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Computational Illustration of Fractional Inequalities via 2D Graphs with Application

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Abstract. In this article, we use generalized (k, s)-Riemann-Liouville fractional integral operator (GRLFIO) to explore the reverse forms of Minkowskis, Holder and Hermite-Hadamard-Fejer type inequalities within an interval-valued $(\imath.\upsilon)$ $(\leftthreetimes^{s+1}, \mho)$ class of convexity. We comprise various existing definitions and propose the novel concept of an $\imath.\upsilon$ $(\leftthreetimes^{s+1}, \mho)$ convexity. Our findings show the remarkable adaptability by adjusting parameter bounds for (k, s)-GRLFIO within structure of an $\imath.\upsilon$ $(\leftthreetimes^{s+1}, \mho)$ convexity presenting broader generalization and new perspective advancements to Hermite-Hadamard-Fejer and Pachpatte-type inequalities. In order to facilitate their applications, we examine the further consequences, constructed specific inequalities and illustrate them through graphical representations. Additionally, we validate the results using tables for various fractional orders. This study establishes the foundation for future research into the mathematical inequalities by emphasizing the importance of fractional integral operators and the expanded concept of convexity.

2020 Mathematics Subject Classifications: 26A33, 26A51, 26D15, 26D20

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1. Introduction

Abel was the first scientist in the history of fractional calculus to use it to solve the Tautochrone problem [1]. To further improve the field, researchers have published their

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work [2, 3]. These studies have made a significant impact on the applications and accomplishments of fractional calculus in mathematical modeling and applied analysis [4, 5]. Without a non-singular kernel, Caputo et al. introduced the well-known Caputo derivative in [6]. Still, there are a number of important research gaps in the idea. Several academics have created their own fractional operators with non-singular kernels to fill in these gaps [7–10]. Non-linear and non-singular extensions of fractional operators and their symmetric features were established by Wu et al. in [11]. A non-linear and non-singular fractional derivative was also introduced by Samraiz et al. in [12], who also examined its uses in applied analysis. In [13, 14], additional advancements in fractional operators using different kinds of kernel functions were introduced. The reader is referred to [15] for further information and applications concerning fractional operators. Convex analysis is based on the idea of convex functions (cf) and sets with convex epigraphs. Convexity is important in many areas of mathematics such as optimization, fixed point theory, topological spaces and advanced analysis. A positive second-order derivative indicates concavity, and derivatives can be used to assess the convexity of functions. Known for its real-world applications, the theory of inequalities encompasses several branches of mathematical analysis. Error limitations for numerical quadrature methods, including the trapezoidal rule, midpoint rule, Ostrowski's rule, and Simpson's rules, are notably refined by integral inequalities, especially when cf and their generalizations are applied. Additionally, these bounds show links between special functions, probability theory, information theory, and other fields. The notion of convexity can be used to generate a number of basic and Hermite-Hadamard-Fejer inequality. Let a function $\varnothing : [r_1, r_2] \subset R \to R$ be continuous, so

$$\varnothing\left(\frac{r_1 + r_2}{2}\right) \leq \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \varnothing(x)dx \leq \frac{\varnothing(r_1) + \varnothing(r_2)}{2}.$$

One could consider this inequality to be an extra standard for cf. Additional information can be found in [16, 17]. Wu redpresented the unified form of convexity, explained as follows.

Definition 1. *[18] If a function* $\leftthreetimes : \circledR \to R$ *be monotonic continuous-*(mc) *function, then* $\circledR \subset R$ *is considered to be* $\leftthreetimes$*-convex set based on* $\leftthreetimes$ *if:*

$$\leftthreetimes^{-1}\left((1-\theta) \leftthreetimes (x) + \theta \leftthreetimes (y)\right) \in \circledR,$$

for all $x, y \in \circledR$ *and* $\theta \in [0, 1]$.

We, now resume the class $\leftthreetimes$-cf.

Definition 2. *A function* $\varnothing : \circledR \to R$ *is stated to be* $\leftthreetimes$*-*cf *with respect to* $(w.r.t)$ *strictly* mc *function* $\leftthreetimes$ *if:*

$$\varnothing(\leftthreetimes^{-1}((1-\theta) \leftthreetimes (x) + \theta \leftthreetimes (y))) \leq (1-\theta)\varnothing(x) + \theta\varnothing(y),$$

for all $x, y \in \circledR$ *and* $\theta \in [0, 1]$.

In order to investigate the several relevant scientific domains, numerous authors have combined fractional calculus with $\imath.\upsilon$ concepts. Working along these lines, Breckner developed the idea of set-valued cf, as seen below.

Definition 3. *[19] A function $\varnothing : [r_1, r_2] \to R_i^+$ is stated to be $\imath.\upsilon$ cf, if:*

$$\varnothing((1-\theta)r_1 + \theta r_2) \supseteq (1-\theta)\varnothing(r_1) + \theta\varnothing(r_2), \theta \in [0,1].$$

The first person to apply inequalities to set-valued functions was Sadowska [20]. He investigated the Hermite-Hadamard-Fejer inequality in the context of set-valued cf. If function $\varnothing : [r_1, r_2] \to R$ be an $\imath.\upsilon$ cf, so

$$\varnothing\left(\frac{r_1 + r_2}{2}\right) \supseteq \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} \varnothing(\theta)d\theta \supseteq \frac{\varnothing(r_1) + \varnothing(r_2)}{2}.$$

If $B([r_1, r_2])$ is a collection of all divisions of $[r_1, r_2]$ and $B(\rho_1, [r_1, r_2])$ be the family of all divisions P in a way that mesh$P < \rho_1$, then $\varnothing : [r_1, r_2] \to R$ is referred to as $\imath.\upsilon$ Riemann integrable on $[r_1, r_2]$, if there exist $\to \in R$, and for each $\epsilon > 0$ there exist $\rho > 0$ such that:

$$d(S(\varnothing, P, \rho), \text{®}) < \epsilon,$$

where $S(\varnothing, P, \rho)$ specifies the Riemann sum of Phi for any $P \in B(\rho, [r_1, r_2])$. The above expression represents that ® is the (IR)-integral of $\varnothing$ such that:

$$\text{®} = (IR)\int_{r_1}^{r_2} \varnothing(\theta)d\theta$$

For the sake of brevity, we specify the space of Riemann integrable functions and $\imath.\upsilon$ Riemann integration on$[r_1, r_2]$ and by $R_{[r_1,r_2]}$ and $IR_{[r_1,r_2]}$ respectively.

Theorem 1. *[21] If a function $\varnothing(\theta) : [r_1, r_2] \to R$ be an $\imath.\upsilon$ continuous, then*

$$\varnothing(\theta) = [\varnothing_*(\theta), \varnothing^*(\theta)], \varnothing(\theta) \in IR_{[r_1,r_2]} \Leftrightarrow \varnothing_*(\theta), \varnothing^*(\theta) \in IR_{[r_1,r_2]},$$

and

$$(IR)\int_{r_1}^{r_2} \varnothing(\theta)d\theta = \left[(R)\int_{r_1}^{r_2} \varnothing_*(\theta)d\theta, (R)\int_{r_1}^{r_2} \varnothing^*(\theta)d\theta\right].$$

The Lebesgue integrable function ($LI\varnothing$) space is defined by $L[r_1, r_2]$. Now, we retrieve the Riemann-Liouville fractional operator, which are provided below.

Definition 4. *[22] Let $\varnothing(\theta) \in [r_1, r_2]$, then*

$$\mathfrak{Z}_{r_1^+}^{\beta}\varnothing(r_2) = \frac{1}{\Gamma(\beta)}\int_{r_1}^{r_2} \varnothing(\theta)(r_2 - \theta)^{\beta-1}d\theta, r_1 < r_2, \beta > 0.$$

Similarly, the right side of the Riemann-Liouville fractional operator is given below

$$\mathfrak{Z}_{r_2^-}^{\beta}\varnothing(r_1) = \frac{1}{\Gamma(\beta)}\int_{r_1}^{r_2} \varnothing(\theta)(\theta - r_1)^{\beta-1}d\theta, r_1 < r_2, \beta > 0.$$

We, now replicated the $\imath.\upsilon$ Riemann-Liouville fractional integral operator.

