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On Weakly Regular Semigroups Characterized in Terms of Cubic Bipolar Fuzzy Ideals

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Abstract. In this paper, we introduce the concept of cubic bipolar fuzzy subsemigroups and cubic bipolar fuzzy ideals in the context of semigroups. We explore their fundamental properties and examine how these structures interact within semigroups. The main thing this study adds is a way to describe weakly regular semigroups using the features of cubic bipolar fuzzy ideals. Through a detailed analysis, we establish several key results that highlight the role of these fuzzy ideals in understanding the algebraic structure of weakly regular semigroups.

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1. Introduction

The theory of fuzzy sets was conceptualized by Zadeh in 1965 [1]. This researchers is used in mathematics and logic but also in medical science, theoretical physics, robotics, computer science, control engineering, information science, etc. After that time, in 1979, Kuroki [2] defined the fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. Later in 1975, Zadeh [3], extended the concept of fuzzy sets by interval valued fuzzy sets as a generalization of the notion of fuzzy sets. In 1994 Zhang [4] introduced the notion of bipolar fuzzy sets with the extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, and used them for modeling and decision analysis. In 2000, Lee [5] used the term bipolar valued fuzzy sets and applied it to algebraic structures. In 2012, Jun et al. [6] introduced a new notion, a cubic set, and investigated several properties of cubic setes as well as introducting cubic

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subsemigroups and cubic left (right) ideals of semigroups. In 2018, Wei et al. [7] studied the concept of interval valued bipolar fuzzy set with the generalization ofa bipolarr fuzzy set. It is a study of the values of positive and negative function. Riaz and Tehrim [8] discussed the concept of cubic bipolar fuzzy sets and some properties.

In this paper, we consider the relationship between of cubic bipolar fuzzy ideals and interior ideals on semigroups. In the goal results, we characterized weak regualr semigroup by using of cubic bipolar fuzzy ideals on semigroups.

2. Preliminaries

In this section, we will give some basic definitions and results needed for the next section.

A *subsemigroup* of a semigroup $\mathfrak{S}$ is a non-empty subset $\mathfrak{T}$ of $\mathfrak{S}$ such that $\mathfrak{T}^2 \subseteq \mathfrak{T}$. A *left (right) ideal* of a semigroup $\mathfrak{S}$ is a non-empty subset $\mathfrak{T}$ of $\mathfrak{S}$ such that $\mathfrak{S}\mathfrak{T} \subseteq \mathfrak{T}$ ($\mathfrak{T}\mathfrak{S} \subseteq \mathfrak{T}$). By an *ideal* of a semigroup $\mathfrak{S}$, we mean a nonempty subset of $\mathfrak{T}$ which is both a left and a right ideal of $\mathfrak{S}$. A *generalized bi-ideal* of a semigroup $\mathfrak{S}$ is a non-empty subset $\mathfrak{T}$ of $\mathfrak{S}$ such that $\mathfrak{T}\mathfrak{S}\mathfrak{T} \subseteq \mathfrak{T}$. A subsemigroup $\mathfrak{T}$ of a semigroup $\mathfrak{S}$ is called a *bi-ideal (interior ideal)* of $\mathfrak{S}$ if $\mathfrak{T}\mathfrak{S}\mathfrak{T} \subseteq \mathfrak{T}$ ($\mathfrak{S}\mathfrak{T}\mathfrak{S} \subseteq \mathfrak{T}$).

2.1. Fuzzy sets and Interval valued fuzzy sets

In this subsection, we review the concept of important fuzzy sets and fuzzy sets valued in intervals.

For any $\mathfrak{h}_i \in [0,1]$, $i \in \mathfrak{F}$, define

$$\bigvee_{i\in\mathcal{F}} \mathfrak{h}_i := \sup_{i\in\mathcal{F}}\{\mathfrak{h}_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{F}} \mathfrak{h}_i := \inf_{i\in\mathcal{F}}\{\mathfrak{h}_i\}.$$

We see that for any $\mathfrak{h}_1, \mathfrak{h}_2 \in [0,1]$, we have

$$\mathfrak{h}_1 \vee \mathfrak{h}_2 = \max\{\mathfrak{h}_1, \mathfrak{h}_2\} \quad \text{and} \quad \mathfrak{h}_1 \wedge \mathfrak{h}_2 = \min\{\mathfrak{h}_1, \mathfrak{h}_2\}.$$

A fuzzy set ω of a non-empty set $\mathfrak{S}$ is a function such that $\omega : \mathfrak{S} \to [0,1]$.

Now, we review the concept of interval valued fuzzy sets.

We use $CS[0,1]$ to denote the set of all closed subintervals in $[0,1]$, i.e.,

$$CS[0,1] = \{\overline{\mathfrak{h}} := [\mathfrak{h}^l, \mathfrak{h}^u] \mid 0 \leq \mathfrak{h}^l \leq \mathfrak{h}^u \leq 1\}.$$

We note that $[\mathfrak{h}, \mathfrak{h}] = \{\mathfrak{h}\}$ for all $\mathfrak{h} \in [0,1]$. For $\mathfrak{h} = 0$ or 1, we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

For any two interval numbers $\overline{\mathfrak{h}}_1$ and $\overline{\mathfrak{h}}_2$ in $CS[0,1]$, define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{\mathfrak{h}}_1 \preceq \overline{\mathfrak{h}}_2$ if and only if $\mathfrak{h}_1^l \leq \mathfrak{h}_2^l$ and $\mathfrak{h}_1^u \leq \mathfrak{h}_2^u$

(2) $\overline{\mathfrak{h}}_1 = \overline{\mathfrak{h}}_2$ if and only if $\mathfrak{h}_1^l = \mathfrak{h}_2^l$ and $\mathfrak{h}_1^u = \mathfrak{h}_2^u$

(3) $\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_2 = [(\mathfrak{h}_1^l \wedge \mathfrak{h}_2^l), (\mathfrak{h}_1^u \wedge \mathfrak{h}_2^u)]$

(4) $\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_2 = [(\mathfrak{h}_1^l \vee \mathfrak{h}_2^l), (\mathfrak{h}_1^u \vee \mathfrak{h}_2^u)]$.

If $\overline{\mathfrak{h}}_1 \succeq \overline{\mathfrak{h}}_2$, we mean $\overline{\mathfrak{h}}_2 \preceq \overline{\mathfrak{h}}_1$.

Proposition 1. *[9] For any elements $\overline{\mathfrak{h}}_1, \overline{\mathfrak{h}}_2$ and $\overline{\mathfrak{h}}_3$ in $CS[0,1]$, the following properties are satisfied:*

(1) $\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_1 = \overline{\mathfrak{h}}_1$ *and* $\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_1 = \overline{\mathfrak{h}}_1$,

(2) $\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_2 = \overline{\mathfrak{h}}_2 \curlywedge \overline{\mathfrak{h}}_1$ *and* $\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_2 = \overline{\mathfrak{h}}_2 \curlyvee \overline{\mathfrak{h}}_1$,

(3) $(\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_2) \curlywedge \overline{\mathfrak{h}}_3 = \overline{\mathfrak{h}}_1 \curlywedge (\overline{\mathfrak{h}}_2 \curlywedge \overline{\mathfrak{h}}_3)$ *and* $(\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_2) \curlyvee \overline{\mathfrak{h}}_3 = \overline{\mathfrak{h}}_1 \curlyvee (\overline{\mathfrak{h}}_2 \curlyvee \overline{\mathfrak{h}}_3)$,

(4) $(\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_2) \curlyvee \overline{\mathfrak{h}}_3 = (\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_3) \curlywedge (\overline{\mathfrak{h}}_2 \curlyvee \overline{\mathfrak{h}}_3)$ *and* $(\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_2) \curlywedge \overline{\mathfrak{h}}_3 = (\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_3) \curlyvee (\overline{\mathfrak{h}}_2 \curlywedge \overline{\mathfrak{h}}_3)$,

(5) *If* $\overline{\mathfrak{h}}_1 \preceq \overline{\mathfrak{h}}_3$, *then,* $\overline{\mathfrak{h}}_1 \curlywedge \overline{\mathfrak{h}}_3 \preceq \overline{\mathfrak{h}}_2 \curlywedge \overline{\mathfrak{h}}_3$ *and* $\overline{\mathfrak{h}}_1 \curlyvee \overline{\mathfrak{h}}_3 \preceq \overline{\mathfrak{h}}_2 \curlyvee \overline{\mathfrak{h}}_3$.

For each interval $\{\overline{\mathfrak{h}}_i := [\mathfrak{h}_i^l, \mathfrak{h}_i^u] \mid i \in \mathcal{F}\}$ be a family of closed subintervals of $[0,1]$. Define $\underset{i\in\mathcal{F}}{\curlywedge} \overline{\mathfrak{h}}_i = [\underset{i\in\mathcal{F}}{\wedge} \mathfrak{h}_i^l, \underset{i\in\mathcal{F}}{\wedge} \mathfrak{h}_i^u]$ and $\underset{i\in\mathcal{F}}{\curlyvee} \overline{\mathfrak{h}}_i = [\underset{i\in\mathcal{F}}{\vee} \mathfrak{h}_i^l, \underset{i\in\mathcal{F}}{\vee} \mathfrak{h}_i^u]$.

