EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5873
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Peak-Shift Control Codes for the L_1 Metric

Nawaf A. Alqwaifly

Department of Electrical Engineering, College of Engineering, Qassim University, Buraydah, Saudi Arabia

Abstract. We propose a new efficient design of q-ary block codes capable of controlling single peak shifts of one direction (left or right shift) of size l. The proposed design is based on elementary symmetric functions. We show that the problem of controlling the $l_L(l_R)$-peak shift is equivalent to the efficient design of some L_1 metric asymmetric error control codes on the natural alphabet $\mathbb{N}$. From the relations with the L_1 distance error control codes and constant weight codes, new improved upper and lower bounds on the size of the optimal single $l_L(l_R)$-peak shift error correcting codes are given. Furthermore, some non-systematic code designs are also given. Decoding can be efficiently performed by algebraic means with the Extended Euclidean Algorithm.

2020 Mathematics Subject Classifications: 11T71, 14G50, 94A60

Key Words and Phrases: Recording codes, Peak-shift correction, L_1 distance, Asymmetric distance, Elementary symmetric functions, Constant weight codes, Design, Algorithm

1. Introduction

In high density magnetic recording systems, peak-shifts (or bit-shifts) and randomly generated errors are considered to be one of the major impairments responsible for most of the errors [1],[2]. Initially, Kuznetsov and Han Vinck [3] related the problem of peak-shift correction to the construction of block codes over the ring of integers modulo q. These codes are capable of correcting specific types of double errors caused by single peak-shifts. The research was exponentially propelled when Levenshtein and Han Vinck proposed a code that can correct a single peak-shift of size l [4]. They proposed perfect (d, k)-codes capable of correcting peak-shift of size l using weight sequences in Abelian groups. Klove [5] proposed a special case of the code in [4] and can correct a single peak-shift or an insertion of a zero and can correct a single transition or transposition. He constructed a large class of perfect constant-weight codes which can correct a single insertion, deletion, or peak-shift. Both [4], [5] restricted their attention to the prefect (d, k) sequences, since many popular recording codes for peak detection channels fall into the class of (d, k) sequences. In [6], a new code design has been proposed that can correct insertions/deletions of the symbol 0 using the elementary symmetric functions. A proposed approach using coset graphs and

DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5873

Email address: nkoiefly@qu.edu.sa (N. A. Alqwaifly)

matrix operations is discussed for secure communication through reliable S-box design in [7] and an efficient S-box design scheme is described for image encryption based on the combination of a coset graph and a matrix transformer in [8].

In this paper, we are interested in the efficient design of q-ary block codes that are capable of correcting a single left (right) peak-shifts of size l or less based on the proposed codes in [6], where $l \in \mathbb{N}$.

Peak-shift errors represent a distinct form of misalignment of symbol positions compared to erased or inserted errors, where the length of the strings changes. This is common in applications like magnetic recording, DNA storage and sequencing, quantum communication channels, etc. Although numerous coding techniques for peak-shift error control exist in the literature, the proposed coding framework is different from those techniques by implementing the constant weight codes and elementary symmetric functions. It is a new approach in this area that allows one to handle $l_L(l_R)$-peak shift errors by utilizing asymmetric distance models. It opens the door for more researchers to look at the misalignment of symbol positions from different perspectives and might lower the redundancy requirements to detect errors and eventually correct them.

In this work, we use l as a representation of the peak-shift error. So, l can be a l_L peak shift or a l_R peak shift error, where l_L and l_R represent left and right peak-shift errors, respectively.

The main contributions of this work are as follows.

(i) We propose a new coding framework to handle $l_L(l_R)$-peak shift errors using constant weight codes and elementary symmetric functions.

(ii) We show that the problem of controlling the $l_L(l_R)$-peak shift is equivalent to the efficient design of some L_1 metric asymmetric error control codes on the natural alphabet $\mathbb{N}$.

(iii) Some efficient non-systematic $l_L(l_R)$-peak shift error of size l correcting codes are designed.

(iv) Some improved upper and lower bounds for the $l_L(l_R)$-peak shift codes are presented.

This paper is organized as follows. Section 2 briefly reviews the basic definitions, notations, and the characterization of l-peak-shift error correcting codes and the L_1 distance. Section 3 focuses on a non-systematic code construction and shows its l-peak shift correcting capability. It also gives the upper and lower bounds and the decoding algorithm. Finally, the paper is concluded in Section 4.

2. Peak-Shifts and the L_1 Metric

In this section, the connection between the peak-shifts control codes and the L_1 metric is shown, but first some basic definitions and background are presented.

Let $X \in \mathbb{Z}_2^n$ be a binary sequence of length n, where $n \geq 1$. A code $\mathcal{C}$ is a set of sequences (codewords), $X_i's$, where $i \in \{1, |\mathcal{C}|\}$. Furthermore, a code $\mathcal{C}$ is a single peak-shifts of size l controlling code if any codeword is uniquely determined from any word that can be obtained from it by a single left (right) peak-shifts in at most l digits, where $l \in \mathbb{N}$. For example, if

$$X = 1000010001 \in \mathbb{Z}_2^{10} \tag{1}$$

is a transmitted binary sequence of length $n = 10$, then

$$Y = 0011000010 \in \mathbb{Z}_2^{10} \tag{2}$$

is the received sequence of length $\hat{n} = 10$ obtained from X by shifting the first-one two positions to the right, shifting the second-one two positions to the left, and shifting the third-one one position to the left, totaling $l = 5$ shifts (Figure 1). The focus is only on a single shift occurs; meaning just only one of any 1's experiences a shift of size l in the previous example.