Definition 5. *[23] Let $\varnothing(x)$ be $\imath.\upsilon$ function such that $\varnothing_1(x), \varnothing_2(x) \in L[r_1, r_2]$, then*

$$\mathfrak{Z}^{\beta}_{x^+}\varnothing(r_2) = \frac{1}{\Gamma(\beta)}\int_x^{r_2} \varnothing(\theta)(r_2-\theta)^{\beta-1}d\theta, x < r_2,$$

and

$$\mathfrak{Z}^{\beta}_{y^-}\varnothing(r_1) = \frac{1}{\Gamma(\beta)}\int_{r_1}^{y} \varnothing(\theta)(\theta-r_1)^{\beta-1}d\theta, r_1 < y,$$

with $\beta > 0$, obviously

$$\mathfrak{Z}^{\beta}_{x^+}\varnothing(r_2) = \left[\mathfrak{Z}^{\beta}_{x^+}\varnothing_1(r_2), \mathfrak{Z}^{\beta}_{x^+}\varnothing_2(r_2)\right],$$

and

$$\mathfrak{Z}^{\beta}_{y^-}\varnothing(r_1) = \left[\mathfrak{Z}^{\beta}_{y^-}\varnothing_1(r_1), \mathfrak{Z}^{\beta}_{y^-}\varnothing_2(r_1)\right].$$

In recent years, $\imath.\upsilon$ functions based on different partial and total ordered relations have been used to refine and generalize a number of integral inequalities. These works provided the groundwork for subsequent developments especially in mathematical inequalities pertaining to set-valued functions and represented the first attempts to improve the practical applications of inequalities. We will derive several fractional variations of reverse Minkowski inequality, Holders inequality, Hermite-Hadamard inequality, its weighted form known as Fejer-Hermite-Hadamard inequality and some inequalities for the product of functions that are known to be Pachpatee's type inclusions as applications of this class. Since the generic class of convexity and its implications in inequalities is a broader space of functions that contains redcf and non-redcf classes, it is the novel aspect of the current proceeding. Our developed results allow for the characterization of large classes of functions. Our findings will also be useful tools for calculating different bounds for the $\imath.\upsilon$ fractional operators. There are a few simulations for numerical examples provided to verify the accuracy of the suggested findings. With this work, we intend to demonstrate further inequalities and related optimization issues to interested readers. Now, we introduce the idea of $\imath.\upsilon$-$(\leftthreetimes^{s+1}, \mho)$ redcf, it indicates how several new generalizations of convexity and the numerous classes of convexity that now exist can be produced as special instances. For our convenient the family of $\imath.\upsilon$-$(\leftthreetimes^{s+1}, \mho)$ redcf, $\imath.\upsilon$-$(\leftthreetimes^{s+1}, \mho)$ concave function, $(\leftthreetimes^{s+1}, \mho)$ redcf and $(\leftthreetimes^{s+1}, \mho)$ concave functions are represented as $SIGX([r_1, r_2], R_i^+)$, $SIGV([r_1, r_2], R_i^+)$, $SGX([r_1, r_2], R)$ and $SGV([r_1, r_2], R)$ respectively.

Definition 6. *Let $s \in R/\{-1\}$, $\leftthreetimes$ be an increasing function and $\varnothing : [r_1, r_2] \to R_i^+$ be an $\imath.\upsilon$-$(\leftthreetimes^{s+1}, \mho)$ function fulfill the condition $\varnothing(x) = [\varnothing_*(x), \varnothing^*(x)]$ and $\mho : [0, 1] \to R$ be a positive function, so*

$$\varnothing\big(\leftthreetimes^{-1}\left((1-\theta)\leftthreetimes^{s+1}(r_1) + \theta\leftthreetimes^{s+1}(r_2)\right)^{\frac{1}{s+1}}\big) \supseteq \mho(1-\theta)\varnothing(r_1) + \mho(\theta)\varnothing(r_2),$$

for all $x \in [r_1, r_2]$ and $\theta \in [0, 1]$.

Remark 1. *(i) If we choose $s = 0$ and $\leftthreetimes(\theta) = \theta$ in (6), then we obtain the following definition presented in [24].*

$$\varnothing\big(\leftthreetimes^{-1}\left((1-\theta)\leftthreetimes(r_1) + \theta\leftthreetimes(r_2)\right)\big) \supseteq \mho(1-\theta)\varnothing(r_1) + \mho(\theta)\varnothing(r_2),$$

(ii) For the choice of $s = 0$, and $\leftthreetimes(\theta) = \mho(\theta) = \theta$ in (6) we acquire $\leftthreetimes$-$\imath.\upsilon$ cf:

$$\varnothing\big(\leftthreetimes^{-1}\left((1-\theta)\leftthreetimes(r_1) + \theta\leftthreetimes(r_2)\right)\big) \supseteq (1-\theta)\varnothing(r_1) + \theta\varnothing(r_2).$$

(iii) If we choose $s = 0$, $\leftthreetimes(\theta) = \mho(\theta) = \theta$ and $\leftthreetimes(x) = \frac{1}{x}$ in (6), then we acquire the $\imath.\upsilon$ hc function define in [25].

$$\varnothing\Big(\frac{r_1 r_2}{\theta r_1 + (1-\theta)r_2}\Big) \supseteq (1-\theta)\varnothing(r_1) + \theta\varnothing(r_2).$$

(iv) If we choose $s = 0$, $\leftthreetimes(\theta) = \mho(\theta) = \theta$ and $\leftthreetimes(x) = x^p$ in (6) then we obtain the $\imath.\upsilon$-p cf define in [26].

$$\varnothing\big(((1-\theta)r_1^p + \theta r_2^p)^{\frac{1}{p}}\big) \supseteq (1-\theta)\varnothing(r_1) + \theta\varnothing(r_2).$$

(v) By selecting $s = 0$, $\leftthreetimes(\theta) = \theta$ and $\leftthreetimes(x) = x$ in (6), we retrieve the definition of $\imath.\upsilon$ cf presented in [24].

Definition 7. *If we fix $s = 0$, $\leftthreetimes(\theta) = \theta$ and $\mho(\theta) = \theta^s$ in (6), we retrieve the $(\leftthreetimes, s)$-$\imath.\upsilon$ cf*

$$\varnothing\big(\leftthreetimes^{-1}\left(\theta\leftthreetimes(r_1) + (1-\theta)\leftthreetimes(r_2)\right)\big) \supseteq (1-\theta)^s\varnothing(r_1) + \theta^s\varnothing(r_2).$$

Remark 2. *The main definitional deductions will now be presented for Definition 7, presented in [24].*

(i) Choosing $\leftthreetimes(x) = \frac{1}{x}$, we retrieve the $\imath.\upsilon$ harmonically $s - cf$, that is

$$\varnothing\Big(\frac{r_1 r_2}{\theta r_1 + (1-\theta)r_2}\Big) \supseteq (1-\theta)^s\varnothing(r_1) + \theta^s\varnothing(r_2).$$

(ii) Choosing $\leftthreetimes(x) = x^p, p \geq -1$, we retrieve the $\imath.\upsilon$-p, $s - cf$, which are as follows:

$$\varnothing\big(((1-\theta)r_1^p + \theta r_2^p)^{\frac{1}{p}}\big) \supseteq (1-\theta)^s\varnothing(r_1) + \theta^s\varnothing(r_2).$$

The authors in [27] used $\imath.\upsilon$ cf and postquantum calculus to investigate novel representations of well-known results. In [28], different forms of trapezoid-type inequalities for non-cf linked sets on fuzzy domains were introduced. A generic convexity framework was used by Cortez et al. [29] to expand Jensen's and Hermite-Hadamard inequalities. Inspired by these investigations, we provide $\imath.\upsilon$ $(\leftthreetimes^{s+1}, \mho)$-cf, defined by weighted arithmetic means, which have a strictly monotone function $\leftthreetimes$ and a non-negative function $\mho$. Using suitable replacements for $\leftthreetimes$ and $\mho$, this framework creates new classes and unifies current convexity notions. In addition to inequalities for function products of the Pachpatte type, we derive several inequalities, such as Minkowski, Holder's, Hermite-Hadamard, and Hermite-Hadamard-Fejer. The scope of convex and non-cf analysis is expanded by this generalized convexity framework, which offers methods for bounding $\imath.\upsilon$ Riemann-Liouville fractional operators. Our results are supported by numerical examples and simulations, providing a basis for more inequality proofs and optimization problems.

Theorem 2. *[24] Let $\varnothing : [r_1, r_2] \to R_i^+$ be $\imath.\upsilon$ function such that $\varnothing(r_1) = [\varnothing_*, \varnothing^*]$ with $\varnothing_* \leq \varnothing^*$ then, $\varnothing \in SIGX([r_1, r_2], R_i^+)$, this implies $\varnothing_* \in SGX([r_1, r_2], R)$ and $\varnothing^* \in SGV([r_1, r_2], R)$.*

Theorem 3. *[24] Let $\varnothing \in SIGX([r_1, r_2], R_i^+)$, then*

$$\varnothing\Bigg(\leftthreetimes^{-1} \left(\frac{1}{w_n} \sum_{i=1}^{n} \theta_i \leftthreetimes (x_i) \right)\Bigg) \supseteq \sum_{i=1}^{n} \mho(\frac{\theta_i}{w_n}) \varnothing(x_i)$$

for $x_i \in [r_1, r_2]$ and $w_n = \sum_{i=1}^{n} \theta_i x_i$.

Definition 8. *[30] If a function $\varnothing$ is continuous on $[a, b]$, $s \in R/\{-1\}$ and $k \geq 0$, then the (k, s)-Riemann-Liouville fractional integral operator for order $\beta > 0$ can be stated as*

$$ {}^{s}_{k}\mathfrak{Z}^{\beta}_{\alpha^+} \varnothing(m_1) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{\alpha^+}^{m_1} (m_1^{s+1} - n_1^{s+1})^{\frac{\beta}{k}-1} n_1^s \varnothing(n_1) dn_1. \tag{1}$$

Now we are going to present generalized form of fractional operator (1).

