\mathrm{X})|\} = g(\mathrm{X}).$$

By a similar fashion, we can argue:

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}) d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}_1}(\mathrm{X}) d(\mathrm{X}).$$

Since $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$ is a closed QNVF, then:

$$\lim_{n\to\infty} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}) = \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}_1^n}(\mathrm{X}).$$

By a similar process, we have:

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}_1^n}(\mathrm{X}) d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}_1}(\mathrm{X}) d(\mathrm{X}),$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}_1^n}(\mathrm{X}) d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}_1}(\mathrm{X}) d(\mathrm{X}),$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}_1^n}(\mathrm{X}) d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}_1}(\mathrm{X}) d(\mathrm{X}),$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}_1^n}(\mathrm{X}) d(\mathrm{X}) = \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}_1}(\mathrm{X}) d(\mathrm{X}).$$

Theorem 3. *If $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$ is a closed bounded QNVF on the closed quadri-neutrosophic bounded interval $[a_1, b_1]$, and the functions $\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{i}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{i}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{k}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{k}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})$, $\tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})$ are all continuous on the interval $[a_1, b_1]$, then we have:*

$$\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI} \textit{ on the closed bounded interval } [a_1, b_1]$$

Thus, the integral is given by:

$$\left(\int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(x)dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \begin{bmatrix} \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{i}}(\mathrm{X})d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{i}}(\mathrm{X})d\mathrm{X} \\ \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})d\mathrm{X} \\ \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{k}}(\mathrm{X})d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{k}}(\mathrm{X})d\mathrm{X} \\ \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})d\mathrm{X} \end{bmatrix}$$

Proof. From Proposition 8, the functions

$$\tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X}), \quad \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})$$

are Riemann integrable on $[a_1, b_1]$. Then, $\tilde{f}_{\mathcal{Q}}(\mathrm{X})$ is Riemann integrable on $[a_1, b_1]$, and we have

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(x)\,dx\right)_{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}} = \begin{bmatrix} \begin{bmatrix} \int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\,d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\,d\mathrm{X} \end{bmatrix} \\ \begin{bmatrix} \int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})\,d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})\,d\mathrm{X} \end{bmatrix} \\ \begin{bmatrix} \int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})\,d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})\,d\mathrm{X} \end{bmatrix} \\ \begin{bmatrix} \int_{a_1}^{b_1} \tilde{f}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})\,d\mathrm{X}, & \int_{a_1}^{b_1} \tilde{f}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})\,d\mathrm{X} \end{bmatrix} \end{bmatrix}$$

Theorem 4. *Let* $\tilde{f}_{\mathcal{Q}}(\mathrm{X})$ *and* $\tilde{g}_{\mathcal{Q}}(\mathrm{X})$ *be closed bounded quadri-neutrosophic valued functions on* $[a_1, b_1]$. *If* $\tilde{f}_{\mathcal{Q}}(\mathrm{X}) \cdot \tilde{g}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI}$, *then* $\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI}$ *and* $\tilde{f}_{\mathcal{Q}}(\mathrm{X}) - \tilde{g}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI}$.
Moreover, we have

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))\, dx = \int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(\mathrm{X})\, dx + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{Q}}(\mathrm{X})\, dx$$

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) - \tilde{g}_{\mathcal{Q}}(\mathrm{X}))\, dx = \int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(\mathrm{X})\, dx - \int_{a_1}^{b_1} \tilde{g}_{\mathcal{Q}}(\mathrm{X})\, dx$$

Proof.
Let $\tilde{h}_{\mathcal{Q}}(\mathrm{X}) = \tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X})$, then $\tilde{h}_{\mathcal{Q}}(\mathrm{X})$ is a closed quadri-neutrosophic valued function.

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(\mathrm{X})\, dx + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{Q}}(\mathrm{X})\, dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

$$\left\langle \Big[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} \Big] \right\rangle$$

$$\Big[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} \Big]$$

$$\Big[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} \Big]$$

$$\Big\langle \Big[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} \Big] \Big\rangle$$

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(\mathrm{X})\, dx + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{Q}}(\mathrm{X})\, dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

$$\Big[\ \Big[\textstyle\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{i}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} \Big] \ \Big]$$

$$\left[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{j}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right]$$

$$\left[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{k}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right]$$

$$\left[\int_{a_1}^{b_1} \tilde{f}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{L}}(\mathrm{X})\, d\mathrm{X}, \quad \int_{a_1}^{b_1} \tilde{f}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X} + \int_{a_1}^{b_1} \tilde{g}^{\mathfrak{l}}_{\mathcal{Q},\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right]$$

$$\langle \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{i}}_{\mathcal{L}}\, dx, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{i}}_{\mathcal{U}}\, dx,$$

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{j}}_{\mathcal{L}}\, dx, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{j}}_{\mathcal{U}}\, dx,$$

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{k}}_{\mathcal{L}}\, dx, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))^{\mathfrak{k}}_{\mathcal{U}}\, dx,$$

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))_{\mathcal{L}}^{\mathfrak{l}}\, dx, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{Q}}(\mathrm{X}) + \tilde{g}_{\mathcal{Q}}(\mathrm{X}))_{\mathcal{U}}^{\mathfrak{l}}\, dx\rangle$$

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{Q}}(\mathrm{X})\, dx + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{Q}}(\mathrm{X})\, d\mathrm{X}\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

Theorem 5. *Let $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$ be a closed bounded quadri-neutrosophic valued function on the closed interval $[a_1, b_1]$.*

If $\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI}$, then $\lambda\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X}) \in \mathcal{QRI}$. Moreover,

$$\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx = \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx$$

where $\lambda \neq 0$ is any real number.

Proof.

For $\lambda \succ 0$, let $\tilde{g}_{\mathcal{Q}}(\mathrm{X}) = \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})$. Also, $\tilde{g}_{\mathcal{Q}}(\mathrm{X})$ is a closed quadri-neutrosophic valued function (QNVF).

$$\left(\lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

$$\Bigg\langle \left[\lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{i}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{i}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{j}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{j}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{k}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{k}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{l}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{l}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right] \Bigg\rangle$$

$$\Bigg\langle \left[\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{i}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{i}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{j}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{j}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{k}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{k}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right],$$

$$\left[\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{l}}^{\mathcal{L}}(\mathrm{X})\, d\mathrm{X},\, \int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}\mathfrak{l}}^{\mathcal{U}}(\mathrm{X})\, d\mathrm{X}\right] \Bigg\rangle$$

$$\left(\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}$$

Since,

$$\int_{a_1}^{b_1} \lambda\tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx = \lambda \int_{a_1}^{b_1} \tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, dx$$

the same holds for $\lambda \prec 0$.

Example 1. *If*

$$\tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X})) = \tilde{n}(\mathrm{X})^2 \quad on \quad [0,1]$$

where

$$\tilde{n} = \left\langle (0,1,2,3); 8\times 10^{-1}, 6\times 10^{-1}, 4\times 10^{-1}, 4\times 10^{-1} \right\rangle$$

is a quadri-neutrosophic valued number. Now, we integrate the quadri-neutrosophic valued function on $[0,1]$ *and try to find*

$$\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}}(\mathrm{X})\, d\mathrm{X}.$$

Now, taking the $(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})$*-cut of the integral, we obtain:*

$$\left(\int_0^1 \tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X}))\, d\mathrm{X}\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})} = \left(\int_0^1 \tilde{n}(\mathrm{X})^2\, d\mathrm{X}\right)_{(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l})}.$$

This results in:

$$\Bigg\langle \left[\int_0^1 \frac{5\mathfrak{i}}{4}\mathrm{X}^2\, d\mathrm{X}, \int_0^1 \frac{8-5\mathfrak{i}}{4}\mathrm{X}^2\, d\mathrm{X}\right], \left[\int_0^1 \frac{5}{2}(1-\mathfrak{j})\mathrm{X}^2\, d\mathrm{X}, \int_0^1 \frac{1}{2}(5\mathfrak{j}-1)\mathrm{X}^2\, d\mathrm{X}\right],$$

$$\left[\int_0^1 \frac{5}{3}(1-\mathfrak{k})\mathrm{X}^2\, d\mathrm{X}, \int_0^1 \frac{1}{3}(5\mathfrak{k}-1)\mathrm{X}^2\, d\mathrm{X}\right], \left[\int_0^1 \frac{5}{3}(1-\mathfrak{l})\mathrm{X}^2\, d\mathrm{X}, \int_0^1 \frac{1}{3}(5\mathfrak{l}-1)\mathrm{X}^2\, d\mathrm{X}\right] \Bigg\rangle.$$

Evaluating the integrals:

$$\left\langle \left[\frac{5\mathfrak{i}}{12}, \frac{8-5\mathfrak{i}}{12}\right], \left[\frac{5}{6}(1-\mathfrak{j}), \frac{1}{6}(5\mathfrak{j}-1)\right], \left[\frac{5}{9}(1-\mathfrak{k}), \frac{1}{9}(5\mathfrak{k}-1)\right], \left[\frac{5}{9}(1-\mathfrak{l}), \frac{1}{9}(5\mathfrak{l}-1)\right] \right\rangle.$$

Thus, we obtain:

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\, d\mathrm{X} = \frac{5\mathfrak{i}}{12}, \quad \int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\, d\mathrm{X} = \frac{8-5\mathfrak{i}}{12},$$

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})\, d\mathrm{X} = \frac{5}{6}(1-\mathfrak{j}), \quad \int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})\, d\mathrm{X} = \frac{1}{6}(5\mathfrak{j}-1).$$

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})\, d\mathrm{X} = \frac{5}{9}(1-\mathfrak{k}), \quad \int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})\, d\mathrm{X} = \frac{1}{9}(5\mathfrak{k}-1).$$

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})\, d\mathrm{X} = \frac{5}{9}(1-\mathfrak{l}), \quad \int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{l}}(\mathrm{X})\, d\mathrm{X} = \frac{1}{9}(5\mathfrak{l}-1).$$

For parameter ranges:

$$\mathfrak{i} \in [0, 8\times 10^{-1}], \quad \mathfrak{j} \in [6\times 10^{-1}, 1], \quad \mathfrak{k} \in [4\times 10^{-1}, 1], \quad \mathfrak{l} \in [4\times 10^{-1}, 1].$$

From the table, it is observed that as the value of α *increases, the value of*

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\, d\mathrm{X}$$

also increases, whereas the value of

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\, d\mathrm{X}$$

decreases. At $\mathfrak{i} = 0.8$*, we obtain:*

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})\, d\mathrm{X}.$$

$\mathfrak{i}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})d\mathrm{X}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{i}}(\mathrm{X})d\mathrm{X}$	$\mathfrak{j}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})d\mathrm{X}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{j}}(\mathrm{X})d\mathrm{X}$	$\mathfrak{k}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})d\mathrm{X}$	$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}\mathfrak{k}}(\mathrm{X})d\mathrm{X}$	$\mathfrak{l}$
0×10^{-1}	0×10^{-1}	0.66667×10^{-5}	6×10^{-1}	0.33333×10^{-5}	0.33333×10^{-5}	4×10^{-1}	0.33333×10^{-5}	0.33333×10^{-5}	4×10^{-1}
1.6667×10^{-5}	5×10^{-1}	8×10^{-1}	0.16667×10^{-5}	5×10^{-1}	6×10^{-1}	0.22222×10^{-5}	0.44444×10^{-5}	6×10^{-1}	0.22222×10^{-5}
0.44444×10^{-5}	6×10^{-1}	0.25×10^{-2}	0.41667×10^{-5}	9×10^{-1}	0.08333×10^{-5}	0.58333×10^{-5}	8×10^{-1}	0.11111×10^{-5}	0.66667×10^{-5}
8×10^{-1}	0.11111×10^{-5}	0.66667×10^{-5}	8×10^{-1}	0.33333×10^{-5}	0.33333×10^{-5}	1×10^{0}	0×10^{-1}	0.66667×10^{-5}	1×10^{-1}

Table 1: Solution for different values of $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l}$ for Example 1

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathfrak{u}}_{\mathcal{Q}_\mathfrak{i}}(\mathrm{X})\, d\mathrm{X}$$

gives the same solution. Again, when the value of $\mathfrak{j}$ increases, the value of

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})\, d\mathrm{X}$$

decreased, and the value of

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathfrak{u}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})\, d\mathrm{X}$$

increased. At $\mathfrak{j} = 0.6$,

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})\, d\mathrm{X}$$

and

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathfrak{u}}_{\mathcal{Q}_\mathfrak{j}}(\mathrm{X})\, d\mathrm{X}$$

give the same solution.

When $\mathfrak{k}$ *increases, the value of*

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_{\|}}(\mathrm{X})\, d\mathrm{X}$$

decreases, and the value of

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_{\|}}(\mathrm{X})\, d\mathrm{X}$$

increases. Similarly, when $\mathfrak{l}$ *increases, the value of*

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{L}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})\, d\mathrm{X}$$

decreases, and the value of

$$\int_0^1 \tilde{\mathfrak{f}}^{\mathcal{U}}_{\mathcal{Q}_\mathfrak{l}}(\mathrm{X})\, d\mathrm{X}$$

increases. This implies that the approximate solution in Table 1 provides a quadri-neutrosophic-valued number. In the following example, we are going to show that how one can use the existing numerical integration methods to solve the quadri-neutrosophic integral. So, in the following example, we consider the same quadri-neutrosophic function, but the parameter was taken in the form of trapezoidal quadri-neutrosophic number.

Example 2. *Let us consider the Quadri-Neutrosophic Valued Function (QNVF) given by:*

$$\tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X})) = \tilde{n}(\mathrm{X})^2 \quad \text{on } [0, 1]$$

where

$$\tilde{n} = \langle (0, 1, 2, 2); 0.8, 0.6, 0.4, 0.4 \rangle$$

is a single-valued triangular quadri-neutrosophic number.

Now, we integrate the quadri-neutrosophic valued function over the interval $[0, 1]$*, aiming to find:*

$$\int_0^1 \tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X}))\, d\mathrm{X}$$

Next, we take the $(\mathfrak{i}, \mathfrak{j}, \|, \mathfrak{l})$*-cut of the integral:*

$$\int_0^1 \tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X}))\, d\mathrm{X}$$

Then, we have:

$$\left(\int_0^1 \tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X}))\, d\mathrm{X}\right)_{(\mathfrak{i},\mathfrak{j},\|,\mathfrak{l})}$$

$$= \begin{bmatrix} \left[\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{i}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{i}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}\right], \\ \left[\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{j}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{j}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}\right], \\ \left[\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\|}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\|}\mathcal{U}(\mathrm{X})\, d\mathrm{X}\right], \\ \left[\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{l}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{l}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}\right] \end{bmatrix}$$

Since each of the integral

$$\begin{aligned} &\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{i}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \quad \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{i}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}, \\ &\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{j}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \quad \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{j}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}, \\ &\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{k}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \quad \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{k}}\mathcal{U}(\mathrm{X})\, d\mathrm{X}, \\ &\int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{l}}\mathcal{L}(\mathrm{X})\, d\mathrm{X}, \quad \int_0^1 \tilde{\mathfrak{f}}_{\mathcal{Q}_\mathfrak{l}}\mathcal{U}(\mathrm{X})\, d\mathrm{X} \end{aligned}$$

are Riemann integrable on $[0, 1]$*. Then we can use the trapezoidal rule to approximate the integral:*

$$\int_0^1 x^2\, d\mathrm{X}$$

with the help of the trapezoidal rule.
Let

$$p = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

be the set of all elements at the endpoints of the sub-intervals, and

$$\Delta\mathrm{X} = \frac{1-0}{4} = \frac{1}{4}.$$

Then,

$$\int_0^1 x^2\, d\mathrm{X} \approx \frac{1}{4} \times \frac{1}{4}\left[\mathfrak{f}(0) + 2\mathfrak{f}\left(\frac{1}{4}\right) + 2\mathfrak{f}\left(\frac{1}{2}\right) + 2\mathfrak{f}\left(\frac{3}{4}\right) + \mathfrak{f}(1)\right]$$

$$= \frac{1}{8}\left(0 + \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 1\right)$$

$$= \frac{11}{32}.$$

Therefore,

$$\left(\int_0^1 \tilde{\mathfrak{f}}(\mathcal{Q}(\mathrm{X}))\,dx\right)_{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}$$

$$= \left(\int_0^1 \tilde{n}(\mathrm{X})^2\,dx\right)_{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}}$$

$$\begin{bmatrix} \int_0^1 \frac{5\mathfrak{i}}{4}(\mathrm{X})^2\,dx, & \int_0^1 \left(2 - \frac{5\mathfrak{i}}{4}\right)(\mathrm{X})^2\,dx \\ \int_0^1 \frac{5}{2}(1-\mathfrak{j})(\mathrm{X})^2\,dx, & \int_0^1 \frac{1}{2}(5\mathfrak{j}-1)(\mathrm{X})^2\,dx \\ \int_0^1 \frac{5}{3}(1-\mathfrak{k})(\mathrm{X})^2\,dx, & \int_0^1 \frac{1}{3}(5\mathfrak{k}-1)(\mathrm{X})^2\,dx \\ \int_0^1 \frac{5}{3}(1-\mathfrak{l})(\mathrm{X})^2\,dx, & \int_0^1 \frac{1}{3}(5\mathfrak{l}-1)(\mathrm{X})^2\,dx \end{bmatrix}$$

$$\begin{bmatrix} \frac{5\mathfrak{i}}{4}\int_0^1 (\mathrm{X})^2\,dx, & \frac{12-5\mathfrak{i}}{4}\int_0^1 (\mathrm{X})^2\,dx \\ \frac{5}{2}(1-\mathfrak{j})\int_0^1 (\mathrm{X})^2\,dx, & \frac{1}{2}(5\mathfrak{j}-1)\int_0^1 (\mathrm{X})^2\,dx \\ \frac{5}{3}(1-\mathfrak{k})\int_0^1 (\mathrm{X})^2\,dx, & \frac{1}{3}(5\mathfrak{k}-1)\int_0^1 (\mathrm{X})^2\,dx \\ \frac{5}{3}(1-\mathfrak{l})\int_0^1 (\mathrm{X})^2\,dx, & \frac{1}{3}(5\mathfrak{l}-1)\int_0^1 (\mathrm{X})^2\,dx \end{bmatrix}$$

