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Certain Subclass of Multivalently Bazilevič and Non-Bazilevič Functions Involving the Lemniscate of Bernoulli

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Abstract. Making use of the principle of subordination, we define a certain subclass of $p-$valently Bazilevič and non-Bazilevič functions associated with the Lemniscate of Bernoulli. Also, subordination results, convolution properties, coefficients estimate and Fekete–Szegö inequalities for this subclass are derived.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of all analytic functions in $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$. For $\chi, \rho \in \mathcal{H}(\mathbb{U})$, we say that $\chi(\xi)$ is subordinate to $\rho(\xi)$, written $\chi \prec \rho$ in $\mathbb{U}$ or $\chi(\xi) \prec \rho(\xi)$ $(\xi \in \mathbb{U})$, if there exists a Schwarz function $\omega(\xi)$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1 (\xi \in \mathbb{U})$ such that $\chi(\xi) = \rho(\omega(\xi))$ $(\xi \in \mathbb{U})$. In addition, if $\rho(\xi)$ is a univalent function in $\mathbb{U}$, then we have the following equivalence (see [1] and [2]):

$$\chi(\xi) \prec \rho(\xi) \quad (\xi \in \mathbb{U}) \Longleftrightarrow \chi(0) = \rho(0) \quad \text{and} \quad \chi(\mathbb{U}) \subset \rho(\mathbb{U}).$$

Also, let $\mathcal{A}_p$ denote the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$\chi(\xi) = \xi^p + \sum_{k=p+1}^{\infty} \varrho_k \xi^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}; \xi \in \mathbb{U}), \tag{1}$$

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which are p-valent in $\mathbb{U}$ with $\mathcal{A}_p = \mathcal{A}$. Sokól and Stankiewicz [3] defined the class $\mathcal{SL}^*$ consisting of analytic functions $\chi \in \mathcal{A}$ satisfying the next inequality

$$\left| \left[\frac{\xi \chi'(\xi)}{\chi(\xi)} \right]^2 - 1 \right| < 1,$$

which is equivalent to

$$\frac{\xi \chi'(\xi)}{\chi(\xi)} \prec q(\xi) = \sqrt{1+\xi}$$

where the function

$$q(\xi) = \sqrt{1+\xi} \quad (\xi \in \mathbb{U}) \tag{2}$$

maps $\mathbb{U}$ into the domain $\mathcal{O} = \left\{ w \in \mathbb{C} : \Re\{w\} > 0, \left| w^2 - 1 \right| < 1 \right\}$ and its boundary $\partial\mathcal{O}$ is the right-half of the lemniscate of Bernoulli $\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) = 0$. Several geometric properties of $\mathcal{SL}^*$ were studied by many authors (see, for example, [4–7]).

Using the principle of differential subordination and the function $q(\xi) = \sqrt{1+\xi}$ of the Bernoulli domain of lemniscate, we now define a new subclass $\mathcal{BN}_p(\lambda, \alpha, \beta)$ of Bazilevič and non-Bazilevič functions as follows:

Definition 1. *A function $\chi \in \mathcal{A}_p$ is said to be the subclass $\mathcal{BN}_p(\lambda, \alpha, \beta)$ when it satisfies the next subordination condition:*

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right) \left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda \frac{\xi\chi'(\xi)}{p\chi(\xi)} \left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \prec \sqrt{1+\xi} \tag{3}$$

all the powers are principal values and throughout the paper unless otherwise mentioned the real parameters λ, α, β are constrained as $\alpha \neq \beta$, $p \in \mathbb{N}$ and $\xi \in \mathbb{U}$.

We note that

(i) $\mathcal{BN}_p(\lambda, \alpha, 0) = \mathcal{B}_p(\lambda, \alpha) = \left\{ \chi \in \mathcal{A}_p : (1-\lambda)\left(\frac{\chi(\xi)}{\xi^p}\right)^\alpha + \lambda \frac{\xi\chi'(\xi)}{p\chi(\xi)} \left(\frac{\chi(\xi)}{\xi^p}\right)^\alpha \prec \sqrt{1+\xi} \right\}$ (see [8]);

(ii) $\mathcal{BN}_p(\lambda, 0, \beta) = \mathcal{N}_p(\lambda, \beta) = \left\{ \chi \in \mathcal{A}_p : (1+\lambda)\left(\frac{\xi^p}{\chi(\xi)}\right)^\beta - \lambda \frac{\xi\chi'(\xi)}{p\chi(\xi)} \left(\frac{\xi^p}{\chi(\xi)}\right)^\beta \prec \sqrt{1+\xi} \right\}$;