Definition 9. *If a function $\varnothing$ is continuous on $[a, b]$, $s \in R/\{-1\}$, $k \geq 0$ and $\leftthreetimes$ be an increasing function then the (k, s)-GRLFIO for order $\beta > 0$ can be stated as*

$$ {}^{s}_{k}\mathfrak{Z}^{\beta}_{\alpha} \varnothing(m_1) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{\alpha}^{m_1} (\leftthreetimes^{s+1}(m_1) - \leftthreetimes^{s+1}(n_1))^{\frac{\beta}{k}-1} \leftthreetimes^s (n_1) \leftthreetimes' (n_1) \varnothing(n_1) dn_1. \tag{2}$$

Definition 10. *[31] If a function $\varnothing$ is continuous on $[a, b]$, $s \in R/\{-1\}$, $k \geq 0$ and $\leftthreetimes$ be an increasing function then left and right sided (k, s)-GRLFIO for order $\beta > 0$ can be stated as*

$$ {}^{s}_{k}\mathfrak{Z}^{\beta}_{\alpha^+} \varnothing(m_1) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{\alpha^+}^{m_1} (\leftthreetimes^{s+1}(m_1) - \leftthreetimes^{s+1}(n_1))^{\frac{\beta}{k}-1} \leftthreetimes^s (n_1) \leftthreetimes' (n_1) \varnothing(n_1) dn_1, \quad m_1 > \alpha^+ \tag{3}$$

and

$$ {}^{s}_{k}\mathfrak{Z}^{\beta}_{\zeta^-} \varnothing(m_1) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{m_1}^{\zeta} (\leftthreetimes^{s+1}(n_1) - \leftthreetimes^{s+1}(m_1))^{\frac{\beta}{k}-1} \leftthreetimes^s (n_1) \leftthreetimes' (n_1) \varnothing(n_1) dn_1, \quad m_1 < \zeta. \tag{4}$$

Now we discuss some applications of defined operators which we used later.

Proposition 1. *If a function $\varnothing$ is continuous on $[a,b]$, $s \in R/\{-1\}$, $k \geq 0$, $\leftthreetimes$ be an increasing function and C be a constant function then for left and right sided (k,s)-GRLFIO of order $\beta > 0$, we have the following results holds:*

$$
{}_k^s\mathfrak{Z}_{r_1^+}^{\beta} \leftthreetimes^{s+1}(r_2) = \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \left(\frac{\leftthreetimes^{s+1}(r_1)(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}{\frac{\beta}{k}(s+1)} \right) + \frac{(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}+1}}{\frac{\beta}{k}(s+1)(\frac{\beta}{k}+1)}
$$

$$
{}_k^s\mathfrak{Z}_{r_2^-}^{\beta} \leftthreetimes^{s+1}(r_1) = \frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left(\leftthreetimes^{s+1}(r_2)(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}} \right) - \frac{(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}+1}}{(\frac{\beta}{k}+1)}
$$

$$
{}_k^s\mathfrak{Z}_{r_1^+}^{\beta} C = \frac{C(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1)^{\frac{\beta}{k}} \right)
$$

$$
{}_k^s\mathfrak{Z}_{r_2^-}^{\beta} C = \frac{C(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1)^{\frac{\beta}{k}} \right).
$$

2. A Class of Some Results

In this session, we will examine the reverse forms of Minkowski, Holder and Hermite-Hadamard-Fejer type inequalities via an $\imath.\upsilon$ cf involving (k,s)-GRLFIO under the framework of convexity. In this result, we want to explore the Minkowski's inequality.

Theorem 4. *Let $s \in R/\{-1\}$, $k \geq 0$, also $\varnothing, \phi : [r_1, r_2] \to R_i^+$ be $\imath.\upsilon$ functions such that $\varnothing(\chi) = [\varnothing_*, \varnothing^*]$ and $\phi(\chi) = [\phi_*, \phi^*]$, $\left({}_k^s\mathfrak{Z}_{r_1^+}^{\beta}\varnothing^p(\chi)\right) < \infty$ and $\left({}_k^s\mathfrak{Z}_{r_1^+}^{\beta}\phi^p(\chi)\right) < \infty$, then the expression* (5) *holds.*

$$
\begin{aligned}
&\left[\frac{1+\vartheta(2+\mu)}{(1+\vartheta)(1+\mu)}, \frac{1+\mu(2+\vartheta)}{(1+\vartheta)(1+\mu)}\right] \left[\left[{}_k^s\mathfrak{Z}_{r_1^+}^{\beta}(\varnothing(\chi)+\phi(\chi))^p\right]^{\frac{1}{p}}\right] \\
&\supseteq \left[{}_k^s\mathfrak{Z}_{r_1^+}^{\beta}\varnothing^p(\chi)\right]^{\frac{1}{p}} + \left[{}_k^s\mathfrak{Z}_{r_1^+}^{\beta}\phi^p(\chi)\right]^{\frac{1}{p}},
\end{aligned} \tag{5}
$$

where, $0 < \vartheta \leq \frac{\varnothing_*(\chi)}{\phi_*(\chi)} \leq \mu$ and $0 < \vartheta \leq \frac{\varnothing^*(\chi)}{\phi^*(\chi)} \leq \mu$ for $\chi \in [r_1, r_2]$, $p \geq 1$ with $\beta > 0$.

Proof. Since $\frac{\varnothing^*(\chi)}{\phi^*(\chi)} \leq \mu$, this implies

$$
(\mu+1)^p \varnothing^{*p}(\chi) \leq \mu^p(\varnothing^*(\chi) + \phi^*(\chi))^p.
$$

After multiplying by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(\chi)\right)^{\frac{\beta}{k}-1}\leftthreetimes^s(\chi)\leftthreetimes'(\chi)$ and applying the integration $w.r.t$ "χ" over $[r_1, r_2]$, we obtain the following expression.

$$
(\mu+1)^p \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{r_1}^{r_2} \left(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(\chi)\right)^{\frac{\beta}{k}-1} \leftthreetimes^s(\chi)\leftthreetimes'(\chi)\varnothing^{*p}(\chi)d\chi
$$

$$\leq \mu^p \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{r_1}^{r_2} \left(\lambda^{s+1}(r_2) - \lambda^{s+1}(\chi) \right)^{\frac{\beta}{k}-1} \lambda^s(\chi) \lambda'(\chi) [\varnothing^*(\chi) + \phi^*(\chi)]^p d\chi. \quad (6)$$

By comparing the expressions (3) and (6), we have

$$\left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \varnothing^{*p}(\chi)\right]^{\frac{1}{p}} \leq \frac{\mu}{\mu+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing^*(\chi) + \phi^*(\chi)]^p\right]^{\frac{1}{p}}. \quad (7)$$

For $\vartheta \leq \frac{\varnothing_*(\chi)}{\phi_*(\chi)}$, we can write

$$\vartheta^p [\varnothing_*(\chi) + \phi_*(\chi)]^p \leq (1+\vartheta)^p (\varnothing_*(\chi))^p.$$

Again multiplying by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \left(\lambda^{s+1}(r_2) - \lambda^{s+1}(\chi) \right)^{\frac{\beta}{k}-1} \lambda^s(\chi) \lambda'(\chi)$ and applying the integration over $[r_1, r_2]$ $w.r.t$ "χ", we have

$$\vartheta^p \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{r_1}^{r_2} \left(\lambda^{s+1}(r_2) - \lambda^{s+1}(\chi) \right)^{\frac{\beta}{k}-1} \lambda^s(\chi) \lambda'(\chi) [\varnothing_*(\chi) + \phi_*(\chi)]^p d\chi$$
$$\leq (\vartheta+1)^p \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_{r_1}^{r_2} \left(\lambda^{s+1}(r_2) - \lambda^{s+1}(\chi) \right)^{\frac{\beta}{k}-1} \lambda^s(\chi) \lambda'(\chi) \varnothing_*^p(\chi) d\chi. \quad (8)$$

Again by comparing the expressions (3) and (8), we have

$$\frac{\vartheta}{\vartheta+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing_*(\chi) + \phi_*(\chi)]^p\right]^{\frac{1}{p}} \leq \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \varnothing_*^p(\chi)\right]^{\frac{1}{p}}. \quad (9)$$

From the expressions (7) and (9)red, we have

$$\left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \varnothing^p(\chi)\right]^{\frac{1}{p}} = \left[\left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \varnothing_*^p(\chi)\right]^{\frac{1}{p}}, \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \varnothing^{*p}(\chi)\right]^{\frac{1}{p}} \right]$$
$$\supseteq \left[\frac{\vartheta}{\vartheta+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing_*(\chi) + \phi_*(\chi)]^p\right]^{\frac{1}{p}}, \frac{\mu}{\mu+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing^*(\chi) + \phi^*(\chi)]^p\right]^{\frac{1}{p}} \right]$$
$$= \left[\frac{\vartheta}{\vartheta+1}, \frac{\mu}{\mu+1}\right] \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing(\chi) + \phi(\chi)]^p\right]^{\frac{1}{p}}. \quad (10)$$

Continuing the same procedure for $\vartheta \leq \frac{\varnothing^*(\chi)}{\phi^*(\chi)}$, then we have

$$\left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \phi^{*p}(\chi)\right]^{\frac{1}{p}} \leq \frac{1}{\vartheta+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing^*(\chi) + \phi^*(\chi)]^p\right]^{\frac{1}{p}}. \quad (11)$$

Also for $\frac{\varnothing_*(\chi)}{\phi_*(\chi)} \leq \mu$, we have

$$\frac{1}{\mu+1} \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} [\varnothing_*(\chi) + \phi_*(\chi)]^p\right]^{\frac{1}{p}} \leq \left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+} \phi_*^p(\chi)\right]^{\frac{1}{p}}. \quad (12)$$

From the expressions (11) and (12), we have

$$\begin{aligned}&\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi^p(\chi)\right]^{\frac{1}{p}} = \left[\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi^p_*(\chi)\right]^{\frac{1}{p}}, \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi^{*p}(\chi)\right]^{\frac{1}{p}}\right]\\&\supseteq \left[\frac{1}{\mu+1}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing_*(\chi)+\phi_*(\chi)]^p\right]^{\frac{1}{p}}, \frac{1}{\vartheta+1}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing^*(\chi)+\phi^*(\chi)]^p\right]^{\frac{1}{p}}\right]\\&= \left[\frac{1}{\mu+1}, \frac{1}{\vartheta+1}\right]\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing(\chi)+\phi(\chi)]^p\right]^{\frac{1}{p}} \end{aligned} \tag{13}$$

Hence by adding (10) and (13) we get the required result (5).