Definition 1. *[10] An* interval valued fuzzy subset *(shortly, IVF subset) of $\mathfrak{S}$ is a function such that $\overline{\mu} : \mathfrak{S} \to CS[0,1]$.*

Definition 2. *[11] For every subset $\mathfrak{K}$ of set $\mathfrak{S}$, an interval valued characteristic function $\overline{\lambda}_{\mathfrak{K}}$ of $\mathfrak{K}$ is defined to be a function $\overline{\lambda}_{\mathfrak{K}} : \mathfrak{S} \to CS[0,1]$ by*

$$\overline{\lambda}_{\mathfrak{K}}(\mathfrak{h}) = \begin{cases} \overline{1} & \text{if} \quad \mathfrak{h} \in \mathfrak{K} \\ \overline{0} & \text{if} \quad \mathfrak{h} \notin \mathfrak{K} \end{cases}$$

for all $\mathfrak{h} \in \mathfrak{S}$.

For two IVF subsets $\overline{\mu}$ and $\overline{\lambda}$ of a non-empty set $\mathfrak{S}$, define

(1) $\overline{\mu} \sqsubseteq \overline{\lambda} \Leftrightarrow \overline{\mu}(h) \preceq \overline{\lambda}(h)$ for all $h \in \mathfrak{F}$,

(2) $\overline{\mu} = \overline{\lambda} \Leftrightarrow \overline{\mu} \sqsubseteq \overline{\lambda}$ and $\overline{\lambda} \sqsubseteq \overline{\mu}$,

(3) $(\overline{\mu} \sqcap \overline{\lambda})(h) = \overline{\mu}(h) \curlywedge \overline{\lambda}(h)$ for all $h \in \mathfrak{F}$.

For $\mathfrak{h} \in \mathfrak{S}$, define $A_{\mathfrak{h}} := \{(\mathfrak{k}, \mathfrak{o}) \in \mathfrak{F} \times \mathfrak{F} \mid \mathfrak{h} = \mathfrak{k}\mathfrak{o}\}$.
For two IVF sets $\overline{\mu}$ and $\overline{\lambda}$ of $\mathfrak{F}$, define the product $\overline{\mu} \bigcirc \overline{\lambda}$ as follows : for all $\mathfrak{h} \in \mathfrak{S}$,

$$(\overline{\mu} \bigcirc \overline{\lambda})(\mathfrak{h}) = \begin{cases} \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}{\curlyvee} \{\overline{\mu}(\mathfrak{k}) \curlywedge \overline{\lambda}(\mathfrak{o})\} & \text{if} \quad A_{\mathfrak{h}} \neq \emptyset \\ \overline{0} & \text{if} \quad A_{\mathfrak{h}} = \emptyset. \end{cases}$$

Definition 3. *[11] An IVF set $\overline{\mu}$ of a semigroup $\mathfrak{S}$ is said to be an*

(1) IVF subsemigroup *of $\mathfrak{S}$, if $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_2)$ for all $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,*

(2) IVF left (right) ideal *of $\mathfrak{S}$, if $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}(\mathfrak{h}_2)$ $(\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}(\mathfrak{h}_1))$for all $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$. An* IVF ideal *of $\mathfrak{S}$ if it is both an IVF left ideal and an IVF right ideal of $\mathfrak{S}$,*

(3) IVF genralized bi-ideal *of* $\mathfrak{S}$, *if* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(4) IVF bi-ideal *of* $\mathfrak{S}$, *if* $\overline{\mu}$ *is an IVF subsemigroup of* $\mathfrak{S}$ *and* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(5) IVF interior ideal *of* $\mathfrak{S}$, *if* $\overline{\mu}$ *is an IVF subsemigroup of* $\mathfrak{S}$ *and* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(6) IVF quasi ideal *of* $\mathfrak{S}$, *if* $(\overline{\mathfrak{S}} \bigcirc \overline{\mu}) \sqcap (\overline{\mu} \bigcirc \overline{\mathfrak{S}}) \sqsubseteq \overline{\mu}$, *where* $\overline{\mathfrak{S}}$ *is an IVF set of* $\mathfrak{S}$ *mapping every element of* $\mathfrak{S}$ *on* $\overline{1}$.

2.2. Bipolar fuzzy set

Definition 4. *[4] A* bipolar fuzzy set *(shortly, BF set)* ω *on* $\mathfrak{S}$ *is an object having the form*

$$\omega := \{(\mathfrak{S}, \omega^p(\mathfrak{h}), \omega^n(\mathfrak{h})) \mid \mathfrak{h} \in \mathfrak{S}\},$$

where $\omega^p : \mathfrak{S} \to [0,1]$ *and* $\omega^n : \mathfrak{S} \to [-1,0]$.

Remark 1. *For the sake of simplicity, we shall use the symbol* $\omega = (\mathfrak{S}; \omega^p, \omega^n)$ *for the BF set* $\omega = \{(\mathfrak{S}, \omega^p(\mathfrak{h}), \omega^n(\mathfrak{h})) \mid \mathfrak{h} \in \mathfrak{S}\}$.

Define products $\omega^p * \psi^p$ and $\omega^n * \psi^n$ as follows: For $\mathfrak{h} \in \mathfrak{S}$

$$(\omega^p * \psi^p)(\mathfrak{h}) = \begin{cases} \bigvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{o})\} & \text{if} \quad A_{\mathfrak{h}} \neq \emptyset \\ 0 & \text{if} \quad A_{\mathfrak{h}} = \emptyset \end{cases}$$

and

$$(\omega^n * \psi^n)(\mathfrak{h}) = \begin{cases} \bigwedge_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\omega^n(\mathfrak{k}) \vee \psi^n(\mathfrak{o})\} & \text{if} \quad A_{\mathfrak{h}} \neq \emptyset \\ 0 & \text{if} \quad A_{\mathfrak{h}} = \emptyset. \end{cases}$$

Definition 5. *[12] A* positive characteristic function and a negative characteristic function *of a non-empty set* $\mathfrak{K}$ *of* $\mathfrak{F}$ *defined by*

$$\chi^p_{\mathfrak{K}} : \mathfrak{S} \to [0,1], \mathfrak{h} \mapsto \chi^p_{\mathfrak{K}}(\mathfrak{h}) := \begin{cases} 1 & \mathfrak{h} \in \mathfrak{K} \\ 0 & \mathfrak{h} \notin \mathfrak{K} \end{cases}$$

and

$$\chi^n_{\mathfrak{K}} : \mathfrak{S} \to [-1,0], \mathfrak{h} \mapsto \chi^n_{\mathfrak{K}}(\mathfrak{h}) := \begin{cases} -1 & \mathfrak{h} \in \mathfrak{K} \\ 0 & \mathfrak{h} \notin \mathfrak{K}. \end{cases}$$

respectively.

Definition 6. *[12] A BF set* $\omega = (\mathfrak{S}; \omega^p, \omega^n)$ *on a semigroup* $\mathfrak{S}$ *is called a*

(1) BF subsemigroup *on* $\mathfrak{S}$, *if* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_2)$ *and* $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,

(2) BF left (right) ideal *on* $\mathfrak{S}$, *if* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_2)$ *(*$\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1)$*) and* $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_2)$ *(*$\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1)$*) for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,

(3) BF generalized bi-ideal *on* $\mathfrak{S}$, *if* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_3)$ *and* $\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(4) BF bi-ideal *on* $\mathfrak{S}$*, if* $\omega = (\mathfrak{S}; \omega^p, \omega^n)$ *is a BF subsemigroup of* $\mathfrak{S}$*,* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_3)$ *and* $\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$*,*

(5) BF interior ideal *on* $\mathfrak{S}$*, if* $\omega = (\mathfrak{S}; \omega^p, \omega^n)$ *is a BF subsemigroup of* $\mathfrak{S}$*,* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_2)$ *and* $\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega^n(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$*,*

(6) BF quasi ideal *on* $\mathfrak{S}$*, if* $(\mathfrak{S}^p * \omega^p) \cap (\omega^p * \mathfrak{S}^p) \subseteq \omega^p$ *and* $(\mathfrak{S}^n * \omega^n) \cap (\omega^n * \mathfrak{S}^n) \supseteq \omega^n$*, where* $\mathfrak{S}$ *is an BF set of* $\mathfrak{S}$ *mapping every element of* $\mathfrak{S}$ *on* $[-1, 1]$*.*

Definition 7. *[7] An* interval valued bipolar fuzzy set *(shortly, IVBF set)*$\mathfrak{C} = (\mathfrak{S}; \overline{\mu}^p, \overline{\mu}^n)$ *of a non-empty set* $\mathfrak{F}$ *if* $\overline{\mu}^p : \mathfrak{S} \to CS[0, 1]$ *and* $\overline{\mu}^n : \mathfrak{S} \to CS[-1, 0]$*.*

2.3. Cubic sets

Definition 8. *[6] A* cubic set $\mathcal{C}$ *of a non-empty set* $\mathfrak{S}$ *is a structure of the form*

$$\mathcal{C} = \{\langle \mathfrak{h}, \overline{\mu}(\mathfrak{h}), \omega(\mathfrak{h})\rangle \mid \mathfrak{h} \in \mathfrak{S}\}$$

and denoted by $\mathcal{C} = \langle \overline{\mu}, \omega\rangle$ *where* $\overline{\mu}$ *is an IVF set and* ω *is a fuzzy set. In this case, we use*

$$\mathcal{C}(\mathfrak{h}) = \langle \overline{\mu}(\mathfrak{h}), \omega(\mathfrak{h})\rangle = \langle [\mu^n(\mathfrak{h}), \mu^p(\mathfrak{h})], \omega(\mathfrak{h})\rangle$$

for all $r \in \mathfrak{S}$*.*

Definition 9. *[6] A cubic set* $\mathcal{C} = \langle \overline{\mu}, \omega\rangle$ *of* $\mathfrak{S}$ *is called*