	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{i}}^{\mathcal{L}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{i}}^{\mathcal{U}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{j}}^{\mathcal{L}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{j}}^{\mathcal{U}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{k}}^{\mathcal{L}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{k}}^{\mathcal{U}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{l}}^{\mathcal{L}}(\mathrm{X})d\mathrm{X}$	$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{l}}^{\mathcal{U}}(\mathrm{X})d\mathrm{X}$
0	0×10^{-1}	0×10^{-1}	1.03125×10^{-5}	6×10^{-1}	0.34375×10^{-5}	0.6875×10^{-4}	4×10^{-1}	0.34375×10^{-5}
0.4	0.34375×10^{-5}	0.6875×10^{-4}	4×10^{-1}	0.34375×10^{-5}	0.6875×10^{-4}	4×10^{-1}	0.171875×10^{-6}	0.859375×10^{-6}
0.8	0.17185×10^{-5}	0.859375×10^{-6}	6×10^{-1}	0.229167×10^{-6}	0.802083×10^{-6}	6×10^{-1}	0.229167×10^{-6}	0.802083×10^{-6}
0.6	0.257813×10^{-6}	0.773438×10^{-6}	9×10^{-1}	0.0859375×10^{-7}	0.945313×10^{-6}	8×10^{-1}	0.114583×10^{-6}	0.916667×10^{-6}
0.8	0.114583×10^{-6}	0.916667×10^{-6}	8×10^{-1}	0.34375×10^{-5}	0.6875×10^{-4}	1×10^{-1}	0×10^{-1}	1.03125×10^{-5}
1.0	0×10^{-1}	1.03125×10^{-5}	1×10^{-1}	0×10^{-1}	1.03125×10^{-5}	1×10^{-1}	0×10^{-1}	1.03125×10^{-5}

Table 2: Value of Example 2 by the Quadri-Trapezoidal Rule for Different Values of $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, and $\mathfrak{l}$

$$\begin{bmatrix} \frac{55\mathfrak{i}}{128}, & \frac{132-55\mathfrak{i}}{128} \\ \frac{55}{64}(1-\mathfrak{j}), & \frac{11}{64}(5\mathfrak{j}-1) \\ \frac{55}{96}(1-\mathfrak{k}), & \frac{11}{96}(5\mathfrak{k}-1) \\ \frac{55}{96}(1-\mathfrak{l}), & \frac{11}{96}(5\mathfrak{l}-1) \end{bmatrix}$$

Where:

$$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{i}}^{\mathcal{L}}(\mathrm{X})\,dx\Big|_0^1 = \frac{55\mathfrak{i}}{128}, \quad \int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{i}}^{\mathcal{U}}(\mathrm{X})\,dx\Big|_0^1 = \frac{132-55\mathfrak{i}}{128}$$

$$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{j}}^{\mathcal{L}}(\mathrm{X})\,dx\Big|_0^1 = \frac{55}{64}(1-\mathfrak{j}), \quad \int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{j}}^{\mathcal{U}}(\mathrm{X})\,dx\Big|_0^1 = \frac{11}{64}(5\mathfrak{j}-1)$$

$$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{k}}^{\mathcal{L}}(\mathrm{X})\,dx\Big|_0^1 = \frac{55}{96}(1-\mathfrak{k}), \quad \int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{k}}^{\mathcal{U}}(\mathrm{X})\,dx\Big|_0^1 = \frac{11}{96}(5\mathfrak{k}-1)$$

$$\int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{l}}^{\mathcal{L}}(\mathrm{X})\,dx\Big|_0^1 = \frac{55}{96}(1-\mathfrak{l}), \quad \int \tilde{\mathfrak{f}}\mathcal{Q}_{\mathfrak{l}}^{\mathcal{U}}(\mathrm{X})\,dx\Big|_0^1 = \frac{11}{96}(5\mathfrak{l}-1).$$

For:

$$\mathfrak{i} \in [0, 0.8], \quad \mathfrak{j} \in [0.6, 1], \quad \mathfrak{k} \in [0.4, 1], \quad \mathfrak{l} \in [0.4, 1]$$

6. Comparative Analysis

The following Table 3 provides a detailed comparative analysis of the proposed methods, contrasting them with the established techniques discussed in [43]. This comparison highlights the strengths and weaknesses of each approach, offering insights into how the proposed methods perform relative to the established techniques across various key factors.

Aspect	**Published Work (Reference [43])**	**Proposed Method/Work**
1. Foundation	Introduces Neutrosophic Riemann Integration, focusing on neutrosophic numbers and functions, extending classical integration theory.	Based on Quadri-Partitioned Neutrosophic Set Theory (QPNST), extending Neutrosophic Set Theory (NST) and Intuitionistic Fuzzy Set Theory (IFST). Introduces a fourth component for refined set representation.
2. Theoretical Basis	Uses neutrosophic numbers with three membership values (truth, indeterminacy, falsity) and (α, β, γ)-level sets.	Extends NST by introducing a fourth component, forming Quadri-Partitioned Neutrosophic Set Theory (QPNST), with Riemann Integral Theory (RIT) adapted to a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ level cut.
3. Conceptual Innovation	Introduces Neutrosophic Riemann Integration to manage uncertainty in classical Riemann integration.	Proposes Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPNRIT), adding a fourth uncertainty dimension for a more comprehensive uncertainty model.
4. Integration Framework	Uses neutrosophic numbers with (α, β, γ)-level sets to define Neutrosophic Riemann Integration.	Incorporates Riemann Integral Theory (RIT) into Quadri-Partitioned Neutrosophic Set Theory (QPNST), allowing for four uncertainty possibilities (true, false, indeterminacy, and a new fourth possibility).
5. Numerical Approach	Uses numerical methods like the trapezoidal rule to compute Neutrosophic Riemann integrals, validating results with examples.	Applies numerical analysis to Quadri-Partitioned Neutrosophic Riemann Integrals (QPNRIT), presenting systematic numerical results in tables to assess the impact of the fourth possibility.
6. Level Set Representation	Uses (α, β, γ)-level sets to represent truth, indeterminacy, and falsity in neutrosophic numbers.	Introduces a four-tuple level cut $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, enhancing the representation of uncertainty within QPNST and providing a more detailed integral calculation framework.
7. Output Representation	Displays results in tables and figures to validate Neutrosophic Riemann Integration, using the trapezoidal rule for approximation.	Presents Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPNRIT) results in tables, showcasing numerical insights and the fourth possibility's impact on integration.
8. Scope and Extensiveness	Introduces and numerically validates Neutrosophic Riemann Integration with graphical representations.	Extends NST and Riemann integration into a more sophisticated framework (QPNST), exploring a deeper theoretical foundation with advanced numerical techniques.

Aspect	**Published Work (Reference [43])**	**Proposed Method/Work**
9. Application Focus	The primary application in the published work is the numerical verification of neutrosophic Riemann integration, focusing on how the trapezoidal rule can be used to approximate the neutrosophic integral.	The proposed method's application focuses on improving the accuracy and handling of uncertainty in Riemann integration, with potential applications in decision-making, engineering, economics, and artificial intelligence. It also lays the groundwork for more sophisticated models and techniques in multi dimensional and dynamic systems.
10. Foundation	The published work introduces Neutrosophic Riemann Integration for the first time, focusing on neutrosophic numbers and functions. It builds on the concept of fuzziness and uncertainty in classical integration theory.	The proposed method is grounded in Quadri-Partitioned Neutrosophic Set Theory (QPNST), an extension of Neutrosophic Set Theory (NST), which itself is an extension of Intuitionistic Fuzzy Set Theory (IFST). It introduces a fourth possibility (along with true, false, and indeterminacy) for set representation, allowing for more refined descriptions of sets and their behavior.
11. Theoretical Basis	Integration framed within neutrosophic numbers with three membership values (truth, indeterminacy, and falsity), using (α, β, γ)-level sets to represent fuzzy membership.	Extends NST with a fourth possibility, defining Quadri-Partitioned Neutrosophic Set Theory (QPNST). Develops Riemann Integral Theory (RIT) within QPNST using a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ level cut.
12. Conceptual Innovation	Introduces neutrosophic Riemann integration, applying neutrosophic numbers to handle uncertainty in classical Riemann integration.	Proposes Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPN-RIT), extending neutrosophic integration by incorporating a fourth dimension of uncertainty for a more comprehensive uncertainty model.
13. Integration Framework	Defines Neutrosophic Riemann Integration using (α, β, γ)-level sets and fuzzy membership functions to represent uncertainty.	Incorporates extended Riemann Integral Theory (RIT) within QPNST, enhancing uncertainty modeling with four possibilities (true, false, indeterminacy, and a new fourth possibility).
14. Numerical Approach	Uses numerical methods like the trapezoidal rule to calculate neutrosophic Riemann integrals, validated with numerical examples, tables, and figures.	Conducts numerical exploration of Quadri-Partitioned Neutrosophic Riemann Integrals (QPNRIT). Systematic numerical results in tables illustrate the impact of the fourth possibility on integral computation.
15. Level Set Representation	Uses (α, β, γ)-level sets, where each parameter represents different membership degrees (truth, indeterminacy, falsity) in the neutrosophic number.	Introduces a more refined four-tuple level cut $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, providing a more granular representation of multiple possibilities within QPNST, enhancing neutrosophic Riemann integral calculation.

Aspect	**Published Work (Reference [43])**	**Proposed Method/Work**
16. Output Representation	Presents numerical results in tables and figures to validate neutrosophic Riemann integration, with the trapezoidal rule used for approximation.	Presents Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPN-RIT) results in tables, offering advanced integral representations to analyze the impact of the fourth possibility on integration.
17. Scope and Extensiveness	Focuses on basic introduction and numerical validation of neutrosophic Riemann integration, including examples and graphical representations.	Extends neutrosophic set theory within Riemann integration to a more complex framework (QPNST), deepening theoretical foundations and numerical techniques for broader applications.
18. Application Focus	Primarily applies to numerical verification of neutrosophic Riemann integration, emphasizing the trapezoidal rule for approximation.	Enhances accuracy and uncertainty handling in Riemann integration with potential applications in decision-making, engineering, economics, and AI, forming the basis for sophisticated multidimensional and dynamic system models.

Table 3: Comparison between Published Work and Proposed Method

7. Operations on Quadri-Partitioned Neutrosophic Soft Sets

Neutrosophic set theory (NST), a generality of vague set theory (VST), is regarded as the most appealing theory since it considers the three possible membership values: true, false, and indeterminacy. The principles are all quite obvious, but the third one is particularly fascinating since it addresses uncertainty, which arises in all aspects of daily life. One can make the situation more certain and free of error if the indeterminacy membership is refined. This can be done by splitting the indeterminacy into five pieces that is possible values. These are relative true, relative false, contradiction, unknown (undefined) and ignorance. This section is devoted to the most basic operations of union, intersection, difference, and absolute null, absolute HPNNNs. Theorems and examples are given for better understanding the situation.

Definition 13. *Let Ω be the set of parameters and X be the key set. Let $P(X)$ represent the power set of X. Then, a Quadri-Partitioned Neutrosophic Set Structure (QPNSS) $(\tilde{F}, \Omega)$ over X is a mapping*

$$\tilde{F} : \Omega \to P(X)$$

where $\tilde{F}$ is the function of QPNSS $(\tilde{F}, \Omega)$. Symbolically,

$$(\tilde{F}, \Omega) = \left[\left(\theta, \left\langle x, AbT_{\tilde{F}(\theta)(x)}, ReT_{\tilde{F}(\theta)(x)}, ReF_{\tilde{F}(\theta)(x)}, AbF_{\tilde{F}(\theta)(x)} : x \in X \right\rangle \right) : \theta \in \Omega \right].$$

Here, $AbT_{\tilde{F}(\theta)(x)}$, $ReT_{\tilde{F}(\theta)(x)}$, $ReF_{\tilde{F}(\theta)(x)}$, and $AbF_{\tilde{F}(\theta)(x)}$ belong to the interval $[0,1]$. These functions are referred to as:

- $AbT_{\tilde{F}(\theta)(x)}$*: Absolute true-membership function,*
- $ReT_{\tilde{F}(\theta)(x)}$*: Relative true-membership function,*
- $ReF_{\tilde{F}(\theta)(x)}$*: Relative false-membership function,*
- $AbF_{\tilde{F}(\theta)(x)}$*: Absolute false-membership function of $\tilde{F}(\theta)$.*

Since the supremum of each function is 1 *and the infimum of each function is* 0*, the following inequality holds automatically:*

$$0 \leq AbT_{\tilde{F}(\theta)(x)} + ReT_{\tilde{F}(\theta)(x)} + ReF_{\tilde{F}(\theta)(x)} + AbF_{\tilde{F}(\theta)(x)} \leq 4.$$

Definition 14. *Let $(\tilde{F}, \Omega)$ be a Quadri-Partitioned Neutrosophic Soft Set (QPNSS) over the key set X. Then, the complement of $(\tilde{F}, \Omega)$ is denoted by $(\tilde{F}, \Omega)^c$ and is defined as follows:*

$$(\tilde{F}, \Omega)^c = \left[\left(\theta, \left\langle x, AbF_{\tilde{F}(\theta)(x)}, ReF_{\tilde{F}(\theta)(x)}, ReT_{\tilde{F}(\theta)(x)}, AbT_{\tilde{F}(\theta)(x)} : x \in X\right\rangle\right) : \theta \in \Omega\right]$$

Furthermore, the double complement satisfies:

$$\left((\tilde{F}, \Omega)^c\right)^c = (\tilde{F}, \Omega).$$

Definition 15. *Let $(\tilde{F}, \Omega)$ and $(\tilde{G}, \Omega)$ be two Quadri-Partitioned Neutrosophic Soft Sets (QPNSSs) over the key set X. Then, $(\tilde{F}, \Omega) \subseteq (\tilde{G}, \Omega)$ if*

$$AbT_{\tilde{F}(\theta)(x)} \preceq AbT_{\tilde{G}(\theta)(x)}, \quad ReT_{\tilde{F}(\theta)(x)} \preceq ReT_{\tilde{G}(\theta)(x)},$$

$$ReF_{\tilde{F}(\theta)(x)} \succeq ReF_{\tilde{G}(\theta)(x)}, \quad AbF_{\tilde{F}(\theta)(x)} \succeq AbF_{\tilde{G}(\theta)(x)},$$

for all $\theta \in \Omega$ and for all $x \in X$.