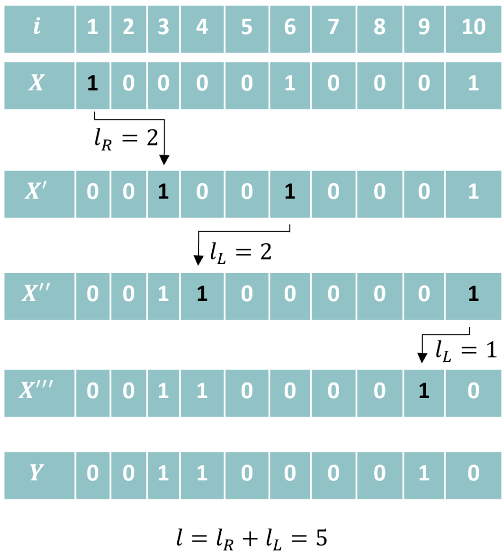

Figure 1: Illustration of Peak-Shift Errors

The following L_1 metric distances between q ary words $X, Y \in \mathbb{Z}_q^n$ are important to describe l peak shift error correction codes in Section 3. Let $x \dot{-} y \stackrel{\text{def}}{=} \max\{0, x - y\}$ for $x, y \in \mathbb{Z}_q$. Then

$$\begin{aligned} &\text{Symmetric } L_1: && d_{L_1}^{sy}(X,Y) \stackrel{\text{def}}{=} |Y \dot{-} X| + |X \dot{-} Y|, \\ &\text{Asymmetric } L_1: && d_{L_1}^{as}(X,Y) \stackrel{\text{def}}{=} \max\{|Y \dot{-} X|, |X \dot{-} Y|\}. \end{aligned} \tag{3}$$

For example, if $q = 5$, $n = 9$, $X = 014230120$, $Y = 432130001$ then $|X \dot{-} Y| = 6$, $|Y \dot{-} X| = 7$, and $d_{L_1}^{sy}(X,Y) = 6 + 7 = 13$, $d_{L_1}^{as}(X,Y) = \max\{6, 7\} = 7$. Note that if X and Y are the transmitted and received words, respectively, then $Y \dot{-} X$ and $X \dot{-} Y$ give the increasing (right shift) and decreasing (left shift) error vectors, respectively.

Constant weight codes play an important role in the code design and hence in the derivations of the upper and lower bounds. So, given $n, w \in \mathbb{N}$ and any numeric set $A \subseteq \mathbb{N}$ as alphabet, let

$$\mathcal{B}(A, n, w) \stackrel{\text{def}}{=} \{X \in A^n : \ w_{L_1}(X) = |X| = w\} \tag{4}$$

be the set of all word over A of length n and constant weight w. From [6], if $A = \mathbb{Z}_\infty = \mathbb{N}$, then

$$|\mathcal{B}(\mathbb{N}, n, w)| = \binom{n + w - 1}{n - 1} \tag{5}$$

Figure 2 illustrates the decomposition into constant weight codes.

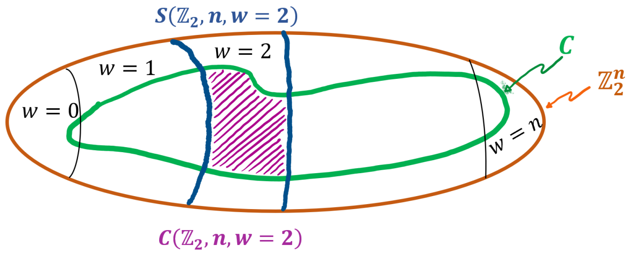

Figure 2: Decomposition into Constant Weight Codes

Levenshtein in [4, 9], introduced the following representation of X. If $X \in \mathbb{Z}_2^*$ then X can be uniquely written as

$$X = 0^{v_1} 1 0^{v_2} 1 0 \ldots 0 1 0^{v_w} 1 0^{v_{w+1}} \tag{6}$$

for all integers $h \in [1, w+1]$, $v_h \stackrel{\text{def}}{=} v_h(X) \in \mathbb{Z}_{n-w+1} \subseteq \mathbb{N}$ is the h-th run of 0's in the word X, where $n \in \mathbb{N}$ is the length of X and $w = w_H(X) \in [0, n]$ is the Hamming weight of X. Note that

$$v_{w+1} = n - w(X) - \sum_{i=1}^{w} v_i. \tag{7}$$

So,

$$V(X) \stackrel{\text{def}}{=} (v_1, v_2, \ldots, v_w, v_{w+1}). \tag{8}$$

Now, restricting the discussion to sequences of fixed length n and fixed Hamming weight w as the representation above, consider the following bijective function.

$$I : \mathbb{Z}_2^n \to \mathbb{Z}_n^w \subset \mathbb{N}^* \tag{9}$$

which associates any $V(X) \in \mathbb{Z}_{w+1}^*$ represented as in (8) with $I(X) \stackrel{\text{def}}{=} (i_0, i_1, \ldots, i_{w-1}) \in \mathbb{N}^*$. The functions $I(X)$ and $V(X)$ are related as follows.

$$\begin{aligned} X &= 0^{v_1} 1 0^{v_2} 1 0 \ldots 0 1 0^{v_w} 1 0^{v_{w+1}} \qquad \leftrightarrow \\ I(X) &\stackrel{\text{def}}{=} (i_1, i_2, \ldots, i_w) \\ &\stackrel{\text{def}}{=} (v_1, v_1 + v_2 + 1, \ldots, v_1 + v_2 + \cdots + v_w + w - 1). \end{aligned} \tag{10}$$

Therefore, the function $I(X) : \mathbb{Z}_2^n \to \mathbb{Z}_n^w \subset \mathbb{N}^*$, which maps a binary sequence to its support, plays a crucial role in explaining that the peak-shift EC problem can be reconsidered as the L_1 distance problem, which becomes easier to solve. Assume $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_2$. The function $I(X)$ can be simply defined as $I(X) = \{i : x_i = 1\}$, returning the set of indices where the binary sequence X has 1. For example, given $n = 22$, $w = 7$. Let $X = 01\,001\,01\,0001\,01\,1\,1\,0000000 \in \mathbb{Z}_2^{22}$, then $V(X) = (1, 2, 1, 3, 1, 0, 0, 7) \in \mathbb{N}^*$, and $I(X) = (1, 4, 6, 10, 12, 13, 14) \in \mathbb{Z}_{22}^7 \subset \mathbb{N}^*$.