(1) *a* cubic subsemigroup *of* $\mathfrak{S}$*, if* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_2)$ *and* $\omega(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega(\mathfrak{h}_1) \vee (\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,

(2) *a* cubic left*(right)*ideal *of* $\mathfrak{S}$*, if* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}(h_2)(\overline{\mu}(h_1h_2) \succeq \overline{\mu}(\mathfrak{h}_1))$ *and* $\omega(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega(\mathfrak{h}_2)(\omega(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega(\mathfrak{h}_1))$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$*. A* cubic ideal *of* $\mathfrak{S}$*, if it is a cubic left ideal and a cubic right ideal of* $\mathfrak{S}$*,*

(3) *a* cubic genralized bi ideal *of* $\mathfrak{S}$*, if* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_3)$ *and* $\omega(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega(\mathfrak{h}_1) \vee \omega(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(4) *a* cubic bi ideal *of* $\mathfrak{S}$*, if* $\overline{\mu}$ *is a cubic subsemigorup of* $\mathfrak{S}$ *and* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_1) \curlywedge \overline{\mu}(\mathfrak{h}_3)$ *and* $\omega(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega(\mathfrak{h}_1) \vee \omega(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(5) *a* cubic interior ideal *of* $\mathfrak{S}$*, if* $\overline{\mu}$ *is a cubic subsemigorup of* $\mathfrak{S}$*,* $\overline{\mu}(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}(\mathfrak{h}_2)$ *and* $\omega(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(6) *a* cubic quasi ideal *of* $\mathfrak{S}$*, if* $(\overline{\mathfrak{S}} \bigcirc \overline{\mu}) \cap (\overline{\mu} \bigcirc \overline{\mathfrak{S}}) \sqsubseteq \overline{\mu}$ *and* $(\mathfrak{S} * \omega) \cap (\omega * \mathfrak{S}) \supseteq \omega$*.*

Riaz and Tehrim [8] discussed the concept of cubic bipolar fuzzy sets and some properties. In this paper, we consider the concepts of cubic bipolar fuzzy subsemigroups and types of cubic bipolar fuzzy ideals. We provide properties of cubic bipolar fuzzy subsemigroups and quasi ideals. In the important results, regular and intra-regular semigroups are characterized in terms of cubic bipolar fuzzy quasi ideals are provided.

3. Cubic bipolar fuzzy subsemigroup and ideals in semigroups

In this part, we give the concepts of cubic bipolar fuzzy subsemigroups and ideals in semigroup. And we study important properties for reference in the next part.

Definition 10. *A* cubic bipolar set *(shortly, CB set)* $\ddot{\mathfrak{C}}$ *of a set* $\mathfrak{S}$ *if*

$$\ddot{\mathfrak{C}} = \{\langle \mathfrak{h}, (\overline{\mu}^p(\mathfrak{h}), \overline{\mu}^n(\mathfrak{h})), \omega(\mathfrak{h})\rangle \mid \mathfrak{h} \in \mathfrak{S}\}$$

and denoted by $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *where* $\overline{\mu}$ *is an IVBF set and* ω *is a BF set.*

Definition 11. *A CB set* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *of a semigroup* $\mathfrak{S}$ *is called a* cubic bipolar fuzzy subsemigroup *(shortly, CBF subsemigroup) of* $\mathfrak{S}$ *if*
$\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mu}^n(\mathfrak{h}_2)$ *and*
$\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^n(\mathfrak{h}_2)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$.

Example 1. *Let* $\mathfrak{S}$ *be a semigroup defined by the following table:*

$\cdot$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

Thus, a CB set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *in* $\mathfrak{F}$ *as follows:* $\overline{\mu}^p(a) = [0.6, 0.7]$, $\overline{\mu}^p(b) = [0.4, 0.5]$, $\overline{\mu}^p(c) = [0.1, 0.2]$, $\overline{\mu}^n(a) = [-0.9, -0.8]$, $\overline{\mu}^n(b) = [-0.7, -0.6]$, $\overline{\mu}^p(c) = [-0.3, -0.2]$ *and* $\omega^p(a) = 0.7$, $\omega^p(b) = 0.4$, $\omega^p(c) = 0.2$, $\omega^n(a) = -0.7$, $\omega^n(b) = -0.3$, $\omega^n(c) = -0.2$. *Thus,* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *is a CBF subsemigroup of* $\mathfrak{S}$.

Definition 12. *A CB set* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *of a semigroup* $\mathfrak{S}$ *is called*

(1) *a* cubic bipolar fuzzy left ideal *(shortly, CBF left ideal) of* $\mathfrak{S}$, *if* $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p(\mathfrak{h}_2)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n(\mathfrak{h}_2)$ *and* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_2)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_2)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,

(2) *a* cubic bipolar fuzzy right ideal *(shortly, CBF rihgt ideal) of* $\mathfrak{S}$, *if* $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p(\mathfrak{h}_1)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n(\mathfrak{h}_1)$ *and* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$,

(3) *a* cubic bipolar fuzzy ideal *(shortly, CBF ideal) of* $\mathfrak{S}$, *if it is a CBF left ideal and a CBF right ideal of* $\mathfrak{S}$,

(4) *a* cubic bipolar fuzzy generalized bi-ideal *(shortly, CBF generalized bi-ideal) of* $\mathfrak{S}$, *if* $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_3)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mu}^n(\mathfrak{h}_3)$ *and* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^n(\mathfrak{h}_3)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$,

(5) *a* cubic bipolar fuzzy bi-ideal *(shortly, CBF bi-ideal) of* $\mathfrak{S}$, *if* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ *is a CBF subsemigroup of* $\mathfrak{S}$, $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_3)$ $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mu}^n(\mathfrak{h}_3)$ *and* $\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^n(\mathfrak{h}_3)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_3)$ *for all* $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$.

Next, we study the subset and product of the CBF set as defined.
Let $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega\rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi\rangle$ are CBF sets of a semigroup $\mathfrak{S}$. Define

(i) $\ddot{\mathfrak{C}}_1 \sqsubseteq \ddot{\mathfrak{C}}_2$ if and only if $\overline{\mu}^p(\mathfrak{h}) \preceq \overline{\lambda}^p(\mathfrak{h})$, $\overline{\mu}^n(\mathfrak{h}) \succeq \overline{\lambda}^n(\mathfrak{h})$ and $\omega^p(\mathfrak{h}) \leq \psi^p(\mathfrak{h})$, $\omega^n(\mathfrak{h}) \geq \psi^n(\mathfrak{h})$, for all $\mathfrak{h} \in \mathfrak{S}$.

(ii) $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 = \langle \overline{\mu} \sqcap \overline{\lambda}, \omega \cap \psi \rangle$ if and only if $(\overline{\mu}^p \sqcap \overline{\lambda}^p)(h) = (\overline{\mu}^p(h) \curlywedge \overline{\lambda}^p(h))$, $(\overline{\mu}^n \sqcap \overline{\lambda}^n)(h) = (\overline{\mu}^n(h) \curlyvee \overline{\lambda}^n(h))$ and $(\omega^p \cap \psi^p)(h) = (\omega^p(h) \wedge \psi^p(h))$, $(\omega^n \cap \psi^n)(h) = (\omega^n(h) \vee \psi^n(h)$ for all $h \in \mathfrak{S}$.

(iii) $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \langle \overline{\mu} \bigcirc \overline{\lambda}, \omega * \psi \rangle$ and define $\overline{\mu} \bigcirc \overline{\lambda}$ as follows. For $\mathfrak{h} \in \mathfrak{S}$

$$(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) = \begin{cases} \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} & \text{if } A_{\mathfrak{h}} \neq \emptyset \\ \overline{0} & \text{if } A_{\mathfrak{h}} = \emptyset, \end{cases}$$

$$(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) = \begin{cases} \curlywedge_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} & \text{if } A_{\mathfrak{h}} \neq \emptyset \\ \overline{0} & \text{if } A_{\mathfrak{h}} = \emptyset, \end{cases}$$

and $\omega * \psi$ is a product of a BF set.

Definition 13. *A cubic biopolar set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ of $\mathfrak{S}$ is called a* cubic biopolar fuzzy quasi ideal *(shortly, CBF quasi ideal) of $\mathfrak{S}$, if $(\overline{\mathfrak{S}} \bigcirc \overline{\mu}) \sqcap (\overline{\mu} \bigcirc \overline{\mathfrak{S}}) \overline{\sqsubset} \overline{\mu}$ and $(\mathfrak{S} * \omega) \cap (\omega * \mathfrak{S}) \subseteq \omega$.*

The follows theorem are basic propeties of CBF ideal of a semigroup $\mathfrak{S}$.

Theorem 1. *Every CBF ideal of a semigroup $\mathfrak{S}$ is a CBF interior ideal of $\mathfrak{S}$.*

Proof. Let $\ddot{\mathcal{C}} = \langle \overline{\mu}, \omega \rangle$ be a CBF ideal of $\mathfrak{S}$ and Let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$. Then, $\ddot{\mathcal{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF left ideal and CBF right ideal of $\mathfrak{S}$. Thus, $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p(\mathfrak{h}_2)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^p(\mathfrak{h}_2)$ and $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^n(\mathfrak{h}_2)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_2)$.

Hence, $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^n(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2)$ and $\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^n(\mathfrak{h}_2) \wedge \omega^n(\mathfrak{h}_2)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_2)$.