If $(\tilde{F}, \Omega) \subseteq (\tilde{G}, \Omega)$ and $(\tilde{F}, \Omega) \supseteq (\tilde{G}, \Omega)$, then

$$(\tilde{F}, \Omega) = (\tilde{G}, \Omega).$$

Definition 16. *Let $(\tilde{F}, \Omega)$ and $(\tilde{G}, \Omega)$ be two QPNSSs over key set X such that $(\tilde{F}, \Omega) \neq (\tilde{G}, \Omega)$. Then their union is denoted by $(\tilde{F}, \Omega)\tilde{\cup}(\tilde{G}, \Omega) = (\tilde{H}, \Omega)$ and is defined as:*

$$(\tilde{H}, \Omega) = \left[\left(\theta, \langle x, AbT_{\tilde{H}(\theta)(x)}, ReT_{\tilde{H}(\theta)(x)}, ReF_{\tilde{H}(\theta)(x)}, AbF_{\tilde{H}(\theta)(x)} : x \in X\rangle\right) : \theta \in \Omega\right]$$

where,

$$\begin{aligned} AbT_{\tilde{H}(\theta)(x)} &= \max\left[AbT_{\tilde{F}(\theta)(x)}, AbT_{\tilde{G}(\theta)(x)}\right], \\ ReT_{\tilde{H}(\theta)(x)} &= \max\left[ReT_{\tilde{F}(\theta)(x)}, ReT_{\tilde{G}(\theta)(x)}\right], \\ ReF_{\tilde{H}(\theta)(x)} &= \min\left[ReF_{\tilde{F}(\theta)(x)}, ReF_{\tilde{G}(\theta)(x)}\right], \\ AbF_{\tilde{H}(\theta)(x)} &= \min\left[AbF_{\tilde{F}(\theta)(x)}, AbF_{\tilde{G}(\theta)(x)}\right]. \end{aligned}$$

Definition 17. *Let $(\tilde{F}, \Omega), (\tilde{G}, \Omega)$ be two QPNSSs over key set X such that $(\tilde{F}, \Omega) \neq (\tilde{G}, \Omega)$, then their intersection is denoted by*

$$(\tilde{F}, \Omega) \cap (\tilde{G}, \Omega) = (\tilde{H}, \Omega)$$

and is defined as

$$(\tilde{H}, \Omega) = \left[\left(\theta, \langle x, AbT_{\tilde{H}(\theta)(x)}, ReT_{\tilde{H}(\theta)(x)}, ReF_{\tilde{H}(\theta)(x)}, AbF_{\tilde{H}(\theta)(x)} : x \in X\rangle\right) : \theta \in \Omega\right]$$

where,

$$\begin{aligned} AbT_{\tilde{H}(\theta)(x)} &= \min\left[AbT_{\tilde{F}(\theta)(x)}, AbT_{\tilde{G}(\theta)(x)}\right], \\ ReT_{\tilde{H}(\theta)(x)} &= \min\left[ReT_{\tilde{F}(\theta)(x)}, ReT_{\tilde{G}(\theta)(x)}\right], \\ ReF_{\tilde{H}(\theta)(x)} &= \max\left[ReF_{\tilde{F}(\theta)(x)}, ReF_{\tilde{G}(\theta)(x)}\right], \\ AbF_{\tilde{H}(\theta)(x)} &= \max\left[AbF_{\tilde{F}(\theta)(x)}, AbF_{\tilde{G}(\theta)(x)}\right]. \end{aligned}$$

Definition 18. *Let $(\tilde{F},\Omega),(\tilde{G},\Omega)$ be two QPNSSs over key set X such that $(\tilde{F},\Omega)\neq(\tilde{G},\Omega)$, then their difference is given by*

$$(\tilde{H},\Omega)=(\tilde{F},\Omega)\setminus(\tilde{G},\Omega)$$

and is defined as

$$(\tilde{H},\Omega)=(\tilde{F},\Omega)\cap(\tilde{G},\Omega)^c$$

such that

$$(\tilde{H},\Omega)=\left[\left(\theta,\langle x,AbT_{\tilde{H}(\theta)(x)},ReT_{\tilde{H}(\theta)(x)},ReF_{\tilde{H}(\theta)(x)},AbF_{\tilde{H}(\theta)(x)}:x\in X\rangle\right):\theta\in\Omega\right]$$

where,

$$\begin{aligned}
AbT_{\tilde{H}(\theta)(x)}&=\min\left[AbT_{\tilde{F}(\theta)(x)},AbT_{\tilde{G}(\theta)(x)}\right],\\
ReT_{\tilde{H}(\theta)(x)}&=\min\left[ReT_{\tilde{F}(\theta)(x)},ReT_{\tilde{G}(\theta)(x)}\right],\\
ReF_{\tilde{H}(\theta)(x)}&=\max\left[ReF_{\tilde{F}(\theta)(x)},ReF_{\tilde{G}(\theta)(x)}\right],\\
AbF_{\tilde{H}(\theta)(x)}&=\max\left[AbF_{\tilde{F}(\theta)(x)},AbF_{\tilde{G}(\theta)(x)}\right].
\end{aligned}$$

Definition 19. *Let $\{(\tilde{F}_i,\Omega):i\in I\}$ be a family of QPNSSs over the key set X. Then,*

$$\bigcup_{i\in I}(\tilde{F}_i,\Omega)\cap\bigcap_{i\in I}(\tilde{F}_i,\Omega)$$

is given by

$$\left[\theta,\left(x,\sup_{i\in I}AbT_{\tilde{F}_i(\theta)(x)},\sup_{i\in I}ReT_{\tilde{F}_i(\theta)(x)},\inf_{i\in I}ReF_{\tilde{F}_i(\theta)(x)},\inf_{i\in I}AbF_{\tilde{F}_i(\theta)(x)}\right):\theta\in\Omega,x\in X\right].$$

Definition 20. *A quadripartitioned neutrosophic soft set $(\tilde{F},\Omega)$ over key set X is said to be a null QPNSS if*

$$\begin{aligned}
AbT_{\tilde{F}(\theta)(x)}=0,\quad ReT_{\tilde{F}(\theta)(x)}=0,\quad \forall\theta\in\Omega,\forall x\in X,\\
ReF_{\tilde{F}(\theta)(x)}=1,\quad AbF_{\tilde{F}(\theta)(x)}=1,\quad \forall\theta\in\Omega,\forall x\in X.
\end{aligned}$$

It is signified as $0(X,\Omega)$.

Definition 21. *A quadripartitioned neutrosophic soft set $(\tilde{F},\Omega)$ over key set X is an absolute QPNSS if*

$$\begin{aligned}
AbT_{\tilde{F}(\theta)(x)}=1,\quad ReT_{\tilde{F}(\theta)(x)}=1,\quad \forall\theta\in\Omega,\forall x\in X,\\
ReF_{\tilde{F}(\theta)(x)}=0,\quad AbF_{\tilde{F}(\theta)(x)}=0,\quad \forall\theta\in\Omega,\forall x\in X.
\end{aligned}$$

Clearly,

$$0(X,\Omega)^c=1(X,\Omega),\quad 1(X,\Omega)^c=0(X,\Omega).$$

Definition 22. *The family of all quadripartitioned neutrosophic soft sets over X is designated as $\mathcal{QPNSS}(X)$. Then, $x_\theta\langle r_1,r_2,r_3,r_4\rangle$ is called a QPNS point for every point $x\in X$, $\theta\in\Omega$, and is defined as follows:*

$$x_\theta\langle r_1,r_2,r_3,r_4\rangle_{\theta'}/(\mathfrak{Y})=\begin{cases}\langle r_1,r_2,r_3,r_4\rangle, & \text{if } \theta'=\theta \text{ and } \mathfrak{Y}=x,\\ (0,0,0,1), & \text{if } \theta'\neq\theta \text{ or } \mathfrak{Y}\neq x.\end{cases}$$

Definition 23. *Let $(\tilde{F},\Omega)$ be a quadripartitioned neutrosophic soft set over key set X. Then, $x_\theta\langle r_1,r_2,r_3,r_4\rangle\in QPNSS(\tilde{F},\Omega)$ if*

$$r_1\preceq AbT_{\tilde{F}(\theta)(x)},\quad r_2\preceq ReT_{\tilde{F}(\theta)(x)},\quad r_3\succeq ReF_{\tilde{F}(\theta)(x)},\quad r_4\succeq AbF_{\tilde{F}(\theta)(x)}.$$

Theorem 6. *Let $(\tilde{F},\Omega)$, $(\tilde{G},\Omega)$, and $(\tilde{H},\Omega)$ be quadripartitioned neutrosophic soft sets over key set X. Then, the following properties hold:*

(i) $(\tilde{F},\Omega)\cup_{\sim}[(\tilde{G},\Omega)\cup_{\sim}(\tilde{H},\Omega)]=[(\tilde{F},\Omega)\cup_{\sim}(\tilde{G},\Omega)]\cup_{\sim}(\tilde{H},\Omega)$.

(ii) $(\tilde{F},\Omega)\cap_{\sim}[(\tilde{G},\Omega)\cap_{\sim}(\tilde{H},\Omega)]=[(\tilde{F},\Omega)\cap_{\sim}(\tilde{G},\Omega)]\cap_{\sim}(\tilde{H},\Omega)$.

(iii) $(\tilde{F},\Omega)\cup_{\sim}[(\tilde{G},\Omega)\cap_{\sim}(\tilde{H},\Omega)]=[(\tilde{F},\Omega)\cup_{\sim}(\tilde{G},\Omega)]\cap_{\sim}[(\tilde{F},\Omega)\cup_{\sim}(\tilde{H},\Omega)]$.

(iv) $(\tilde{F},\Omega)\cap_{\sim}[(\tilde{G},\Omega)\cup_{\sim}(\tilde{H},\Omega)]=[(\tilde{F},\Omega)\cap_{\sim}(\tilde{G},\Omega)]\cup_{\sim}[(\tilde{F},\Omega)\cap_{\sim}(\tilde{H},\Omega)]$.

(v) $(\tilde{F},\Omega)\cup_{\sim}0(X,\Omega)=(\tilde{F},\Omega)$.

(vi) $(\tilde{F},\Omega)\cap_{\sim}0(X,\Omega)=0(X,\Omega)$.

(vii) $(\tilde{F},\Omega)\cup_{\sim}1(X,\Omega)=1(X,\Omega)$.

(viii) $(\tilde{F},\Omega)\cap_{\sim}1(X,\Omega)=(\tilde{F},\Omega)$.

Proof. Obvious.

Theorem 7. *Let $(\tilde{F},\Omega)$ and $(\tilde{G},\Omega)$ be QPNSSs over key set X. Then, the following De Morgan's laws hold:*

(i) $[(\tilde{F},\Omega)\cup_{\sim}(\tilde{G},\Omega)]^c=(\tilde{F},\Omega)^c\cap_{\sim}(\tilde{G},\Omega)^c$.

(ii) $[(\tilde{F},\Omega)\cap_{\sim}(\tilde{G},\Omega)]^c=(\tilde{F},\Omega)^c\cup_{\sim}(\tilde{G},\Omega)^c$.

Proof. Obvious.

Theorem 8. *Let $(\tilde{F},\Omega)$ and $(\tilde{G},\Omega)$ be QPNSSs over key set X. Then, the following De Morgan's laws hold:*

(i) $[(\tilde{F},\Omega)\vee_{\sim}(\tilde{G},\Omega)]^c=(\tilde{F},\Omega)^c\wedge_{\sim}(\tilde{G},\Omega)^c$.

(ii) $[(\tilde{F},\Omega)\cap_{\sim}(\tilde{G},\Omega)]^c=(\tilde{F},\Omega)^c\cup_{\sim}(\tilde{G},\Omega)^c$.

Proof. **1.** $\forall(\theta_1,\theta_2)\in\Omega\times\Omega,\forall x\in X,$

$$
\begin{aligned}
(\tilde{F},\Omega)\vee(\tilde{G},\Omega)=\Big\{\Big(&x,\max\left[\mathrm{AbT}_{\tilde{F}(\theta)(x)},\mathrm{AbT}_{G(\theta)(x)}\right],\\
&\max\left[\mathrm{ReT}_{\tilde{F}(\theta)(x)},\mathrm{ReT}_{G(\theta)(x)}\right],\\
&\min\left[\mathrm{ReF}_{\tilde{F}(\theta)(x)},\mathrm{ReF}_{G(\theta)(x)}\right],\\
&\min\left[\mathrm{AbF}_{\tilde{F}(\theta)(x)},\mathrm{AbF}_{G(\theta)(x)}\right]\Big)\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
[(\tilde{F},\Omega)\vee(\tilde{G},\Omega)]^c=\Big\{\Big(&x,\min\left[\mathrm{AbF}_{\tilde{F}(\theta)(x)},\mathrm{AbF}_{G(\theta)(x)}\right],\\
&\min\left[\mathrm{ReF}_{\tilde{F}(\theta)(x)},\mathrm{ReF}_{G(\theta)(x)}\right],\\
&\max\left[\mathrm{ReT}_{\tilde{F}(\theta)(x)},\mathrm{ReT}_{G(\theta)(x)}\right],
\end{aligned}
$$

$$\max\left[\mathrm{AbT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{G(\theta)(x)}\right]\Big)\Bigg\}.$$

Now,

$$(\tilde{F},\Omega)^c = \left\{\langle x, \mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{\tilde{F}(\theta)(x)}\rangle\right\},$$

$$(\tilde{G},\Omega)^c = \left\{\langle x, \mathrm{AbF}_{\tilde{G}(\theta)(x)}, \mathrm{ReF}_{\tilde{G}(\theta)(x)}, \mathrm{ReT}_{\tilde{G}(\theta)(x)}, \mathrm{AbT}_{\tilde{G}(\theta)(x)}\rangle\right\}.$$

Thus,

$$\begin{aligned}(\tilde{F},\Omega)^c \wedge (\tilde{G},\Omega)^c = \Bigg\{\Big(x, &\min\left[\mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{AbF}_{G(\theta)(x)}\right],\\ &\min\left[\mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{G(\theta)(x)}\right],\\ &\max\left[\mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{G(\theta)(x)}\right],\\ &\max\left[\mathrm{AbT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{G(\theta)(x)}\right]\Big)\Bigg\}.\end{aligned}$$

Therefore,

$$[(\tilde{F},\Omega) \vee (\tilde{G},\Omega)]^c = (\tilde{F},\Omega)^c \wedge (\tilde{G},\Omega)^c.$$

2. $\forall(\theta_1,\theta_2) \in \Omega \times \Omega, \forall x \in X$

$$\begin{aligned}(\tilde{F},\Omega) \wedge (\tilde{G},\Omega) = \Bigg\{\Big(x, &\min\left[\mathrm{T}_{\tilde{F}(\theta)(x)}, \mathrm{T}_{G(\theta)(x)}\right],\\ &\min\left[\mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{G(\theta)(x)}\right],\\ &\max\left[\mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{G(\theta)(x)}\right],\\ &\max\left[\mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{AbF}_{G(\theta)(x)}\right]\Big)\Bigg\}.\end{aligned}$$

$$\begin{aligned}[(\tilde{F},\Omega) \wedge (\tilde{G},\Omega)]^c = \Bigg\{\Big(x, &\max\left[\mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{AbF}_{G(\theta)(x)}\right],\\ &\max\left[\mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{G(\theta)(x)}\right],\\ &\min\left[\mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{G(\theta)(x)}\right],\\ &\min\left[\mathrm{AbT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{G(\theta)(x)}\right]\Big)\Bigg\}.\end{aligned}$$

Now,

$$(\tilde{F},\Omega)^c = \left\{\langle x, \mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{\tilde{F}(\theta)(x)}\rangle\right\},$$

$$(\tilde{G},\Omega)^c = \left\{\langle x, \mathrm{AbF}_{\tilde{G}(\theta)(x)}, \mathrm{ReF}_{\tilde{G}(\theta)(x)}, \mathrm{ReT}_{\tilde{G}(\theta)(x)}, \mathrm{AbT}_{\tilde{G}(\theta)(x)}\rangle\right\}.$$

Thus,

$$(\tilde{F},\Omega)^c \vee (\tilde{G},\Omega)^c = \Big\{ \Big(x, \max\big[\mathrm{AbF}_{\tilde{F}(\theta)(x)}, \mathrm{AbF}_{G(\theta)(x)}\big], \max\big[\mathrm{ReF}_{\tilde{F}(\theta)(x)}, \mathrm{ReF}_{G(\theta)(x)}\big], \min\big[\mathrm{ReT}_{\tilde{F}(\theta)(x)}, \mathrm{ReT}_{G(\theta)(x)}\big], \min\big[\mathrm{AbT}_{\tilde{F}(\theta)(x)}, \mathrm{AbT}_{G(\theta)(x)}\big] \Big) \Big\}.$$

Therefore,

$$[(\tilde{F},\Omega) \wedge (\tilde{G},\Omega)]^c = (\tilde{F},\Omega)^c \vee (\tilde{G},\Omega)^c.$$

8. A New Approach to Operations on Quadripartitioned Neutrosophic Soft Topological Space

The notion of QPNSTS is presented in this section. The terms QPNS semi-open, QPNS pre-open, and QPNS $*b$-open sets are defined. One of these intriguing QPNS generalized open sets, referred to as the QPNS pre-open set, is selected, and certain fundamentals are then produced based on this description. These consist of the QPNS closure, QPNS exterior, QPNS boundary, and QPNS interior.