In this result, redwe are going to present the fractional reverse Holder's inequality involving (k,s)-GRLFIO.

Theorem 5. *Let $s \in R/\{-1\}$, $k \geq 0$, also $\varnothing, \phi : [r_1, r_2] \to R_i^+$ be $\imath.\upsilon$ functions such that $\varnothing(\chi) = [\varnothing_*, \varnothing^*]$ and $\phi(\chi) = [\phi_*, \phi^*]$, $\left({}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing^p(\chi)\right) < \infty$ and $\left({}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi^p(\chi)\right) < \infty$, then the expression (14) holds.*

$$\left[\left(\frac{\vartheta}{\mu}\right)^{\frac{1}{pq}}, \left(\frac{\mu}{\vartheta}\right)^{\frac{1}{pq}}\right]\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing^{\frac{1}{p}}(\chi)\phi^{\frac{1}{q}}(\chi)]\right] \supseteq \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing(\chi)\right]^{\frac{1}{p}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi(\chi)\right]^{\frac{1}{q}}, \tag{14}$$

where, $0 < \vartheta \leq \frac{\varnothing_*(\chi)}{\phi_*(\chi)} \leq \mu$ and $0 < \vartheta \leq \frac{\varnothing^*(\chi)}{\phi^*(\chi)} \leq \mu$ for $\chi \in [r_1, r_2]$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $\beta > 0$.

Proof. Since $\frac{\varnothing^*(\chi)}{\phi^*(\chi)} \leq \mu$, implies

$$\varnothing^*(\chi) \leq \mu^{\frac{1}{q}}\varnothing^{*\frac{1}{p}}(\chi)\phi^{*\frac{1}{q}}(\chi).$$

Multiplying by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(\chi)\right)^{\frac{\beta}{k}-1}\leftthreetimes^s(\chi)\leftthreetimes'(\chi)$ and applying the integration over $[r_1, r_2]$ $w.r.t$ "χ"red, so we have

$$\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing^*(\chi)\right]^{\frac{1}{p}} \leq \mu^{\frac{1}{pq}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\right]\left[\varnothing^{*\frac{1}{p}}(\chi)\phi^{*\frac{1}{q}}(\chi)\right]^{\frac{1}{p}}. \tag{15}$$

Analogously for $\frac{\varnothing_*(\chi)}{\phi_*(\chi)} \geq \vartheta$, we have

$$\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing_*(\chi)\right]^{\frac{1}{p}} \geq \vartheta^{\frac{1}{pq}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\right]\left[\varnothing_*^{\frac{1}{p}}(\chi)\phi_*^{\frac{1}{q}}(\chi)\right]^{\frac{1}{p}}. \tag{16}$$

From the expressions (15) and (16), we can write

$$\begin{aligned}&\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing(\chi)\right]^{\frac{1}{p}} = \left[\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing_*(\chi)\right]^{\frac{1}{p}}, \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing^*(\chi)\right]^{\frac{1}{p}}\right]\\&\supseteq \left[\vartheta^{\frac{1}{pq}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\right]\left[\varnothing_*^{\frac{1}{p}}(\chi)\phi_*^{\frac{1}{q}}(\chi)\right]^{\frac{1}{p}}, \mu^{\frac{1}{pq}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\right]\left[\varnothing^{*\frac{1}{p}}(\chi)\phi^{*\frac{1}{q}}(\chi)\right]^{\frac{1}{p}}\right]\end{aligned}$$

$$= \left[\vartheta^{\frac{1}{pq}}, \mu^{\frac{1}{pq}}\right] \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing^{\frac{1}{p}}(\chi)\phi^{\frac{1}{q}}(\chi)]^{\frac{1}{p}}\right]. \tag{17}$$

We can retrieve the following confinement for $\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi(\chi)\right]^{\frac{1}{q}}$.

$$\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi(\chi)\right]^{\frac{1}{q}} = \left[\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi_*(\chi)\right]^{\frac{1}{q}}, \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\phi^*(\chi)\right]^{\frac{1}{q}}\right]$$
$$\supseteq \left[\frac{1}{\mu^{\frac{1}{pq}}}[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}]\left[\varnothing_*^{\frac{1}{p}}(\chi)\phi_*^{\frac{1}{q}}(\chi)\right]^{\frac{1}{q}}, \frac{1}{\vartheta^{\frac{1}{pq}}}[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}]\left[\varnothing^{*\frac{1}{p}}(\chi)\phi^{*\frac{1}{q}}(\chi)\right]^{\frac{1}{q}}\right]$$
$$= \left[\frac{1}{\mu^{\frac{1}{pq}}}, \frac{1}{\vartheta^{\frac{1}{pq}}}\right]\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}[\varnothing^{\frac{1}{p}}(\chi)\phi^{\frac{1}{q}}(\chi)]^{\frac{1}{q}}\right]. \tag{18}$$

Adding expressions (17) and (18) we get our required relation (14).

Theorem 6. *Let $s \in R/\{-1\}$, $k \geq 0$ and $\varnothing \in SIGX([r_1, r_2], R_i^+)$, then for $\beta > 0$ the following redinequality hold*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{\mho(\frac{1}{2})\beta\Gamma_k(\beta)}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)$$
$$\supseteq \frac{1}{\left(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1)\right)^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\varnothing(r_1)\right]$$
$$\supseteq \left[\varnothing(r_1)+\varnothing(r_2)\right]\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho(\theta)+\mho(1-\theta)]d\theta, \forall x, y \in [r_1, r_2].$$

Proof. Since $\varnothing$ is $\imath.\upsilon$-$(\leftthreetimes^{s+1}, \mho)$ cf and for $\theta = \frac{1}{2}$, we have

$$\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(x)+\leftthreetimes^{s+1}(y)}{2}\right)^{\frac{1}{s+1}}\right) \supseteq \mho(\frac{1}{2})\left[\varnothing(x)+\varnothing(y)\right]. \tag{19}$$

By substituting $x = \leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}$ and $y = \leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}$ in above inequality 19red. Also by multiplying both sides by $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}$ also taking the integration over $[0,1]$ $w.r.t$ "θ"red, we have

$$\frac{1}{\mho(\frac{1}{2})}\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)d\theta$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\left(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta$$
$$+ \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\left(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta. \tag{20}$$

Now we are proceeding with the left part of the inclusion (20).

$$\begin{aligned}
&\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big)d\theta\\
&=\Bigg[\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing_*\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big)d\theta,\\
&\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing^*\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big)d\theta\Bigg]\\
&=\Bigg[\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing_*\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big)\\
&+\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing^*\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big)\Bigg]\\
&=\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing\Big(\curlywedge^{-1}\Big(\frac{\curlywedge^{s+1}(r_1)+\curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\Big). \qquad (21)
\end{aligned}$$

For the right part of the inclusion (20).

$$\begin{aligned}
&\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\Big(\curlywedge^{-1}\left(\theta\curlywedge^{s+1}(r_1)+(1-\theta)\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta\\
&+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\Big(\curlywedge^{-1}\left((1-\theta)\curlywedge^{s+1}(r_1)+\theta\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta\\
&=\Bigg[\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing_*\Big(\curlywedge^{-1}\left(\theta\curlywedge^{s+1}(r_1)+(1-\theta)\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta\\
&+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing_*\Big(\curlywedge^{-1}\left((1-\theta)\curlywedge^{s+1}(r_1)+\theta\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta,\\
&\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing^*\Big(\curlywedge^{-1}\left(\theta\curlywedge^{s+1}(r_1)+(1-\theta)\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta\\
&+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing^*\Big(\curlywedge^{-1}\left((1-\theta)\curlywedge^{s+1}(r_1)+\theta\curlywedge^{s+1}(r_2)\right)^{\frac{1}{s+1}}\Big)d\theta\Bigg].
\end{aligned}$$