This shows that $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of $\mathfrak{S}$. Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$. Then,

$$\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) = \overline{\mu}^p(\mathfrak{h}_1(\mathfrak{h}_2\mathfrak{h}_3)) \succeq \overline{\mu}^p(\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}^p(\mathfrak{h}_2),$$

$$\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) = \overline{\mu}^n(\mathfrak{h}_1(\mathfrak{h}_2\mathfrak{h}_3)) \preceq \overline{\mu}^n(\mathfrak{h}_2\mathfrak{h}_3) \preceq \overline{\mu}^n(\mathfrak{h}_2)$$

and

$$\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) = \omega^p(\mathfrak{h}_1(\mathfrak{h}_2\mathfrak{h}_3)) \geq \omega^p(\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^p(\mathfrak{h}_2),$$

$$\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) = \omega^n(\mathfrak{h}_1(\mathfrak{h}_2\mathfrak{h}_3)) \leq \omega^n(\mathfrak{h}_2\mathfrak{h}_3) \geq \omega^n\omega^n(\mathfrak{h}_2).$$

Thus, $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF interior ideal of $\mathfrak{S}$.

Theorem 2. *Every CBF quasi ideal of a semigroup $\mathfrak{S}$ is a CBF bi-ideal of $\mathfrak{S}$.*

Proof. Assume that $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF quasi ideal of $\mathfrak{S}$ and $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$. Then,

$$\begin{aligned} \overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) &\succeq (\overline{\mu}^p \bigcirc \overline{\mathfrak{S}}^p)(\mathfrak{h}_1\mathfrak{h}_2) \curlywedge (\overline{\mathfrak{S}}^p \bigcirc \overline{\mu}^p)(\mathfrak{h}_1\mathfrak{h}_2) \\ &= \curlyvee_{(\mathfrak{i},\mathfrak{j}) \in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\overline{\mu}^p(\mathfrak{i}) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{j})\} \curlywedge \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\overline{\mathfrak{S}}^p(\mathfrak{k}) \curlywedge \overline{\mu}^p(\mathfrak{o})\} \\ &\succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{h}_2) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2) \\ &= \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{1} \curlywedge \overline{1} \curlywedge \overline{\mu}^p(\mathfrak{h}_2) = \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2), \end{aligned}$$

$$\begin{aligned} \overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) &\preceq (\overline{\mu}^n \bigcirc \overline{\mathfrak{S}}^n)(\mathfrak{h}_1\mathfrak{h}_2) \curlyvee (\overline{\mathfrak{S}}^n \bigcirc \overline{\mu}^n)(\mathfrak{h}_1\mathfrak{h}_2) \\ &= \curlywedge_{(\mathfrak{i},\mathfrak{j}) \in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\overline{\mu}^n(\mathfrak{i}) \curlyvee \overline{\mathfrak{S}}^n(\mathfrak{j})\} \curlyvee \curlywedge_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\overline{\mathfrak{S}}^n(\mathfrak{k}) \curlyvee \overline{\mu}^n(\mathfrak{o})\} \\ &\preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mathfrak{S}}^n(\mathfrak{h}_2) \curlywedge \overline{\mathfrak{S}}^n(\mathfrak{h}_1) \curlywedge \overline{\mu}^n(\mathfrak{h}_2) \\ &= \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{-1} \curlywedge \overline{-1} \curlyvee \overline{\mu}^p(\mathfrak{h}_2) = \overline{\mu}^p(\mathfrak{h}_1) \curlyvee \overline{\mu}^p(\mathfrak{h}_2). \end{aligned}$$

And

$$\begin{array}{rcl}\omega^p(\mathfrak{h}_1\mathfrak{h}_2) & \geq & (\omega^p * \mathfrak{S}^p)(\mathfrak{h}_1\mathfrak{h}_2) \wedge (\mathfrak{S}^p * \omega^p)(\mathfrak{h}_1\mathfrak{h}_2) \\ & = & \bigvee_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\omega^p(\mathfrak{i}) \wedge \mathfrak{F}^p(\mathfrak{j})\} \wedge \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\mathfrak{S}^p(\mathfrak{k}) \wedge \omega^p(\mathfrak{o})\} \\ & \geq & \omega^p(\mathfrak{h}_1) \wedge \mathfrak{S}^p(\mathfrak{h}_2) \wedge \mathfrak{S}^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_2) \\ & = & \omega^p(\mathfrak{h}_1) \wedge 1 \wedge 1 \wedge \omega^p(\mathfrak{h}_2) = \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_2),\end{array}$$

$$\begin{array}{rcl}\omega^n(\mathfrak{h}_1\mathfrak{h}_2) & \leq & (\omega^n * \mathfrak{S}^n)(\mathfrak{h}_1\mathfrak{h}_2) \vee (\mathfrak{S}^n * \omega^n)(\mathfrak{h}_1\mathfrak{h}_2) \\ & = & \bigwedge_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\omega^p(\mathfrak{i}) \vee \mathfrak{S}^n(\mathfrak{j})\} \vee \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2}} \{\mathfrak{S}^n(\mathfrak{k}) \vee \omega^n(\mathfrak{o})\} \\ & \leq & \omega^n(\mathfrak{h}_1) \vee \mathfrak{S}^n(\mathfrak{h}_2) \vee \mathfrak{S}^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_2) \\ & = & \omega^n(\mathfrak{h}_1) \vee -1 \vee -1 \vee \omega^n(\mathfrak{h}_2) = \omega^n(\mathfrak{h}_1) \wedge \omega^n(\mathfrak{h}_2).\end{array}$$

Thus, $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_2)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mu}^n(\mathfrak{h}_2)$ and
$\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_2)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_2)$.
Hence, $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of $\mathfrak{S}$.

In a similar way, let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathfrak{S}$ we get that

$$\begin{array}{rcl}\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) & \succeq & (\overline{\mu}^p \bigcirc \overline{\mathfrak{S}}^p)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \curlywedge (\overline{\mathfrak{S}}^p \bigcirc \overline{\mu}^p)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \\ & = & \curlyvee_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\overline{\mu}^p(\mathfrak{i}) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{j})\} \curlywedge \curlyvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\overline{\mathfrak{S}}^p(\mathfrak{k}) \curlywedge \overline{\mu}^p(\mathfrak{o})\} \\ & \succeq & \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{h}_2\mathfrak{h}_3) \curlywedge \overline{\mathfrak{S}}^p(\mathfrak{h}_1\mathfrak{h}_2) \curlywedge \overline{\mu}^p(\mathfrak{h}_3) \\ & = & \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{1} \curlywedge \overline{1} \curlywedge \overline{\mu}^p(\mathfrak{h}_3) = \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_3),\end{array}$$

$$\begin{array}{rcl}\overline{\mu}^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) & \preceq & (\overline{\mu}^n \bigcirc \overline{\mathfrak{S}}^n)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \curlyvee (\overline{\mathfrak{S}}^n \bigcirc \overline{\mu}^n)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \\ & = & \curlywedge_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\overline{\mu}^n(\mathfrak{i}) \curlyvee \overline{\mathfrak{F}}^n(\mathfrak{j})\} \curlyvee \curlywedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\overline{\mathfrak{F}}^n(\mathfrak{k}) \curlyvee \overline{\mu}^n(\mathfrak{o})\} \\ & \preceq & \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mathfrak{S}}^n(\mathfrak{h}_2\mathfrak{r}_3) \curlywedge \overline{\mathfrak{S}}^n(\mathfrak{h}_1\mathfrak{h}_2) \curlywedge \overline{\mu}^n(\mathfrak{h}_3) \\ & = & \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{-1} \curlywedge \overline{-1} \curlyvee \overline{\mu}^p(\mathfrak{h}_3) = \overline{\mu}^p(\mathfrak{h}_1) \curlyvee \overline{\mu}^p(\mathfrak{h}_3).\end{array}$$

And

$$\begin{array}{rcl}\omega^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) & \geq & (\omega^p * \mathfrak{F}^p)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \wedge (\mathfrak{S}^p * \omega^p)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \\ & = & \bigvee_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\omega^p(\mathfrak{i}) \wedge \mathfrak{S}^p(\mathfrak{j})\} \wedge \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\mathfrak{F}^p(\mathfrak{k}) \wedge \omega^p(\mathfrak{o})\} \\ & \geq & \omega^p(\mathfrak{h}_1) \wedge \mathfrak{S}^p(\mathfrak{h}_2\mathfrak{h}_3) \wedge \mathfrak{S}^p(\mathfrak{h}_1\mathfrak{h}_2) \wedge \omega^p(\mathfrak{h}_3) \\ & = & \omega^p(\mathfrak{h}_1) \wedge 1 \wedge 1 \wedge \omega^p(\mathfrak{h}_3) = \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_3),\end{array}$$

$$\begin{array}{rcl}\omega^n(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) & \leq & (\omega^n * \mathfrak{S}^n)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \vee (\mathfrak{S}^n * \omega^n)(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \\ & = & \bigwedge_{(\mathfrak{i},\mathfrak{j})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\omega^p(\mathfrak{i}) \vee \mathfrak{F}^n(\mathfrak{j})\} \vee \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3}} \{\mathfrak{S}^n(\mathfrak{k}) \vee \omega^n(\mathfrak{o})\} \\ & \leq & \omega^n(\mathfrak{h}_1) \vee \mathfrak{F}^n(\mathfrak{h}_2\mathfrak{h}_3) \vee \mathfrak{S}^n(\mathfrak{h}_1\mathfrak{h}_2) \vee \omega^n(\mathfrak{h}_3) \\ & = & \omega^n(\mathfrak{h}_1) \vee -1 \vee -1 \vee \omega^n(\mathfrak{h}_3) = \omega^n(\mathfrak{h}_1) \wedge \omega^n(\mathfrak{h}_3).\end{array}$$

Thus, $\overline{\mu}^p(\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_3) \succeq \overline{\mu}^p(\mathfrak{h}_1) \curlywedge \overline{\mu}^p(\mathfrak{h}_3)$, $\overline{\mu}^n(\mathfrak{h}_1\mathfrak{r}_2\mathfrak{h}_3) \preceq \overline{\mu}^n(\mathfrak{h}_1) \curlyvee \overline{\mu}^n(\mathfrak{h}_3)$ and
$\omega^p(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p(\mathfrak{h}_1) \wedge \omega^p(\mathfrak{h}_3)$, $\omega^n(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n(\mathfrak{h}_1) \vee \omega^n(\mathfrak{h}_3)$.
Hence, $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF bi-ideal of $\mathfrak{S}$.