Definition 24. *Let $QNSS(\tilde{X},\Omega)$ be the family of all QPNSSs and $\tau \subset QPNSS(\tilde{X},\Omega)$, then τ is a quadri-partitioned neutrosophic soft topology (QPNST) on $\tilde{X}$ if*

(i) $0_{(\langle X\rangle,\Omega)}, 1_{(\langle X\rangle,\Omega)} \in \tau$,

(ii) The union of any number of QPNSSs in τ belongs to τ,

(iii) The intersection of a finite number of QPNSSs in τ belongs to τ.

Then, $(\tilde{X},\tau,\Omega)$ is said to be a QPNSTS over $\tilde{X}$.

Definition 25. *$(\tilde{X},\tau,\Omega)$ is a QPNSTS over X. A QPNSS $(\tilde{F},\Omega)$ is a QPNS neighborhood of a QPNS point $x^{\lambda}_{\langle r_1,r_2,r_3,r_4\rangle} \in (\tilde{F},\Omega)$, if there is a QPNS open set $(\tilde{G},\Omega)$ such that $x^{\lambda}_{\langle r_1,r_2,r_3,r_4\rangle} \in (\tilde{G},\Omega)$.*

Definition 26. *Let (X,τ_1,Ω) and (X,τ_2,Ω) be two QPNSBTSs. Then, (X,τ_1,τ_2,Ω) is a QPNSBTS.*

If (X,τ_1,τ_2,Ω) is a QPNSBTS, a QPNSS subset $(\tilde{F},\Omega)$ is open in (X,τ_1,τ_2,Ω) if there exists a QPNSS open set $(\tilde{G},\Omega)$ belonging to τ_1 and a QPNSS open set $(\tilde{H},\Omega)$ belonging to τ_2 such that

$$(\tilde{F},\Omega) = (\tilde{G},\Omega) \cup (\tilde{H},\Omega).$$

Example 3. *Let $X = \{x_1,x_2,x_3\}$ and $\Omega = \{\theta_1,\theta_2\}$, $\tau_1 = \{0_{(X,\Omega)}, 1_{(X,\Omega)}, (\tilde{F},\Omega), (\tilde{G},\Omega)\}$ and $\tau_2 = \{0_{(X,\Omega)}, 1_{(X,\Omega)}, (\tilde{H},\Omega), (\tilde{I},\Omega)\}$,*

where $(\tilde{F},\Omega)$, $(\tilde{G},\Omega)$, $(\tilde{H},\Omega)$, and $(\tilde{I},\Omega)$ being QPNSSs are as follows:

$$(\tilde{F},\Omega) = \begin{bmatrix} \theta_1 = \langle x_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle x_2, \frac{4}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle x_3, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{2}{10}\rangle, \\ \theta_2 = \langle x_1, \frac{3}{10}, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle x_2, \frac{1}{10}, \frac{5}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle x_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{5}{10}\rangle \end{bmatrix}$$

$$(\tilde{G},\Omega)=\begin{bmatrix}\theta_1=\langle x_1,\frac{4}{10},\frac{3}{10},\frac{6}{10},\frac{6}{10}\rangle,\langle x_2,\frac{4}{10},\frac{5}{10},\frac{6}{10},\frac{3}{10}\rangle,\langle x_3,\frac{3}{10},\frac{5}{10},\frac{6}{10},\frac{2}{10}\rangle,\\ \theta_2=\langle x_1,\frac{3}{10},\frac{4}{10},\frac{6}{10},\frac{5}{10}\rangle,\langle x_2,\frac{2}{10},\frac{6}{10},\frac{6}{10},\frac{4}{10}\rangle,\langle x_3,\frac{4}{10},\frac{6}{10},\frac{6}{10},\frac{3}{10}\rangle\end{bmatrix}$$

$$(\tilde{H},\Omega)=\begin{bmatrix}\theta_1=\langle x_1,\frac{6}{10},\frac{6}{10},\frac{6}{10},\frac{2}{10}\rangle,\langle x_2,\frac{6}{10},\frac{6}{10},\frac{6}{10},\frac{2}{10}\rangle,\langle x_3,\frac{4}{10},\frac{6}{10},\frac{6}{10},\frac{1}{10}\rangle,\\ \theta_2=\langle x_1,\frac{5}{10},\frac{6}{10},\frac{6}{10},\frac{2}{10}\rangle,\langle x_2,\frac{6}{10},\frac{7}{10},\frac{6}{10},\frac{2}{10}\rangle,\langle x_3,\frac{5}{10},\frac{5}{10},\frac{6}{10},\frac{1}{10}\rangle\end{bmatrix}$$

$$(\tilde{I},\Omega)=\begin{bmatrix}\theta_1=\langle x_1,\frac{1}{10},\frac{2}{10},\frac{6}{10},\frac{7}{10}\rangle,\langle x_2,\frac{4}{10},\frac{4}{10},\frac{6}{10},\frac{3}{10}\rangle,\langle x_3,\frac{2}{10},\frac{4}{10},\frac{6}{10},\frac{2}{10}\rangle,\\ \theta_2=\langle x_1,\frac{3}{10},\frac{2}{10},\frac{6}{10},\frac{5}{10}\rangle,\langle x_2,\frac{1}{10},\frac{5}{10},\frac{6}{10},\frac{5}{10}\rangle,\langle x_3,\frac{4}{10},\frac{3}{10},\frac{6}{10},\frac{5}{10}\rangle\end{bmatrix}$$

Theorem 9. *Let (X,τ_1,τ_2,Ω) be a QPNSBTS. Then $\tau_1\cap\tau_2$ is a QPNSBTS on X.*

Proof. The first and third requirements are clear, and we move forward as follows for the second condition.

Let $\{(\tilde{F_i},\Omega); i\in I\}\in\tau_1\cap\tau_2$, then $(\tilde{F_i},\Omega)\in\tau_1$ and $(\tilde{F_i},\Omega)\in\tau_2$.

As τ_1 and τ_2 are QPNSBTSs on X, then $\bigcup_i(\tilde{F_i},\Omega)\in\tau_1$ and $\bigcup_i(\tilde{F_i},\Omega)\in\tau_2$.

So, $\bigcup_i(\tilde{F_i},\Omega)\in\tau_1\cap\tau_2$.

Definition 27. *Let (X,τ_1,τ_2,Ω) be a QPNSBTS over X, and let $(\tilde{\ddot{\Upsilon}},\Omega)$ be a QPNSS. Then,*

(i) $(\tilde{\ddot{\Upsilon}},\Omega)$ is QPNS semi-open if

$$(\tilde{\ddot{\Upsilon}},\Omega)\subseteq NScl(NSint(\tilde{\ddot{\Upsilon}},\Omega))$$

(ii) $(\tilde{\ddot{\Upsilon}},\Omega)$ is QPNS pre-open (p-open) if

$$(\tilde{\ddot{\Upsilon}},\Omega)\subseteq NSint(NScl(\tilde{\ddot{\Upsilon}},\Omega))$$

(iii) $(\tilde{\ddot{\Upsilon}},\Omega)$ is QPNS $*$b-open if

$$(\tilde{\ddot{\Upsilon}},\Omega)\subseteq NScl(NSint(\tilde{\ddot{\Upsilon}},\Omega))\cup NSint(NScl(\tilde{\ddot{\Upsilon}},\Omega))$$

(iv) $(\tilde{\ddot{\Upsilon}},\Omega)$ is QPNS $*$b-close if

$$(\tilde{\ddot{\Upsilon}},\Omega)\supseteq NScl(NSint(\tilde{\ddot{\Upsilon}},\Omega))\cap NSint(NScl(\tilde{\ddot{\Upsilon}},\Omega))$$

Definition 28. *Let (X,τ,Ω) be a quadripartitioned neutrosophic soft topological space over X and $(\tilde{F},\Omega)\in QPNSS(X,\Omega)$. Then, the collection*

$$\tau(\tilde{F},\Omega)=\{(\tilde{F},\Omega)\cap(\tilde{G},\Omega):(\tilde{G},\Omega)\in\tau \text{ for } i\in I\}$$

is called a quadripartitioned neutrosophic soft subspace topology on $(\tilde{F},\Omega)$, and $(\tau(F,\Omega),\tau(\tilde{F},\Omega),\Omega)$ is called a quadripartitioned neutrosophic soft topological subspace of (X,τ,Ω).

In order for the above definition to be consistent, we must prove that $(\tau(F,\Omega),\tau(\tilde{F},\Omega),\Omega)$ is actually a quadripartitioned neutrosophic soft topology for $(\tilde{F},\Omega)$.

Theorem 10. *Let (X,τ,Ω) be a quadripartitioned neutrosophic soft topological space over X and $(\tilde{F},\Omega)\in QPNSS(X,\Omega)$. Then, the collection*

$$\tau(\tilde{F},\Omega)=\{(\tilde{F},\Omega)\cap(\tilde{G},\Omega):(\tilde{G},\Omega)\in\tau\}$$

is a quadripartitioned neutrosophic soft topology on $(\tilde{F},\Omega)$, and $(X(F,\Omega),\tau(\tilde{F},\Omega),\Omega)$ is a quadripartitioned neutrosophic soft topological space of (X,τ,Ω).

Proof. (1) Since $0(\tilde{F},\Omega), 1(\tilde{F},\Omega) \in (X,\tau,\Omega)$, so by definition:

$$0(\tilde{F},\Omega) \cap (\tilde{F},\Omega) = 0(\tilde{F},\Omega),$$
$$1(\tilde{F},\Omega) \cap (\tilde{F},\Omega) = (\tilde{F},\Omega).$$

Thus, $0(\tilde{F},\Omega), (\tilde{F},\Omega) \in \tau(\tilde{F},\Omega)$.

(2) Let $\{(\tilde{H}_i,\Omega) : i \in I\}$ be a quadripartitioned neutrosophic soft sub-collection of $\tau(\tilde{F},\Omega)$. Then $(\tilde{H}_i,\Omega) \in \tau(\tilde{F},\Omega)$ for all $i \in I$, so by definition, this implies that:

$$(\tilde{H}_i,\Omega) = (\tilde{F},\Omega) \cap (\tilde{G}_i,\Omega) \quad \text{for some } (\tilde{G}_i,\Omega) \in \tau.$$

Taking the union, we obtain:

$$\bigcup_{i\in I}(\tilde{H}_i,\Omega) = \bigcup_{i\in I}\Big((\tilde{F},\Omega) \cap (\tilde{G}_i,\Omega)\Big)$$
$$= (\tilde{F},\Omega) \cap \left(\bigcup_{i\in I}(\tilde{G}_i,\Omega)\right).$$

Since $(\tilde{G}_i,\Omega) \in \tau$ for all $i \in I$ and τ is a quadripartitioned neutrosophic soft topological space, we conclude that:

$$\bigcup_{i\in I}(\tilde{G}_i,\Omega) \in \tau.$$

Thus,

$$(\tilde{F},\Omega) \cap \left(\bigcup_{i\in I}(\tilde{G}_i,\Omega)\right) \in \tau(\tilde{F},\Omega),$$

implying that $\bigcup_{i\in I}(\tilde{H}_i,\Omega) \in \tau(\tilde{F},\Omega)$.

(3) Now, let $(\tilde{H}_1,\Omega), (\tilde{H}_2,\Omega), \ldots, (\tilde{H}_n,\Omega) \in \tau(\tilde{F},\Omega)$. Then, for all $i = 1,2,\ldots,n$, we have:

$$(\tilde{H}_i,\Omega) = (\tilde{F},\Omega) \cap (\tilde{G}_i,\Omega) \quad \text{for some } (\tilde{G}_i,\Omega) \in \tau.$$

Taking the intersection, we obtain:

$$\bigcap_{i=1}^{n}(\tilde{H}_i,\Omega) = \bigcap_{i=1}^{n}\Big((\tilde{F},\Omega) \cap (\tilde{G}_i,\Omega)\Big)$$
$$= (\tilde{F},\Omega) \cap \left(\bigcap_{i=1}^{n}(\tilde{G}_i,\Omega)\right).$$

Since $(\tilde{G}_i,\Omega) \in \tau$ for all $i = 1,2,\ldots,n$ and τ is a quadripartitioned neutrosophic soft topology, we conclude that:

$$\bigcap_{i=1}^{n}(\tilde{G}_i,\Omega) \in \tau.$$

Therefore,

$$(\tilde{F},\Omega) \cap \left(\bigcap_{i=1}^{n}(\tilde{G}_i,\Omega)\right) \in \tau(\tilde{F},\Omega),$$

implying that $\bigcap_{i=1}^{n}(\tilde{H}_i,\Omega) \in \tau(\tilde{F},\Omega)$.

Thus,

$$\tau(\tilde{F},\Omega) = \{(\tilde{F},\Omega) \cap (\tilde{G},\Omega) : (\tilde{G},\Omega) \in \tau\}$$

is a quadripartitioned neutrosophic soft topology on $(\tilde{F},\Omega)$.

Definition 29. *Let $(X, \tau_1, \tau_2, \Omega)$ be a QPNSBTS over X and $(\tilde{\ddot{\Upsilon}}, \Omega)$ be a QPNS. Then the interior of $(\tilde{\ddot{\Upsilon}}, \Omega)$, designated by $(\tilde{\ddot{\Upsilon}}, \Omega)^\circ$, is the union of all QPNS s-open sets of $(\tilde{\ddot{\Upsilon}}, \Omega)$. Clearly, $(\tilde{\ddot{\Upsilon}}, \Omega)^\circ$ is the largest QPNS s-OS that is contained in $(\tilde{\ddot{\Upsilon}}, \Omega)$.*

Definition 30. *Let $(X, \tau_1, \tau_2, \Omega)$ be a QPNSBTS, and let $(\tilde{\ddot{\Upsilon}}, \Omega)$ be a QPNS. The frontier of $(\tilde{\ddot{\Upsilon}}, \Omega)$, denoted as $Fr((\tilde{\ddot{\Upsilon}}, \Omega))$, is a QPNS point ${x_1}^{\lambda}_{\langle r_1, r_2, r_3, r_4\rangle}$. A point $x_1\lambda_{\langle r_1, r_2, r_3, r_4\rangle}$ is in the frontier of $(\tilde{\ddot{\Upsilon}}, \Omega)$ if every QPNS s-open set containing $x_1\lambda_{\langle r_1, r_2, r_3, r_4\rangle}$ contains at least one point of $(\tilde{\ddot{\Upsilon}}, \Omega)$ and at least one QPNS point of $(\tilde{\ddot{\Upsilon}}, \Omega)^c$.*

Definition 31. *If $(X, \tau_1, \tau_2, \Omega)$ is a QPNSBTS and $(\tilde{\ddot{\Upsilon}}, \Omega)$ is a QPNS, then the exterior of $(\tilde{\ddot{\Upsilon}}, \Omega)$, denoted by $Ext((\tilde{\ddot{\Upsilon}}, \Omega))$, is a QPNS point ${x_1}^{\lambda}_{\langle r_1, r_2, r_3, r_4\rangle}$.*

A QPNS point ${x_1}^{\lambda}_{\langle r_1, r_2, r3, r4\rangle}$ is in the exterior of $(\tilde{\ddot{\Upsilon}}, \Omega)$ if and only if it is in the interior of $(\tilde{\ddot{\Upsilon}}, \Omega)^c$, meaning there exists a QPNS s-open set $(\tilde{g}, \Omega)$ such that

$$ {x_1}^{\lambda}_{\langle r_1, r_2, r_3, r_4\rangle} \in (\tilde{g}, \Omega) \subseteq (\tilde{\ddot{\Upsilon}}, \Omega)^c. $$

Definition 32. *If $(\tilde{X}, \tau_1, \tau_2, \Omega)$ and $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \Omega)$ are QPNSBTSs, and $(\{, \varphi) : (\tilde{X}, \tau_1, \tau_2, \Omega) \to (\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \Omega)$ is a QPNS mapping, then if the image $(\{, \varphi)(\tilde{\ddot{\Upsilon}}, \Omega)$ of each QPNS s-closed set $(\tilde{\ddot{\Upsilon}}, \Omega)$ over $\tilde{X}$ is a QPNS s-closed set in $\langle\tilde{Y}\rangle$, the mapping $(\{, \varphi)$ is said to be a QPNS s-closed mapping.*

Theorem 11. *Let $(X, \tau_1, \tau_2, \Omega)$ be a QPNSBTS over X and $(\tilde{\ddot{\Upsilon}}, \Omega)$ be a QPNS subset. Then, $(\tilde{\ddot{\Upsilon}}, \Omega)$ is a QPNS s-open set if and only if $(\tilde{\ddot{\Upsilon}}, \Omega) = (\tilde{\ddot{\Upsilon}}, \Omega)^\circ$.*

Proof. Let $(\tilde{\ddot{\Upsilon}}, \Omega)$ be a HQPNS s-open set. Then, the largest QPNS s-open set that is contained within $(\tilde{\ddot{\Upsilon}}, \Omega)$ is equal to $(\tilde{\ddot{\Upsilon}}, \Omega)$. Hence, $(\tilde{\ddot{\Upsilon}}, \Omega) = (\tilde{\ddot{\Upsilon}}, \Omega)^\circ$.