By substituting $\curlywedge^{s+1}(\chi)=\theta\curlywedge^{s+1}(r_1)+(1-\theta)\curlywedge^{s+1}(r_2)$red, we have

$$\begin{aligned}
&=\frac{1}{\left(\curlywedge^{s+1}(r_2)-\curlywedge^{s+1}(r_1)\right)^{\frac{\beta}{k}}}\\
&\times\Bigg[\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_{r_1}^{r_2}\left(\curlywedge^{s+1}(r_2)-\curlywedge^{s+1}(\chi)\right)^{\frac{\beta}{k}-1}\curlywedge^{s}(\chi)\curlywedge'(\chi)\varnothing(\chi)d\chi\\
&+\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_{r_1}^{r_2}(\curlywedge^{s+1}(\chi)-\curlywedge^{s+1}(r_1))^{\frac{\beta}{k}-1}\curlywedge^{s}(\chi)\curlywedge'(\chi)\varnothing(\chi)d\chi\Bigg].
\end{aligned}$$

This implies

$$= \frac{1}{\left(\curlywedge^{s+1}(r_2) - \curlywedge^{s+1}(r_1)\right)^{\frac{\beta}{k}}} \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+} \varnothing(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-} \varnothing(r_1) \right]. \tag{22}$$

We get first half of our relation from the inclusions (21) and (22) for the second half employing $\imath.\upsilon$-$(\curlywedge^{s+1}, \mho)$ convexity of function $\varnothing$, we have

$$\varnothing\left(\curlywedge^{-1} \left(\theta \curlywedge^{s+1}(r_1) + (1-\theta) \curlywedge^{s+1}(r_2) \right)^{\frac{1}{s+1}} \right) \supseteq \mho(\theta)\varnothing(r_1) + \mho(1-\theta)\varnothing(r_2), \tag{23}$$

and

$$\varnothing\left(\curlywedge^{-1} \left((1-\theta) \curlywedge^{s+1}(r_1) + \theta \curlywedge^{s+1}(r_2) \right)^{\frac{1}{s+1}} \right) \supseteq \mho(1-\theta)\varnothing(r_1) + \mho(\theta)\varnothing(r_2). \tag{24}$$

Adding (23) and (24) inclusions and multiplying both sides by $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}$ also taking the integration over $[0,1]$ $w.r.t$ "θ"red, then we acquired our required relation.

Corollary 1. *If we fix $\mho(\theta) = \theta$ in Theorem 6, we possess*

$$\begin{aligned} &\frac{2(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \varnothing\left(\curlywedge^{-1} \left(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2} \right)^{\frac{1}{s+1}} \right) \\ &\supseteq \frac{1}{\left(\curlywedge^{s+1}(r_2) - \curlywedge^{s+1}(r_1)\right)^{\frac{\beta}{k}}} \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+} \varnothing(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-} \varnothing(r_1) \right] \\ &\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left[\varnothing(r_1) + \varnothing(r_2) \right]. \end{aligned} \tag{25}$$

Example 1. *If we choose $\varnothing(r) = [4 - \curlywedge^{s+1}(r), 8 + \curlywedge^{s+1}(r)]$ and $\curlywedge(r) = r$ in (25) and using proposition 1, we have*

$$\begin{aligned} &\frac{2(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left(4 - \frac{r_1^{s+1} + r_2^{s+1}}{2}, 8 + \frac{r_1^{s+1} + r_2^{s+1}}{2} \right) \\ &\supseteq \frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)} \left(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1}) \right) \\ &\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} \left(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1}) \right). \end{aligned}$$

Example 2. *For graphical representation if we choose $\varnothing(r) = [4 - \curlywedge^{s+1}(r), 8 + \curlywedge^{s+1}(r)]$, $\mho(r) = \frac{1}{8}$ and $\curlywedge(r) = \sin r$ in (25) and using the proposition 1, we have*

$$\frac{8(s+1)^{-\frac{\beta}{k}}}{(\beta)\Gamma_k(\beta)} \left(4 - \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2}, 8 + \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2} \right)$$

$$\supseteq \frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\Big(8-(\sin^{s+1}(r_1)+\sin^{s+1}(r_2)),16+(\sin^{s+1}(r_1)+\sin^{s+1}(r_2))\Big)$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{4\beta\Gamma_k(\beta)}\big(8-(\sin^{s+1}(r_1)+\sin^{s+1}(r_2)),16+(\sin^{s+1}(r_1)+\sin^{s+1}(r_2))\big).$$

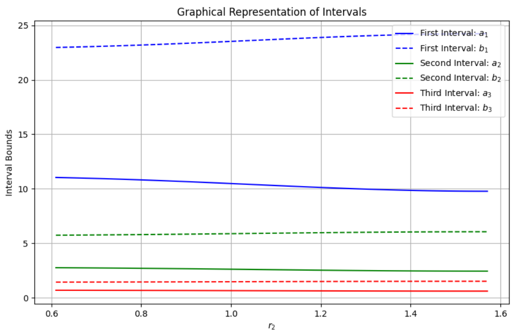

Figure 1: Graphical representation of Theorem 6 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

Also for tabular form we have

r_2	(a_1, b_1)	(a_2, b_2)	(a_3, b_3)
0.610000	(11.037156, 22.964070)	(2.759289, 5.741017)	(0.6898221.435254)
0.802159	(10.811418, 23.189807)	(2.702855, 5.797452)	(0.675714, 1.449363)
0.994319	(10.489794, 23.511432)	(2.622448, 5.877858)	(0.655612, 1.469465)
1.186478	(10.143322, 23.857903)	(2.535831, 5.964476)	(0.633958, 1.491119)
1.378637	(9.874478, 24.126747)	(2.468620, 6.031687)	(0.617155, 1.507922)
1.570796	(9.773019, 24.228207)	(2.443255, 6.057052)	(0.610814, 1.514263)

Table 1: Interval bounds of Theorem 6 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$

Corollary 2. *If we fix $s = 0$, $k = 1$, $\leftthreetimes(r) = r$ and $\mho(\theta) = \theta$ in Theorem 6, we possess Hermite-Hadamard-Fejer inequality for ı.υ-$(\leftthreetimes^{s+1}, \mho)$ cf that is provided in [23].*

$$\frac{1}{\beta\Gamma(\beta)}\varnothing\left(\frac{r_1+r_2}{2}\right)$$
$$\supseteq \frac{1}{(r_2-r_1)^{\beta}}\left[3^{\beta}_{r_1^+}\varnothing(r_2)+3^{\beta}_{r_2^-}\varnothing(r_1)\right]$$
$$\supseteq \frac{1}{\beta\Gamma(\beta)}\big[\varnothing(r_1)+\varnothing(r_2)\big].$$

Theorem 7. *Let $s \in R/\{-1\}$, $k \geq 0$, $\varnothing \in SIGX([r_1, r_2], R_i^+)$, and $\Game : [r_1, r_2] \to R$ be symmetric w.r.t $\frac{r_1+r_2}{2}$, then for $\beta > 0$ the following inclusions holds.*

$$\begin{aligned}
&\frac{1}{2\mho(\frac{1}{2})}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)\left[{}_{k}^{s}\mathfrak{Z}^{\beta}_{r_1^+}\Game(r_2)+{}_{k}^{s}\mathfrak{Z}^{\beta}_{r_2^-}\Game(r_1)\right]\\
&\supseteq\left[{}_{k}^{s}\mathfrak{Z}^{\beta}_{r_1^+}\Game\varnothing(r_2)+{}_{k}^{s}\mathfrak{Z}^{\beta}_{r_2^-}\Game\varnothing(r_1)\right]\\
&\supseteq\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(\int_0^1\theta^{\frac{\beta}{k}-1}[\mho(\theta)+\mho(1-\theta)]\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]\right.\\
&\left.\times[\varnothing(r_1)+\varnothing(r_2)]d\theta\right).
\end{aligned}$$

Proof. *Since $\varnothing$ is $\imath.\upsilon$-$(\leftthreetimes^{s+1},\mho)$ cf and for $\theta=\frac{1}{2}$, we have*

$$\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(x)+\leftthreetimes^{s+1}(y)}{2}\right)^{\frac{1}{s+1}}\right)\supseteq\mho(\frac{1}{2})\left[\varnothing(x)+\varnothing(y)\right].$$

By substituting $x=\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}$ and $y=\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}$ in above expression. Multiplying both sides by $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]$ and taking the integration over $[0,1]$ w.r.t "θ", we have

$$\begin{aligned}
&\frac{1}{\mho(\frac{1}{2})}\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]\\
&\times\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)d\theta\\
&\supseteq\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]\\
&\times\varnothing\left(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta\\
&+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]\\
&\times\varnothing\left(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta. \qquad (26)
\end{aligned}$$

Taking benefit of the fact $\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})=\Game(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})$ for the left part of (26).

$$\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\left[\Game(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\right]$$

$$
\begin{aligned}
&\times \varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)d\theta \\
&= \frac{1}{2}\Bigg[\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\big[\eth(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\big] \\
&\times \varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)d\theta \\
&+ \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\big[\eth(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\big] \\
&\times \varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)d\theta\Bigg].
\end{aligned}
$$

By substituting $\leftthreetimes^{s+1}(\chi)=\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2)$*red, we have*

$$
\begin{aligned}
&= \frac{1}{2(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right) \\
&\times \Bigg[\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_{r_1}^{r_2}\left(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(\chi)\right)^{\frac{\beta}{k}-1}\leftthreetimes^{s}(\chi)\leftthreetimes'(\chi)\eth(\chi)d\chi \\
&+ \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_{r_1}^{r_2}(\leftthreetimes^{s+1}(\chi)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}-1}\leftthreetimes^{s}(\chi)\leftthreetimes'(\chi)\eth(\chi)d\chi\Bigg] \\
&= \frac{1}{2(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right) \\
&\times \left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^{+}}\eth(r_2)+{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^{-}}\eth(r_1)\right].
\end{aligned} \tag{27}
$$