Next, we review the definition of the characteristic cubic bipolar fuzzy function. Let $\mathfrak{T}$ be a non-empty subset of $\mathfrak{S}$. The characteristic cubic bipolar fuzzy set (shortly, CCBF set) $\chi_{\mathfrak{T}} = \langle \overline{\mu}_{\chi_{\mathfrak{T}}}, \omega_{\chi_{\mathfrak{T}}} \rangle$ is defined as follows:

$$\overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}) = \begin{cases} \overline{1} & \text{if} \quad \mathfrak{h} \in \mathfrak{T} \\ \overline{0} & \text{if} \quad \mathfrak{h} \notin \mathfrak{T} \end{cases}, \quad \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}) = \begin{cases} \overline{-1} & \text{if} \quad \mathfrak{h} \in \mathfrak{T} \\ \overline{0} & \text{if} \quad \mathfrak{h} \notin \mathfrak{T} \end{cases}$$

for all $\mathfrak{h} \in \mathfrak{S}$ and $\omega_{\chi_{\mathfrak{T}}}$ is a characteristic bipolar fuzzy set.

In the following theorems, we give a relationship between a left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) and the CCBF function.

Theorem 3. *Let $\mathfrak{T}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then, $\mathfrak{T}$ is a left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{T}} = \langle \overline{\mu}_{\chi_{\mathfrak{T}}}, \omega_{\chi_{\mathfrak{T}}} \rangle$ is a CBF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of $\mathfrak{S}$.*

Proof. ($\Rightarrow$) Suppose that $\mathfrak{T}$ is a left ideal of $\mathfrak{S}$ and let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{S}$.

If $\mathfrak{h}_2 \in \mathfrak{T}$, then, $\mathfrak{h}_1\mathfrak{h}_2 \in \mathfrak{T}$. Thus, $\overline{1} = \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$, $\overline{-1} = \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$ and $1 = \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$, $-1 = \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$.
Hence, $\overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$, $\overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$ and $\omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$, $\omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \leq \vee\omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$.

If $\mathfrak{h}_2 \notin \mathfrak{T}$, then, $\overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$, $\overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$ and $\omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$, $\omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2)$.
Thus, $\chi_{\mathfrak{T}} = \langle \overline{\mu}_{\chi_{\mathfrak{T}}}, \omega_{\chi_{\mathfrak{T}}} \rangle$ is a CBF left ideal of $\mathfrak{S}$.

$\Leftarrow$ Suppose that $\chi_{\mathfrak{T}} = \langle \overline{\mu}_{\chi_{\mathfrak{T}}}, \omega_{\chi_{\mathfrak{T}}} \rangle$ is a CBF left ideal of $\mathfrak{S}$ and let $\mathfrak{h}_2 \in \mathfrak{T}$. Then, $\overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{1}$, $\overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{-1}$ and $\omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = 1$, $\omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = -1$. If $\mathfrak{h}_1\mathfrak{h}_2 \notin \mathfrak{T}$, then, $\overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) = \overline{0} = \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$ and $\omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) = 0 = \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2)$. Thus,

$$\overline{0} = \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \succeq \overline{\mu}^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{1},\ \overline{0} = \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \preceq \overline{\mu}^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = \overline{-1}$$

and

$$0 = \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \geq \omega^p_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = 1,\ 0 = \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_1\mathfrak{h}_2) \leq \omega^n_{\chi_{\mathfrak{T}}}(\mathfrak{h}_2) = -1.$$

It is a contradiction. Hence, $\mathfrak{h}_1\mathfrak{h}_2 \in \mathfrak{T}$. Therefore $\mathfrak{T}$ is a left ideal of $\mathfrak{S}$.

4. Characterizations of weakly regular semigroups in terms of cubic bipolar fuzzy ideals.

In this section, we will characterize weakly regular semigroups in terms of CBF subsemigroups.

Theorem 4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a non-empty subsets of a semigroup $\mathfrak{S}$. Then,*

(1) $\chi_{\mathfrak{M}} \circledast \chi_{\mathfrak{N}} = \chi_{\mathfrak{M}\mathfrak{N}}$ *i.e.* $\langle \overline{\mu}_{\chi_{\mathfrak{M}}} \bigcirc \overline{\mu}_{\chi_{\mathfrak{N}}}, \omega_{\chi_{\mathfrak{M}}} * \omega_{\chi_{\mathfrak{N}}} \rangle = \langle \overline{\mu}_{\chi_{\mathfrak{M}\mathfrak{N}}}, \omega_{\chi_{\mathfrak{M}\mathfrak{N}}} \rangle$

(2) $\chi_{\mathfrak{M}} \overline{\sqcap} \chi_{\mathfrak{N}} = \chi_{\mathfrak{M} \overline{\sqcap} \mathfrak{N}}$ *i.e.* $\langle \overline{\mu}_{\chi_{\mathfrak{M}}} \sqcap \overline{\mu}_{\chi_{\mathfrak{N}}}, \omega_{\chi_{\mathfrak{M}}} \cap \omega_{\chi_{\mathfrak{N}}} \rangle = \langle \overline{\mu}_{\chi_{\mathfrak{M} \sqcap \mathfrak{N}}}, \omega_{\chi_{\mathfrak{M} \cap \mathfrak{N}}} \rangle$

On the basis of Lemma 1, we can prove Theorem 5.

Lemma 1. *If $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ is a CBF right ideal and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ is a CBF left ideal of $\mathfrak{S}$, then, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$.*

Proof. Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ is a CBF right ideal and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ is a CBF left ideal of $\mathfrak{S}$ and let $\mathfrak{h} \in \mathfrak{S}$.

If $A_{\mathfrak{h}} = \emptyset$, then, it is easy to verify that, $(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) \preceq (\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h})$, $(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) \succeq (\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h})$ and $(\omega^p * \psi^p)(\mathfrak{h}) \leq (\omega^p \cap \psi^p)(\mathfrak{h})$, $(\omega^n * \psi^n)(\mathfrak{h}) \geq (\omega^n \cap \psi^n)(\mathfrak{h})$.

If $A_{\mathfrak{h}} \neq \emptyset$, then,

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) &= \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} \preceq \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^p(\mathfrak{k}\mathfrak{o}) \curlywedge \overline{\lambda}^p(\mathfrak{k}\mathfrak{o})\} \\ &= \overline{\mu}^p(\mathfrak{h}) \curlywedge \overline{\lambda}^p(\mathfrak{h}) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h}),\end{aligned}$$

$$\begin{aligned}(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) &= \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}{\curlywedge}\{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} \succeq \underset{(\mathfrak{k},\mathfrak{o})\in A_h}{\curlywedge}\{\overline{\mu}^n(\mathfrak{ko}) \curlyvee \overline{\lambda}^n(\mathfrak{ko})\} \\ &= \overline{\mu}^n(\mathfrak{h}) \curlyvee \overline{\lambda}^n(\mathfrak{h}) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h}),\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \psi^p)(\mathfrak{h}) &= \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{o})\} \leq \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^p(\mathfrak{ko}) \wedge \psi^p(\mathfrak{ko})\} \\ &= \omega^p(\mathfrak{h}) \wedge \psi^p(\mathfrak{h}) = (\omega^p \cap \psi^p)(\mathfrak{h}), \\ (\omega^n * \psi^n)(\mathfrak{h}) &= \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^n(\mathfrak{k}) \vee \psi^n(\mathfrak{o})\} \geq \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^p(\mathfrak{ko}) \vee \psi^n(\mathfrak{ko})\} \\ &= \omega^n(\mathfrak{h}) \vee \psi^n(\mathfrak{h}) = (\omega^n \cap \psi^n)(\mathfrak{h}).\end{aligned}$$

Thus, $(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) \preceq (\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h})$, $(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) \succeq (\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h})$ and $(\omega^p * \psi^p)(\mathfrak{h}) \leq (\omega^p \cap \psi^p)(\mathfrak{h})$, $(\omega^n * \psi^n)(\mathfrak{h}) \geq (\omega^n \cap \psi^n)(\mathfrak{h})$. Hence, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \bar{\sqcap} \ddot{\mathfrak{C}}_2$.

Any way, in the proof of Theorem 5, these are used.

Definition 14. *[13] A semigroup $\mathfrak{S}$ is called* weakly regular *if for every $\mathfrak{h} \in \mathfrak{S}, \mathfrak{h} \in (\mathfrak{h}\mathfrak{S})^2$.*

Lemma 2. *[13] A monoid $\mathfrak{S}$ is weakly regular if and only if $\mathfrak{R} \cap \mathfrak{J} = \mathfrak{R}\mathfrak{J}$ for every right ideal $\mathfrak{R}$ and every ideal $\mathfrak{J}$ of $\mathfrak{S}$.*

Now we characterize weakly regular semigroups in terms of generalized IVF ideals.

