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Singular Value Inequalities for Matrix Sums and Products

Ahmad Al-Natoor[1,*], Rawan Al-smadi[2], Aliaa Burqan[2]

[1] *Department of Mathematics, Faculty of Sciences, Isra University, Amman 11622, Jordan*
[2] *Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan*

Abstract. In this paper, we discuss some inequalities involving singular values for matrix sums and products. In some of our results, we prove inequalities for functions of matrices and this enables us to give a generalization of known recent result.
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1. Introduction

In this paper, the symbol $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. The numbers $s_1(A) \geq, ..., \geq s_n(A) \geq 0$ are called the singular values of $A \in \mathbb{M}_n(\mathbb{C})$ which are the eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in decreasing order and counted according to multiplicity.

The spectral norm of $A \in M_n(\mathbb{C})$, denoted by $\|A\|$, can be expressed as the largest singular value of A, i.e, $\|A\| = s_1$.

It is known (see [1] or [2]) that if $X \in M_m$ and $Y \in M_n$ are such that $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \geq 0$, then

$$s_j(Z) \leq \frac{1}{2} s_j\left(\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}\right) \tag{1}$$

for $j = 1, 2, ..., r$, where $r = \min(n, m)$.

If A and B are positive semidefinite matrices, then by letting $Z = AB$, $X = A^2$, and $Y = B^2$ in inequality (1), we have

$$s_j(AB) \leq \frac{1}{2} s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right). \tag{2}$$

Among other results in this paper, we give a general version of inequality (2). For more singular value and norm inequalities for matrices, we refer the reader to [3], [4], [5], [6], and [7].

*Corresponding author.
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Email addresses: ahmad.alnatoor@iu.edu.jo (A. Al-Natoor), rawansmadi2@gmail.com (R. Al-smadi), aliaaburqan@zu.edu.jo (A. Burqan)

2. Main results

To start our analysis, we need the fact that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(A) \leq s_j(B) \text{ iff } s_j(A \oplus A) \leq s_j(B \oplus B) \tag{3}$$

for $j = 1, ..., 2n$.

Lemma 1. *Let $A, B, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then for $j = 1, ..., 2n$, we have*

$$s_j\left((AY - YB) \oplus 0\right) \leq \max\left(\|A\|, \|B\|\right) s_j\left(Y \oplus Y\right).$$

In particular, if $B = A$, then

$$s_j\left((AY - YA) \oplus 0\right) \leq \|A\| \, s_j\left(Y \oplus Y\right).$$

Theorem 1. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite and let $g(t), h(t)$ be real polynomials. Then for $j = 1, ..., 2n$, we have*

$$s_j\left((ABh(B) + g(A)AB) \oplus 0\right) \leq \left\|A^2 + B^2\right\| s_j\left(g(A) \oplus h(B)\right).$$

Proof. Let $Q_1 = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix} = \begin{bmatrix} A^2 g(A) & ABh(B) \\ BAg(A) & B^2 h(B) \end{bmatrix}$, and $Q_2 = \begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\begin{bmatrix} A & 0 \\ -B & 0 \end{bmatrix}\begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} g(A)A^2 & -g(A)AB \\ -h(B)BA & h(B)B^2 \end{bmatrix}$.

So, $Q_1 - Q_2 = \begin{bmatrix} A^2 g(A) - g(A)A^2 & ABh(B) + g(A)AB \\ BAg(A) + h(B)BA & B^2 h(B) - h(B)B^2 \end{bmatrix} = \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}$, where $Q = ABh(B) + g(A)AB$.

Now,

$$\begin{aligned} & s_j\left((ABh(B) + g(A)AB) \oplus (ABh(B) + g(A)AB)\right) \\ &= s_j((ABh(B) + g(A)AB) \oplus (BAg(A) + h(B)BA)) \\ &= s_j\left(Q \oplus Q^*\right) \\ &= s_j\left(\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}\right) \\ &= s_j\left(Q_1 - Q_2\right). \end{aligned} \tag{4}$$