(iii) $\mathcal{BN}_1(\lambda, \alpha, 0) = \mathcal{B}(\lambda, \alpha) = \left\{ \chi \in \mathcal{A} : (1-\lambda)\left(\frac{\chi(\xi)}{\xi}\right)^\alpha + \lambda \frac{\xi\chi'(\xi)}{\chi(\xi)} \left(\frac{\chi(\xi)}{\xi}\right)^\alpha \prec \sqrt{1+\xi} \right\}$ (see [8]);

(iv) $\mathcal{BN}_1(\lambda, 0, \beta) = \mathcal{N}(\lambda, \beta) = \left\{ \chi \in \mathcal{A} : (1+\lambda)\left(\frac{\xi}{\chi(\xi)}\right)^\beta - \lambda \frac{\xi\chi'(\xi)}{\chi(\xi)} \left(\frac{\xi}{\chi(\xi)}\right)^\beta \prec \sqrt{1+\xi} \right\}$;

(v) $\mathcal{BN}_p(\lambda, 1, 0) = \mathcal{B}_p(\lambda) = \left\{ \chi \in \mathcal{A}_p : (1-\lambda)\frac{\chi(\xi)}{\xi^p} + \lambda \frac{\chi'(\xi)}{p\xi^{p-1}} \prec \sqrt{1+\xi} \right\}$ and $\mathcal{B}_1(\lambda) = \mathcal{B}(\lambda) = \left\{ \chi \in \mathcal{A} : (1-\lambda)\frac{\chi(\xi)}{\xi} + \lambda\chi'(\xi) \prec \sqrt{1+\xi} \right\}$ (see [8]);

(vi) $\mathcal{BN}_p(\lambda,0,1)=\mathcal{N}_p(\lambda)=\left\{\chi\in\mathcal{A}_p:(1+\lambda)\frac{\xi^p}{\chi(\xi)}-\lambda\frac{\xi^{p+1}\chi'(\xi)}{p\chi^2(\xi)}\prec\sqrt{1+\xi}\right\}$ and $\mathcal{N}_1(\lambda)=\mathcal{N}(\lambda)=\left\{\chi\in\mathcal{A}:(1+\lambda)\frac{\xi}{\chi(\xi)}-\lambda\frac{\xi^2\chi'(\xi)}{\chi^2(\xi)}\prec\sqrt{1+\xi}\right\}$;

(vii) $\mathcal{BN}_p(1,\alpha,0)=\mathcal{B}_p(\alpha)=\left\{\chi\in\mathcal{A}_p:\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left(\frac{\chi(\xi)}{\xi^p}\right)^\alpha\prec\sqrt{1+\xi}\right\}$ and $\mathcal{B}_1(\alpha)=\mathcal{B}(\alpha)=\left\{\chi\in\mathcal{A}:\frac{\xi\chi'(\xi)}{\chi(\xi)}\left(\frac{\chi(\xi)}{\xi}\right)^\alpha\prec\sqrt{1+\xi}\right\}$ (see [8]);

(viii) $\mathcal{BN}_p(-1,0,\beta)=\mathcal{N}_p(\beta)=\left\{\chi\in\mathcal{A}_p:\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left(\frac{\xi^p}{\chi(\xi)}\right)^\beta\prec\sqrt{1+\xi}\right\}$ and $\mathcal{N}_1(\beta)=\mathcal{N}(\beta)=\left\{\chi\in\mathcal{A}:\frac{\xi\chi'(\xi)}{\chi(\xi)}\left(\frac{\xi}{\chi(\xi)}\right)^\beta\prec\sqrt{1+\xi}\right\}$;

(ix) $\mathcal{BN}_p(1,0,0)=\mathcal{SL}_p^*=\left\{\chi\in\mathcal{A}_p:\frac{\xi\chi'(\xi)}{p\chi(\xi)}\prec\sqrt{1+\xi}\right\}$ and $\mathcal{SL}_1^*=\mathcal{SL}^*=\left\{\chi\in\mathcal{A}:\frac{\xi\chi'(\xi)}{\chi(\xi)}\prec\sqrt{1+\xi}\right\}$.

In order to establish our main results, we need the following lemmas.