Theorem 5. *A monoid $\mathfrak{S}$ is weakly regular if and only if $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \bar{\sqcap} \ddot{\mathfrak{C}}_2$ for every CBF right ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and every CBF ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of $\mathfrak{S}$.*

Proof. Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ is a CBF right ideal and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ is a CBF ideal of $\mathfrak{S}$. Let $\mathfrak{h} \in \mathfrak{S}$. Since $\mathfrak{S}$ is weakly regular, there exist $\mathfrak{p}, \mathfrak{q} \in \mathfrak{S}$ such that $\mathfrak{h} = \mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}$. Thus,

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) &= \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}{\curlyvee}\{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} = \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{hphq}}}{\curlyvee}\{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} \\ &\succeq \overline{\mu}^p(\mathfrak{hp}) \curlywedge \overline{\lambda}^p(\mathfrak{hq}) \succeq \overline{\mu}^p(\mathfrak{h}) \curlywedge \overline{\lambda}^p(\mathfrak{h}) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h}), \\ (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) &= \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}{\curlywedge}\{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} = \underset{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{hphq}}}{\curlywedge}\{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} \\ &\preceq \overline{\mu}^n(\mathfrak{hp}) \curlyvee \overline{\lambda}^n(\mathfrak{hq}) \preceq \overline{\mu}^n(\mathfrak{h}) \curlyvee \overline{\lambda}^n(\mathfrak{h}) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h}),\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \psi^p)(\mathfrak{h}) &= \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{o})\} = \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{hphq}}}\{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{k})\} \\ &\geq \omega^p(\mathfrak{hp}) \wedge \psi^p(\mathfrak{hq}) \geq \omega^p(\mathfrak{h}) \wedge \psi^p(\mathfrak{h}) = (\omega^p \cap \psi^p)(\mathfrak{h}), \\ (\omega^n * \psi^n)(\mathfrak{h}) &= \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}}\{\omega^n(\mathfrak{k}) \vee \psi^n(\mathfrak{o})\} = \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{hphq}}}\{\omega^n(\mathfrak{ko}) \vee \psi^n(\mathfrak{ko})\} \\ &\leq \omega^n(\mathfrak{hp}) \vee \psi^n(\mathfrak{hq}) \leq \omega^n(\mathfrak{h}) \vee \psi^n(\mathfrak{h}) = (\omega^n \cap \psi^n)(\mathfrak{h}).\end{aligned}$$

Hence, $(\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h}) \preceq (\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h})$, $(\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h}) \succeq (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h})$ and $(\omega^p \cap \psi^p)(\mathfrak{h}) \leq (\omega^p * \psi^p)(\mathfrak{h})$, $(\omega^n \cap \psi^n)(\mathfrak{h}) \geq (\omega^n * \psi^n)(\mathfrak{h})$. Therefore, $\ddot{\mathfrak{C}}_1 \bar{\sqcap} \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$.

On the other hand, since $\ddot{\mathfrak{C}}_2$ is a CBF ideal of $\mathfrak{S}$ we see that $\ddot{\mathfrak{C}}_2$ is a CBF left ideal of $\mathfrak{S}$. Thus, by Lemma 1, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \bar{\sqcap} \ddot{\mathfrak{C}}_2$. Hence, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \bar{\sqcap} \ddot{\mathfrak{C}}_2$.

Conversely, let $\mathfrak{R}$ and $\mathfrak{J}$ be a right ideal and ideal of $\mathfrak{S}$. Then, by Theorem 3, $\chi_{\mathfrak{R}}$ and $\chi_{\mathfrak{J}}$ CBF right ideal and CBF ideal of $\mathfrak{S}$. By supposition and Lemma 4, we have

$$\overline{\mu}^p_{\chi_{\mathfrak{R}\mathfrak{J}}}(\mathfrak{h}) = (\overline{\mu}^p_{\chi_{\mathfrak{R}}} \bigcirc \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = (\overline{\mu}^p_{\chi_{\mathfrak{R}}} \sqcap \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \overline{\mu}^p_{\chi_{\mathfrak{R}\cap\mathfrak{J}}}(\mathfrak{h}) = \overline{1},$$
$$\overline{\mu}^n_{\chi_{\mathfrak{R}\mathfrak{J}}}(\mathfrak{h}) = (\overline{\mu}^p_{\chi_{\mathfrak{R}}} \bigcirc \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = (\overline{\mu}^n_{\chi_{\mathfrak{R}}} \sqcap \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \overline{\mu}^n_{\chi_{\mathfrak{R}\cap\mathfrak{J}}}(\mathfrak{h}) = \overline{-1},$$

and

$$\omega^p_{\chi_{\mathfrak{R}\mathfrak{J}}}(\mathfrak{h}) = (\omega^p_{\chi_{\mathfrak{R}}} * \omega^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = (\omega^p_{\chi_{\mathfrak{R}}} \cap \omega^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \omega^p_{\chi_{\mathfrak{R}\cap\mathfrak{J}}}(\mathfrak{h}) = 1,$$
$$\omega^n_{\chi_{\mathfrak{R}\mathfrak{J}}}(\mathfrak{h}) = (\omega^n_{\chi_{\mathfrak{R}}} * \omega^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = (\omega^n_{\chi_{\mathfrak{R}}} \cap \omega^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \omega^n_{\chi_{\mathfrak{R}\cap\mathfrak{J}}}(\mathfrak{h}) = -1.$$

Thus, $\mathfrak{h} \in \mathfrak{R}\mathfrak{J}$. Hence, $\mathfrak{R} \cap \mathfrak{J} = \mathfrak{R}\mathfrak{J}$. Therefore, by Lemma 2, $\mathfrak{S}$ is weakly regular.

Lemma 3. *[13] Let $\mathfrak{S}$ be a monoid. Then, the following statements are equivalent:*

(1) *$\mathfrak{S}$ is weakly regular.*

(2) *$\mathfrak{Q} \cap \mathfrak{J} \subseteq \mathfrak{Q}\mathfrak{J}$ for every quasi-ideal $\mathfrak{Q}$ and every ideal $\mathfrak{J}$ of $\mathfrak{S}$.*

On the basis of Lemma 3, we can prove Theorem 6.

Theorem 6. *For a monoid $\mathfrak{S}$, the following statements are equivalent.*

(1) *$\mathfrak{S}$ is weakly regular.*

(2) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$ for every CBF quasi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and every CBF ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of $\mathfrak{S}$.*

(3) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$ for every CBF bi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and every CBVF ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of $\mathfrak{S}$.*

(4) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$ for every CBF generalized bi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and every CBF interior ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of $\mathfrak{S}$.*

Proof. (1) $\Rightarrow$ (4) Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ is a CBF generalized bi-ideal and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ is a CBF interior ideal of $\mathfrak{S}$. Let $\mathfrak{h} \in \mathfrak{S}$. Then, there exist $\mathfrak{p}, \mathfrak{q} \in \mathfrak{S}$ such that $\mathfrak{h} = \mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}$. Thus,

$$\begin{aligned}
(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h}) &= \curlyvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} = \curlyvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge \overline{\lambda}^p(\mathfrak{o})\} \\
&\succeq \overline{\mu}^p(\mathfrak{h}) \curlywedge \overline{\lambda}^p(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \succeq \overline{\mu}^p(\mathfrak{h}) \curlywedge \overline{\lambda}^p(\mathfrak{h}) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h}), \\
(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h}) &= \curlywedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} = \curlywedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\overline{\mu}^n(\mathfrak{k}) \curlyvee \overline{\lambda}^n(\mathfrak{o})\} \\
&\preceq \overline{\mu}^n(\mathfrak{h}) \curlyvee \overline{\lambda}^n(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \preceq \overline{\mu}^n(\mathfrak{h}) \curlyvee \overline{\lambda}^n(\mathfrak{h}) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
(\omega^p * \psi^p)(\mathfrak{h}) &= \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{o})\} = \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\omega^p(\mathfrak{k}) \wedge \psi^p(\mathfrak{k})\} \\
&\geq \omega^p(\mathfrak{h}) \wedge \psi^p(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \geq \omega^p(\mathfrak{h}) \wedge \psi^p(\mathfrak{h}) = (\omega^p \cap \psi^p)(\mathfrak{h}), \\
(\omega^n * \psi^n)(\mathfrak{h}) &= \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\omega^n(\mathfrak{k}) \vee \psi^n(\mathfrak{o})\} = \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\omega^n(\mathfrak{k}) \vee \psi^n(\mathfrak{k})\} \\
&\leq \omega^n(\mathfrak{h}) \vee \psi^n(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \leq \omega^n(\mathfrak{h}) \vee \psi^n(\mathfrak{h}) = (\omega^n \cap \psi^n)(\mathfrak{h}).
\end{aligned}$$

Hence, $(\overline{\mu}^p \sqcap \overline{\lambda}^p)(\mathfrak{h}) \preceq (\overline{\mu}^p \bigcirc \overline{\lambda}^p)(\mathfrak{h})$, $(\overline{\mu}^n \sqcap \overline{\lambda}^n)(\mathfrak{h}) \succeq (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(\mathfrak{h})$ and $(\omega^p \cap \psi^p)(\mathfrak{h}) \leq (\omega^p * \psi^p)(\mathfrak{h})$, $(\omega^n \cap \psi^n)(\mathfrak{h}) \geq (\omega^n * \psi^n)(\mathfrak{h})$. Therefore, $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) This is obvious because every CBF bi-ideal is a CBF generalized bi-ideal of $\mathfrak{S}$, every CBF ideal is a CBF interior ideal of $\mathfrak{S}$ and every CBF quasi-ideal is a CBF bi-ideal of $\mathfrak{S}$.