Conversely, it is known that $(\tilde{\ddot{\Upsilon}}, \Omega)^\circ$ is a QPNS s-open set, and if $(\tilde{\ddot{\Upsilon}}, \Omega) = (\tilde{\ddot{\Upsilon}}, \Omega)^\circ$, then $(\tilde{\ddot{\Upsilon}}, \Omega)$ is a QPNS p-open set.

Theorem 12. *Let $(X, \tau_1, \tau_2, \Omega)$ be a QPNSBTS over X, and let $(\tilde{F}, \Omega)$, $(\tilde{G}, \Omega)$ be QPNS subsets. Then,*

1. $\left[(\tilde{F}, \Omega)^\circ\right]^\circ = (\tilde{F}, \Omega)^\circ$,
2. $\left(0_{(\langle X\rangle,\Omega)}\right)^\circ = 0_{(\langle X\rangle,\Omega)}$ *and* $\left(1_{(\langle X\rangle,\Omega)}\right)^\circ = 1_{(\langle X\rangle,\Omega)}$,
3. $(\tilde{F}, \Omega) \subseteq (\tilde{G}, \Omega) \Rightarrow (\tilde{F}, \Omega)^\circ \subseteq (\tilde{G}, \Omega)^\circ$,
4. $\left[(\tilde{F}, \Omega) \cap (\tilde{G}, \Omega)\right]^\circ = (\tilde{F}, \Omega)^\circ \cap (\tilde{G}, \Omega)^\circ$,
5. $(\tilde{F}, \Omega)^\circ \cup (\tilde{G}, \Omega)^\circ \subseteq \left[(\tilde{F}, \Omega) \cup (\tilde{G}, \Omega)\right]^\circ$.

Proof. Let $(X, \tau_1, \tau_2, \Omega)$ be a QPNSBTS over X, and let $(\tilde{F}, \Omega)$, $(\tilde{G}, \Omega)$ be QPNS subsets. Then,

1. If $(\tilde{F}, \Omega)^\circ = (\tilde{G}, \Omega)$, then $(\tilde{G}, \Omega) \in \tau$ if and only if $(\tilde{G}, \Omega) = (\tilde{F}, \Omega)^\circ$. Thus, $\left[(\tilde{F}, \Omega)^\circ\right]^\circ = (\tilde{F}, \Omega)^\circ$.

2. Since $0_{(\langle X\rangle,\Omega)}$ and $1_{(\langle X\rangle,\Omega)}$ are always QPNS s-open sets, we have

$$ \left(0_{(\langle X\rangle,\Omega)}\right)^\circ = 0_{(\langle X\rangle,\Omega)} \quad \text{and} \quad \left(1_{(\langle X\rangle,\Omega)}\right)^\circ = 1_{(\langle X\rangle,\Omega)}. $$

3. It is known that $(\tilde{F}, \Omega)^\circ \subseteq (\tilde{F}, \Omega) \subseteq (\tilde{G}, \Omega)$ and $(\tilde{G}, \Omega)^\circ \subseteq (\tilde{G}, \Omega)$. Since $(\tilde{G}, \Omega)^\circ$ is the largest QPNS s-open set contained in $(\tilde{G}, \Omega)$, it follows that

$$ (\tilde{F}, \Omega)^\circ \subseteq (\tilde{G}, \Omega)^\circ. $$

4. Since $(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\subseteq(\tilde{F},\Omega)$ and $(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\subseteq(\tilde{G},\Omega)$, we have

$$\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}\subseteq(\tilde{F},\Omega)^{\circ}\quad\text{and}\quad\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}\subseteq(\tilde{G},\Omega)^{\circ}.$$

Therefore,

$$\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}\subseteq(\tilde{F},\Omega)^{\circ}\cap(\tilde{G},\Omega)^{\circ}.$$

Conversely, since $(\tilde{F},\Omega)^{\circ}\subseteq(\tilde{F},\Omega)$ and $(\tilde{G},\Omega)^{\circ}\subseteq(\tilde{G},\Omega)$, we obtain

$$(\tilde{F},\Omega)^{\circ}\cap(\tilde{G},\Omega)^{\circ}\subseteq(\tilde{F},\Omega)\cap(\tilde{G},\Omega).$$

Since $\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}$ is the largest QPNS s-open set contained in $(\tilde{F},\Omega)\cap(\tilde{G},\Omega)$, it follows that

$$(\tilde{F},\Omega)^{\circ}\cap(\tilde{G},\Omega)^{\circ}\subseteq\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}.$$

Thus,

$$\left[(\tilde{F},\Omega)\cap(\tilde{G},\Omega)\right]^{\circ}=(\tilde{F},\Omega)^{\circ}\cap(\tilde{G},\Omega)^{\circ}.$$

5. Since $(\tilde{F},\Omega)\subseteq(\tilde{F},\Omega)\cup(\tilde{G},\Omega)$ and $(\tilde{G},\Omega)\subseteq(\tilde{F},\Omega)\cup(\tilde{G},\Omega)$, we have

$$(\tilde{F},\Omega)^{\circ}\subseteq\left[(\tilde{F},\Omega)\cup(\tilde{G},\Omega)\right]^{\circ}\quad\text{and}\quad(\tilde{G},\Omega)^{\circ}\subseteq\left[(\tilde{F},\Omega)\cup(\tilde{G},\Omega)\right]^{\circ}.$$

Thus,

$$(\tilde{F},\Omega)^{\circ}\cup(\tilde{G},\Omega)^{\circ}\subseteq\left[(\tilde{F},\Omega)\cup(\tilde{G},\Omega)\right]^{\circ}.$$

Theorem 13. *Let (X,τ_1,τ_2,Ω) be a QPNSBTS over X, and let $(\tilde{F},\Omega)$ be a QPNS subset. Then, $(\tilde{F},\Omega)$ is a QPNS p-closer set if and only if*

$$(\tilde{F},\Omega)=\overline{(\tilde{F},\Omega)}.$$

Proof.
Let $(\tilde{F},\Omega)$ be a QPNS p-closer set. Then,

$$(\tilde{F},\Omega)^{d}=(\tilde{F},\Omega)$$

which implies

$$(\tilde{F},\Omega)\tilde{\cup}(\tilde{F},\Omega)^{d}\cong(\tilde{F},\Omega)$$
$$\Rightarrow\overline{(\tilde{F},\Omega)}\cong(\tilde{F},\Omega)$$

Conversely, let

$$\overline{(\tilde{F},\Omega)}\cong(\tilde{F},\Omega)$$

this implies

$$(\tilde{F},\Omega)\tilde{\cup}(\tilde{F},\Omega)^{d}\cong(\tilde{F},\Omega)$$
$$\Rightarrow(\tilde{F},\Omega)^{d}\cong(\tilde{F},\Omega)$$

which implies that $(\tilde{F},\Omega)$ is a QPNS p-closer set.

Theorem 14. *Let (X,τ_1,τ_2,Ω) be a QPNSBTS over X, and let $(\tilde{F},\Omega)$ and $(\tilde{G},\Omega)$ be QPNS subsets. Then:*

(i) $\overline{\overline{(\tilde{F},\Omega)}}=\overline{(\tilde{F},\Omega)}$,

(ii) $\overline{0_{(\langle \tilde{X},\Omega\rangle)}} = 0_{(\langle \tilde{X},\Omega\rangle)}$ *and* $\overline{1_{(\langle \tilde{X},\Omega\rangle)}} = 1_{(\langle \tilde{X},\Omega\rangle)}$,

(iii) $(\tilde{F},\Omega) \subseteq \langle(\tilde{G},\Omega)\rangle \Rightarrow \overline{(\tilde{F},\Omega)} \subseteq \overline{(\tilde{G},\Omega)}$,

(iv) $\overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)} = \overline{(\tilde{F},\Omega)} \cup \overline{(\tilde{G},\Omega)}$,

(v) $\overline{(\tilde{F},\Omega) \cap (\tilde{G},\Omega)} \subseteq \overline{(\tilde{F},\Omega)} \cap \overline{(\tilde{G},\Omega)}$.

Proof. Let (X,τ_1,τ_2,Ω) be a QPNSBTS over X, and let $(\tilde{F},\Omega)$ and $(\tilde{G},\Omega)$ be QPNS subsets. Then:

(i) If $\overline{(\tilde{F},\Omega)} = (\tilde{G},\Omega)$, then $(\tilde{G},\Omega)$ is a QPNS s-CS. Hence, if $(\tilde{G},\Omega)$ and $\overline{(\tilde{G},\Omega)}$ are equal, then:
$$\overline{\overline{(\tilde{F},\Omega)}} = \overline{(\tilde{F},\Omega)}$$

(ii) Since $0_{(\langle \tilde{X},\Omega\rangle)}$ and $1_{(\langle \tilde{X},\Omega\rangle)}$ are always QPNS s-CS, by the above result (1), we get:
$$\overline{0_{(\langle \tilde{X},\Omega\rangle)}} = 0_{(\langle \tilde{X},\Omega\rangle)} \quad \text{and} \quad \overline{1_{(\langle \tilde{X},\Omega\rangle)}} = 1_{(\langle \tilde{X},\Omega\rangle)}.$$

(iii) Since $(\tilde{F},\Omega) \subseteq \overline{(\tilde{F},\Omega)}$ and $(\tilde{G},\Omega) \subseteq \overline{(\tilde{G},\Omega)}$, it follows that:
$$(\tilde{F},\Omega) \subseteq (\tilde{G},\Omega) \subseteq \overline{(\tilde{G},\Omega)}.$$
Since $\overline{(\tilde{F},\Omega)}$ is the smallest QPNS p-CS covering $(\tilde{F},\Omega)$, we obtain:
$$\overline{(\tilde{F},\Omega)} \subseteq \overline{(\tilde{G},\Omega)}.$$

(iv) Since $(\tilde{F},\Omega) \subseteq (\tilde{F},\Omega) \cup (\tilde{G},\Omega)$ and $(\tilde{G},\Omega) \subseteq (\tilde{F},\Omega) \cup (\tilde{G},\Omega)$, we have:
$$\overline{(\tilde{F},\Omega)} \subseteq \overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)}$$
and
$$\overline{(\tilde{G},\Omega)} \subseteq \overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)}.$$
Thus,
$$\overline{(\tilde{F},\Omega)} \cup \overline{(\tilde{G},\Omega)} \subseteq \overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)}.$$
Conversely, since $(\tilde{F},\Omega) \subseteq \overline{(\tilde{F},\Omega)}$ and $(\tilde{G},\Omega) \subseteq \overline{(\tilde{G},\Omega)}$, we have:
$$(\tilde{F},\Omega) \cup (\tilde{G},\Omega) \subseteq \overline{(\tilde{F},\Omega)} \cup \overline{(\tilde{G},\Omega)}.$$
Since $\overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)}$ is the smallest QPNS p-closed set enclosing $(\tilde{F},\Omega) \cup (\tilde{G},\Omega)$, we obtain:
$$\overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)} \subseteq \overline{(\tilde{F},\Omega)} \cup \overline{(\tilde{G},\Omega)}.$$
Hence,
$$\overline{(\tilde{F},\Omega) \cup (\tilde{G},\Omega)} = \overline{(\tilde{F},\Omega)} \cup \overline{(\tilde{G},\Omega)}.$$

(v) Since $\langle 0_{(\langle \tilde{X},\Omega\rangle)}\rangle \cap (\tilde{G},\Omega) \subseteq \overline{(\tilde{F},\Omega)} \cap \overline{(\tilde{G},\Omega)}$ and $\overline{(\tilde{F},\Omega) \cap (\tilde{G},\Omega)}$ is the smallest QPNS p-closed set enclosing $(\tilde{F},\Omega) \cap (\tilde{G},\Omega)$, we conclude:
$$\overline{(\tilde{F},\Omega) \cap (\tilde{G},\Omega)} \subseteq \overline{(\tilde{F},\Omega)} \cap \overline{(\tilde{G},\Omega)}.$$

9. Characterization of few more results in terms of basis concerning P-open sets

Definition 33. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X, and let $\mathcal{B}^{NSS}$ be a sub-family of τ^{QPNSS}.*

$\mathcal{B}^{NSS}$ is said to be a quadripartitioned neutrosophic soft base (or p-open base or basis) for the quadripartitioned neutrosophic soft topology τ^{QPNSS} if for any non-empty quadripartitioned neutrosophic soft set $(\tilde{G}, \Omega) \in \tau^{QPNSS}$, there exists $\mathcal{B}_1 \subseteq \mathcal{B}^{NSS}$ such that:

$$(\tilde{G}, \Omega) = \bigcup \{B : B \in \mathcal{B}_1\}.$$

In other words, $\mathcal{B}^{NSS}$ is said to be a base for the quadripartitioned neutrosophic soft topology if for every $x_{(r_1,r_2,r_3,r_4)} \in (G, \Omega)$, there exists $B \in \mathcal{B}$ such that:

$$x_{(r_1,r_2,r_3,r_4)} \in B \subseteq (\tilde{G}, \Omega).$$

Moreover, if $(\tilde{G}, \Omega) \in \tau^{QPNSS}$, then there must exist some base element $B \in \mathcal{B}^{NSS}$ satisfying:

$$x_{(r_1,r_2,r_3,r_4)} \in B \subseteq (\tilde{G}, \Omega).$$

Definition 34. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X, and let $\mathcal{S}^{NSS}$ be a sub-family of τ^{QPNSS}.*

$\mathcal{S}^{NSS}$ is said to be a quadripartitioned neutrosophic soft sub-base (or p-open sub-base or sub-basis) for the quadripartitioned neutrosophic soft topology τ^{QPNSS} on X if finite intersections of the members of $\mathcal{S}^{NSS}$ form a base for the quadripartitioned neutrosophic soft topology τ^{QPNSS} on X.

That is, the union of the members of $\mathcal{S}^{NSS}$ generates all the members of τ^{QPNSS}. The elements of $\mathcal{S}^{NSS}$ are referred to as sub-basic quadripartitioned neutrosophic soft p-open sets.

If, for any non-empty quadripartitioned neutrosophic soft set $(\tilde{G}, \Omega) \in \tau^{QPNSS}$, there exists $\mathcal{B}_1 \subseteq \mathcal{B}^{NSS}$ such that:

$$(\tilde{G}, \Omega) = \bigcup \{B : B \in \mathcal{B}_1\},$$

then $\mathcal{B}^{NSS}$ is said to be a base for the quadripartitioned neutrosophic soft topology. In other words, $\mathcal{B}^{NSS}$ is a base for the quadripartitioned neutrosophic soft topology if, for every point $x_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$, there exists $B \in \mathcal{B}^{NSS}$ such that:

$$x_{(r_1,r_2,r_3,r_4)} \in B \subseteq (\tilde{G}, \Omega).$$

Definition 35. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X.*

A family $\mathcal{B}^{\alpha}$ of quadripartitioned neutrosophic soft p-open subsets of X is said to be a quadripartitioned neutrosophic soft local base at $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ in the neutrosophic soft topology on X if:

(i) For any $B \in \mathcal{B}_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$, we have $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B$.

(ii) For any $(\tilde{G}, \Omega) \in \tau^{QPNSS}$ with $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B \subseteq (\tilde{G}, \Omega)$, there exists $B \in \mathcal{B}_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ such that:

$$y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B \subseteq (\tilde{G}, \Omega).$$

Definition 36. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X. The space X satisfies the first axiom of quadripartitioned neutrosophic soft countability if X has a quadripartitioned neutrosophic soft countable local base at each $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$. A quadripartitioned neutrosophic soft space X satisfying this condition is called a first quadripartitioned neutrosophic soft countable space.*

Definition 37. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X. The space X satisfies the second axiom of quadripartitioned neutrosophic soft countability if there exists a quadripartitioned neutrosophic soft countable base for τ^{QPNSS} on X.*

A quadripartitioned neutrosophic soft space X satisfying this condition is called a second quadripartitioned neutrosophic soft countable space.