Now, the peak-shift errors occur when the peaks (positions of ones) shift either left or right, as stated in the above example. Since the scope of this discussion is restricted to sequences of fixed length n and fixed Hamming weight w, the mapping I in (10) becomes a bijection function from the set of all binary words of any finite length $n \in \mathbb{N}$ and Hamming weight w (= number of 1' of the binary words) into words over $\mathbb{N}$ of length w. Therefore, it enables this transformation and preserves the distance as stated in Theorem 1. Thus, it establishes the equivalence between the spaces $\mathcal{B}(\mathbb{Z}_2, n, w)$ and $\mathcal{B}(\mathbb{Z}_n^w, n, w)$ and hence between the metrics $d_{bs}(X, Y)$ and $d_{L_1}^{sy}(I(X), (Y))$, where d_{bs} stands for the bit-shift distance. Therefore, the peak-shift error correction problem is reformulated with the L_1 distance problem.

Throughout this work, we define this mapping as "the index mapping", and it is worth mentioning that $I(X)$ is always a strictly increasing sequence. For example, for $n = 4$, the mapping I acts on $\mathbb{Z}_2^4$ is reported in Table 1; the mapping I acts on binary sequences X of length $n = 4$, producing output sequences in $\mathbb{Z}_n^w$. In this context, $m(X)$ indicates the length of any $I(X) \in \mathbb{Z}_n^w$, while $w_{L1}(I(X))$ denotes the L_1 weight (i.e., the sum of values of entries) of the codeword $I(X)$.

The following Theorem 1 connects the peak-shifts controlling codes and the L_1 metric. This connection is similar to some extent to the ideas in [6].

Theorem 1 (Isometry between $(\mathcal{B}(\mathbb{Z}_2, n, w), d_{bs})$ and $\mathcal{B}(\mathbb{Z}_n^w, w, n), d_{L_1}^{sy})$)**.** *For all $X, Y \in \mathbb{Z}_2^*$,*

$$d_{bs}(X, Y) = \begin{cases} d_{L_1}^{sy}(I(X), I(Y)) & \text{if } w(X) = w(Y), \\ \infty & \text{if } w(X) \neq w(Y). \end{cases} \tag{11}$$

Note that if we extend the domain of $d_{L_1}^{sy}$ to include the case when $m(A) \neq m(B)$, then

$$d_{bs}(X, Y) < \infty \iff w(X) = w(Y)$$

and for all $X, Y \in \mathbb{Z}_2^*$,

$$d_{bs}(X, Y) = d_{L_1}^{sy}(I(X), I(Y)).$$

This means that the mapping I in (10) is an isometry (a distance-preserving transformation) between the metric spaces $(\mathbb{Z}_2^*, d_{bs})$ and $(\mathbb{N}^*, d_{L_1}^{sy})$. The d_{bs} stands for the bit-shift distance.

For example, **Case 1:** $w_H(X) = w_H(Y)$. Assume $X = 0101100$ and $Y = 0011100$. Thus, $V(X) = 1102 \sim I(X) = 134$ and $V(Y) = 2002 \sim I(Y) = 234$. Now, a $\ell_R = 1$ peak shift of size 1 will transfer X to Y. Thus, $d_{\text{bs}}(X, Y) = 1$ and $d_{L_1}^{\text{as}}(I(X), I(Y)) = 1$.

Table 1: Action of the Mapping I on Binary Sequences of Length $n = 4$

$\mathbb{Z}_2^4$ space		$\mathbb{Z}_4^w$ space		
Weight $w(X)$	Sequence X	$I(X)$	$m(I(X))$	$w_{L1}(I(X))$
0	0000	–	0	0
	0001	3		3
	0010	2	1	2
	0100	1		1
	1000	0		0
	0011	23		5
	0101	13		4
	0110	12	2	3
	1001	03		3
	1010	02		2
	1100	01		1
	0111	123		6
	1011	023	3	5
	1101	013		4
	1110	012		3
4	1111	0123	4	6

Case 2: $w_H(X) \neq w_H(Y)$. Assume $X = 01011001$ and $Y = 0011100$. Thus, $V(X) = 11020 \sim I(X) = 1347$ and $V(Y) = 2002 \sim I(Y) = 234$. Now, since $w_H(X) \neq w_H(Y)$, X will never transfer to Y. Thus, $d_{\text{bs}}(X, Y) = \infty$ and $d_{L_1}^{\text{as}}(I(X), I(Y)) = \infty$.

The function I, and therefore I^{-1}, is a one-to-one mapping such that $I(\mathcal{B}(\mathbb{Z}_2, n, w)) = \mathcal{B}(\mathbb{Z}_n^w, w, n)$, and $\mathcal{B}(\mathbb{Z}_2, n, w) = I^{-1}(\mathcal{B}(\mathbb{Z}_n^w, w, n))$. This one-to-one correspondence between binary words and vectors of non-negative increasing integers plays an important role in the following discussion. Since a single l-peak-shift in a word X does not change the number, but only the values of components in the vector $I(X)$, the code design in Section 3 is based on the I codomain.

3. Non-systematic Code Design

In this section, the code construction, lower and upper bounds, and the decoding algorithm for a single l peak-shift code are presented.

3.1. Code Design

The proposed code design is based on the L_1 metric error control σ-codes over $\mathbb{Z}_q$ in [10],[6], however, first we need the following definition of the sigma polynomials of a word before introducing the σ-codes.