(2) ⇒ (1) Let $\mathfrak{Q}$ be a quasi-ideal and $\mathfrak{J}$ be an ideal of $\mathfrak{S}$. Then, by Theorem 3, $\overline{\chi}_{\mathfrak{Q}}$ is a CBF quasi-ideal and $\overline{\chi}_{\mathfrak{J}}$ is a CBF ideal of $\mathfrak{S}$. By supposition and Lemma 4, we have

$$\overline{\mu}^p_{\chi_{\mathfrak{Q}\mathfrak{J}}}(\mathfrak{h}) = (\overline{\mu}^p_{\chi_{\mathfrak{Q}}} \bigcirc \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) \succeq (\overline{\mu}^p_{\chi_{\mathfrak{Q}}} \sqcap \overline{\mu}^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \overline{\mu}^p_{\chi_{\mathfrak{Q}\cap\mathfrak{J}}}(\mathfrak{h}) = \overline{1},$$

$$\overline{\mu}^n_{\chi_{\mathfrak{Q}\mathfrak{J}}}(\mathfrak{h}) = (\overline{\mu}^n_{\chi_{\mathfrak{Q}}} \bigcirc \overline{\mu}^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) \preceq (\overline{\mu}^n_{\chi_{\mathfrak{Q}}} \sqcap \overline{\mu}^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \overline{\mu}^n_{\chi_{\mathfrak{Q}\cap\mathfrak{J}}}(\mathfrak{h}) = \overline{-1},$$

and

$$\omega^p_{\chi_{\mathfrak{Q}\mathfrak{J}}}(\mathfrak{h}) = (\omega^p_{\chi_{\mathfrak{Q}}} * \omega^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) \geq (\omega^p_{\chi_{\mathfrak{Q}}} \cap \omega^p_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \omega^p_{\chi_{\mathfrak{Q}\cap\mathfrak{J}}}(\mathfrak{h}) = 1,$$

$$\omega^n_{\chi_{\mathfrak{Q}}}(\mathfrak{h}) = (\omega^n_{\chi_{\mathfrak{Q}}} * \omega^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) \leq (\omega^n_{\chi_{\mathfrak{Q}}} \cap \omega^n_{\chi_{\mathfrak{J}}})(\mathfrak{h}) = \omega^n_{\chi_{\mathfrak{Q}\cap\mathfrak{J}}}(\mathfrak{h}) = -1.$$

Thus, $\mathfrak{h} \in \mathfrak{Q}\mathfrak{J}$. Hence, $\mathfrak{Q} \cap \mathfrak{J} \sqsubseteq \mathfrak{Q}\mathfrak{J}$. Therefore, by Lemma 3, $\mathfrak{S}$ is weakly regular.

Lemma 4. *[13] Let $\mathfrak{S}$ be a monoid. Then, the following statements are equivalent:*

(1) *$\mathfrak{S}$ is weakly regular.*

(2) *$\mathfrak{Q} \cap \mathfrak{J} \cap \mathfrak{R} \subseteq \mathfrak{Q}\mathfrak{J}\mathfrak{R}$ for every quasi-ideal $\mathfrak{Q}$, every ideal $\mathfrak{J}$ and every right ideal $\mathfrak{R}$ of $\mathfrak{S}$.*

On the basis of Lemma 4, we can prove Theorem 7.

Theorem 7. *Let $\mathfrak{S}$ be a monoid. Then, the following statements are equivalent:*

(1) *$\mathfrak{S}$ is weakly regular.*

(2) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqcap} \ddot{\mathfrak{C}}_3 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \circledast \ddot{\mathfrak{C}}_3$, for every CBF quasi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$, every CBF ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ and every CBF right ideal $\ddot{\mathfrak{C}}_3 = \langle \overline{\nu}, \kappa \rangle$ of $\mathfrak{S}$.*

(3) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqcap} \ddot{\mathfrak{C}}_3 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \circledast \ddot{\mathfrak{C}}_3$, for every CBF bi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$, every CBF ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ and every CBF right ideal $\ddot{\mathfrak{C}}_3 = \langle \overline{\nu}, \kappa \rangle$ of $\mathfrak{S}$.*

(4) *$\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqcap} \ddot{\mathfrak{C}}_3 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \circledast \ddot{\mathfrak{C}}_3$, for every CBF generalized bi-ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$, every CBF interior ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ and every CBF right ideal $\ddot{\mathfrak{C}}_3 = \langle \overline{\nu}, \kappa \rangle$ of $\mathfrak{S}$.*

Proof. (1) ⇒ (4) Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ is a CBF generalized bi-ideal, $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ is a CBF interior ideal and $\ddot{\mathfrak{C}}_3 = \langle \overline{\nu}, \kappa \rangle$ is a CBF right ideal of S. Let $\mathfrak{h} \in \mathfrak{S}$. Then, there exist $\mathfrak{p}, \mathfrak{q} \in \mathfrak{S}$ such that $\mathfrak{h} = \mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}$. Thus,

$$\begin{aligned}
(\overline{\mu}^p \bigcirc \overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{h}) &= \overline{\mu}^p \bigcirc (\overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{h}) = \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge (\overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{o})\} \\
&= \curlyvee_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\overline{\mu}^p(\mathfrak{k}) \curlywedge (\overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{o})\} \succeq \overline{\mu}^p(\mathfrak{h}) \curlywedge (\overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \\
&= \overline{\mu}^p(\mathfrak{h}) \curlywedge \curlyvee_{(\mathfrak{y},\mathfrak{z}) \in A_{\mathfrak{p}\mathfrak{h}\mathfrak{q}}} (\overline{\lambda}^p(\mathfrak{y}) \curlywedge \overline{\nu}^p(\mathfrak{z})) \\
&= \overline{\mu}^p(\mathfrak{h}) \curlywedge \curlyvee_{(\mathfrak{y},\mathfrak{z}) \in A_{\mathfrak{p}\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}\mathfrak{q}}} (\overline{\lambda}^p(\mathfrak{y}) \curlywedge \overline{\nu}^p(\mathfrak{z})) = \overline{\mu}^p(\mathfrak{h}) \curlywedge (\overline{\lambda}^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \curlywedge \overline{\nu}^p(\mathfrak{h}\mathfrak{q}\mathfrak{q})) \\
&= \overline{\mu}^p(\mathfrak{h}) \curlywedge (\overline{\lambda}^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \curlywedge \overline{\nu}^p(\mathfrak{h}\mathfrak{q}^2)) \succeq \overline{\mu}^p(\mathfrak{h}) \curlywedge (\overline{\lambda}^p(\mathfrak{h}) \curlywedge \overline{\nu}^p(\mathfrak{h})) \\
&= \overline{\mu}^p(\mathfrak{h}) \curlywedge (\overline{\lambda}^p \sqcap \overline{\nu}^p)(\mathfrak{h}) = (\overline{\mu}^p \sqcap \overline{\lambda}^p \sqcap \overline{\nu}^p)(\mathfrak{h}),
\end{aligned}$$

$$\begin{aligned}
(\overline{\mu}^n \bigcirc \overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{h}) &= \overline{\mu}^n \bigcirc (\overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{h}) = \curlywedge_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}}} \{\overline{\mu}^n(\mathfrak{k}) \curlyvee (\overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{o})\} \\
&= \curlywedge_{(\mathfrak{k},\mathfrak{o}) \in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\overline{\mu}^n(\mathfrak{k}) \curlyvee (\overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{o})\} \preceq \overline{\mu}^n(\mathfrak{h}) \curlyvee (\overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \\
&= \overline{\mu}^n(\mathfrak{h}) \curlyvee \curlywedge_{(\mathfrak{y},\mathfrak{z}) \in A_{\mathfrak{p}\mathfrak{h}\mathfrak{q}}} (\overline{\lambda}^n(\mathfrak{y}) \curlyvee \overline{\nu}^n(\mathfrak{z})) \\
&= \overline{\mu}^n(\mathfrak{h}) \curlyvee \curlyvee_{(\mathfrak{y},\mathfrak{z}) \in A_{\mathfrak{p}\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}\mathfrak{q}}} (\overline{\lambda}^n(\mathfrak{y}) \curlyvee \overline{\nu}^n(\mathfrak{z})) = \overline{\mu}^n(\mathfrak{h}) \curlyvee (\overline{\lambda}^n(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \curlyvee \overline{\nu}^n(\mathfrak{h}\mathfrak{q}\mathfrak{q})) \\
&= \overline{\mu}^n(\mathfrak{h}) \curlyvee (\overline{\lambda}^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \curlyvee \overline{\nu}^n(\mathfrak{h}\mathfrak{q}^2)) \preceq \overline{\mu}^n(\mathfrak{h}) \curlyvee (\overline{\lambda}^n(\mathfrak{h}) \curlyvee \overline{\nu}^n(\mathfrak{h})) \\
&= \overline{\mu}^n(\mathfrak{h}) \curlyvee (\overline{\lambda}^n \sqcap \overline{\nu}^n)(\mathfrak{h}) = (\overline{\mu}^n \sqcap \overline{\lambda}^n \sqcap \overline{\nu}^n)(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \psi^p * \kappa^p)(\mathfrak{h}) &= \omega^p * (\psi^p * \kappa^p)(\mathfrak{h}) = \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\omega^p(\mathfrak{k}) \wedge (\psi^p * \kappa^p)(\mathfrak{o})\} \\ &= \bigvee_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\omega(\mathfrak{k}) \wedge (\psi^p * \kappa^p)(\mathfrak{o})\} \geq \omega^p(\mathfrak{h}) \wedge (\psi^p * \kappa^p)(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \\ &= \omega^p(\mathfrak{h}) \wedge \bigvee_{(\mathfrak{y},\mathfrak{z})\in A_{\mathfrak{p}\mathfrak{h}\mathfrak{q}}} (\psi^p(\mathfrak{y}) \wedge \kappa^p(\mathfrak{z})) \\ &= \omega^p(\mathfrak{h}) \wedge \bigvee_{(\mathfrak{y},\mathfrak{z})\in A_{\mathfrak{p}\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}\mathfrak{q}}} (\psi^p(\mathfrak{y}) \wedge \kappa^p(\mathfrak{z})) = \omega^p(\mathfrak{h}) \wedge (\psi^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \wedge \kappa^p(\mathfrak{h}\mathfrak{q}\mathfrak{q})) \\ &= \omega^p(\mathfrak{h}) \wedge (\psi^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \wedge \kappa^p(\mathfrak{h}\mathfrak{q}^2)) \geq \omega^p(\mathfrak{h}) \wedge (\psi^p(\mathfrak{h}) \wedge \kappa^p(\mathfrak{h})) \\ &= \omega^p(\mathfrak{h}) \wedge (\psi^p \cap \kappa^p)(\mathfrak{h}) = (\omega^p \cap \psi^p \cap \kappa^p)(\mathfrak{h}),\end{aligned}$$