Using Lemma 1 and the fact that $\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}$ and $\begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}$ are unitarily equivalent via $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ yield that

$$\begin{aligned}
s_j\left((Q_1-Q_2)\oplus 0\right) &= s_j\left(\begin{array}{c} \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix} \\ -\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\begin{bmatrix} A & 0 \\ -B & 0 \end{bmatrix}\begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} \end{array}\right) \\
&= s_j\left(\left(\begin{array}{c} \begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix} \\ -\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix} \end{array}\right)\oplus 0\right) \\
&\leq \max\left(\left\|\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right\|, \left\|\begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right\|\right) \\
&\quad \times s_j\left(\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\oplus\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\right) \\
&= \left\|\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right\| s_j\left(\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\oplus\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\right) \\
&= \left\|\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right\| s_j\left(\left(g(A)\oplus h(B)\right)\oplus\left(g(A)\oplus h(B)\right)\right).
\end{aligned}$$

Now, by using the fact that $\|T^*T\| = \|TT^*\|$ for any $T \in M_n(\mathbb{C})$, we have

$$\begin{aligned}
\left\|\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right\| &= \left\|\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} A^2+B^2 & 0 \\ 0 & 0 \end{bmatrix}\right\| \\
&= \left\|A^2+B^2\right\|.
\end{aligned}$$

So,

$$\begin{aligned}
&s_j\left(\left(ABh(B)+g(A)AB\right)\oplus\left(ABh(B)+g(A)AB\right)\oplus 0\right) \\
&\qquad \leq \left\|A^2+B^2\right\| s_j\left(\left(g(A)\oplus h(B)\right)\oplus\left(g(A)\oplus h(B)\right)\right). \qquad (5)
\end{aligned}$$

Now, the desired result follows by inequalities (3), (4), and (5).

Lemma 2. *[8]Let $A, B, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then for $j = 1, ..., 2n$, we have*

$$s_j\left((AY - YB)\oplus 0\right) \leq \|Y\|\, s_j(A\oplus B).$$

Theorem 2. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite and let $g(t), h(t)$ be real polynomials. Then for $j = 1, ..., 2n$, we have*

$$s_j\left(ABh(B) + g(A)AB\right) \leq \max\left(\|g(A)\|, \|h(B)\|\right) s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right). \tag{6}$$

Proof. Let Q_1, Q_2, Q, and U be as in the proof of Theorem 1. Then by equation (4), we have

$$s_j\left((ABh(B) + g(A)AB) \oplus (ABh(B) + g(A)AB)\right)$$

$$\begin{aligned} &= s_j(Q_1 - Q_2) \\ &\leq \left\|\begin{bmatrix} g(A) & 0 \\ 0 & h(B) \end{bmatrix}\right\| s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right) \\ &\qquad \text{(by Lemma 2)} \\ &= \max\left(\|g(A)\|, \|h(B)\|\right) s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \left(U\begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix} U^*\right)\right) \\ &= \max\left(\|g(A)\|, \|h(B)\|\right) s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right). \end{aligned} \tag{7}$$

Now, the desired result follows by inequalities (3) and (7).

Inequality (6) represents a general version of inequality (2). In fact, letting $h(t) = g(t) = 1$ in Corollary 2, we have

$$s_j(AB) \leq \frac{1}{2} s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right),$$

which is inequality (2).

An application of Theorem 2 can be seen in the following corollary, which depends on the following lemma .

Lemma 3. *[9]Let $A, B \in M_n(\mathbb{C})$. Then for $j = 1, ..., 2n$, we have*

$$s_j((A \pm B) \oplus 0) \leq s_j(A \oplus B) + \frac{1}{2}\|A \pm B\|.$$

Corollary 1. *Let $A, B, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then for $j = 1, ..., 2n$, we have*

$$s_j\left((ABh(B) + g(A)AB) \oplus 0\right)$$

$$\begin{aligned} \leq\ & \max\left(\|g(A)\|, \|h(B)\|\right) s_j\left(A^2 \oplus B^2 \oplus AB \oplus AB\right) \\ & + \frac{1}{2}\left\|A^2 + B^2\right\|. \end{aligned}$$