Definition 1 ([11]: The σ-polynomial of a word)**.** *Let $q \in \mathbb{N} \cup \{\infty\}$, $\mathbb{F}$ be* **any field** *and $\partial S \subseteq \mathbb{F}$ be a set of $n \in \mathbb{N}$ distinct elements in $\mathbb{F}$. The σ-polynomial associated with a word*

$X \in \mathbb{Z}_q^n$ *is*

$$\begin{aligned} \sigma_X(z) &\stackrel{\text{def}}{=} z^{x_0} \prod_{s \in \partial S - \{0\}} (1 - sz)^{x_s} \\ &= z^{x_0} \left(1 + \sigma_1(X)z + \sigma_2(X)z^2 + \ldots\right) \in \mathbb{F}[z]. \end{aligned} \tag{12}$$

For example, if $n = 7$, $\partial S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\} \subseteq \mathbb{F} - \{0\}$ and $X = 4121000 = \{s_0, s_0, s_0, s_0, s_1, s_2, s_2, s_3\}$, then

$$\begin{aligned} \sigma_X(z) &= (1 - s_0 z)^4 (1 - s_1 z)(1 - s_2 z)^2 (1 - s_3 z) \\ &= 1 - (4s_0 + s_1 + 2s_2 + s_3)z + (6s_0^2 + 4s_0 s_1 + 8s_0 s_2 + 4s_0 s_3 + 2s_1 s_2 \\ &\quad + s_1 s_3 + s_2^2 + 2s_2 s_3)z^2 + \ldots + (s_0^4 s_1 s_2^2 s_3)z^8. \end{aligned}$$

To understand how the σ-polynomial associated with a word $X \in \mathbb{Z}_q^n$ is calculated, consider the following example. Assume that $X = 00111$ and the field is $GF(4)$. Then $I(X) = 234$. The elementary symmetric functions, $\sigma_0\sigma_1\sigma_2\sigma_3$, associated with 234 is $(1 - z)^2 + (1 - 2z)^3 + (1 - 3z)^4 = 1032$.

Note that $\sigma_X(z)$ is a polynomial of degree $\deg(\sigma_X) = w_{L_1}(X) = |X|$ that has $w_H(X) = |\partial X|$ distinct roots in $\mathbb{F}$, each with multiplicity x_s, for $s \in \partial S \subseteq \mathbb{F}$. In particular, X coincides with the multiset of all inverses of the roots of $\sigma_X(z)$, where we let $1/0 \stackrel{\text{def}}{=} 0$. Hence, its coefficient sequence is given by the elementary symmetric functions, 1, $\sigma_1(X)$, $\sigma_2(X), \ldots \in \mathbb{F}$, of the elements in the multiset $X - \{0\}$ ordered in increasing order of their degrees and eventually shifted to the right by $x_0 \in \mathbb{Z}_q \subseteq \mathbb{N}$ if $0 \in \partial S \subseteq \mathbb{F}$. For example, the elementary symmetric functions for $\mathbb{Z}_2^5$ and $\mathbb{Z}_2^7$ are shown in Table 2 and Table 6, respectively. In these two examples, the choice of the field is the smallest field, $\mathbb{F}$, whose cardinality is $|\mathbb{F}| > w$. Now, the general definition of σ-code as follows:

Definition 2 ([11], [6]: The σ-Code)**.** *For all polynomials $g(z), \sigma(z) \in \mathbb{F}[z]$, the q-ary σ-code of length n associated with g and σ is*

$$\mathcal{C}_{g,\sigma}(\mathbb{Z}_q, n) \stackrel{\text{def}}{=} \left\{ X \in \mathbb{Z}_q^n \;\middle|\; \begin{array}{l} \sigma_X(z) \;=\; c_X \sigma(z) \bmod \\ g(z), \text{ with } c_X \in \mathbb{F} - \{0\} \end{array} \right\}. \tag{13}$$

For clarity, we choose $g(z) = z^{l+1}$.

Table 2: The mapping I acting on $\mathbb{Z}_2^5$ and the elementary symmetric functions, $(\sigma_0\sigma_1\ldots\sigma_{|\mathbb{F}|-1})$, associated with $I(X)$.

$w(X)$	X	$I(X)$	$\mathbb{F}_w$	$\sigma_0\sigma_1\ldots\sigma_{\lvert\mathbb{F}\rvert-1}$
0	00000	–	$GF(2)$	10
1	00001	4	$GF(2)$	10
	00010	3		11
	00100	2		10
	01000	1		11
	10000	0		10
2	00011	34	$GF(3)$	110
	00101	24		122
	00110	23		111
	01001	14		102
	01010	13		120
	01100	12		112
	11000	01		110
	10100	02		121
	10010	03		100
	10001	04		110
3	00111	234	$GF(2^2)$	1032
	01011	134		1102
	01101	124		1322
	01110	123		1202
	10110	023		1331
	11010	013		1113
	11100	012		1012
	10011	034		1220
	10101	024		1020
	11001	014		1220
4	01111	1234	$GF(5)$	10020
	10111	0234		11133
	11011	0134		13222
	11101	0124		11023
	11110	0123		10021
5	11111	01234	$GF(7)$	1210030

Now, to define a l-peak-shift error correcting code, $\mathcal{C} \subseteq \mathbb{Z}_2^n$, the σ-codes in (13) are used in the function I codomain; where I is given in (10). Thus, $X \in \mathcal{C}$ if and only if $\sigma_{I(X)0^*}(z) = \sigma(z) \bmod z^{l+1}$, where $\sigma(z)$ is a monic polynomial of degree l. Note that under the mapping $X \to \sigma_{I(X)0^*}(z) \bmod z^{l+1}$, the set of constant weight w vectors of length n over $\mathbb{Z}_2$ is partitioned into $|\mathbb{F}|^l$ classes, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\mathbb{F}^l}$. Now, X and Y are in $\mathcal{C}_i$ if, and only if, $\sigma_{I(X)0^*}(z) = \sigma_{I(Y)0^*}(z) \bmod z^{l+1}$ and from the σ-code theory, each of the $I(\mathcal{C}_i)$'s is an asymmetric L_1 distance $(l+1)$ code. Thus, by pigeon-hole principle, one of the classes, for example, $\tilde{\mathcal{C}}(\mathbb{F}; n, w)$ should have at least $\binom{n}{w}/|\mathbb{F}|^l$ codewords. The l-peak-shift error correction code, $\mathcal{C}$, can be simply defined as $\mathcal{C}_w \stackrel{\text{def}}{=} \tilde{\mathcal{C}}(\mathbb{F}; n, w) \subseteq \mathcal{B}(\mathbb{Z}_2, n, w)$ for all $w \in [0, n]$. In order to maximize $|\mathcal{C}|$, the algebraic structure $\mathbb{F}$ is chosen to be the smallest possible field if $l > 1$ or the smallest group if $l = 1$. Figure 3 provides a flowchart that illustrates the proposed coding framework for peak-shift control.