Lemma 1. *[9] Let $h(\xi)$ be univalent and convex the function in $\mathbb{U}$ with $h(0)=1$. Suppose also that $\rho(\xi)$ given by*

$$\rho(\xi)=1+c_1\xi+c_2\xi^2+... \tag{4}$$

is analytic in $\mathbb{U}$. If

$$\rho(\xi)+\frac{\xi\rho'(\xi)}{\gamma}\prec h(\xi)\quad(\Re(\gamma)\geq 0;\gamma\neq 0;\xi\in\mathbb{U}), \tag{5}$$

then

$$\rho(\xi)\prec q(\xi)=\gamma\xi^{-\gamma}\int_0^\xi h(t)\,t^{\gamma-1}\,dt\prec h(\xi),$$

and $q(\xi)$ is the best dominant.

Lemma 2. *[10] For real or complex numbers $a,b,c(c\neq 0,-1,-2,...)$ and $\xi\in\mathbb{U}$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\xi)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\;{}_2\Omega_1(a,b;c;\xi)\;\;(\Re(c)>\Re(b)>0); \tag{6}$$

$${}_2\Omega_1(a,b;c;\xi)=(1-\xi)^{-a}\;{}_2\Omega_1\left(a,c-b;c;\frac{\xi}{\xi-1}\right); \tag{7}$$

Lemma 3. *[11] Let $\chi(\xi)=\sum_{k=1}^{\infty}\varrho_k\xi^k$ be analytic in $\mathbb{U}$ and $\rho(\xi)=\sum_{k=1}^{\infty}b_k\xi^k$ be analytic and convex in $\mathbb{U}$. If $\chi\prec\rho$, then*

$$|\varrho_k|<|b_1|\quad(k\in\mathbb{N}).$$

Lemma 4. *[12] Let* $\rho(\xi) = 1 + \sum_{k=1}^{\infty} c_k \xi^k \in \mathcal{P}$*, i.e., let* ρ *be analytic in* $\mathbb{U}$ *and satisfy* $\Re\{\rho(\xi)\} > 0$ *for* $\xi \in \mathbb{U}$*, then the following sharp estimate holds*

$$\left|c_2 - v c_1^2\right| \leq 2 \max\{1, |2v - 1|\} \quad for\ all\ v \in \mathbb{C}. \tag{8}$$

The result is sharp for the functions given by

$$\rho(\xi) = \frac{1+\xi^2}{1-\xi^2} \quad or \quad \rho(\xi) = \frac{1+\xi}{1-\xi}.$$

Lemma 5. *[12] If* $\rho(\xi) = 1 + \sum_{k=1}^{\infty} c_k \xi^k \in \mathcal{P}$*, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0, \\ 2 & if \quad 0 \leq \nu \leq 1, \\ 4\nu - 2 & if \quad \nu \geq 1, \end{cases} \tag{9}$$

when $v < 0$ *or* $\nu > 1$*, the equality holds if and only if* $\rho(\xi) = (1+\xi)/(1-\xi)$ *or one of its rotations. If* $0 < \nu < 1$*, then the equality holds if and only if* $\rho(\xi) = (1+\xi^2)/(1-\xi^2)$ *or one of its rotations. If* $\nu = 0$*, the equality holds if and only if*

$$\rho(\xi) = \left(\frac{1+\lambda}{2}\right)\frac{1+\xi}{1-\xi} + \left(\frac{1-\lambda}{2}\right)\frac{1-\xi}{1+\xi} \qquad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $\nu = 1$*, the equality holds if and only if* ρ *is the reciprocal of one of the functions such that equality holds in the case of* $\nu = 0$*.*

Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$*:*

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \leq 2 \qquad \left(0 \leq \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

In some literature, we found many works related to the subclasses of Bazilevi č or non-Bazilevič analytic functions which are sometimes defined by linear operators. For example, we can see those subclasses in the papers in [13–22]. The novelty in our paper is that we have combined Bazilevič and non-Bazilevič analytic functions in one subclass $\mathcal{BN}_p(\lambda, \alpha, \beta)$ to study some geometric properties such as subordination properties, inclusion relationship, convolution result, coefficients estimate and Fekete–Szegö inequalities.

2. Geometric Properties for $\mathcal{BN}_p(\lambda, \alpha, \beta)$

Theorem 1. *If* $\chi \in \mathcal{BN}_p(\lambda, \alpha, \beta)$ *with* $\frac{\lambda}{\alpha+\beta} > 0$*, then*

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \prec Q(\xi) = (1+\xi)^{\frac{1}{2}}\ {}_2\Omega_1\left(-\frac{1}{2}, 1; \frac{p(\alpha+\beta)}{\lambda} + 1; \frac{\xi}{1+\xi}\right) \prec \sqrt{1+\xi}, \tag{10}$$

where the function $Q(\xi)$ *is the best dominant.*

Proof. Let

$$\rho(\xi) = \left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \quad (\xi \in \mathbb{U}). \tag{11}$$