A second quadripartitioned neutrosophic soft countable space is also called a quadripartitioned neutrosophic soft completely separable space.

Definition 38. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X. A property P^{QPNSS} of X is said to be hereditary if the property is possessed by every subspace of X.*

For example, quadripartitioned neutrosophic soft first countability and quadripartitioned neutrosophic soft second countability are hereditary properties. However, quadripartitioned neutrosophic soft p-closed sets and quadripartitioned neutrosophic soft p-open sets are not hereditary properties.

Definition 39. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X. This space is said to be quadripartitioned neutrosophic soft separable if and only if X contains a quadripartitioned neutrosophic soft countable dense subset.*

That is, there exists a quadripartitioned neutrosophic soft countable subset $(\tilde{k}, \Omega)$ of X such that

$$\overline{(\tilde{k}, \Omega)} = X.$$

Theorem 15. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X, and let B^{QPNSS} be a quadripartitioned neutrosophic soft basis for τ^{QPNSS}.*

Then, the topology τ^{QPNSS} is given by the collection of all quadripartitioned neutrosophic soft unions of elements of B^{QPNSS}, that is,

$$\tau^{QPNSS} = \left\{ \bigcup_{\alpha \in I} B_\alpha \mid B_\alpha \in B^{QPNSS}, I \text{ is an index set} \right\}.$$

Proof. This is easily seen from the definition of quadripartitioned neutrosophic soft basis.

Theorem 16. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X. A sub-collection B^{QPNSS} of τ_{QPNSS} is a basis for τ^{QPNSS} if and only if, for each quadripartitioned neutrosophic soft p-open set $(\tilde{G}, \Omega) \in \tau^{QPNSS}$ and for each quadripartitioned neutrosophic soft point $x^\theta_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$, there exists a basis element $B_{x^\theta_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$ such that*

$$x^\theta_{(r_1,r_2,r_3,r_4)} \in B_{x^\theta_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega).$$

Proof. Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space, and let B^{QPNSS} be a collection of quadripartitioned neutrosophic soft p-open sets. Suppose that B^{QPNSS} forms a basis for the quadripartitioned neutrosophic soft topology τ^{QPNSS}. Then, for every quadripartitioned neutrosophic soft p-open set $(\tilde{G}, \Omega) \in \tau^{QPNSS}$ and for every quadripartitioned neutrosophic soft point $x^\theta_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$, there exists a basis element $B_{x^\theta_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$ such that

$$x^\theta_{(r_1,r_2,r_3,r_4)} \in B_{x^\theta_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega).$$

Given that $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space and B^{QPNSS} is a collection of quadripartitioned neutrosophic soft p-open sets. Let B^{QPNSS} be a base for a quadripartitioned neutrosophic soft topology τ^{QPNSS}, then by definition every quadripartitioned neutrosophic soft p-open set $(\tilde{G}, \Omega)$ is the union of some members of B^{QPNSS}, i.e.,

$$(\tilde{G}, \Omega) = \bigcup_{i \in I} B_i,$$

where $B_i \in B^{QPNSS}$ for all $i \in I$. Let $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ be an arbitrary quadripartitioned neutrosophic soft point of $(\tilde{G}, \Omega)$. We are to prove that there exists a quadripartitioned neutrosophic soft basis element $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ containing $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ such that

$$B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega).$$

Since $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$ but $(\tilde{G}, \Omega) = \bigcup_{i \in I} B_i$, it follows that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in \bigcup_{i \in I} B_i,$$

which implies that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_i$ for some $i \in I$.

Let $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_i$ for $i = x^{\theta}_{(r_1,r_2,r_3,r_4)}$, then

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq \bigcup B_i, \quad i \in I.$$

Since $B_i \subseteq \bigcup B_i$ for all i, it implies that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq \bigcup B_i, \quad i \in I.$$

This further implies that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega),$$

where $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B^{QPNSS}$.

Conversely, suppose for each quadripartitioned neutrosophic soft point $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ of a quadripartitioned neutrosophic soft p-open set $(\tilde{G}, \Omega)$, there exists a quadripartitioned neutrosophic soft set $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B^{QPNSS}$ such that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega).$$

We are to prove that B^{QPNSS} is a quadripartitioned neutrosophic soft basis for the quadripartitioned neutrosophic soft topology τ^{QPNSS}. For this, we will prove that every quadripartitioned neutrosophic soft p-open set $(\tilde{G}, \Omega)$ can be written as a union of some members of B^{QPNSS}.

Since $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{e}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega)$, it follows that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}},$$

and

$$B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G}, \Omega).$$

Thus, we have

$$\bigcup \{x^{\theta}_{(r_1,r_2,r_3,r_4)}\} \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)} (\tilde{G}, \Omega).$$

This implies that

$$(\tilde{G}, \Omega) \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}},$$

and

$$\bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in(\tilde{G},\Omega)} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq (\tilde{G},\Omega).$$

Therefore,

$$(\tilde{G},\Omega) = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in(\tilde{G},\Omega)} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

Since each $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$, it follows that $(\tilde{G},\Omega)$ is the union of some members of B^{QPNSS}. Since $(\tilde{G},\Omega)$ is an arbitrary quadripartitioned neutrosophic soft p-open set, we conclude that every quadripartitioned neutrosophic soft p-open set is the union of some members of B^{QPNSS}.

Thus, B^{QPNSS} is a quadripartitioned neutrosophic soft basis for the quadripartitioned neutrosophic soft topology τ^{QPNSS}.

Theorem 17. *Let (X,τ^{QPNSS},Ω) be a quadripartitioned neutrosophic soft topological space over X. A sub-collection B^{QPNSS} of τ^{QPNSS} is a base for τ^{QPNSS} if and only if:*

(i) Every quadripartitioned neutrosophic soft point of X is in some $B \in B^{QPNSS}$.

(ii) For $B_1, B_2 \in B^{QPNSS}$ and $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_1 \cap B_2$, there is a $B \in B^{QPNSS}$ such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B \subseteq B_1 \cap B_2$.

Proof. Let B^{QPNSS} be a base for τ^{QPNSS}.

1. Let $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ be an arbitrary quadripartitioned neutrosophic soft point of X.
2. Since X is a quadripartitioned neutrosophic soft p-open set, there exists

$$B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$$

such that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq X.$$

3. This implies that:

$$\bigcup\{x^{\theta}_{(r_1,r_2,r_3,r_4)} \mid x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X\} \subseteq X.$$

4. Furthermore, since $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ is a base element, we get:

$$\bigcup_{x(\alpha,\beta,\gamma)\in X} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq X.$$

5. Therefore,

$$X = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in X} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

This shows that each quadripartitioned neutrosophic soft point of X belongs to some $B \in B^{QPNSS}$. 2. Let $B_1, B_2 \in B^{QPNSS}$ and $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_1 \cap B_2$.

Since B_1 and B_2 are quadripartitioned neutrosophic p-open sets, it follows that $B_1 \cap B_2$ is also a quadripartitioned neutrosophic p-open set. Therefore, there exists $B \in B^{QPNSS}$ such that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B \subseteq B_1 \cap B_2.$$

Conversely, suppose that (1) and (2) hold. We now prove that a sub-collection B^{QPNSS} of τ^{QPNSS} forms a base for τ^{QPNSS}.

To establish this, we show that the sub-collection B^{QPNSS} of τ^{QPNSS} satisfies the three conditions of quadripartitioned neutrosophic soft topology. We proceed as follows:

- The quadripartitioned neutrosophic soft null set $0_{(X,\Omega)}$, being the union of a quadripartitioned neutrosophic soft null collection of quadripartitioned neutrosophic soft subsets in B^{QPNSS}, belongs to τ^{QPNSS}. - Since X is quadripartitioned neutrosophic soft p-open, for any

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$$

the condition (1) provides a

$$B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$$

such that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq X.$$

- This implies:

$$\bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq X.$$

- Since $X \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq X$, it follows that:

$$X = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

This implies that

$$X = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X} B_{x^{e}_{(r_1,r_2,r_3,r_4)}}.$$

This shows that X, being the union of members of B^{QPNSS}, is in τ^{QPNSS}.

Next, we proceed with the second condition as follows: The union of any number of members of τ^{QPNSS}, being the union of members of B^{QPNSS}, is in τ^{QPNSS}.

Now, we proceed with the third condition as follows: Let

$$(G\ ,\Omega)_1, (G\ ,\Omega)_2, (G\ ,\Omega)_3, (G\ ,\Omega)_4 \in \tau^{QPNSS}.$$

By the definition of τ^{QPNSS}, we have

$$(G\ ,\Omega)_1 = \bigcup B_{r_1}, \quad (G\ ,\Omega)_2 = \bigcup B_{r_2}, \quad (G\ ,\Omega)_3 = \bigcup B_{r_3}, \quad (G\ ,\Omega)_4 = \bigcup B_{r_4}$$

for some α, γ ranging over a sub-collection of B^{QPNSS}.

Therefore,

$$(G\ ,\Omega)_1 \cap (G\ ,\Omega)_2 \cap (G\ ,\Omega)_3 \cap (G\ ,\Omega)_4 = \bigcup (B_{r_1} \cap B_{r_2} \cap B_{r_3} \cap B_{r_4}). \tag{i}$$

By condition (2), for any $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_1 \cap B_2$, there exists

$$B_{x^{e}_{(r_1,r_2,r_3,r_4)}} \in B^{QPNSS}$$

such that

$$B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq B_{r_1} \cap B_{r_2} \cap B_{r_3} \cap B_{r_4}.$$

This implies that

$$\bigcup \{x^{\theta}_{(r_1,r_2,r_3,r_4)}\} \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_\alpha \cap B_\gamma} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

Since

$$\bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in(B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4})} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4},$$

it follows that

$$B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4} \subseteq \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in B_\alpha\cap B_\gamma} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4}.$$

Thus, we conclude that

$$B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4} = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in B_\alpha\cap B_\gamma} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

Substituting this value in equation (i), we obtain:

$$(G\ ,\Omega)_1\cap(G\ ,\Omega)_2\cap(G\ ,\Omega)_3\cap(G\ ,\Omega)_4 = \bigcup(B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4}).$$

Since

$$B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4} = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4}} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$$

we conclude that

$$(G\ ,\Omega)_1\cap(G\ ,\Omega)_2\cap(G\ ,\Omega)_3\cap(G\ ,\Omega)_4 = \bigcup_{x^{\theta}_{(r_1,r_2,r_3,r_4)}\in B_{r_1}\cap B_{r_2}\cap B_{r_3}\cap B_{r_4}} B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

This shows that

$$(G\ ,\Omega)_1\cap(G\ ,\Omega)_2\cap(G\ ,\Omega)_3\cap(G\ ,\Omega)_4$$

is the union of members of B^{QPNSS}, which implies it is in τ^{QPNSS}.

Similarly, we can prove that the intersection of any finite number of members of B^{QPNSS} is in τ^{QPNSS}. Since all the conditions of quadripartitioned neutrosophic soft topology are satisfied, we conclude that τ^{QPNSS} is a quadripartitioned neutrosophic soft topology on X.

Consequently, B^{QPNSS} is a quadripartitioned neutrosophic soft base for τ^{QPNSS}.

Theorem 18. *Let (X,τ^{QPNSS},Ω) be a quadripartitioned neutrosophic soft topological space over X. A quadripartitioned neutrosophic soft point $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ in a quadripartitioned neutrosophic soft topological space is a quadripartitioned neutrosophic soft limit point of $(\tilde{F},\Omega)\subseteq X$ if and only if every member of any quadripartitioned neutrosophic soft local base $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ at $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ contains a point of $(\tilde{F},\Omega)$ different from $x^{\theta}_{(r_1,r_2,r_3,r_4)}$.*

Proof.

Let (X,τ^{QPNSS},Ω) be a quadripartitioned neutrosophic soft topological space over X, and let $(\tilde{F},\Omega)\subseteq X$. Let $x^{\theta}_{(r_1,r_2,r_3,r_4)}\in X$ be a quadripartitioned neutrosophic soft limit point of $(\tilde{F},\Omega)$. Let $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ be a local base at $x^{\theta}_{(r_1,r_2,r_3,r_4)}$. We need to prove that:

$$(B - x^{\theta}_{(r_1,r_2,r_3,r_4)})\cap(\tilde{F},\Omega)\neq\emptyset \quad \forall B\in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$$

Since $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ is a quadripartitioned neutrosophic soft limit point of $(\tilde{F},\Omega)$, we have:

$$((\tilde{G},\Omega) - x^{\theta}_{(r_1,r_2,r_3,r_4)})\cap(\tilde{F},\Omega)\neq\emptyset \quad \forall(\tilde{G},\Omega)\in\tau^{QPNSS}.$$

By the definition of a quadripartitioned neutrosophic soft local base, we know that:

$$(\tilde{G},\Omega) \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \implies (\tilde{G},\Omega) \in \tau^{QPNSS}.$$

Thus, since $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G},\Omega)$, it follows that:

$$((\tilde{G},\Omega) - x^{\theta}_{(r_1,r_2,r_3,r_4)}) \cap (\tilde{F},\Omega) \neq \emptyset.$$

Since $B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ consists of members of τ^{QPNSS}, we conclude:

$$(B - x^{\theta}_{(r_1,r_2,r_3,r_4)}) \cap (\tilde{F},\Omega) \neq \emptyset \quad \forall B \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

Conversely, suppose that for some quadripartitioned neutrosophic soft topology τ^{QPNSS} on X, we have:

$$(B - x^{\theta}_{(r_1,r_2,r_3,r_4)}) \cap (\tilde{F},\Omega) \neq \emptyset \quad \forall B \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}.$$

Let $(\tilde{G},\Omega) \in \tau^{QPNSS}$ be an arbitrary set such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G},\Omega)$. By the definition of a quadripartitioned neutrosophic soft local base, there exists $B \in B_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ such that:

$$B \subseteq (\tilde{G},\Omega).$$

Consequently, we have:

$$((\tilde{G},\Omega) - x^{\theta}_{(r_1,r_2,r_3,r_4)}) \cap (\tilde{F},\Omega) \neq \emptyset.$$

Thus, $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ is a quadripartitioned neutrosophic soft limit point of $(\tilde{F},\Omega)$, completing the proof.

Theorem 19. *Let τ_1^{QPNSS} and τ_2^{QPNSS} be two quadripartitioned neutrosophic soft topologies over X generated by the quadripartitioned neutrosophic soft bases B_1^{QPNSS} and B_2^{QPNSS}, respectively. Then $\tau_1^{QPNSS} \subseteq \tau_2^{QPNSS}$ if and only if for each $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in QPNSS(X,\Omega)$ and for each $(\tilde{B}_1,\Omega) \in B_1^{QPNSS}$ containing $x^{\theta}_{(r_1,r_2,r_3,r_4)}$, there exists $(\tilde{B}_2,\Omega) \in B_2^{QPNSS}$ such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_2,\Omega) \subseteq (\tilde{B}_1,\Omega)$.*

Proof. Suppose that $\tau_1^{QPNSS} \subseteq \tau_2^{QPNSS}$ and $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in QPNSS(X,\Omega)$, $(\tilde{B}_1,\Omega) \in B_1^{QPNSS}$ such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_1,\Omega)$. Since B_1^{QPNSS} is a quadripartitioned neutrosophic soft basis for quadripartitioned neutrosophic soft topology τ_1^{QPNSS} over X, then $B_1^{QPNSS} \subseteq \tau_1^{QPNSS}$.

Thus, $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_1,\Omega) \in B_2^{QPNSS} \subseteq \tau_1^{QPNSS}$, i.e., $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_1,\Omega) \in \tau_2^{QPNSS}$. Since B_2^{QPNSS} is a quadripartitioned neutrosophic soft basis for τ_2^{QPNSS}, there exists $(\tilde{B}_2,\Omega) \in B_2^{QPNSS}$ such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_2,\Omega) \subseteq (\tilde{B}_1,\Omega)$.

Conversely, assume that the hypothesis holds. Let $(\tilde{F},\Omega) \in \tau_1^{QPNSS}$. Since B_1^{QPNSS} is a quadripartitioned neutrosophic soft basis for quadripartitioned neutrosophic soft topology τ_1^{QPNSS}, then for $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{F},\Omega)$, there exists $(\tilde{B}_1,\Omega) \in B_1^{QPNSS}$ such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{B}_1,\Omega) \subseteq (\tilde{F},\Omega)$.

By hypothesis, there exists $(\tilde{B}_2,\Omega) \in B_2^{QPNSS}$ such that $(\tilde{B}_2,\Omega) \subseteq (\tilde{B}_1,\Omega) \subseteq (\tilde{F},\Omega)$. This implies $(\tilde{B}_2,\Omega) \subseteq (\tilde{F},\Omega)$, which means $(\tilde{F},\Omega) \in \tau_2^{QPNSS}$. Thus, we conclude that $\tau_1^{QPNSS} \subseteq \tau_2^{QPNSS}$.