For the right part of (26)*red, we have*

$$
\begin{aligned}
&\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\big[\eth(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\big] \\
&\times \varnothing\left(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta \\
&+ \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\big[\eth(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}})\big] \\
&\times \varnothing\left(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\right)d\theta.
\end{aligned}
$$

Substituting $\leftthreetimes^{s+1}(\chi)=\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2)$*red, we have*

$$
= \frac{1}{(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^{+}}\eth\varnothing(r_2)+{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^{-}}\eth\varnothing(r_1)\right]. \tag{28}
$$

We achieve our first desired inclusion from (27) and (28). In this way for the other inclusion employing $\imath.\upsilon$-$(\curlywedge^{s+1}, \mho)$ convexity of function $\varnothing$, we have

$$\varnothing\bigg(\curlywedge^{-1} \Big(\theta \curlywedge^{s+1} (r_1) + (1-\theta) \curlywedge^{s+1} (r_2)\Big) \supseteq \mho(\theta)\varnothing(r_1) + \mho(1-\theta)\varnothing(r_2), \tag{29}$$

and

$$\varnothing\bigg(\curlywedge^{-1} \Big((1-\theta) \curlywedge^{s+1} (r_1) + \theta \curlywedge^{s+1} (r_2)\Big) \supseteq \mho(1-\theta)\varnothing(r_1) + \mho(\theta)\varnothing(r_2). \tag{30}$$

Adding (29) and (30) inclusions and multiplying both sides by $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}\big[\eth(\curlywedge^{-1}(\theta\curlywedge^{s+1}(r_1) + (1-\theta)\curlywedge^{s+1}(r_2))^{\frac{1}{s+1}})\big]$ *also taking the integration over* $[0,1]$ *w.r.t "θ"red, then we acquired our required relation.*

Corollary 3. *If we fix $\mho(\theta) = \theta$ in Theorem 7, we possess*

$$\begin{aligned}
&\varnothing\bigg(\curlywedge^{-1} \Big(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2}\Big)^{\frac{1}{s+1}}\bigg)\Big[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\eth(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\eth(r_1)\Big] \\
&\supseteq \Big[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\eth\varnothing(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\eth\varnothing(r_1)\Big] \\
&\supseteq [\varnothing(r_1) + \varnothing(r_2)]\Big[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\eth(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\eth(r_1)\Big].
\end{aligned} \tag{31}$$

Example 3. *If we choose $\varnothing(r) = [4 - \curlywedge^{s+1}(r), 8 + \curlywedge^{s+1}(r)]$ and $\curlywedge(r) = r$ and $\eth(r) = 1$ in (31) and utilizing proposition 1, we have*

$$\begin{aligned}
&\frac{8(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\bigg(4 - \frac{r_1^{s+1} + r_2^{s+1}}{2}, 8 + \frac{r_1^{s+1} + r_2^{s+1}}{2}\bigg)\big(r_2^{s+1} - r_1^{s+1}\big)^{\frac{\beta}{k}} \\
&\supseteq \frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\big(r_2^{s+1} - r_1^{s+1}\big)^{\frac{\beta}{k}}\bigg(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1})\bigg) \\
&\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{2\delta\Gamma_k(\beta)}\big(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1})\big)\big(r_2^{s+1} - r_1^{s+1}\big)^{\frac{\beta}{k}}.
\end{aligned}$$

Example 4. *For graphical representation if we choose $\varnothing(r) = [4 - \curlywedge^{s+1}(r), 8 + \curlywedge^{s+1}(r)]$, $\mho(r) = \frac{1}{8}$ and $\curlywedge(r) = \sin r$ and $\eth(r) = 1$ in (31) and utilizing proposition 1, we have*

$$\begin{aligned}
&\frac{4(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\bigg(4 - \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2}, 8 + \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2}\bigg)\big(\sin^{s+1}(r_2) - \sin^{s+1}(r_1)\big)^{\frac{\beta}{k}} \\
&\supseteq \frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\big(\sin^{s+1}(r_2) - \sin^{s+1}(r_1)\big)^{\frac{\beta}{k}}\bigg(8 - (\sin^{s+1}(r_1) + \sin^{s+1}(r_2)), 16 + (\sin^{s+1}(r_1) + \sin^{s+1}(r_2))\bigg) \\
&\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{4\delta\Gamma_k(\beta)}\big(8 - (\sin^{s+1}(r_1) + \sin^{s+1}(r_2)), 16 + (\sin^{s+1}(r_1) + \sin^{s+1}(r_2))\big)\big(\sin^{s+1}(r_2) - \sin^{s+1}(r_1)\big)^{\frac{\beta}{k}}.
\end{aligned}$$

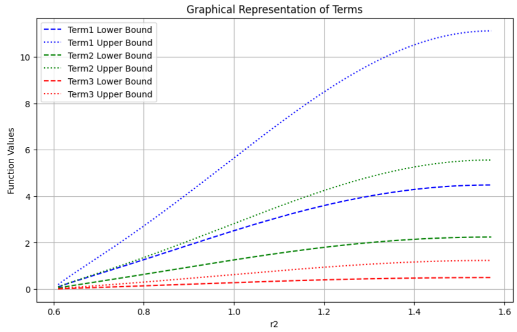

Figure 2: Graphical representation of Theorem 7 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

r_2	(a_1, b_1)	(a_2, b_2)	(a_3, b_3)
0.6100	$(0.0928, 0.1930)$	$(0.0464, 0.0965)$	$(0.0116, 0.0241)$
0.8022	$(1.2813, 2.7484)$	$(0.6407, 1.3742)$	$(0.1602, 0.3435)$
0.9943	$(2.4816, 5.5622)$	$(1.2408, 2.7811)$	$(0.3102, 0.6953)$
1.1865	$(3.5355, 8.3157)$	$(1.7677, 4.1578)$	$(0.4419, 1.0395)$
1.3786	$(4.2401, 10.3601)$	$(2.1201, 5.1800)$	$(0.5300, 1.2950)$
1.5708	$(4.4849, 11.1186)$	$(2.2425, 5.5593)$	$(0.5606, 1.3898)$

Table 2: Interval bounds of Theorem 7 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

For tabular form we have

Corollary 4. *If we fix $s = 0$, $k = 1$, $\leftthreetimes(r) = r$ and $\mho(\theta) = \theta$ in Theorem 7, we possess*

$$\begin{aligned}
&\varnothing\left(\frac{r_1 + r_2}{2}\right)\left[\mathfrak{Z}^{\beta}_{r_1^+}\eth(r_2) + \mathfrak{Z}^{\beta}_{r_2^-}\eth(r_1)\right] \\
&\supseteq \left[\mathfrak{Z}^{\beta}_{r_1^+}\eth\varnothing(r_2) + \mathfrak{Z}^{\beta}_{r_2^-}\eth\varnothing(r_1)\right] \\
&\supseteq [\varnothing(r_1) + \varnothing(r_2)]\left[\mathfrak{Z}^{\beta}_{r_1^+}\eth(r_2) + \mathfrak{Z}^{\beta}_{r_2^-}\eth(r_1)\right].
\end{aligned}$$

Theorem 8. *Let $s \in R/\{-1\}$, $\varnothing, \eth \in SIGX([r_1, r_2], R_i^+)$ and $k \geq 0$, then for $\beta > 0$ the following redinequality fulfilled*

$$\begin{aligned}
&\frac{1}{(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\left[{}^s_k\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2) + {}^s_k\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\right] \\
&\supseteq P(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)]d\theta \\
&+ Q(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta) + \mho_1(1-\theta)\mho_2(\theta)]d\theta, \\
&\textit{where, } P(r_1, r_2) = \varnothing(r_1)\eth(r_1) + \varnothing(r_2)\eth(r_2), Q(r_1, r_2) = \varnothing(r_1)\eth(r_2) + \varnothing(r_2)\eth(r_1).
\end{aligned}$$

Proof. Since $\varnothing, \eth \in SIGX([r_1, r_2], R^+)$, *then*

$$\varnothing\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big) \supseteq \mho_1(\theta)\varnothing(r_1) + \mho_1(1-\theta)\varnothing(r_2)$$
$$\eth\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}} \supseteq \mho_2(\theta)\eth(r_1) + \mho_2(1-\theta)\eth(r_2).$$

By multiplying, we have

$$\varnothing\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\supseteq \mho_1(\theta)\mho_2(\theta)\varnothing(r_1)\eth(r_1) + \mho_1(\theta)\mho_2(1-\theta)\varnothing(r_1)\eth(r_2)$$
$$+ \mho_1(1-\theta)\mho_2(\theta)\varnothing(r_2)\eth(r_1) + \mho_1(1-\theta)\mho_2(1-\theta)\varnothing(r_2)\eth(r_2). \tag{32}$$

Similarly,

$$\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1) + \theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1) + \theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\supseteq \mho_1(1-\theta)\mho_2(1-\theta)\varnothing(r_1)\eth(r_1) + \mho_1(1-\theta)\mho_2(\theta)\varnothing(r_1)\eth(r_2)$$
$$+ \mho_1(\theta)\mho_2(1-\theta)\varnothing(r_2)\eth(r_1) + \mho_1(\theta)\mho_2(\theta)\varnothing(r_2)\eth(r_2). \tag{33}$$

Adding (32) *and* (33) *inclusions and multiplying both sides by* $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}$ *also taking the integration over* $[0,1]$ *w.r.t "*θ*"red, then we have*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\,\eth\big(\leftthreetimes^{-1}(\theta \leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)d\theta$$
$$+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1) + \theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\,\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1) + \theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)d\theta$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)][\varnothing(r_1)\eth(r_1) + \varnothing(r_2)\eth(r_2)]d\theta$$
$$+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta) + \mho_1(1-\theta)\mho_2(\theta)][\varnothing(r_1)\eth(r_2) + \varnothing(r_2)\eth(r_1)]d\theta.$$

Substituting $\leftthreetimes^{s+1}(\chi) = \theta\leftthreetimes^{s+1}(r_1) + (1-\theta)\leftthreetimes^{s+1}(r_2)$*red, then we have*

$$= \frac{1}{(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\Big[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\Big]$$
$$\supseteq P(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)]d\theta$$
$$+ Q(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta) + \mho_1(1-\theta)\mho_2(\theta)]d\theta.$$

This complete our required relation.