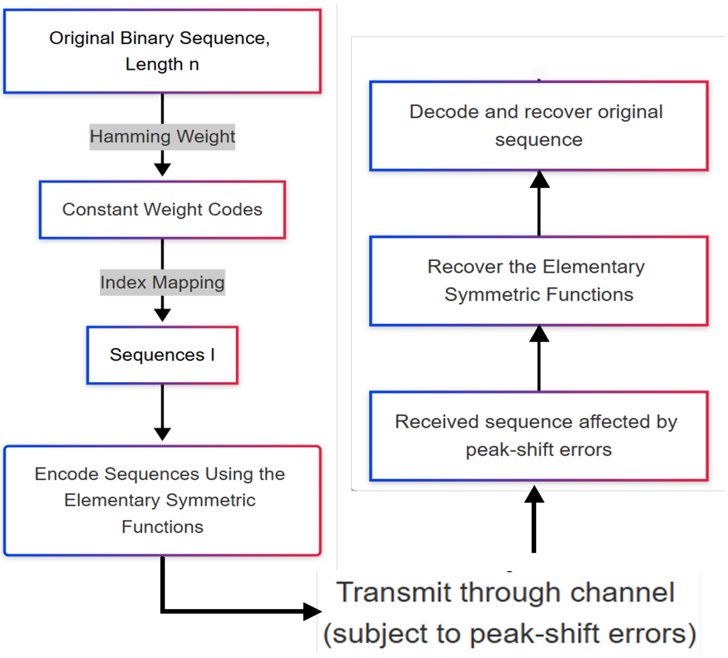

Figure 3: Flowchart Illustrating the Proposed Coding Framework for Peak-Shift Control

3.2. Lower and Upper Bounds

Since the proposed code is a constant weight q-ary code, we can use the lower and upper L_1 bounds for the q-ary codes derived in [6] as stated in the following Theorem 2.

Theorem 2 (lower bound on l-Peak-Shift Error Correcting Codes). *For all $n, l \in \mathbb{N}$ there exists a l-peak-shift error correcting binary code $\mathcal{C} \subseteq \mathbb{Z}_2^n$ of length n whose cardinality is*

$$|\mathcal{C}| \geq 1 + \left\lceil \frac{n}{l+1} \right\rceil + \sum_{w=2}^{n} \left\lceil \binom{n}{w} \Big/ |\mathbb{F}_w|^l \right\rceil \tag{14}$$

where $\mathbb{F}_w$ *is the smallest field,* $\mathbb{F}$*, whose cardinality is* $|\mathbb{F}| > w$*, when* $l > 1$*; and* $\mathbb{F}_w = (\mathbb{Z}_{w+1}, + \bmod (w+1))$ *when* $l = 1$*. Note that if* $l = 1$*, then* $|\mathbb{F}_w| = w + 1$*.*

We can have some improvements in terms of the numbers of codewords if we relax the restriction on the choice of the field, but for clarity, we did not.

The upper L_1 bound for q-ary codes as follows; given $n, l \in \mathbb{N}$, then the upper L_1 bound,

$$D(n,l) \leq \sum_{w=0}^{n} \left\lfloor \binom{n+2l}{w+l} \Big/ \binom{n+2l}{l} \right\rfloor . \tag{15}$$

Consider the following two examples; Table 3 shows a code of length $n = 5$ which is capable of correcting a single left (right) peak shift of size $l = 1$ and Table 4 presents a code of length $n = 7$ which is capable of correcting a single left (right) peak shift of size $l = 2$. Note that, in Table 3, the minimum asymmetric L_1 distance between any two codewords $X, Y \in \mathcal{C}_w$ is 2. Thus, this code is capable of correcting a single peak shift of size $l = 1$. In Table 4, the minimum asymmetric L_1 distance between any two codewords $X, Y \in \mathcal{C}_w$ is 3 and therefore this code is capable of correcting a single peak shift of size $l = 2$. In Table 5, the numbers of codewords for the proposed code for the codelength, $n = 1, \ldots, 20$ and the peak shift of sizes $l = l_L(l_R) = 1, \ldots, 8$ are shown. The following is an example of how the lower and upper bounds are calculated in Table 3. Let $n = 5$ and $\ell = 1$. Thus,

$$|C| \geq 1 + \left\lceil \frac{n}{\ell+1} \right\rceil + \sum_{w=2}^{n} \left\lceil \frac{\binom{n}{w}}{|F_w|^{\ell}} \right\rceil$$

$$|C| \geq 1 + \left\lceil \frac{5}{2} \right\rceil + \left\lceil \frac{10}{3} \right\rceil + \left\lceil \frac{10}{4} \right\rceil + 1 + \left\lceil \frac{1}{7} \right\rceil = 13.$$

$$D(n,\ell) \leq \sum_{w=0}^{n} \left\lfloor \frac{\binom{n+2\ell}{w+\ell}}{\binom{n+2\ell}{\ell}} \right\rfloor \left\lfloor \frac{\binom{7}{1}}{\binom{7}{1}} \right\rfloor + \left\lfloor \frac{\binom{7}{2}}{\binom{7}{1}} \right\rfloor + \left\lfloor \frac{\binom{7}{3}}{\binom{7}{1}} \right\rfloor + \left\lfloor \frac{\binom{7}{4}}{\binom{7}{1}} \right\rfloor + \left\lfloor \frac{\binom{7}{5}}{\binom{7}{1}} \right\rfloor + \left\lfloor \frac{\binom{7}{6}}{\binom{7}{1}} \right\rfloor$$

$$= \left\lfloor \frac{7}{7} \right\rfloor + \left\lfloor \frac{21}{7} \right\rfloor + \left\lfloor \frac{35}{7} \right\rfloor + \left\lfloor \frac{35}{7} \right\rfloor + \left\lfloor \frac{21}{7} \right\rfloor + \left\lfloor \frac{7}{7} \right\rfloor = 1 + 3 + 5 + 5 + 3 + 1 = 18$$

as stated in the table.