Proof. By inequality (6), we have

$$
\begin{aligned}
&s_j\left((ABh(B)+g(A)AB)\oplus 0\right)\\
&\le \max\left(\|g(A)\|,\|h(B)\|\right)s_j\left(\begin{bmatrix} A^2 & AB\\ BA & B^2\end{bmatrix}\right)\\
&= \max\left(\|g(A)\|,\|h(B)\|\right)s_j\left(\begin{bmatrix} A^2 & 0\\ 0 & B^2\end{bmatrix}+\begin{bmatrix} 0 & AB\\ BA & 0\end{bmatrix}\right)\\
&\le \max\left(\|g(A)\|,\|h(B)\|\right)\left(\begin{array}{c} s_j\left(\begin{bmatrix} A^2 & 0\\ 0 & B^2\end{bmatrix}\oplus\begin{bmatrix} 0 & AB\\ BA & 0\end{bmatrix}\right)\\ +\frac{1}{2}\left\|\begin{bmatrix} A^2 & 0\\ 0 & B^2\end{bmatrix}+\begin{bmatrix} 0 & AB\\ BA & 0\end{bmatrix}\right\|\end{array}\right)\\
&\qquad\qquad\text{(by Lemma 3)}\\
&= \max\left(\|g(A)\|,\|h(B)\|\right)\left(\begin{array}{c} s_j\left(\begin{bmatrix} A^2 & 0\\ 0 & B^2\end{bmatrix}\oplus\begin{bmatrix} 0 & AB\\ BA & 0\end{bmatrix}\right)\\ +\frac{1}{2}\left\|\begin{bmatrix} A^2 & AB\\ BA & B^2\end{bmatrix}\right\|\end{array}\right)\\
&= \max\left(\|g(A)\|,\|h(B)\|\right)\left(s_j\left(A^2\oplus B^2\oplus AB\oplus AB\right)+\frac{1}{2}\left\|A^2+B^2\right\|\right),
\end{aligned}
$$

as required.

Theorem 3. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite and let $g(t), h(t)$ be real polynomials. Then for $j = 1, 2, ..., 2n$, we have*

$$
s_j\left(ABh(B)+g(A)AB\oplus 0\right) \le s_j\left(\begin{bmatrix} A^2g(A) & ABh(B)\\ BAg(A) & B^2h(B)\end{bmatrix}\right)+\frac{1}{2}\left\|ABh(B)+g(A)AB\right\|. \tag{8}
$$

In particular, if $j = 1$, then

$$
\|ABh(B)+g(A)AB\| \le 2\left\|\begin{bmatrix} A^2g(A) & ABh(B)\\ BAg(A) & B^2h(B)\end{bmatrix}\right\|.
$$

Proof. Let Q_1, Q_2, Q, and U be as in the proof of Theorem 1. Then by equation (4), we have

$$
\begin{aligned}
&s_j\left((ABh(B)+g(A)AB)\oplus(ABh(B)+g(A)AB)\right)\\
&= s_j(Q_1-Q_2)\\
&\le s_j(Q_1\oplus Q_2)+\frac{1}{2}\|Q_1-Q_2\|
\end{aligned}
$$

$$\begin{aligned}
& \text{(by Lemma 3)} \\
\leq\ & s_j\left(\begin{bmatrix} A^2 g(A) & ABh(B) \\ BAg(A) & B^2 h(B) \end{bmatrix} \oplus \begin{bmatrix} g(A)A^2 & -g(A)AB \\ -h(B)BA & h(B)B^2 \end{bmatrix}\right) \\
& +\frac{1}{2}\left\|\begin{bmatrix} 0 & ABhB + g(A)AB \\ BAg(A) + h(B)BA & 0 \end{bmatrix}\right\| \\
=\ & s_j\left(\begin{bmatrix} A^2 g(A) & ABh(B) \\ BAg(A) & B^2 h(B) \end{bmatrix} \oplus \left(U\begin{bmatrix} g(A)A^2 & -g(A)AB \\ -h(B)BA & h(B)B^2 \end{bmatrix}U^*\right)\right) \\
& +\frac{1}{2}\max\left(\|ABh(B) + g(A)AB\|, \|BAg(A) + h(B)BA\|\right) \\
=\ & s_j\left(\begin{bmatrix} A^2 g(A) & ABh(B) \\ BAg(A) & B^2 h(B) \end{bmatrix} \oplus \begin{bmatrix} g(A)A^2 & g(A)AB \\ h(B)BA & h(B)B^2 \end{bmatrix}\right) \\
& +\frac{1}{2}\|ABh(B) + g(A)AB\|.
\end{aligned}$$

Consequently,

$$s_j((ABh(B) + g(A)AB) \oplus (ABh(B) + g(A)AB))$$

$$\begin{aligned}
\leq\ & s_j\left(\begin{bmatrix} A^2 g(A) & ABh(B) \\ BAg(A) & B^2 h(B) \end{bmatrix} \oplus \begin{bmatrix} g(A)A^2 & g(A)AB \\ h(B)BA & h(B)B^2 \end{bmatrix}\right) \\
& +\frac{1}{2}\|ABh(B) + g(A)AB\|. \qquad (9)
\end{aligned}$$

Now, the desired result follows from inequalities (3) and (9).