Then the function $\rho(\xi)$ is of the form (4), analytic in $\mathbb{U}$ and $\rho(0) = 1$. By taking the derivatives in the both sides of (11), we get

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} = \rho(\xi) + \frac{\lambda\xi\rho'(\xi)}{p(\alpha+\beta)}. \tag{12}$$

Since $\chi \in \mathcal{BN}_p(\lambda,\alpha,\beta)$, we have

$$\rho(\xi) + \frac{\lambda\xi\rho'(\xi)}{p(\alpha+\beta)} \prec \sqrt{1+\xi}.$$

Now, by applying Lemma 1 for $\gamma = \frac{p(\alpha+\beta)}{\lambda}$, we derive that

$$\begin{aligned}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \prec Q(\xi) &= \frac{p(\alpha+\beta)}{\lambda}\,\xi^{-\frac{p(\alpha+\beta)}{\lambda}}\int_0^{\xi} t^{\frac{p(\alpha+\beta)}{\lambda}-1}(1+t)^{\frac{1}{2}}\,dt \\ &= \frac{p(\alpha+\beta)}{\lambda}\int_0^1 u^{\frac{p(\alpha+\beta)}{\lambda}-1}(1+\xi u)^{\frac{1}{2}}\,du \\ &= (1+\xi)^{\frac{1}{2}}\,{}_2\Omega_1\left(-\frac{1}{2},1;\frac{p(\alpha+\beta)}{\lambda}+1;\frac{\xi}{1+\xi}\right),\end{aligned} \tag{13}$$

where we have made a change of variables followed by the use of identities in Lemma 2 with $a = -\frac{1}{2}$, $b = \frac{p\alpha}{\lambda n}$ and $c = b+1$. This finishes the proof of Theorem 1.

Taking $\beta = 0$ in Theorem 1, we get

Corollary 1. *If $\chi \in \mathcal{B}_p(\lambda,\alpha)$ with $\frac{\lambda}{\alpha} > 0$, then*

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha} \prec Q_2(\xi) = (1+\xi)^{\frac{1}{2}}\,{}_2\Omega_1\left(-\frac{1}{2},1;\frac{p\alpha}{\lambda}+1;\frac{\xi}{1+\xi}\right) \prec \sqrt{1+\xi},$$

where $Q_2(\xi)$ is the best dominant.

Taking $\alpha = 0$ in Theorem 1, we get

Corollary 2. *If $\chi \in \mathcal{N}_p(\lambda,\beta)$ with $\frac{\lambda}{\beta} > 0$, then*

$$\left[\frac{\xi^p}{\chi(\xi)}\right]^{\beta} \prec Q_3(\xi) = (1+\xi)^{\frac{1}{2}}\,{}_2\Omega_1\left(-\frac{1}{2},1;\frac{p\beta}{\lambda}+1;\frac{\xi}{1+\xi}\right) \prec \sqrt{1+\xi},$$

where $Q_3(\xi)$ is the best dominant.

For a function $\chi \in \mathcal{A}_p$ given by (1), the generalized Bernardi-Libera-Livingston integral operator $L_{p,\mu} : \mathcal{A}_p \to \mathcal{A}_p$, with $\mu > -p$, is defined by (see [23–26])

$$L_{p,\mu}\chi(\xi) = \frac{\mu + p}{\xi^{\mu}} \int_0^{\xi} t^{\mu-1}\chi(t)\, dt \quad (\mu > -p)\,. \tag{14}$$

It is easy to verify that for all $\chi \in \mathcal{A}_p$ we have

$$\xi \left(L_{p,\mu}\chi(\xi)\right)' = (\mu + p)\chi(\xi) - \mu L_{p,\mu}\chi(\xi). \tag{15}$$

Theorem 2. *If the function $\chi \in \mathcal{A}_p$ satisfies the next subordination condition*

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\chi(\xi)}{L_{p,\mu}\chi(\xi)}\left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \prec \sqrt{1+\xi}, \tag{16}$$

with $\frac{\lambda}{\alpha+\beta} > 0$ and $L_{p,\mu}$ is the integral operator defined by (14), then

$$\left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \prec K(\xi) = (1+\xi)^{\frac{1}{2}}\ {}_2\Omega_1\left(-\frac{1}{2}, 1; \frac{(\alpha+\beta)(p+\mu)}{\lambda} + 1; \frac{\xi}{1+\xi}\right) \prec \sqrt{1+\xi},$$

where the function K is the best dominant of (16).