Theorem 20. *Let (X,τ^{QPNSS},E) be a quadripartitioned neutrosophic soft topological space over $(\tilde{F},\Omega),(\tilde{K},\Omega) \in QPNSS(X,\Omega)$.*

(i) If B^{QPNSS} *is a quadripartitioned neutrosophic soft base for* τ^{QPNSS}*, then*

$$B(\tilde{F},E)^{QPNSS} = \{(\tilde{B},\Omega) \cap (\tilde{F},\Omega) : (\tilde{B},\Omega) \in B^{QPNSS}\}$$

is a quadripartitioned neutrosophic soft base for the quadripartitioned neutrosophic soft sub-topology $\tau^{QPNSS}_{(\tilde{F},E)}$.

(ii) If $(\tilde{G},\Omega)$ *is a quadripartitioned neutrosophic soft p-closed set in* $\tau^{QPNSS}_{(\tilde{F},E)}$ *and* $(\tilde{F},\Omega)$ *is a quadripartitioned neutrosophic soft p-closed set in* $\tau^{QPNSS}_{(\tilde{F},E)}$*, then* $(\tilde{F},\Omega)$ *is a quadripartitioned neutrosophic soft p-closed set in* $\tau^{QPNSS}_{(\tilde{F},E)}$.

(iii) Let $(\tilde{F},\Omega) \subseteq (\tilde{K},\Omega)$*. If* $(\tilde{G},\Omega) \in \tau^{QPNSS}$*, then* $(\tilde{G},\Omega) \cap (\tilde{F},\Omega)$ *is the quadripartitioned neutrosophic soft closure in* $(X_{(\tilde{F},E)}, \tau^{QPNSS}_{(\tilde{F},E)}, \Omega)$.

Proof. **1.** Since B^{QPNSS} is a quadripartitioned neutrosophic soft base for τ^{QPNSS}, for arbitrary $(\tilde{U},\Omega) \in \tau^{QPNSS}$, we have:

$$(\tilde{U},\Omega) = \bigcup_{(\tilde{B},\Omega)\in B^{QPNSS}} (\tilde{B},\Omega).$$

Thus,

$$(\tilde{U},\Omega) \cap (\tilde{F},\Omega) = \bigcup_{(\tilde{B},\Omega)\in B^{QPNSS}} (\tilde{B},\Omega) \cap (\tilde{F},\Omega).$$

Since $(\tilde{U},\Omega) \cap (\tilde{F},\Omega) \in \tau^{QPNSS}_{(\tilde{F},E)}$, it follows that

$$\bigcup_{(\tilde{B},\Omega)\in B^{QPNSS}} ((\tilde{B},\Omega) \cap (\tilde{F},\Omega)) \in \tau^{QPNSS}_{(\tilde{F},E)}.$$

Since an arbitrary member of $\tau^{QPNSS}_{(\tilde{F},E)}$ can be expressed as the union of members of $B^{QPNSS}_{(\tilde{F},E)}$, it follows that $B^{QPNSS}_{(\tilde{F},E)}$ is a quadripartitioned neutrosophic soft base for $\tau^{QPNSS}_{(\tilde{F},E)}$.

2. We first show that if $(\tilde{G},\Omega)$ is a quadripartitioned neutrosophic soft p-closed set in $\tau^{QPNSS}_{(\tilde{F},E)}$ then there exists a p-closed set $(\tilde{V},\Omega) \subseteq (\tilde{K},\Omega)$ i.e., $(\tilde{V},\Omega) \notin \tau^{QPNSS}$ such that

$$(\tilde{G},\Omega) = (\tilde{V},\Omega) \cap (\tilde{F},\Omega).$$

Let $(\tilde{G},\Omega)$ be p-closed in $\tau^{QPNSS}_{(\tilde{F},E)}$. Then $(\tilde{G}_i,\Omega)^c$ is a quadripartitioned neutrosophic soft p-open set in $\tau^{QPNSS}_{(\tilde{F},E)}$, i.e., $(\tilde{G}_i,\Omega)^c$ can be written as

$$(\tilde{G}',\Omega)^c = (\tilde{U},\Omega) \cap (\tilde{F},\Omega),$$

where $(\tilde{U},\Omega) \in \tau^{QPNSS}$.

Thus,

$$((\tilde{G}',\Omega)^c)^c = (\tilde{F},\Omega) \cap ((\tilde{U},\Omega) \cap (\tilde{F},\Omega))^c = (\tilde{U}',\Omega)^c \cap (\tilde{F},\Omega).$$

Here, $(\tilde{U},\Omega)^c$ is p-closed in τ^{QPNSS}. So, it acts as $(\tilde{V},\Omega) \subseteq (\tilde{K},\Omega)$. Conversely, suppose that

$$(\tilde{G},\Omega) = (\tilde{V},\Omega) \cap (\tilde{F},\Omega),$$

where $(\tilde{F},\Omega) \subseteq (\tilde{K},\Omega)$ and $(\tilde{V},\Omega)$ is p-closed in $\tau^{QPNSS}_{(\tilde{K},E)}$. Clearly, $(\tilde{V},\Omega)^c \in \tau^{QPNSS}$, so that

$$(\tilde{V},\Omega)^c \cap (\tilde{F},\Omega) \in \tau^{QPNSS}(\tilde{K},E).$$

Now,

$$(\tilde{V},\Omega)^c \cap (\tilde{F},\Omega) = ((\tilde{K},\Omega) \setminus (\tilde{V},\Omega)) \cap (\tilde{F},\Omega)$$

$$= ((\tilde{K},\Omega) \cap (\tilde{F},\Omega)) \setminus ((\tilde{V},\Omega) \cap (\tilde{F},\Omega)) = (\tilde{F},\Omega) \setminus (\tilde{G},\Omega).$$

This implies that $(\tilde{F},\Omega) \setminus (\tilde{G},\Omega)$ is a quadripartitioned neutrosophic soft set in $(\tilde{F},\Omega)$, i.e., $(\tilde{G},\Omega)$ is a neutrosophic soft p-closed set in $\tau^{QPNSS}_{(\tilde{K},E)}$.

$$\bigcap\{(\tilde{G_i},\Omega) \mid (\tilde{G_i},\Omega) \text{ is closed and } (\tilde{G_i},\Omega) \supseteq (\tilde{G},\Omega)\}$$

is the quadripartitioned neutrosophic soft closure of $(\tilde{G},\Omega)$ and so $(\tilde{G},\Omega)$ is a quadripartitioned neutrosophic soft p-closed set. Now,

$$(\tilde{G},\Omega) \cap (\tilde{F},\Omega) = \bigcap\{(\tilde{G_i},\Omega) \mid (\tilde{G_i},\Omega) \text{ is closed and } (\tilde{G_i},\Omega) \supseteq (\tilde{G},\Omega)\} \cap (\tilde{F},\Omega)$$

$$= \bigcap((\tilde{G_i},\Omega) \cap (\tilde{F},\Omega)).$$

Since each $(\tilde{G_i},\Omega)$ is p-closed, then each $(\tilde{G_i},\Omega) \cap (\tilde{F},\Omega)$ is p-closed in $\tau^{QPNSS}_{(\tilde{F},E)}$. Now,

$$(G,\Omega) \subseteq (\tilde{G_i},\Omega) \quad \text{and} \quad (G,\Omega) \subseteq (\tilde{F},\Omega).$$

So,

$$(\tilde{G},\Omega) \cap (\tilde{F},\Omega) \subseteq (\tilde{G_i},\Omega) \cap (\tilde{F},\Omega) \Rightarrow (\tilde{G},\Omega) \subseteq (\tilde{G_i},\Omega) \cap (\tilde{F},\Omega).$$

Therefore,

$$(\tilde{G},\Omega)\cap(\tilde{F},\Omega) = \bigcap\{(\tilde{G_i},\Omega)\cap(\tilde{F},\Omega) \mid (\tilde{G_i},\Omega)\cap(\tilde{F},\Omega) \text{ is p-closed and } (\tilde{G_i},\Omega)\cap(\tilde{F},\Omega) \supseteq (\tilde{G_i},\Omega)\}.$$

Thus, $(\tilde{G},\Omega) \cap (\tilde{F},\Omega)$ is a quadripartitioned neutrosophic soft closure of $(\tilde{G},\Omega)$ in $\tau^{QPNSS}_{(\tilde{F},E)}$.

Theorem 21. *Let $(X_(\tilde{F},E),\tau^{QPNSS},\Omega)$ be a quadripartitioned neutrosophic soft subspace of a quadripartitioned neutrosophic soft topological space (X,τ^{QPNSS},E) over X. If $(\tilde{F},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},E) if and only if $(\tilde{F_1},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},Ω).*

Proof. Suppose that $(\tilde{F},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},E) such that a quadripartitioned neutrosophic soft subset $(\tilde{F_1},\Omega)$ of $(\tilde{F},\Omega)$ is a p-open set in $(X_{(\tilde{F},E)},\tau^{QPNSS}_{(\tilde{F},E)},\Omega)$.

Then $(\tilde{F_1},\Omega) \in \tau^{QPNSS}_{(\tilde{F},E)}$ and so $(\tilde{F_1},\Omega) = (\tilde{U},\Omega) \cap (\tilde{F},\Omega)$ for some $(\tilde{U},\Omega) \in \tau^{QPNSS}$.

But $(\tilde{F_1},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},E) as $(\tilde{U},\Omega)$ and $(\tilde{F},\Omega)$ are both quadripartitioned neutrosophic soft p-open sets in (X,τ^{QPNSS},E).

Conversely, assume that $(\tilde{F_1},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},E) when $(\tilde{F},\Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X,τ^{QPNSS},E) and $(\tilde{F_1},\Omega) \subseteq (\tilde{F},\Omega)$.

Then $(\tilde{F_1},\Omega) \in \tau^{QPNSS}$. But $(\tilde{F_1},\Omega)\cap(\tilde{F},\Omega) = (\tilde{F_1},\Omega)$, and so $(\tilde{F_1},\Omega)$ is a quadripartitioned neutrosophic soft set in (X,τ^{QPNSS},E).

Therefore, the first part is proved.

Theorem 22. *Let $(X_{(\tilde{K},E)}, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft subspace of a quadripartitioned neutrosophic soft topological space (X, τ^{QPNSS}, E) over X.*

If $(\tilde{K}, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E), then a quadripartitioned neutrosophic soft set $(\tilde{K}_1, \Omega) \subseteq (\tilde{K}, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in $(X_{(\tilde{K},E)}, \tau^{QPNSS}_{(\tilde{K},E)}, \Omega)$

if and only if $(\tilde{K}_1, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E).

Proof. Suppose that $(\tilde{K}, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E) such that a quadripartitioned neutrosophic soft subset $(\tilde{K}_1, \Omega)$ or $(\tilde{K}, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in $(X_{(\tilde{K},E)}, \tau^{QPNSS}_{(\tilde{K},E)}, \Omega)$.

Since $(\tilde{K}_1, \Omega)$ is p-closed in $(X_{(\tilde{K},E),} \tau^{QPNSS}_{(\tilde{K},E)}, \Omega)$, it follows that

$$(\tilde{K}_1, \Omega) = (\tilde{V}, \Omega) \cap (\tilde{K}, \Omega)$$

for some $(\tilde{V}, \Omega)$, which is a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E).

But $(\tilde{K}_1, \Omega)$ is also a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E) because both $(\tilde{V}, \Omega)$ and $(\tilde{K}, \Omega)$ are quadripartitioned neutrosophic soft p-closed sets in (X, τ^{QPNSS}, E).

Conversely, assume that $(\tilde{F}_1, \Omega)$ is a quadripartitioned neutrosophic soft p-open set in (X, τ^{QPNSS}, E), where $(\tilde{K}, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in (X, τ^{QPNSS}, E) and $(\tilde{K}_1, \Omega) \subseteq (\tilde{K}, \Omega)$.

Then

$$(\tilde{K}_1, \Omega) \cap (\tilde{K}, \Omega) = (\tilde{K}_1, \Omega)$$

which implies that $(\tilde{K}_1, \Omega)$ is a quadripartitioned neutrosophic soft p-closed set in $(X_{(\tilde{K},E)}, \tau^{QPNSS}_{(\tilde{K},E)}, \Omega)$.

Hence, the first part is proved.

Theorem 23. *Every quadripartitioned neutrosophic soft second countable space is always quadripartitioned Neutrosophic soft first countable space.*

Proof. Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X which satisfies the second axiom of quadripartitioned neutrosophic soft countability. That is, $(X, \tau^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft second countable. To prove that $(X, \tau^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft first countable, we proceed as follows. By hypothesis, there exists a quadripartitioned neutrosophic soft countable base B^{QPNSS} for the quadripartitioned neutrosophic soft topology τ^{QPNSS} on X. The countability of B^{QPNSS} implies that B^{QPNSS} can be expressed as

$$B^{QPNSS} = \{B_n : n \in \mathbb{N}\}.$$

Let $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$ be arbitrary. Define the set

$$L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} = \{B_n \in B^{QPNSS} : x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_n\}.$$

(i) Since $L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ is a quadripartitioned neutrosophic soft subset of the countable set B^{QPNSS}, it follows that $L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ is also countable.

(ii) Since members of B^{QPNSS} are τ^{QPNSS}-open sets, the members of $L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ are also contained in τ_{QPNSS}, i.e.,

$$L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}} \subseteq \tau^{QPNSS}.$$

(iii) Any $(\tilde{G}, \Omega) \in L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ implies that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$.

(iv) Let $(\tilde{G}, \Omega) \in \tau^{QPNSS}$ be arbitrary such that $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in (\tilde{G}, \Omega)$. Then, by the definition of a quadripartitioned neutrosophic soft base, there exists some $B_r \in B_{QPNSS}$ such that

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_r \subseteq (\tilde{G}, \Omega).$$

Since $B_r \in B^{QPNSS}$ and contains $x^{\theta}_{(r_1,r_2,r_3,r_4)}$, it follows that $B_r \in L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$.

Thus, $L_{x^{\theta}_{(r_1,r_2,r_3,r_4)}}$ forms a quadripartitioned neutrosophic soft countable local base at $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$. Hence, by definition, $(X, \tau^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft first countable.

Theorem 24. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X such that it is quadripartitioned neutrosophic soft first countable, and let $(Y, \mathcal{J}^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft subspace of $(X, \tau^{QPNSS}, \Omega)$. Then $(Y, \mathcal{J}^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft first countable.*

Proof. Let $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in Y$ be arbitrary, then $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in Y$ as $Y \subseteq X$. Since X is quadripartitioned neutrosophic soft first countable, it guarantees that there exists a quadripartitioned neutrosophic soft countable local base at $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$ and hence, in particular, there exists a quadripartitioned neutrosophic soft countable local base B^{QPNSS} at $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)}$.

Members of B^{QPNSS} can be enumerated as $B_1, B_2, B_3, B_4, B_5, \ldots$, that is,

$$B^{QPNSS} = \{B_n : n \in \mathbb{N}\}.$$

Evidently, $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in X$. Since $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_n$ for all $n \in \mathbb{N}$, we write

$$B_1 = \{Y \cap B_n : n \in \nu/\mathbb{N}\} \quad (1).$$

Since $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in Y$ and $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_n$ for all $n \in \mathbb{N}$, it follows that

$$y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_n \text{ for all } n \in \mathbb{N} \quad (2).$$

Since $B_n \in B^{QPNSS}$ for all $n \in \mathbb{N}$, we have $B_n \in \tau^{QPNSS}$, which implies that

$$Y \cap B_n \in \mathcal{J}^{QPNSS}.$$

Theorem 25. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X such that it is quadripartitioned neutrosophic soft second countable, and let $(Y, \mathcal{J}^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft subspace of $(X, \tau^{QPNSS}, \Omega)$. Then $(Y, \mathcal{J}^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft second countable.*

Proof. Let $(Y, \mathcal{J}^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft subspace of $(X, \tau^{QPNSS}, \Omega)$, a quadripartitioned neutrosophic soft topological space over X which is quadripartitioned neutrosophic soft second countable. This implies that there exists a quadripartitioned neutrosophic soft countable base B^{QPNSS} for τ^{QPNSS}. If we prove that $(Y, \mathcal{J}^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft countable, the result will automatically follow.