Lemma 4. *[10]Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then for $r \geq 0$, we have*

$$\left\|A^{1/2}(A+B)^r B^{1/2}\right\| \leq \frac{1}{2}\left\|(A+B)^{r+1}\right\|.$$

Corollary 2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then for $j = 1, ..., 2n$, we have*

$$s_j\left(\left(A^{1/2}B^{3/2} + A^{3/2}B^{12}\right) \oplus 0\right) \leq s_j\left(\begin{bmatrix} A^2 & A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2} & B^2 \end{bmatrix}\right) + \frac{1}{4}\left\|(A+B)^2\right\|$$

Proof. Let $g(t) = h(t) = t^2$ in inequality (8). Then

$$\begin{aligned}
s_j\left((AB^3 + A^3B) \oplus 0\right) &\leq s_j\left(\begin{bmatrix} A^4 & AB^3 \\ BA^3 & B^4 \end{bmatrix}\right) + \frac{1}{2}\left\|AB^3 + A^3B\right\| \\
&\leq s_j\left(\begin{bmatrix} A^4 & AB^3 \\ BA^3 & B^4 \end{bmatrix}\right) + \frac{1}{2}\left\|A(B^2 + A^2)B\right\|.
\end{aligned}$$

Replacing A by $A^{1/2}$ and B by $B^{1/2}$, we have

$$s_j\left(\left(A^{1/2}B^{3/2} + A^{3/2}B^{12}\right) \oplus 0\right) \leq s_j\left(\begin{bmatrix} A^2 & A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2} & B^2 \end{bmatrix}\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left\|A^{1/2}(B+A)B^{1/2}\right\| \\
\leq\ & s_j\left(\left[\begin{array}{cc} A^2 & A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2} & B^2 \end{array}\right]\right)+\frac{1}{4}\left\|(A+B)^2\right\| \\
&\text{(by Lemma 4)}
\end{aligned}
$$

as required.

The author in [11] proved that if $A, B, Y \in \mathbb{M}_n(\mathbb{C})$ are such that Y is positive semidefinite, then

$$s_j\left(AYB^*\right) \leq \frac{1}{2}\|Y\| s_j\left(A^*A+B^*B\right) \tag{10}$$

for $j=1,2,...,n$. Based on this inequality, we have the following lemma.

Lemma 5. *Let $A, B, Y \in \mathbb{M}_n(\mathbb{C})$ be such that Y is positive semidefinite. Then*

$$s_j\left(AYB^*\right) \leq \frac{1}{2}\|Y\|\|A\|\|B\| s_j\left(\frac{A^*A}{\|A\|^2}+\frac{B^*B}{\|B\|^2}\right) \tag{11}$$

for $j=1,...,n$.

Proof. In inequality (10), replacing A and B by $\sqrt{\frac{\|B\|}{\|A\|}}A$ and $\sqrt{\frac{\|A\|}{\|B\|}}B$ respectively, we have

$$
\begin{aligned}
s_j\left(AYB^*\right) & \leq \frac{1}{2}\|Y\| s_j\left(\frac{\|B\| A^*A}{\|A\|}+\frac{\|A\| B^*B}{\|B\|}\right) \\
& =\frac{1}{2}\|Y\|\|A\|\|B\| s_j\left(\frac{A^*A}{\|A\|^2}+\frac{B^*B}{\|B\|^2}\right),
\end{aligned}
$$

as required.

Theorem 4. *Let $A, B, C, D, E, F \in \mathbb{M}_n(\mathbb{C})$ be such that C and D are positive semidefinite. Then for $j=1,...,2n$, we have*

$$
\begin{aligned}
& s_j\left(\left(ACE^*+BDF^*\right) \oplus 0\right) \\
& \leq \frac{\max(\|C\|,\|D\|)}{2\sqrt{k_1k_2}} s_j\left(\left[\begin{array}{cc} k_2A^*A+k_1E^*E & k_2A^*B+k_1E^*F \\ k_2B^*A+k_1F^*E & k_2B^*B+k_1F^*F \end{array}\right]\right),
\end{aligned} \tag{12}
$$

where $k_1=\|AA^*+BB^*\|$ and $k_2=\|EE^*+FF^*\|$.