Proof. Let

$$\rho(\xi) = \left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \quad (\xi \in \mathbb{U})\,, \tag{17}$$

then ρ is analytic function in $\mathbb{U}$. Differentiating (17) with respect to ξ and using (16) in the resulting relation, we get

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\chi(\xi)}{L_{p,\mu}\chi(\xi)}\left[\frac{L_{p,\mu}\chi(\xi)}{\xi^p}\right]^{\alpha-\beta}$$
$$= \rho(\xi) + \frac{\lambda\xi\rho'(\xi)}{(\alpha+\beta)(p+\mu)} \prec \sqrt{1+\xi}.$$

Using the same method we used to prove Theorem 1, the remaining part of this theorem can be derived in a similar way.

Theorem 3. $\chi \in \mathcal{BN}_p(\lambda, \alpha, \beta)$ *if and only if*

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} * \left(\frac{1-\left[\left(1+\frac{\lambda}{p(\alpha+\beta)}\right)e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2\right]\xi+\left[e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1\right]\xi^2}{(1-\xi)^2}\right) \neq 0. \tag{18}$$

Proof. For any function $\chi \in \mathcal{A}_p$, we can confirm that

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} = \left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} * \frac{1}{1-\xi} \tag{19}$$

and

$$\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} = \left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} * \frac{1-\left(1-\frac{1}{p(\alpha-\beta)}\right)\xi}{(1-\xi)^2}. \tag{20}$$

First, in order to prove that (18) holds, we will write (3) by using the principle of subordination, that is,

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} = \sqrt{1+w(\xi)},$$

where $w(\xi)$ is a Schwarz function, hence

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \neq \sqrt{1+e^{i\theta}}, \tag{21}$$

for all $\xi \in \mathbb{U}$ and $0 \leq \theta < 2\pi$. From (19) and (20), the relation (21) may be written as

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} * \left[\frac{1-\sqrt{1+e^{i\theta}}-\left(1-\frac{\lambda}{p(\alpha+\beta)}-2\sqrt{1+e^{i\theta}}\right)\xi-\sqrt{1+e^{i\theta}}\xi^2}{(1-\xi)^2}\right] \neq 0,$$

which is equivalent to

$$\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} * \left[\frac{1-\left[\left(1+\frac{\lambda}{p(\alpha+\beta)}\right)e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2\right]\xi+\left[e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1\right]\xi^2}{(1-\xi)^2}\right] \neq 0,$$

that is (18).

Reversely, let $\chi \in \mathcal{A}_p$ satisfy the condition (18). Like it was previously shown, the assumption (18) is equivalent to (20), that is,

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \neq \sqrt{1+e^{i\theta}} \quad (\xi \in \mathbb{U}). \tag{22}$$

Denoting

$$\varphi(\xi) = \left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} \quad \text{and} \quad \psi(\xi) = \sqrt{1+\xi},$$

the relation (22) could be written as $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Therefore, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From this fact, using that $\varphi(0) = \psi(0) = 1$ together with the univalence of the function ψ, it follows that $\varphi(\xi) \prec \psi(\xi)$, that is $\chi \in \mathcal{BN}_p(\lambda, \alpha, \beta)$.

Taking $\beta = 0$ in Theorem 1, we get

Corollary 3. *$\chi \in \mathcal{B}_p(\lambda,\alpha)$ if and only if*

$$\left(\frac{\chi(\xi)}{\xi^p}\right)^{\alpha} * \left(\frac{1-\left[\left(1+\frac{\lambda}{p\alpha}\right)e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2\right]\xi+\left[e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1\right]\xi^2}{(1-\xi)^2}\right) \neq 0.$$

Taking $\alpha = 0$ in Theorem 1, we get

Corollary 4. *$\chi \in \mathcal{N}_p(\lambda,\beta)$ if and only if*

$$\left(\frac{\xi^p}{\chi(\xi)}\right)^{\beta} * \left(\frac{1-\left[\left(1+\frac{\lambda}{p\beta}\right)e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2\right]\xi+\left[e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1\right]\xi^2}{(1-\xi)^2}\right) \neq 0.$$

Theorem 4. *If $\chi(\xi)$ given by (1) belongs to $\mathcal{BN}_p(\lambda,\alpha,\beta)$, then*

$$|\varrho_{p+1}| \leq \frac{|\alpha+\beta|\, p}{2\,|\alpha-\beta|\,|p(\alpha+\beta)+\lambda|}. \tag{23}$$

Proof. Combining (1) and (3), we obtain

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^p}\right]^{\alpha-\beta}$$

$$= 1+\frac{(\alpha-\beta)\left[p(\alpha+\beta)+\lambda\right]}{p(\alpha+\beta)}\varrho_{p+1}\xi+\ldots. \prec \sqrt{1+\xi} = 1+\frac{1}{2}\xi-\frac{1}{8}\xi^2+\ldots\ . \tag{24}$$

An application of Lemma 3 to (24) yields

$$\left|\frac{(\alpha-\beta)\left[p(\alpha+\beta)+\lambda\right]}{p(\alpha+\beta)}\varrho_{p+1}\right| < \frac{1}{2}. \tag{25}$$

Thus, from (25), we easily obtain (23) asserted by Theorem 4.