Table 5 shows the cardinality of the proposed code versus the peak-shift errors for different code lengths, n, and various numbers of errors, l_L (l_R). As expected, for fixed n, the number of codewords decreases as l_L (l_R)-peak shift errors increase. It is worth mentioning that for fixed n, the numbers of codewords tends to increase with weight increases until it reaches the middle weight and then it starts decreasing. For example, for $n = 15$ and $l_L(l_R) = 2$, the highest numbers of codewords occur in weight 6 which is around 85 codewords. In particular, cardinality increases with respect to n, which is consistent with standard coding theory principles and reflects the increasing capability and capacity of the proposed σ-code as the code length grows.

Table 3: Non-systematic code parameters with $n = 5$ and $l = 1$. Here, the lower bound in (14) gives 13, however, the actual code defining the lower bound in (14) has $|\mathcal{C}| = 15$. Also, the upper bound value obtained with (15) is 18.

$w(X)$	X	$I(X)$	$m(I(X))$	$\mathbb{F}_w$	$\|\mathcal{C}\| = 15$
0	00000	–	0	$GF(2)$	1
1	00001	4	1	$GF(2)$	3
	00100	2			
	10000	0			
2	00011	34	2	$GF(3)$	5
	00110	23			
	11000	01			
	01100	12			
	10001	04			
3	00111	234	3	$GF(2^2)$	3
	11100	012			
	10101	024			
4	01111	1234	4	$GF(5)$	2
	11110	0123			
5	11111	01234	5	$GF(7)$	1

3.3. l-Peak-Shift Decoding Algorithm

Since the problem is transferred from peak-shift EC problem into the constant weight L_1 problem, we can implement the General l-SyEC/$(l+1)$-SyED/AUED decoding algorithm for constant weight codes as the one in [6]. The algorithm 1 is a general efficient error control algorithm for any q-ary constant weight w code, $\mathcal{A}$, of length n with a minimum L_1 distance $d_{L_1}^{sy}(\mathcal{A}) \geq 2(l+1)$.

Algorithm 1. General l-SyEC/$(l+1)$-SyED/AUED decoding algorithm for Constant Weight codes
Input:

1) The code $\mathcal{A} \stackrel{\text{def}}{=} \hat{\mathcal{A}}\, x_{n-1} \subseteq \mathcal{S}(\mathbb{Z}_m, n, w)$, where

$$x_{n-1} \stackrel{\text{def}}{=} w - w_{L_1}(\hat{X}) \in \mathbb{Z}_m, \quad \hat{X} \in \hat{\mathcal{A}},$$

is the parity digit.

2) Any efficient (l_L, l_R)-EC decoding algorithms, $\mathcal{D}ec(\hat{\mathcal{A}}, l_L, l_R)$, for $\hat{\mathcal{A}}$, for all $l \in \mathbb{N}$ such that $l < d_{L_1}^{as}(\hat{\mathcal{A}})$.

3) The (received) word $Y = \hat{Y}\, y_n \in \mathbb{Z}_m^n$ with $\hat{Y} \in \mathbb{Z}_m^{n-1}$ and $y_n \in \mathbb{Z}_m$.

Table 4: Non-systematic code parameters with $n = 7$ and $l = 2$. Here, the lower bound in (14) gives 15, however, the actual code defining the lower bound in (14) has $|\mathcal{C}| = 18$. Also, the upper bound value obtained with (15) is 36.

$w(X)$	X	$I(X)$	$m(I(X))$	$\mathbb{F}_w$	$\|\mathcal{C}\| = 18$
0	0000000	0	–	$GF(2)$	1
1	0000001 1000000	6 0	1	$GF(2)$	2
2	0001100 1000100 1100000	34 04 01	2	$GF(3)$	3
3	0011100 1000011 1100001	234 056 016	3	$GF(2^2)$	3
4	0001111 0011110 0111100 1010101	3456 2345 1234 0246	4	$GF(5)$	4
5	1101011 1111100	01356 01234	5	$GF(7)$	2
6	1111110 0111111	012345 123456	6	$GF(7)$	2
7	1111111	0123456	7	$GF(2^3)$	1

Output:

1) A word $X' = \hat{X}' x'_n \in \mathbb{Z}_m^n$, where $\hat{X}' \in \mathbb{Z}_m^{n-1}$ and $x'_n \in \mathbb{Z}_m$ (**the estimation of the original codeword**).

2) **An indication signal** $cor \in \{0, 1\}$ such that if $cor = 1$ then errors are corrected ($X' = X$).

Execute the following steps.
S1: **Assessing the error correction capability**.
Compute

$$\Delta(X,Y) \stackrel{\text{def}}{=} |Y \dot{-} X| - |X \dot{-} Y| = |Y| - w. \tag{16}$$

S2: If $|\Delta(X,Y)| \geq l+1$ then set $cor = 0$, set X' to be any word, output cor, output X' and **exit**.
S3: Otherwise, if $|\Delta(X,Y)| \leq l$ then execute the following steps.
S3.1: Compute

$$l_L \stackrel{\text{def}}{=} \left\lfloor \frac{l - \Delta(X,Y)}{2} \right\rfloor \text{ and } l_R \stackrel{\text{def}}{=} \left\lfloor \frac{l + \Delta(X,Y)}{2} \right\rfloor. \tag{17}$$

Note that $0 \leq l_L, l_R \leq l$ (because $|\Delta(X,Y)| \leq l$) and

$$l_L + l_R \leq \frac{l_L - \Delta(X,Y)}{2} + \frac{l_R + \Delta(X,Y)}{2} = l. \tag{18}$$

Table 5: The number of codewords for the proposed method for codelength, $n = 1, \dots, 20$ and peak shifts, $l_L(l_R) = 1, \dots, 8$ where l_L and l_R represent the left and right shifts, respectively.