Proof. Let $T_1=\left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]$, $T_2=\left[\begin{array}{cc} C & 0 \\ 0 & D \end{array}\right]$, and $T_3=\left[\begin{array}{cc} E & F \\ 0 & 0 \end{array}\right]$. Then

$$\|T_1\|=\sqrt{\|T_1T_1^*\|}=\sqrt{\|AA^*+BB^*\|}=\sqrt{k_1},$$

$$\|T_3\|=\sqrt{\|T_3T_3^*\|}=\sqrt{\|EE^*+FF^*\|}=\sqrt{k_2}$$

and

$$\|T_2\| = \max\left(\|C\|, \|D\|\right).$$

So,

$$\begin{aligned}
s_j\left(ACE^* + BDF^* \oplus 0\right) &= s_j\left(T_1T_2T_3^*\right) \\
&\leq \frac{1}{2}\|T_1\|\,\|T_2\|\,\|T_3\|\, s_j\left(\frac{T_1^*T_1}{\|T_1\|^2} + \frac{T_3^*T_3}{\|T_3\|^2}\right) \\
&= \frac{1}{2}\sqrt{k_1k_2}\max\left(\|C\|, \|D\|\right) s_j\left(\frac{\begin{bmatrix} A^*A & A^*B \\ B^*A & BB \end{bmatrix}}{k_1} + \frac{\begin{bmatrix} E^*E & E^*F \\ F^*E & F^*F \end{bmatrix}}{k_2}\right) \\
&= \frac{1}{2}\sqrt{k_1k_2}\max\left(\|C\|, \|D\|\right) s_j\left(\frac{k_2\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} + k_1\begin{bmatrix} E^*E & E^*F \\ F^*E & F^*F \end{bmatrix}}{k_1k_2}\right) \\
&= \frac{\max\left(\|C\|, \|D\|\right)}{2\sqrt{k_1k_2}} s_j\left(\begin{bmatrix} k_2A^*A + k_1E^*E & k_2A^*B + k_1E^*F \\ k_2B^*A + k_1F^*E & k_2B^*B + k_1F^*F \end{bmatrix}\right),
\end{aligned}$$

as required.

Corollary 3. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$ be such that C and D are positive semidefinite. Then for $j = 1, ..., 2n$, we have*

$$s_j\left(\left(ACB^* + BDA^*\right) \oplus 0\right) \leq \frac{max\left(\|C\|, \|D\|\right)}{2} s_j^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right).$$

In particular, if $C = D = I$, we have

$$s_j\left(\left(\operatorname{Re}\left(AB^*\right)\right) \oplus 0\right) \leq \frac{1}{4} s_j^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)$$

where $\operatorname{Re}(T) = \frac{T+T^*}{2}$ *denotes the real part of $T \in \mathbb{M}_n(\mathbb{C})$.*

Proof. Letting $E = B$ and $F = A$ in inequality (12), we have

$$\begin{aligned}
s_j\left(\left(ACB^* + BDA^*\right) \oplus 0\right) & \\
&\leq \frac{\max\left(\|C\|, \|D\|\right)}{2k_1} s_j\left(\begin{bmatrix} k_1A^*A + k_1B^*B & k_1A^*B + k_1B^*A \\ k_1B^*A + k_1A^*B & k_1B^*B + k_1A^*A \end{bmatrix}\right) \\
&= \frac{\max\left(\|C\|, \|D\|\right)}{2} s_j\left(\begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ B^*A + A^*B & B^*B + A^*A \end{bmatrix}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\max\left(\|C\|,\|D\|\right)}{2} s_j\left(\begin{bmatrix} A^* & B^* \\ B^* & A^* \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\
&= \frac{\max\left(\|C\|,\|D\|\right)}{2} s_j\left(\left|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right|^2\right) \\
&= \frac{\max\left(\|C\|,\|D\|\right)}{2} s_j^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\
&= \frac{\max\left(\|C\|,\|D\|\right)}{2} s_j^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right).
\end{aligned}$$

We end this paper by the following result, which gives a lower bound for singular values of products and sums of matrices