Taking $\beta = 0$ in Theorem 1, we get

Corollary 5. *If $\chi(\xi)$ given by (1) belongs to $\mathcal{B}_p(\lambda,\alpha)$, then*

$$|\varrho_{p+1}| \leq \frac{p}{2\,|p\alpha+\lambda|}.$$

Taking $\alpha = 0$ in Theorem 1, we get

Corollary 6. *If $\chi(\xi)$ given by (1) belongs to $\mathcal{N}_p(\lambda,\beta)$, then*

$$|\varrho_{p+1}| \leq \frac{p}{2\,|p\beta+\lambda|}.$$

3. Fekete-Szegö Problem for $\mathcal{BN}_p(\lambda,\alpha,\beta)$

In this section we study the Fekete–Szegö inequalities for the class $\mathcal{BN}_p(\lambda,\alpha,\beta)$. It is worth noting that many authors have been investigated the Fekete-Szegö problem for several subclasses of analytic functions (see, for instance [27–32]).

Theorem 5. *If χ given by (1) belongs to the class $\mathcal{BN}_p(\lambda,\alpha,\beta)$, then*

$$\left|\varrho_{p+2}-\mu a_{p+1}^{2}\right| \leq \frac{p|\alpha+\beta|}{2|\alpha-\beta||p(\alpha+\beta)+2\lambda|}\max\left\{1;\frac{1}{4}\left|1+\frac{p(\alpha+\beta)[p(\alpha+\beta)+2\lambda](\alpha-\beta+2\mu-1)}{(\alpha-\beta)[p(\alpha+\beta)+\lambda]^{2}}\right|\right\}. \tag{26}$$

Proof. If $\chi\in\mathcal{BN}_p(\lambda,\alpha,\beta)$, then there is a Schwarz function ω in $\mathbb{U}$ such that

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^{p}}\right]^{\alpha-\beta}+\frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^{p}}\right]^{\alpha-\beta}=\sqrt{1+\omega(\xi)}, \tag{27}$$

Define the function $g(\xi)$ by

$$g(\xi)=\frac{1+\omega(\xi)}{1-\omega(\xi)}=1+c_1\xi+c_2\xi^2+\dots . \tag{28}$$

Since $\omega(\xi)$ is a Schwarz function, we see that $g\in\mathcal{P}$ with $g(0)=1$. Therefore,

$$\sqrt{1+\omega(\xi)}=\sqrt{\frac{2g(\xi)}{g(\xi)+1}}=1+\frac{1}{4}c_1\xi+\left(\frac{1}{4}c_2-\frac{5}{32}c_1^2\right)\xi^2+\dots . \tag{29}$$

Now by substituting (29) in (27), we have

$$\left(1-\frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\chi(\xi)}{\xi^{p}}\right]^{\alpha-\beta}+\frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\xi\chi'(\xi)}{p\chi(\xi)}\left[\frac{\chi(\xi)}{\xi^{p}}\right]^{\alpha-\beta}=1+\frac{c_1}{4}\xi+\left(\frac{c_2}{4}-\frac{5c_1^2}{32}\right)\xi^2+\dots .$$

Equating the coefficients of ξ and ξ^2 we obtain

$$\varrho_{p+1}=\frac{p(\alpha+\beta)}{4(\alpha-\beta)[p(\alpha+\beta)+\lambda]}c_1.$$

$$\varrho_{p+2}=\frac{p(\alpha+\beta)}{4(\alpha-\beta)[p(\alpha+\beta)+2\lambda]}\left[c_2-\frac{1}{8}\left(5+\frac{p(\alpha+\beta)(\alpha-\beta-1)[p(\alpha+\beta)+2\lambda]}{(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}\right)c_1^2\right].$$

Therefore,

$$\varrho_{p+2}-\mu\varrho_{p+1}^2=\frac{p(\alpha+\beta)}{4(\alpha-\beta)[p(\alpha+\beta)+2\lambda]}\left\{c_2-vc_1^2\right\}, \tag{30}$$

where

$$\nu=\frac{1}{8}\left[5+\frac{p(\alpha+\beta)[p(\alpha+\beta)+2\lambda](\alpha-\beta+2\mu-1)}{(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}\right]. \tag{31}$$

Our result now follows by an application of Lemma 4. This completes the proof of Theorem 5.