Corollary 5. *If we fix $\mho(\theta)=\theta$ in Theorem 8, we possess*

$$\frac{1}{(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2)+{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\right]$$
$$\supseteq\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left[P(r_1,r_2)\frac{\frac{\beta^2}{k^2}+\frac{\beta}{k}+1}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+1)(\frac{\beta}{k})}+Q(r_1,r_2)\frac{2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+1)}\right],$$

where, $P(r_1,r_2)=\varnothing(r_1)\eth(r_1)+\varnothing(r_2)\eth(r_2), Q(r_1,r_2)=\varnothing(r_1)\eth(r_2)+\varnothing(r_2)\eth(r_1).$

Example 5. *If we choose $\varnothing(r)=[4-\leftthreetimes^{s+1}(r),8+\leftthreetimes^{s+1}(r)]$ and $\leftthreetimes(r)=r$, $\eth(r)=1$, $\mho(\theta)=\frac{1}{4}$ in Theorem 8 and utilizing proposition 1, we have*

$$\frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\left(8-(r_1^{s+1}+r_2^{s+1}),16+(r_1^{s+1}+r_2^{s+1})\right)$$
$$\supseteq\frac{(s+1)^{-\frac{\beta}{k}}}{8\beta\Gamma_k(\beta)}\left(8-(r_1^{s+1}+r_2^{s+1}),16+(r_1^{s+1}+r_2^{s+1})\right).$$

Example 6. *For graphical representation if we choose $\varnothing(r)=[4-\leftthreetimes^{s+1}(r),8+\leftthreetimes^{s+1}(r)]$ and $\leftthreetimes(r)=\sin r$, $\eth(r)=1$, $\mho(\theta)=\frac{1}{8}$ in Theorem 8 and utilizing proposition 1, we have*

$$\frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\left(8-(\sin^{s+1}(r_1)+\sin^{s+1}(r_2)),16+(\sin^{s+1}(r_1)+\sin^{s+1}(r_2))\right)$$
$$\supseteq\frac{(s+1)^{-\frac{\beta}{k}}}{32\beta\Gamma_k(\beta)}\left(8-(\sin^{s+1}(r_1)+\sin^{s+1}(r_2)),16+(\sin^{s+1}(r_1)+\sin^{s+1}(r_2))\right).$$

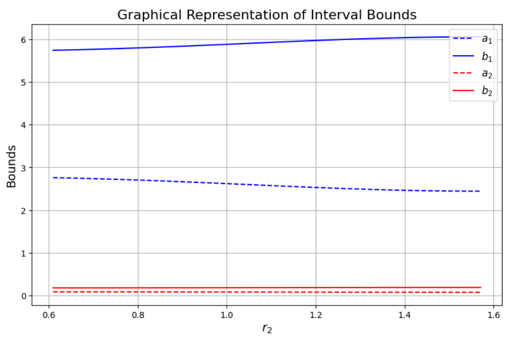

Figure 3: Graphical representation of Theorem 8 corresponding to the choice of parameters $r_1=0.6$, $r_1<r_2\le\frac{\pi}{2}$, $k=1$, $s=3$ and $\beta=0.8$.

For tabular form we have

Corollary 6. *If we fix $s=0$, $k=1$, $\leftthreetimes(\theta)=\theta$, and $\mho(\theta)=\theta$ in Theorem 8, we possess*

$$\frac{1}{(r_2-r_1)^{\beta}}\left[\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2)+\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\right]$$
$$\supseteq\frac{1}{\Gamma(\beta)}\left[P(r_1,r_2)\frac{\beta^2+\beta+1}{(\beta+2)(\beta+1)(\beta)}+Q(r_1,r_2)\frac{2}{(\beta+2)(\beta+1)}\right].$$

r_2	(a_1, b_1)	(a_2, b_2)
0.6100	(2.7593, 5.7410)	(0.0862, 0.1794)
0.8022	(2.7029, 5.7975)	(0.0845, 0.1812)
0.9943	(2.6224, 5.8779)	(0.0820, 0.1837)
1.1865	(2.5358, 5.9645)	(0.0792, 0.1864)
1.3786	(2.4686, 6.0317)	(0.0771, 0.1885)
1.5708	(2.4433, 6.0571)	(0.0764, 0.1893)

Table 3: Interval bounds of Theorem 8 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

Corollary 7. *If we fix $\curlywedge(\theta) = \theta$ and $s + 1 \geq 0$ in Theorem 8, we possess*

$$\frac{1}{(r_2^{s+1} - r_1^{s+1})^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{(r_1^{s+1})^+}\varnothing\eth(r_2^{s+1}) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{(r_2^{s+1})^-}\varnothing\eth(r_2^{s+1})\right]$$
$$\supseteq P(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)]d\theta$$
$$+ Q(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta) + \mho_1(1-\theta)\mho_2(\theta)]d\theta$$

Corollary 8. *If we fix $\curlywedge(\theta) = \theta$, $s + 1 \geq 0$ and $\mho(\theta) = \theta$ in Theorem 8, we possess*

$$\frac{1}{(r_2^{s+1} - r_1^{s+1})^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{(r_1^{s+1})^+}\varnothing\eth(r_2^{s+1}) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{(r_2^{s+1})^-}\varnothing\eth(r_2^{s+1})\right]$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left[P(r_1, r_2)\frac{\frac{\beta^2}{k^2} + \frac{\beta}{k} + 1}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+1)(\frac{\beta}{k})} + Q(r_1, r_2)\frac{2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+1)}\right].$$

Theorem 9. *Let $s \in R/\{-1\}$, $\varnothing, \eth \in SIGX([r_1, r_2], R_i^+)$ and $k \geq 0$, then for $\beta > 0$ the following redinequality fulfill*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing\left(\curlywedge^{-1}\left(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)\eth\left(\curlywedge^{-1}\left(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)$$
$$\supseteq \frac{\mho_1(\frac{1}{2})\mho_2(\frac{1}{2})}{(\curlywedge^{s+1}(r_2) - \curlywedge^{s+1}(r_1))^{\frac{\beta}{k}}}\left[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2) + {}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\right]$$
$$+ \mho_1(\frac{1}{2})\mho_2(\frac{1}{2})\Bigg(P(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta) + \mho_1(1-\theta)\mho_2(\theta)]d\theta$$
$$+ Q(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)]d\theta\Bigg),$$

where, $P(r_1, r_2) = \varnothing(r_1)\eth(r_1) + \varnothing(r_2)\eth(r_2), Q(r_1, r_2) = \varnothing(r_1)\eth(r_2) + \varnothing(r_2)\eth(r_1).$

Proof. Since $\varnothing, \eth \in SIGX([r_1, r_2], R^+)$, then

$$\varnothing\left(\curlywedge^{-1}\left(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)\eth\left(\curlywedge^{-1}\left(\frac{\curlywedge^{s+1}(r_1) + \curlywedge^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)$$

$$\supseteq \mho_1(\frac{1}{2})\mho_2(\frac{1}{2})$$
$$\Big[\varnothing\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+((1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$+\varnothing\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$+\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}(\theta(r_1)+(1-\theta)r_2)^{\frac{1}{s+1}}\big)$$
$$+\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)\Big].$$

Multiplying preceding inclusion by $\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\theta^{\frac{\beta}{k}-1}$ *and taking the integration on* $[0,1]$ *w.r.t "*θ*" red, we have*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\varnothing\Bigg(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\Bigg)\eth\Bigg(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\Bigg)d\theta$$
$$\supseteq \mho_1(\frac{1}{2})\mho_2(\frac{1}{2})\Bigg[\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\eth\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)d\theta$$
$$+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}r_2)^{\frac{1}{s+1}}\big)d\theta$$
$$+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\eth\big(\leftthreetimes^{-1}(\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)d\theta$$
$$+\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}\varnothing\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)$$
$$\times\eth\big(\leftthreetimes^{-1}((1-\theta)\leftthreetimes^{s+1}(r_1)+\theta\leftthreetimes^{s+1}(r_2))^{\frac{1}{s+1}}\big)d\theta\Bigg].$$

By definition of $\imath.\upsilon$-$(\leftthreetimes^{s+1},\mho)$ *and substituting* $\leftthreetimes^{s+1}(\chi)=\theta\leftthreetimes^{s+1}(r_1)+(1-\theta)\leftthreetimes^{s+1}(r_2)$*red, we have*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing\Bigg(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\Bigg)\eth\Bigg(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1)+\leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\Bigg)$$
$$\supseteq\frac{\mho_1(\frac{1}{2})\mho_2(\frac{1}{2})}{(\leftthreetimes^{s+1}(r_2)-\leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\Big[{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2)+{}^{s}_{k}\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\Big]$$
$$+\mho_1(\frac{1}{2})\mho_2(\frac{1}{2})\Bigg(P(r_1,r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_0^1\theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(1-\theta)+\mho_1(1-\theta)\mho_2(\theta)]d\theta$$

$$+ Q(r_1, r_2)\frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_0^1 \theta^{\frac{\beta}{k}-1}[\mho_1(\theta)\mho_2(\theta) + \mho_1(1-\theta)\mho_2(1-\theta)]d\theta\Bigg).$$

Hence we have proved our result.