$n\backslash l$	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	3	3	3	3	3	3	3	3
3	5	4	4	4	4	4	4	4
4	7	6	5	5	5	5	5	5
5	13	8	7	6	6	6	6	6
6	19	10	8	8	7	7	7	7
7	33	15	9	9	9	8	8	8
8	57	20	11	10	10	10	9	9
9	103	28	15	11	11	11	11	10
10	181	43	16	12	12	12	12	12
11	334	69	22	14	13	13	13	13
12	611	108	28	15	14	14	14	14
13	1136	180	40	18	16	15	15	15
14	2119	308	58	21	17	16	16	16
15	3972	530	86	26	18	18	17	17
16	7470	926	135	32	19	19	18	18
17	14096	1636	219	43	20	20	20	19
18	26657	2907	326	59	22	21	21	20
19	50542	5208	591	87	27	22	22	22
20	96039	9369	995	129	34	23	23	23

S3.2: With the word $\hat{Y} \in \mathbb{Z}_m^{n-1}$ as input, execute the algorithm $\mathcal{D}ec(\hat{\mathcal{A}}, l_L, l_R)$ for $\hat{\mathcal{A}}$. Let $\hat{X}' \in \mathbb{Z}_m^{n-1}$ be its output word.
S3.3: Set $X' = \hat{X}'\, x'_{n-1} \in \mathcal{A}$ if $\hat{X}' \in \hat{\mathcal{A}}$, and $X' =$ any word if $\hat{X}' \notin \hat{\mathcal{A}}$; where

$$x'_{n-1} = w - w_{L_1}(\hat{X}') \tag{19}$$

is the parity digit of $\hat{X}'$.
S3.4: Set

$$cor = \begin{cases} 1 & \text{if } X' \in \mathcal{A} \text{ and } d^{as}_{L_1}(X', Y) \le l, \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

S3.5: Output X', output cor and **exit**.

4. Conclusions

Even though the proposed code solves only the one-direction shift, it provides a new scheme in designing a peak-shift code using elementary symmetric functions (σ-codes). Based on the results developed in this paper and [12],[13], [6, 10, 11, 14–19], whenever it is possible to define an isometry from the metric space which characterizes a given coding problem to the L_1 metric (as the mapping I in (10)), any information on codes for the L_1 metric is reflected in the analogous information for that coding problem.

Table 6: The mapping I acting on $\mathbb{Z}_2^7$ and the elementary symmetric functions, $(\sigma_0\sigma_1 \dots \sigma_{|\mathbb{F}|-1})$, associated with $I(X)$.

$w(X)$	X	$I(X)$	$\mathbb{F}_w$	$\sigma_0\sigma_1\dots\sigma_{\mid\mathbb{F}\mid-1}$	$w(X)$	X	$I(X)$	$\mathbb{F}_w$	$\sigma_0\sigma_1\dots\sigma_{\mid\mathbb{F}\mid-1}$
0	0000000	–	$GF(2)$	10	7	1111111	0123456	$GF(2^3)$	10647400
	0000001	6		10		1111110	012345		1000031
	0000010	5		11		1111101	012346		1100034
	0000100	4		10		1111011	012356		1320033
1	0001000	3	$GF(2)$	11	6	1110111	012456	$GF(7)$	1646035
	0010000	2		10		1101111	013456		1301333
	0100000	1		11		1011111	023456		1111144
	1000000	0		10		0111111	123456		1000030
	0000011	56		111		1111100	01234		1210030
	0000101	46		120		1110101	01235		1452036
	0000110	45		112		1111001	01236		1665435
	0001001	36		100		1110110	01245		1033631
	0001010	35		121		1110101	01246		1232510
	0001100	34		110		1110011	01256		1524423
	0010001	26		111		1101110	01345		1431466
	0010010	25		101		1101101	01346		1640604
	0010100	24		122		1101011	01356		1215644
	0011000	23		111		1100111	01456		1622506
2	0100001	16	$GF(3)$	120	5	1011110	02345	$GF(7)$	1222251
	0100010	15		112		1011101	02346		1466424
	0100100	14		102		1011011	02356		1043363
	0101000	13		120		1010111	02456		1445146
	0110000	12		112		1001111	03456		1234525
	1000001	06		100		0111110	12345		1100033
	1000010	05		121		0111101	12346		1320032
	1000100	04		110		0111011	12356		1646034
	1001000	03		100		0110111	12456		1301332
	1010000	02		121		0101111	13456		1111143
	1100000	01		110		0011111	23456		1000036
	0000111	456		1012		0001111	3456		10023
	0001011	356		1120		0010111	2456		11131
	0001101	346		1300		0011011	2356		13220
	0001110	345		1222		0011101	2346		11021
	0010011	256		1200		0011110	2345		10024
	0100101	246		1000		0100111	1456		12312
	0100110	245		1313		0101011	1356		14133
	0011001	236		1200		0101101	1346		12240
	0011010	235		1131		0101110	1345		11132
	0011100	234		1032		0110011	1256		11341
	0100011	156		1333		0110101	1246		14043
	0100101	146		1111		0110110	1245		13221
	0100110	145		1010		0111001	1236		12124
	0101001	136		1333		0111010	1235		11022
	0101010	135		1212		0111100	1234		10020
	0101100	134		1102		1000111	0456		13124
3	0110001	126	$GF(2^2)$	1111	4	1001011	0356	$GF(5)$	10142
	0110010	125		1010		1001101	0346		13044
	0110100	124		1322		1001110	0345		12313
	0111000	123		1202		1010011	0256		12040
	1000011	056		1032		1010101	0246		10042
	1000101	046		1230		1010110	0245		14134
	1000110	045		1122		1011001	0236		13410
	1001001	036		1032		1011010	0235		12241
	1001010	035		1302		1011100	0234		11133
	1001100	034		1220		1100011	0156		14300
	1010001	026		1230		1100101	0146		12421
	1010010	025		1122		1100110	0145		11342
	1010100	024		1020		1101001	0136		10443
	1011000	023		1331		1101010	0135		14044
	1100001	016		1032		1101100	0134		13222
	1100010	015		1302		1110001	0126		13332
	1100100	014		1220		1110010	0125		12120
	1101000	013		1113		1110100	0124		11023
	1110000	012		1012		1111000	0123		10021