Putting $\beta=0$ in Theorem 5, we obtain the following.

Corollary 7. *If χ given by (1) belongs to the class $\mathcal{B}_p(\lambda,\alpha)$, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|\leq\frac{p}{2\left|p\alpha+2\lambda\right|}\max\left\{1;\frac{1}{4}\left|1+\frac{p\left[p\alpha+2\lambda\right]\left(\alpha+2\mu-1\right)}{\left(p\alpha+\lambda\right)^2}\right|\right\}.$$

Putting $\alpha=0$ in Theorem 5, we obtain the following.

Corollary 8. *If χ given by (1) belongs to the class $\mathcal{N}_p(\lambda,\beta)$, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|\leq\frac{p}{2\left|p\beta+2\lambda\right|}\max\left\{1;\frac{1}{4}\left|1+\frac{p\left(p\beta+2\lambda\right)\left(\beta-2\mu+1\right)}{\left(p\beta+\lambda\right)^2}\right|\right\}.$$

Theorem 6. *Let*

$$\begin{aligned}\sigma_1&=\frac{1}{2}\left(1-\alpha+\beta-\frac{5\left(\alpha-\beta\right)\left[p\left(\alpha+\beta\right)+\lambda\right]^2}{p\left(\alpha+\beta\right)\left[p\left(\alpha+\beta\right)+2\lambda\right]}\right),\\ \sigma_2&=\frac{1}{2}\left(1-\alpha+\beta+\frac{3\left(\alpha-\beta\right)\left[p\left(\alpha+\beta\right)+\lambda\right]^2}{p\left(\alpha+\beta\right)\left[p\left(\alpha+\beta\right)+2\lambda\right]}\right),\\ \sigma_3&=\frac{1}{2}\left(1-\alpha+\beta-\frac{\left(\alpha-\beta\right)\left[p\left(\alpha+\beta\right)+\lambda\right]^2}{p\left(\alpha+\beta\right)\left[p\left(\alpha+\beta\right)+2\lambda\right]}\right).\end{aligned}$$

If χ given by (1) belongs to the class $\mathcal{BN}_p(\lambda,\alpha,\beta)$, then

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|\leq\begin{cases}\frac{p(\alpha+\beta)}{8(\alpha-\beta)}\left[-\frac{1}{[p(\alpha+\beta)+2\lambda]}-\frac{p(\alpha+\beta)(\alpha-\beta+2\mu-1)}{(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}\right] & (\mu\leq\sigma_1)\\ \frac{p(\alpha+\beta)}{2(\alpha-\beta)[p(\alpha+\beta)+2\lambda]} & (\sigma_1\leq\mu\leq\sigma_2)\\ \frac{p(\alpha+\beta)}{8(\alpha-\beta)}\left[\frac{1}{[p(\alpha+\beta)+2\lambda]}+\frac{p(\alpha+\beta)(\alpha-\beta+2\mu-1)}{(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}\right] & (\mu\geq\sigma_2)\end{cases}$$

Further, if $\sigma_1\leq\mu\leq\sigma_3$, then

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[\frac{5(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}{p(\alpha+\beta)[p(\alpha+\beta)+2\lambda]}+\alpha-\beta+2\mu-1\right]\left|\varrho_{p+1}\right|^2\leq\frac{p(\alpha+\beta)}{2(\alpha-\beta)[p(\alpha+\beta)+2\lambda]}.$$

If $\sigma_3\leq\mu\leq\sigma_2$, then

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[\frac{3(\alpha-\beta)[p(\alpha+\beta)+\lambda]^2}{p(\alpha+\beta)[p(\alpha+\beta)+2\lambda]}-\alpha+\beta-2\mu+1\right]\left|\varrho_{p+1}\right|^2\leq\frac{p(\alpha+\beta)}{2(\alpha-\beta)[p(\alpha+\beta)+2\lambda]}.$$

Proof. Applying Lemma 5 to (30) and (31), we can get our results of Theorem 6.

Putting $\beta=0$ in Theorem 6, we obtain the following.