Corollary 9. *If we fix $\mho(\theta) = \theta$ in Theorem 9, we possess*

$$\frac{(s+1)^{-\frac{\beta}{k}}}{\beta\Gamma_k(\beta)}\varnothing\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1) + \leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)\eth\left(\leftthreetimes^{-1}\left(\frac{\leftthreetimes^{s+1}(r_1) + \leftthreetimes^{s+1}(r_2)}{2}\right)^{\frac{1}{s+1}}\right)$$
$$\supseteq \frac{1}{(\leftthreetimes^{s+1}(r_2) - \leftthreetimes^{s+1}(r_1))^{\frac{\beta}{k}}}\left[{}_k^s\mathfrak{Z}^{\beta}_{r_1^+}\varnothing\eth(r_2) + {}_k^s\mathfrak{Z}^{\beta}_{r_2^-}\varnothing\eth(r_1)\right]$$
$$+ \frac{(s+1)^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left[P(r_1, r_2)\frac{2}{(\beta+2)(\beta+1)} + Q(r_1, r_2)\frac{\beta^2 + \beta + 1}{(\beta+2)(\beta+1)(\beta)}\right].$$

Example 7. *If we choose $\varnothing(r) = [4 - \leftthreetimes^{s+1}(r), 8 + \leftthreetimes^{s+1}(r)]$ and $\leftthreetimes(r) = r$, $\eth(r) = 1$, $\mho_1(r) = \mho_2(r) = \frac{1}{4}$ in Theorem 9 and utilizing proposition 1, we have*

$$\frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\left(4 - \frac{r_1^{s+1} + r_2^{s+1}}{2}, 8 + \frac{r_1^{s+1} + r_2^{s+1}}{2}\right)$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{\delta\Gamma_k(\beta)}\left(\frac{1}{64}(r_2^{s+1} - r_1^{s+1})^{\frac{\beta}{k}-1}\left(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1})\right)\right.$$
$$\left. + \frac{1}{8}\big(8 - (r_1^{s+1} + r_2^{s+1}), 16 + (r_1^{s+1} + r_2^{s+1})\big)\right).$$

Example 8. *For graphical representation if we choose $\varnothing(r) = [4 - \leftthreetimes^{s+1}(r), 8 + \leftthreetimes^{s+1}(r)]$, $\leftthreetimes(r) = \sin r$, $\eth(r) = 1$ and $\mho_1(r) = \mho_2(r) = \frac{1}{4}$ in Theorem 9 and utilizing proposition 1, we have*

$$\frac{(s+1)^{\frac{-\beta}{k}}}{\beta\Gamma_k(\beta)}\left(4 - \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2}, 8 + \frac{\sin^{s+1}(r_1) + \sin^{s+1}(r_2)}{2}\right)$$
$$\supseteq \frac{(s+1)^{-\frac{\beta}{k}}}{2048\beta\Gamma_k(\beta)}\left(\left(8 - (\sin^{s+1}(r_1) + \sin^{s+1}(r_2)), 16 + (\sin^{s+1}(r_1) + \sin^{s+1}(r_2))\right)\right.$$
$$\left. + \big(8 - (\sin^{s+1}(r_1) + \sin^{s+1}(r_2)), 16 + (\sin^{s+1}(r_1) + \sin^{s+1}(r_2))\big)\right).$$

For tabular value

Corollary 10. *If we fix $s = 0$, $k = 1$, $\leftthreetimes(r) = r$ and $\mho(\theta) = \theta$ in Theorem 9, we possess*

$$\frac{1}{\beta\Gamma(\beta)}\varnothing\left(\frac{r_1 + r_2}{2}\right)\eth\left(\frac{r_1 + r_2}{2}\right)$$

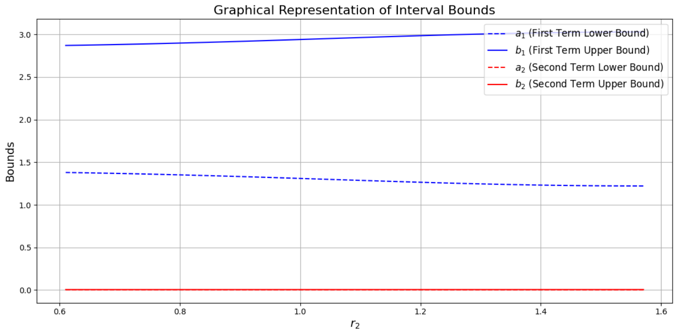

Figure 4: Graphical representation of Theorem 9 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

r_2	(a_1, b_1)	(a_2, b_2)
0.6100	$(1.3796e+00, 2.8705e+00)$	$(2.6946e-03, 5.6065e-03)$
0.8022	$(1.3514e+00, 2.8987e+00)$	$(2.6395e-03, 5.6616e-03)$
0.9943	$(1.3112e+00, 2.9389e+00)$	$(2.5610e-03, 5.7401e-03)$
1.1865	$(1.2679e+00, 2.9822e+00)$	$(2.4764e-03, 5.8247e-03)$
1.3786	$(1.2343e+00, 3.0158e+00)$	$(2.4108e-03, 5.8903e-03)$
1.5708	$(1.2216e+00, 3.0285e+00)$	$(2.3860e-03, 5.9151e-03)$

Table 4: Interval bounds of Theorem 9 corresponding to the choice of parameters $r_1 = 0.6$, $r_1 < r_2 \leq \frac{\pi}{2}$, $k = 1$, $s = 3$ and $\beta = 0.8$.

$$\supseteq \frac{1}{4(r_2 - r_1)^{\beta}} \left[3^{\beta}_{r_1^+} \varnothing\eth(r_2) + 3^{\beta}_{r_2^-} \varnothing\eth(r_1) \right] + \frac{1}{k\Gamma(\beta)} \left[P(r_1, r_2) \frac{2}{(\beta+2)(\beta+1)} + Q(r_1, r_2) \frac{\beta^2 + \beta + 1}{(\beta+2)(\beta+1)(\beta)} \right].$$

3. Conclusion

redFractional inequalities, which extend classical inequalities to fractional-order settings, play a crucial role in various industrial applications by providing precise analytical tools for optimization, stability analysis, and error estimation. In control systems and automation, fractional inequalities help establish stability criteria for fractional-order controllers, improving system robustness in robotics, aerospace and process control industries. In signal processing and telecommunications, they aid in error bounds estimation and performance analysis of fractional filters, enhancing data transmission and noise reduction. In materials science and mechanical engineering, fractional inequalities contribute to modeling stress-strain relationships in viscoelastic and complex materials, optimizing designs in structural engineering and manufacturing. The energy sector benefits from these inequalities in analyzing fractional diffusion processes, optimizing heat conduction models, and improving energy storage systems like batteries and supercapacitors. Additionally, in biomedical engineering, they assist in developing fractional-order models for physiological systems, ensuring accurate predictions in drug delivery and neural activity analysis. By refining analytical bounds and improving modeling accuracy, fractional inequalities significantly contribute to industrial advancements across multiple domains. In this study, we introduce the novel Hermite-Hadamard-Fejer type inequalities within the structure of $\imath.v$ $(\leftthreetimes^{s+1}, \mho)$ class of convexity. Our analysis provides comprehensive bounds for several well-known fractional problems. Specifically, we investigate the interplay between the classical Hermite-Hadamard-Fejer inequality and unified forms of Minkowskis and Holder inequalities within the class of convexity. We provide a versatile structure for mathematical inequalities related to generalized fractional operators by extending and generalizing the reverse forms of these inequalities with in the unified class of convexity. To enhance their practical applications, we investigate the additional implications, derive specific inequalities, and illustrate them through graphical representations. We also check the results by using tables for different fractional orders. The sharpness of our inequalities is confirmed by these graphical comparison. We urge readers to investigate more generalized fractional operators in order to create bigger classes of inequalities for future research. Also, by comparing recently hypothesized disparities with those that already exist, future research could evaluate the adaptability of their findings by graphical analysis.

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Declarations:

Availability of data and material

The data used to support the findings of this study are available from the corresponding author upon request.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final version.

Competing interests

The authors declare that they have no conflicts of interest.

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