$$\begin{aligned}(\omega^n * \psi^n * \kappa^n)(\mathfrak{h}) &= \omega^n * (\psi^n * \kappa^n)(\mathfrak{h}) = \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}}} \{\omega^n(\mathfrak{k}) \vee (\psi^n * \kappa^n)(\mathfrak{o})\} \\ &= \bigwedge_{(\mathfrak{k},\mathfrak{o})\in A_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}}} \{\omega(\mathfrak{k}) \wedge (\psi^n * \kappa^n)(\mathfrak{o})\} \leq \omega^n(\mathfrak{h}) \vee (\psi^n * \kappa^n)(\mathfrak{p}\mathfrak{h}\mathfrak{q}) \\ &= \omega^n(\mathfrak{h}) \vee \bigwedge_{(\mathfrak{y},\mathfrak{z})\in A_{\mathfrak{p}\mathfrak{h}\mathfrak{q}}} (\psi^n(\mathfrak{y}) \vee \kappa^n(\mathfrak{z})) \\ &= \omega^n(\mathfrak{h}) \vee \bigwedge_{(\mathfrak{y},\mathfrak{z})\in A_{\mathfrak{p}\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{q}\mathfrak{q}}} (\psi^n(\mathfrak{y}) \vee \kappa^n(\mathfrak{z})) = \omega^n(\mathfrak{h}) \vee (\psi^p(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \vee \kappa^n(\mathfrak{h}\mathfrak{q}\mathfrak{q})) \\ &= \omega^n(\mathfrak{h}) \vee (\psi^n(\mathfrak{p}\mathfrak{h}\mathfrak{p}) \vee \kappa^n(\mathfrak{h}\mathfrak{q}^2)) \leq \omega^n(\mathfrak{h}) \vee (\psi^n(\mathfrak{h}) \vee \kappa^n(\mathfrak{h})) \\ &= \omega^n(\mathfrak{h}) \vee (\psi^n \cap \kappa^n)(\mathfrak{h}) = (\omega^n \cap \psi^n \cap \kappa^n)(\mathfrak{h}).\end{aligned}$$

Hence, $(\overline{\mu}^p \sqcap \overline{\lambda}^p \sqcap \overline{\nu}^p)(\mathfrak{h}) \preceq (\overline{\mu}^p \bigcirc \overline{\lambda}^p \bigcirc \overline{\nu}^p)(\mathfrak{h})$, $(\overline{\mu}^n \sqcap \overline{\lambda}^n \sqcap \overline{\nu}^n)(\mathfrak{h}) \succeq (\overline{\mu}^n \bigcirc \overline{\lambda}^n \bigcirc \overline{\nu}^n)(\mathfrak{h})$ and $(\omega^p \cap \psi^p \cap \kappa^p)(\mathfrak{h}) \leq (\omega^p * \psi^p * \kappa^p)(\mathfrak{h})$, $(\omega^n \cap \psi^n \cap \kappa^n)(\mathfrak{h}) \geq (\omega^n * \psi^n * \kappa^n)(\mathfrak{h})$ Therefore, $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqcap} \ddot{\mathfrak{C}}_3 \overline{\sqsubseteq} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \circledast \ddot{\mathfrak{C}}_3$.

It is obvious that (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Let $\mathfrak{Q}$ be a quasi-ideal, $\mathfrak{H}$ be an ideal and $\mathfrak{R}$ be a right ideal of $\mathfrak{S}$. Then, by Theorem 3, $\overline{\chi}_{\mathfrak{Q}}$ is a CBF quasi-ideal, $\overline{\chi}_{\mathfrak{J}}$ is a CBF ideal and $\overline{\chi}_{\mathfrak{R}}$ is a CBF right ideal of S. By supposition and Lemma 4, we have

$$\overline{\mu}^p_{\chi_{\mathfrak{Q}\mathfrak{J}\mathfrak{R}}}(\mathfrak{h}) = (\overline{\mu}^p_{\chi_{\mathfrak{Q}}} \bigcirc \overline{\mu}^p_{\chi_{\mathfrak{J}}} \bigcirc \overline{\mu}^p_{\chi_{\mathfrak{R}}})(\mathfrak{h}) \succeq (\overline{\mu}^p_{\chi_{\mathfrak{Q}}} \sqcap \overline{\mu}^p_{\chi_{\mathfrak{J}}} \sqcap \overline{\mu}^p_{\chi_{\mathfrak{R}}})(\mathfrak{h}) = \overline{\mu}^p_{\chi_{\mathfrak{Q}\sqcap\mathfrak{J}\sqcap\mathfrak{R}}}(\mathfrak{h}) = \overline{1},$$

$$\overline{\mu}^n_{\chi_{\mathfrak{Q}\mathfrak{J}\mathfrak{R}}}(\mathfrak{h}) = (\overline{\mu}^n_{\chi_{\mathfrak{Q}}} \bigcirc \overline{\mu}^n_{\chi_{\mathfrak{J}}} \bigcirc \overline{\mu}^n_{\chi_{\mathfrak{R}}})(\mathfrak{h}) \preceq (\overline{\mu}^n_{\chi_{\mathfrak{Q}}} \sqcap \overline{\mu}^n_{\chi_{\mathfrak{J}}} \sqcap \overline{\mu}^n_{\chi_{\mathfrak{R}}})(\mathfrak{h}) = \overline{\mu}^n_{\chi_{\mathfrak{Q}\sqcap\mathfrak{J}\sqcap\mathfrak{R}}}(\mathfrak{h}) = \overline{-1},$$

and

$$\omega^p_{\chi_{\mathfrak{Q}\mathfrak{J}\mathfrak{R}}}(\mathfrak{h}) = (\omega^p_{\chi_{\mathfrak{Q}}} * \omega^p_{\chi_{\mathfrak{J}}} * \omega^p_{\chi_{\mathfrak{R}}})(\mathfrak{h}) \geq (\omega^p_{\chi_{\mathfrak{Q}}} \cap \omega^p_{\chi_{\mathfrak{J}}} \cap \omega^n_{\chi_{\mathfrak{R}}})(\mathfrak{h}) = \omega^p_{\chi_{\mathfrak{Q}\cap\mathfrak{J}\cap\mathfrak{R}}}(\mathfrak{h}) = 1,$$

$$\omega^n_{\chi_{\mathfrak{Q}\mathfrak{J}\mathfrak{R}}}(\mathfrak{h}) = (\omega^n_{\chi_{\mathfrak{Q}}} * \omega^n_{\chi_{\mathfrak{J}}} * \omega^n_{\chi_{\mathfrak{R}}})(\mathfrak{h}) \leq (\omega^n_{\chi_{\mathfrak{Q}}} \cap \omega^n_{\chi_{\mathfrak{J}}} \cap \omega^p_{\chi_{\mathfrak{R}}})(\mathfrak{h}) = \omega^n_{\chi_{\mathfrak{Q}\cap\mathfrak{J}\cap\mathfrak{R}}}(\mathfrak{h}) = -1.$$

Thus, $\mathfrak{h} \in \mathfrak{Q}\mathfrak{J}\mathfrak{R}$. Hence, $\mathfrak{Q} \cap \mathfrak{J} \cap \mathfrak{R} \sqsubseteq \mathfrak{Q}\mathfrak{J}\mathfrak{R}$. Therefore, by Lemma 4, $\mathfrak{S}$ is weakly regular.

5. Conclusion

In this article, we extend the concept of cubic fuzzy sets and bipolar fuzzy sets by introducing the notion of cubic bipolar fuzzy sets, which serve as a more generalized framework for dealing with uncertainty in algebraic structures. This extended concept provides a powerful tool for analyzing and characterizing various subsemigroups, offering a new perspective on their structural properties. One of the main contributions of this paper is the characterization of weakly regular semigroups in terms of cubic bipolar fuzzy ideals. By exploring the fundamental properties and interactions of these fuzzy ideals within semigroups, we establish key results that enhance our understanding of weakly regular semigroups and their algebraic behavior. For future research, we aim to extend our findings by characterizing certain classes of subsemigroups using cubic bipolar fuzzy ideals.

This will further enrich the theoretical framework and provide deeper insights into the role of fuzzy structures in semigroup theory. Additionally, we plan to investigate potential applications of cubic bipolar fuzzy sets in other mathematical and computational domains, particularly in decision-making processes and algebraic systems with uncertainty.

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