Corollary 9. *Let*

$$\begin{aligned}\sigma_4 &= \frac{1}{2}\left(1-\alpha-\frac{5\left(p\alpha+\lambda\right)^2}{p\left(p\alpha+2\lambda\right)}\right), \sigma_5=\frac{1}{2}\left(1-\alpha+\frac{3\left(p\alpha+\lambda\right)^2}{p\left(p\alpha+2\lambda\right)}\right),\\ \sigma_6 &= \frac{1}{2}\left(1-\alpha-\frac{\left(p\alpha+\lambda\right)^2}{p\left(p\alpha+2\lambda\right)}\right).\end{aligned}$$

If χ *given by (1) belongs to the class* $\mathcal{B}_p\left(\lambda,\alpha\right)$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right| \leq \begin{cases} -\frac{p}{8}\left[\frac{1}{p\alpha+2\lambda}+\frac{p(\alpha+2\mu-1)}{(p\alpha+\lambda)^2}\right] & (\mu\leq\sigma_4)\\ \frac{p}{2(p\alpha+2\lambda)} & (\sigma_4\leq\mu\leq\sigma_5)\\ \frac{p}{8}\left[\frac{1}{p\alpha+2\lambda}+\frac{p(\alpha+2\mu-1)}{(p\alpha+\lambda)^2}\right] & (\mu\geq\sigma_5)\end{cases}$$

Further, if $\sigma_4\leq\mu\leq\sigma_6$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[\frac{5\left(p\alpha+\lambda\right)^2}{p\left(p\alpha+2\lambda\right)}+\alpha+2\mu-1\right]\left|\varrho_{p+1}\right|^2\leq\frac{p}{2\left(p\alpha+2\lambda\right)}.$$

If $\sigma_6\leq\mu\leq\sigma_5$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[\frac{3\left(p\alpha+\lambda\right)^2}{p\left(p\alpha+2\lambda\right)}-\alpha-2\mu+1\right]\left|\varrho_{p+1}\right|^2\leq\frac{p}{2\left(p\alpha+2\lambda\right)}.$$

Putting $\alpha=0$ in Theorem 6, we obtain the following result.

Corollary 10. *Let*

$$\begin{aligned}\sigma_7 &= \frac{1}{2}\left(1+\beta+\frac{5\left(p\beta+\lambda\right)^2}{p\left(p\beta+2\lambda\right)}\right), \sigma_8=\frac{1}{2}\left(1+\beta-\frac{3\left(p\beta+\lambda\right)^2}{p\left(p\beta+2\lambda\right)}\right),\\ \sigma_9 &= \frac{1}{2}\left(1+\beta+\frac{\left(p\beta+\lambda\right)^2}{p\left(p\beta+2\lambda\right)}\right).\end{aligned}$$

If χ *given by (1) belongs to the class* $\mathcal{B}_p\left(\lambda,\beta\right)$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right| \leq \begin{cases} \frac{p}{8}\left[\frac{1}{p\beta+2\lambda}+\frac{p(\beta-2\mu+1)}{(p\beta+\lambda)^2}\right] & (\mu\leq\sigma_7)\\ -\frac{p}{2(p\beta+2\lambda)} & (\sigma_7\leq\mu\leq\sigma_8)\\ -\frac{p}{8}\left[\frac{1}{p\beta+2\lambda}+\frac{p(\beta-2\mu+1)}{(p\beta+\lambda)^2}\right] & (\mu\geq\sigma_8)\end{cases}$$

Further, if $\sigma_7\leq\mu\leq\sigma_9$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[-\frac{5\left(p\beta+\lambda\right)^2}{p\left(p\beta+2\lambda\right)}-\beta+2\mu-1\right]\left|\varrho_{p+1}\right|^2\leq-\frac{p}{2\left(p\beta+2\lambda\right)}.$$

If $\sigma_9\leq\mu\leq\sigma_8$*, then*

$$\left|\varrho_{p+2}-\mu\varrho_{p+1}^2\right|+\frac{1}{2}\left[-\frac{3\left(p\beta+\lambda\right)^2}{p\left(p\beta+2\lambda\right)}+\beta-2\mu+1\right]\left|\varrho_{p+1}\right|^2\leq-\frac{p}{2\left(p\beta+2\lambda\right)}.$$

4. Conclusion

In this presentation, we have defined the subclass of multivalently Bazilevič and Non-Bazilevič functions that are subordinate to the function of the Bernoulli domain lemniscate $\mathcal{BN}_p(\lambda, \alpha, \beta)$. We have investigated some interesting properties such as subordination results, convolution properties, coefficients estimate and Fekete-Szegö inequalities for functions belonging to this subclass. This paper provides significant contributions to the study of some geometric properties of the Bazilevič and Non-Bazilevič functions. It also highlights the potential for future research to explore important geometric properties for similar subclasses of analytic functions involving linear operators.

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