Since B^{QPNSS} is quadripartitioned neutrosophic soft countable, it follows that $B^{QPNSS} \sim \mathbb{N}$, which implies that B^{QPNSS} can be expressed as

$$B^{QPNSS} = \{B_n : n \in \mathbb{N}\}.$$

Define

$$B_1 = \{Y \cap B_n : n \in \mathbb{N}\} \quad \text{(i)}.$$

Evidently, $B_1 \sim \mathbb{N}$ under the quadripartitioned neutrosophic soft map $Y \cap B_n \to n$. Therefore, B_1 is quadripartitioned neutrosophic soft countable.

(ii) B_1 is a quadripartitioned neutrosophic soft family of all $\mathcal{J}^{QPNSS}$-quadripartitioned neutrosophic soft p-open sets. Since $B_n \in B^{QPNSS}$ implies $B_n \in \tau^{QPNSS}$, it follows that $B \subseteq \tau^{QPNSS}$, implying that $Y \cap B_n \in \mathcal{J}^{QPNSS}$.

(iii) For any $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in (\tilde{G},\Omega) \in \mathcal{J}^{QPNSS}$, there exists $B_r \cap Y \in B_1$ such that $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_r \cap Y \subseteq (\tilde{G},\Omega)$.

To prove this, let $(\tilde{G},\Omega) \in \mathcal{J}^{QPNSS}$ such that $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in (\tilde{G},\Omega)$. Then there exists $(\tilde{H},\Omega) \in \tau^{QPNSS}$ such that $(\tilde{G},\Omega) = (\tilde{H},\Omega) \cap Y$. Since $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in (\tilde{H},\Omega) \cap Y$, we have $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in (\tilde{G},\Omega)$.

By the definition of the quadripartitioned neutrosophic soft base, any $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in Y$ belonging to $\mathcal{J}^{QPNSS}$ implies that there exists $B_r \in B^{QPNSS}$ such that $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_r \subseteq (\tilde{H},\Omega)$.

From the above, it follows that B_1 is a quadripartitioned neutrosophic soft countable base for the quadripartitioned neutrosophic soft topology $\mathcal{J}^{QPNSS}$ on Y. Consequently, $(Y, \mathcal{J}^{QPNSS}, \Omega)$ is quadripartitioned neutrosophic soft second countable.

Theorem 26. *Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X such that it is a quadripartitioned neutrosophic soft second countable space. Then it also has the characteristics of another quadripartitioned neutrosophic soft space known as a quadripartitioned neutrosophic soft separable space. Interestingly, the converse is not always true.*

Proof. Let $(X, \tau^{QPNSS}, \Omega)$ be a quadripartitioned neutrosophic soft topological space over X such that it is a quadripartitioned neutrosophic soft second countable space. Since X is a quadripartitioned neutrosophic soft second countable space, there exists a quadripartitioned neutrosophic soft countable base B_{QPNSS} for the quadripartitioned neutrosophic soft topology τ_{QPNSS}. Members of B^{QPNSS} may be enumerated as $B_1, B_2, B_3, B_4, \ldots$

Choose any quadripartitioned neutrosophic soft point $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ from each B_i and take $\mathfrak{Y}$ as a collection of all these quadripartitioned neutrosophic soft points:

$$\mathfrak{Y} = \{x^{\theta}_{(r_1,r_2,r_3,r_4)} \mid \forall i \in \mathbb{N}\}. \tag{1}$$

That is to say,

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_i \in B^{QPNSS}, \quad \forall i \in \mathbb{N}. \tag{2}$$

Evidently, $\mathbb{N} \sim \mathfrak{Y}$ under the quadripartitioned neutrosophic soft map $i \to x^{\theta}_{(r_1,r_2,r_3,r_4)}$, therefore $\mathfrak{Y}$ is enumerable. Clearly, $\mathfrak{Y} \subseteq X$.

We claim that $\overline{\mathfrak{Y}} = X$. Suppose not. Then $X - \overline{\mathfrak{Y}} \neq \emptyset$:

$$X - \overline{\mathfrak{Y}} \neq \emptyset. \tag{3}$$

Let $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in X - \overline{\mathfrak{Y}}$ be arbitrary. Since $\overline{\mathfrak{Y}}$ is quadripartitioned neutrosophic soft p-closed, $X - \overline{\mathfrak{Y}}$ is quadripartitioned neutrosophic soft p-open. That is,

$$y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in X - \overline{\mathfrak{Y}} \in \tau^{QPNSS}. \tag{4}$$

By the definition of the quadripartitioned neutrosophic soft base, there exists $B_{n_0} \in B^{QPNSS}$ such that

$$y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)_{n_0}} \subseteq X - \overline{\mathfrak{Y}}. \tag{5}$$

But $x^{\theta}_{(r_1,r_2,r_3,r_4)_{n_0}} \in \mathfrak{Y}$ according to (1) and (2), contradicting (4). Thus, our assumption $X - \overline{\mathfrak{Y}} \neq \emptyset$ was incorrect. Consequently,

$$X - \overline{\mathfrak{Y}} = \emptyset \Rightarrow X = \overline{\mathfrak{Y}}. \tag{6}$$

Thus, we have proved that there exists $\mathfrak{Y} \subseteq X$ such that $X = \overline{\mathfrak{Y}}$ and $\mathfrak{Y}$ is quadripartitioned neutrosophic soft enumerable. By definition, this proves that X is quadripartitioned neutrosophic soft separable. Let us now discuss the converse. Suppose X be an infinite quadripartitioned neutrosophic soft set. Let τ^{QPNSS} be a family consisting of $0_{(X,\Omega)}$ and all those quadripartitioned neutrosophic soft subsets $\mathfrak{Y}$ of X such that $\mathfrak{Y}^c$ is quadripartitioned neutrosophic soft finite. Then τ^{QPNSS} is a quadripartitioned neutrosophic soft topology.

We claim that τ^{QPNSS} is quadripartitioned neutrosophic soft separable. Since X is an infinite quadripartitioned neutrosophic soft set, there exists $\mathfrak{Y} \subseteq X$ such that $\mathfrak{Y}$ is quadripartitioned neutrosophic soft enumerable.

To prove that $X = \overline{\mathfrak{Y}}$, we note that $\mathfrak{Y} \subseteq X$, and hence all the quadripartitioned neutrosophic soft closure points of $\mathfrak{Y}$ will reside in X, which implies that $\overline{\mathfrak{Y}} \subseteq X$.

If $(\tilde{G}, \Omega) \in \tau^{QPNSS}$, then $(\tilde{G}, \Omega)^c$ is quadripartitioned neutrosophic soft p-closed. Thus, $(\tilde{G}, \Omega) \in \tau^{QPNSS}$ implies that $(\tilde{G}, \Omega)^c$ is quadripartitioned neutrosophic soft finite and quadripartitioned neutrosophic soft p-closed. This means that all the quadripartitioned neutrosophic soft p-closed subsets of X are finite quadripartitioned neutrosophic subsets of X and X itself.

Thus, the only infinite quadripartitioned neutrosophic soft p-closed subset of X is X, which contains $\mathfrak{Y}$. Therefore, $X = \overline{\mathfrak{Y}}$. Since $\overline{\mathfrak{Y}}$ is the smallest quadripartitioned neutrosophic soft p-closed set containing $\mathfrak{Y}$ and X is the only such set, we conclude that:

$$X = \overline{\mathfrak{Y}}.$$

To prove that $(X, \tau^{QPNSS}, \Omega)$ is not quadripartitioned neutrosophic soft second countable, suppose the contrary. Then X is quadripartitioned neutrosophic soft second countable, so there exists a quadripartitioned neutrosophic soft countable base B^{QPNSS} for the quadripartitioned neutrosophic soft topology on X. The members of B^{QPNSS} may be enumerated as $B_1, B_2, B_3, B_4, \ldots$.

Let $x^{\theta}_{(r_1,r_2,r_3,r_4)} \in X$ be arbitrary but fixed. Define:

$$B_0^{QPNSS} = \{B_r \in B^{QPNSS} \mid x^{\theta}_{(r_1,r_2,r_3,r_4)} \in B_r\} \quad (1)$$

$$\mathfrak{Y} = \bigcap \{B_r \mid B_r \in B_0^{QPNSS}\} \quad (2)$$

From (2), it is clear that $x^{\theta}_{(r_1,r_2,r_3,r_4)}$ is common in all the members of B_0^{QPNSS} and hence:

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \in \mathfrak{Y}.$$

We claim that:

$$\mathfrak{Y} = \{x^{\theta}_{(r_1,r_2,r_3,r_4)}\} \quad (3).$$

Let $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in X$ be arbitrary such that:

$$x^{\theta}_{(r_1,r_2,r_3,r_4)} \neq y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)}.$$

Since $\{x^{\theta}_{(r_1,r_2,r_3,r_4)}\}$ is a quadripartitioned neutrosophic soft finite set, its complement is in τ^{QPNSS}, i.e.,

$$\{x^{\theta}_{(r_1,r_2,r_3,r_4)}\}^c \in \tau^{QPNSS}.$$

Clearly, $y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in \{x^{\theta}_{(r_1,r_2,r_3,r_4)}\}^c$. By the definition of the quadripartitioned neutrosophic soft base, there exists $B_{y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)}} \in B^{QPNSS}$ such that:

$$y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)} \in B_{y^{\theta'}_{(r'_1,r'_2,r'_3,r'_4)}} \subseteq \{x^{\theta}_{(r_1,r_2,r_3,r_4)}\}^c.$$

This implies that:

$$y^{\theta'}_{(r_1',r_2',r_3',r_4')} \notin B_0^{QPNSS}.$$

From (2), this means:

$$y^{\theta'}_{(r_1',r_2',r_3',r_4')} \notin \mathfrak{Y}.$$

Thus, we have shown that:

$$y^{\theta'}_{(r_1',r_2',r_3',r_4')} \in X, \quad y^{\theta'}_{(r_1',r_2',r_3',r_4')} \notin \mathfrak{Y}.$$

This proves (3), and consequently:

$$\mathfrak{Y}^c \text{ is quadripartitioned neutrosophic soft countable.} \quad (4)$$

Since X is quadripartitioned neutrosophic soft uncountable and $\mathfrak{Y}$ is a quadripartitioned neutrosophic soft singleton set, we get:

$$\mathfrak{Y}^c = X - \bigcap\{B_r \mid B_r \in B_0^{QPNSS}\} = \bigcup\{B_r^c \mid B_r \in B_0^{QPNSS}\} \quad (5).$$

Since $B_r \in B_0^{QPNSS}$, it follows that $B_r \in B^{QPNSS}$, so:

$$B_0^{QPNSS} \subseteq B^{QPNSS} \subseteq \tau^{QPNSS}.$$

Thus, $B_r \in \tau^{QPNSS}$ implies that B_r^c is a quadripartitioned neutrosophic soft finite set. Being a quadripartitioned neutrosophic soft countable union of finite quadripartitioned neutrosophic soft sets, $\bigcup\{B_r^c \mid B_r \in B_0^{QPNSS}\}$ is quadripartitioned neutrosophic soft countable, which means:

$$\mathfrak{Y}^c \text{ is quadripartitioned neutrosophic soft countable, in accordance with (5).}$$

This contradicts (4). Hence, our supposition was wrong, and we conclude that:

$$(X, \tau^{QPNSS}, \Omega) \text{ is not quadripartitioned neutrosophic soft second countable.}$$

10. Advantages of Quadri-Partitioned Neutrosophic Set Theory

(i) **More Complex and Comprehensive Framework:** By adding a fourth partition, QPNST provides a more detailed structure for handling sets, capturing more possibilities and allowing for a more comprehensive exploration of complex systems. This expanded framework is particularly useful in situations where traditional fuzzy logic or intuitionistic sets may fall short.

(ii) **Improved Mathematical Tools:** The introduction of Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPNRIT) offers new insights into the Riemann integral, an essential concept in analysis. By extending the theory into the QPNST context, this work opens new avenues for the study of integration and its properties under uncertainty, helping to reveal behaviors and characteristics not previously observable.

(iii) **Level Cut (Four-Tuple Representation):** The definition of the level cut as a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$ captures the multiple possibilities inherent in QPNST. This enables more precise modeling and analysis, reflecting the different dimensions of truth, indeterminacy, and falsity in set membership more clearly. This richer structure supports more accurate decision-making and problem-solving in uncertain contexts.

(iv) **Numerical Study and Practical Insights:** The numerical study conducted within the QPNST framework provides a concrete understanding of the behavior of the Riemann integral in this extended context. Organizing the findings in tabular form makes it easier for researchers and practitioners to grasp the implications and applications of the new theory.

11. Limitations of Quadri-Partitioned Neutrosophic Set Theory

(i) **Increased Complexity:** While the added complexity of QPNST allows for more detailed and accurate representations of uncertainty, it also makes the theory more challenging to apply. Researchers may find it difficult to work with the four-part structure, especially in cases where the additional partitions add little value in practical applications.

(ii) **Computational Challenges:** With the introduction of additional partitions and the need to calculate more complex level cuts, the computational effort required to implement QPNST and QPNRIT could be significantly higher. This might limit its use in large-scale or real-time applications where computational efficiency is critical.

(iii) **Interpretation and Practical Implementation:** The four-dimensional nature of the model may pose interpretative challenges. While it provides a richer model for uncertainty, translating the abstract concepts into real-world decision-making processes can be difficult. Practitioners may struggle to interpret the results or apply the theory in practical situations, especially in industries not traditionally familiar with higher-order logic systems.

(iv) **Limited Existing Tools and Resources:** The extension of Riemann integral theory into the QPNST context is a relatively new development, and as such, there may be limited resources, tools, and research available to fully support its application. This could slow down its adoption and the development of practical applications based on QPNRIT.

(v) **Need for Further Theoretical Validation:** Although the framework holds promise, the full range of theoretical properties of QPNST and QPNRIT needs to be further explored. Many questions regarding the consistency, stability, and broader applicability of these theories remain unanswered, requiring further research and validation.

12. Conclusion and Future Work

Finally, by adding a third option for set representation, Neutrosophic Set Theory (NST) expands on Intuitionistic Fuzzy Set Theory (IFST) and improves the theory's ability to handle uncertainty. By introducing Quadri-Partitioned Neutrosophic Set Theory (QPNST), which adds a fourth alternative and offers a more complex and all-encompassing framework for set representation, this work significantly develops NST. Within this framework, we define the Riemann Integral Theory (RIT), opening new avenues for exploring the properties and characteristics of the Riemann integral in an extended context. A central concept in this work is the level cut, defined as a four-tuple $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l})$, which captures the multiple possibilities inherent in QPNST. Furthermore, we conduct a numerical study of the Quadri-Partitioned Neutrosophic Riemann Integral Theory (QPNRIT) and provide the findings in an organized tabular manner. This numerical study enhances our comprehension of the integral's behavior within the QPNST framework and offers insightful information about its characteristics. This study explores quadripartitioned neutrosophic soft topological spaces, extending neutrosophic set theory (NST), which incorporates three membership values: true, false, and indeterminacy. The study introduces new concepts such as QPNS semi-open, QPNS pre-open, and QPNS $*_{\mathfrak{b}}$ open sets, and builds on these to define QPNS closure, exterior, boundary, and interior. A key development is the definition of a quadripartitioned neutrosophic soft base, which plays a central role in these topological structures. The paper also explores the concept of a quadripartitioned neutrosophic soft sub-base and discusses local bases, as well as the first- and second-countability axioms. The study further examines hereditary properties of these spaces, distinguishing between inherited and non-inherited properties. Key results include that a quadripartitioned neutrosophic soft subspace of a first-countable space is also first-countable, and a second countable subspace of a second-countable space remains second-countable. It also highlights the relationship between second countability and separability in these spaces, asserting that a second-countable quadripartitioned neutrosophic soft space is separable, though the converse is not always true. This work lays the foundation for further research in neutrosophic soft topologies. With the possibility of incorporating it into other sophisticated theoretical frameworks, future studies will seek to broaden the use of Quadri-Partitioned Neutrosophic Set Theory (QPNST) in a variety of mathematical fields. Investigating a generalized Riemann Integral Theory (RIT) employing QPNST in more intricate contexts, like multi-dimensional spaces and dynamic systems, is one exciting avenue. In order to increase computational accuracy and efficiency, more effort will be done to enhance numerical methods for computing Quadri-Partitioned Neutrosophic Riemann Integrals (QPNRIT). The creation of decision-making models that incorporate QPNST will be another crucial topic. These models could provide more reliable frameworks for dealing with uncertainty in real-world applications including artificial intelligence, engineering, and economics. Furthermore, it is crucial to conduct additional research on the theoretical characteristics of QPNST, such as its algebraic and topological features. These studies could provide deeper insights into the behavior of neutrosophic sets in various contexts, enriching the overall understanding of this extended theory.

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