In addition, lower bounds, upper bounds, code designs, and decoding algorithms can be given for l-Sy0EC codes which satisfy the $RLL(d,k)$ constraint [4, 20] using the results in this work, [6] and L_1 error control codes over $\mathbb{Z}_q$, with $q \in \mathbb{N} \cup \{\infty\}$. It is because the set of all $RLL(d,k)$ binary words of length n and weight w with the $d_{0\text{-}D/I}$ metric can be put in bijection with $\left(\mathbb{Z}^{w+1}_{k-d+1}, d^{sy}_{L_1}\right)$ through the following isometry

$$0^{v_1}10^{v_2}1\ldots0^{v_w}10^{v_{w+1}} \quad \leftrightarrow \quad (v_1 - d, v_2 - d, \ldots, v_w - d).$$

For future direction, the results of this work can be extended to include multi-error correction. In addition, this method can be applied to the (d,k) sequences and practical storage systems.

Acknowledgements

The researcher would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

References

[1] T.D. Howell. Analysis of correctable errors in the ibm 3380 disk file. *IBM Journal of Research and Development*, 28(2):206–211, 1984.

[2] P. Siegel. Recording codes for digital magnetic storage. *IEEE Transactions on Magnetics*, 21(5):1344–1349, 1985.

[3] A.V. Kuznetsov and A.J.H. Vinck. A coding scheme for single peak-shift correction in (d, k)-constrained channels. *IEEE Transactions on Information Theory*, 39(4):1444–1450, 1993.

[4] V.I. Levenshtein and A.J.H. Vinck. Perfect (d,k)-codes capable of correcting single peak-shifts. *IEEE Transactions on Information Theory*, 39(2):656–662, 1993.

[5] T. Klove. Codes correcting a single insertion/deletion of a zero or a single peak-shift. *IEEE Transactions on Information Theory*, 41(1):279–283, 1995.

[6] L.G. Tallini, N. Alqwaifly, and B. Bose. Deletions and insertions of the symbol ″0″ and asymmetric/unidirectional error control codes for the L metric. *IEEE Transactions on Information Theory*, 69(1):86–106, 2023.

[7] A. Razaq, G. Alhamzi, S. Abbas, M. Ahmad, and A. Razzaque. Secure communication through reliable S-box design: A proposed approach using coset graphs and matrix operations. *Heliyon*, 9(5):e15902, 2023.

[8] A. Razzaque, A. Razaq, S.M. Farooq, I. Masmali, and M.I. Faraz. An efficient s-box design scheme for image encryption based on the combination of a coset graph and a matrix transformer. *Electronic Research Archive*, 31(5):2708–2732, 2023.

[9] V. Levenshtein. Binary codes with correction for deletions and insertions of the symbol 1. *Problems of Information Transmission*, 1(1):8–17, 1965.

[10] L.G. Tallini and B. Bose. On l_1 metric asymmetric/unidirectional error control codes. In *2013 International Symposium on Information Theory, Istanbul, Turke.*, pages 694–698. 2013.

[11] L.G. Tallini and B. Bose. On l_1-distance error control codes. In *2011 IEEE International Symposium on Information Theory Proceedings, St. Petersburg, Russia.*, pages 1026–1030. 2011.

[12] B. Bose, N. Elarief, and L.G. Tallini. On codes achieving zero error capacities in limited magnitude error channels. *IEEE Trans. on Inform. Theory*, 64:257–273, 2018.

[13] B. Bose and T.R.N. Rao. Theory of unidirectional error correcting/detecting codes. *IEEE Trans. on Comput.*, 31:521–530, 1982.

[14] J.H. Weber and D.E. Boekee C. de Vroedt. Necessary and sufficient conditions on block codes correcting/detecting errors of various types. *IEEE Transactions on Computers*, 41(9):1189–1193, 1992.

[15] L.G. Tallini and B. Bose. On a new class of error control codes and symmetric functions. In *2008 IEEE International Symposium on Information Theory, Toronto, ON, Canada.*, pages 980–984. 2008.

[16] L.G. Tallini and B. Bose. On decoding some error control codes using the elementary symmetric functions. In N. Melone F. Mazzocca and D. Olanda, editors, *Trends in Incidence and Galois Geometries: a Tribute to Giuseppe Tallini - Quaderni di Matematica, Vol 19.*, pages 265–297. Caserta, Dipartimento di Matematica, Seconda Università di Napoli, 2010.

[17] L.G. Tallini, N. Elarief, and B. Bose. On efficient repetition error correcting codes. In *2010 IEEE International Symposium on Information Theory, Austin, TX, USA.*, pages 1012–1016. 2010.

[18] L.G. Tallini and B. Bose. On symmetric l_1 distance error control codes and elementary symmetric functions. In *2012 IEEE International Symposium on Information Theory Proceedings, Cambridge, MA, USA.*, pages 741–745. 2012.

[19] L.G. Tallini and B. Bose. On some new $\iint_m$ linear codes based on elementary symmetric functions. In *2018 IEEE International Symposium on Information Theory (ISIT), Vail, CO, USA.*, pages 1665–1669,. 2018.

[20] F. Palunčić, K.A.S. Abdel-Ghaffar, H. C. Ferreira, and W. A. Clarke. A multiple insertion/deletion correcting code for run-length limited sequences. *IEEE Transactions on Information Theory*, 58(3):1809–1824, 2012.