Lemma 6. *[12]Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $j = 1, ..., n$, we have*

$$s_j\left(AB\right) \geq s_n\left(A\right)s_j\left(B\right) \tag{13}$$

and

$$s_j\left(AB\right) \leq s_j\left(A\right)s_1\left(B\right). \tag{14}$$

Theorem 5. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j\left((AC+BD)\oplus 0\right) \geq s_n\left(\left(\sqrt{AA^*+BB^*}\right)\oplus 0\right)s_j\left(\left(\sqrt{C^*C+D^*D}\right)\oplus 0\right) \tag{15}$$

and

$$s_j\left((AC+BD)\oplus 0\right) \leq s_1\left(\left(\sqrt{AA^*+BB^*}\right)\oplus 0\right)s_j\left(\left(\sqrt{C^*C+D^*D}\right)\oplus 0\right). \tag{16}$$

Proof. We have

$s_j\left((AC+BD)\oplus 0\right)$

$$\begin{aligned}
&= s_j\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right) \\
&\geq s_n\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right)s_j\left(\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right) \\
&\qquad \text{(by inequality (13))} \\
&= s_n\left(\left|\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right|\right)s_j\left(\left|\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right|\right) \\
&= s_n\left(\left(\begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right)^{1/2}\right)s_j\left(\left(\begin{bmatrix} C^* & D^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right)^{1/2}\right) \\
&= s_n^{1/2}\left(\begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right)s_j^{1/2}\left(\begin{bmatrix} C^* & D^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right)
\end{aligned}$$

$$\begin{aligned}
&= s_n^{1/2}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\right) s_j^{1/2}\left(\begin{bmatrix} C^* & D^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\right) \\
&= s_n^{1/2}\left(\begin{bmatrix} AA^*+BB^* & 0 \\ 0 & 0 \end{bmatrix}\right) s_j^{1/2}\left(\begin{bmatrix} C^*C+D^*D & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= s_n\left(\begin{bmatrix} \sqrt{AA^*+BB^*} & 0 \\ 0 & 0 \end{bmatrix}\right) s_j\left(\begin{bmatrix} \sqrt{C^*C+D^*D} & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= s_n\left(\left(\sqrt{AA^*+BB^*}\right)\oplus 0\right) s_j\left(\left(\sqrt{C^*C+D^*D}\right)\oplus 0\right).
\end{aligned}$$

which proves inequality (15). The inequality (16) follows by applying inequality (14) and using the same argument that we use in proving inequality (15).

References

[1] Y. Tao. More results on singular value inequalities of matrices. *Linear Algebra and its Applications*, 416(2-3):724–729, 2006.

[2] A. Al-Natoor, O. Hirzallah, and F. Kittaneh. Singular value inequalities for convex functions of positive semidefinite matrices. *Annals of Functional Analysis*, 14(1):7, 2023.

[3] A. Al-Natoor and F. Alrimawi. Singular value inequalities for concave and convex functions of matrix sums and products. *European Journal of Pure and Applied Mathematics*, 18(1):558–578, 2025.

[4] A. Al-Natoor, M. A. Amleh, B. Abughazaleh, and A. Burqan. Generalization of some unitarily invariant norm inequalities for matrices. *Journal of Mathematical Inequalities*, 17(2):581–589, 2023.

[5] A. Al-Natoor, A. Burqan, M. A. Amleh, and C. Conde. Some singular value inequalities for matrices. *Journal of Mathematical Inequalities*, 18(3):911–919, 2024.

[6] A. Burqan and F. Kittaneh. Singular value and norm inequalities associated with 2 × 2 positive semidefinite block matrices. *Electronic Journal of Linear Algebra*, 32:116–124, 2017.

[7] F. Kittaneh, H. R. Moradi, and M. Sababheh. Singular values of compact operators via operator matrices. *Mathematical Inequalities & Applications*, 27(3):759–774, 2024.

[8] A. Al-Natoor and F. Kittaneh. Singular value and norm inequalities for positive semidefinite matrices. *Linear and Multilinear Algebra*, 70(21):4498–4509, 2022.

[9] A. Al-Natoor, O. Hirzallah, and F. Kittaneh. Singular value and unitarily invariant norm inequalities for matrices. *Annals of Functional Analysis*, 15(2):21, 2024.

[10] R. Bhatia and F. Kittaneh. The matrix arithmetic-geometric mean inequality revisited. *Linear Algebra and its Applications*, 428(8-9):2177–2191, 2008.

[11] H. Albadawi. Singular values and arithmetic-geometric mean inequalities for operators. *Annals of Functional Analysis*, 3(2):10–18, 2012.

[12] F. Zhang. *Matrix theory: basic results and techniques.* Springer, New York, 2 edition, 2011.