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Some Characterizations of Quasi-Curves in Galilean 3-Space

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Abstract. This study investigates the theoretical basis of the quasi-frame in three-dimensional Galilean geometry. We derive mathematical expressions for the position vectors of curves defined in relation to this quasi-frame and establish the quasi equations that govern their behavior. Our findings demonstrate the absence of normal curves in Galilean 3-space, challenging existing theories in the field and providing new insights into the geometric structure of the Galilean 3-space. We explore the geometric properties of quasi-rectifying and quasi-osculating curves, establishing the necessary and sufficient conditions for their classification. A curve is identified as quasi-rectifying if its position vector can be represented as a linear combination of its tangent and quasi-binormal vectors. In contrast, a curve is classified as quasi-osculating if it remains entirely within its quasi-osculating plane, determined by its tangent and quasi-normal vectors.
The quasi-frame serves as a generalization of the classical Frenet frame, particularly useful in scenarios where the curvature vanishes and the Frenet frame becomes undefined. By introducing the quasi curvatures, we provide a robust framework for analyzing curves in Galilean 3-space. We derive explicit expressions for the position vectors of curves with respect to the Quasi frame and solve for their components under specific conditions. Furthermore, we prove that normal curves cannot exist in Galilean space, a result that clarifies the limitations of certain geometric classifications in this context.

2020 Mathematics Subject Classifications: 51A05, 53A35

Key Words and Phrases: Quasi-frame, Galilean 3-space, quasi-normal curves, quasi-rectifying curves, quasi-osculating curves.

1. Introduction

Galilean geometry, as articulated by Cayley and Klein, encompasses transformations that are fundamental to both classical and modern physics. The group of Galilean transformations is pivotal within these theoretical frameworks [1]. Notably, the conventional Frenet frame becomes inapplicable at points where curvature approaches zero, specifically at locations where the normal and binormal vectors are undefined [2–4]. In response to

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this limitation, various researchers have devised alternative frames that effectively handle such scenarios in Euclidean space [5, 6], Minkowski space [7–9], and Galilean space [10–14]. These include the equiform frame [15, 16], the Bishop frame [17], the Darboux frame [17, 18], the modified frame [2, 3, 12, 13], and the quasi frame [5, 6, 10, 11].

In the context of plane curves, three primary classifications can be identified: osculating, normal, and rectifying curves. An osculating curve is characterized by the tangent and normal vectors residing within the same plane defined by its position vector at all instances. In contrast, a normal curve is defined by a position vector that consistently maintains a normal orientation. The plane in question is formed by the curve's normal and binormal vectors. Previous studies have explored the characteristics of normal, osculating, and rectifying curves across various geometric frameworks [19–24]. A recent study by Dede et al. [25] introduced an innovative approach by constructing an adapted frame that precisely follows a space curve, moving beyond reliance on the traditional Serret Frenet frame. This newly developed framework termed the quasi-frame (Q-frame), enhances precision and applicability, thereby serving as an expanded interpretation of the Frenet frame. The Q-frame is distinguished by a fixed vector and the angle between the quasi-normal vector and the principal normal of the Frenet frame. At points where the curvature is zero, this frame undergoes rotation by the specified angle, establishing the Q-normal as orthogonal to both the tangent vector and the fixed vector. The Q-binormal vector is defined as the unit vector orthogonal to both the tangent and Q-normal vectors. Numerous studies have examined the Q-frame within Euclidean and Minkowski spaces [26–30], while more recent investigations have focused on position vectors in Galilean three- and four-dimensional spaces using the Frenet frame [31–34].

The structure of this paper is organized as follows: Section 2 details the Q-frame and its relationship to the Frenet frame. Section 3 delves into the analysis of quasi-formulas within Galilean 3-space. Section 4 investigates position vectors in Galilean 3-space and determines coefficients under specific conditions, covering quasi-rectifying and quasi-osculating curves. Furthermore, we establish the non-existence of Q-normal curves in Galilean 3-space, outlining the necessary and sufficient criteria for classifying a curve as either quasi-rectifying or quasi-osculating.

2. Preliminaries

In this section, we present essential concepts and definitions that will be crucial for our subsequent analysis. The three-dimensional Galilean space, denoted as $\mathbb{G}_3$, is a real vector space structured according to the Cayley-Klein model, characterized by a projective metric with signature $(0, 0, +, +)$. The absolute structure of this three-dimensional Galilean space can be represented by an ordered triple $\{\Omega, L, J\}$, where Ω signifies the absolute plane within $\mathbb{G}_3$, L denotes the absolute line contained in Ω, and J represents a fixed elliptic involution of the points along L.

A vector $\mathbf{p} = (p_1, p_2, p_3)$ in $\mathbb{G}_3$ is classified as non-isotropic if $p_1 \neq 0$; otherwise, it is termed isotropic [31, 32]. Vectors of the form $\mathbf{p} = (1, p_2, p_3)$ are considered unit non-isotropic vectors. The Galilean metric g for vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{G}_3$ is defined as follows:

$$g(\mathbf{p},\mathbf{q}) = \begin{cases} p_1q_1, & \text{if } p_1 \neq 0 \text{ or } q_1 \neq 0,\\ p_2q_2 + p_3q_3, & \text{if } p_1 = 0 \text{ and } q_1 = 0.\end{cases}$$

Consequently, the Galilean norm of the vector $\mathbf{q}$ is given by:

$$\|\mathbf{q}\| = \begin{cases} |q_1|, & \text{if } q_1 \neq 0,\\ \sqrt{q_2^2 + q_3^2}, & \text{if } q_1 = 0.\end{cases}$$

The Galilean vector product of vectors $\mathbf{p}$ and $\mathbf{q}$ is defined as:

$$\mathbf{p}\times\mathbf{q} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3\\ p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\end{vmatrix}, & \text{if } p_1 \neq 0 \text{ or } q_1 \neq 0,\\ \begin{vmatrix} e_1 & e_2 & e_3\\ p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\end{vmatrix}, & \text{if } p_1 = 0 \text{ and } q_1 = 0,\end{cases}$$

where (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$ [11, 35].

In $\mathbb{G}_3$, a curve is defined as a mapping from an open interval J in $\mathbb{R}$ to $\mathbb{G}_3$, represented as:

$$\gamma : J \to \mathbb{G}_3, \quad t \mapsto \gamma(t) = (x(t), y(t), z(t)).$$

A curve is considered admissible if it has no inflection points (i.e., $\dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$) and does not possess isotropic tangents (i.e., $\dot{x}(t) \neq 0$ for all $t \in J$) [11, 31]. For an admissible differentiable curve $\gamma : J \subset \mathbb{R} \to \mathbb{G}_3$, parameterized by the Galilean invariant arc length s, the curve can be expressed as:

$$\gamma(s) = (s, y(s), z(s)).$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve $\gamma(s)$ are given by the formulas:

$$\kappa(s) = \|\gamma''(s)\| = \sqrt{y''^2(s) + z''^2(s)},$$

$$\tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa^2(s)}.$$

The moving Frenet frame $\{T(s), N(s), B(s)\}$ for the curve $\gamma(s)$ is defined as:

$$T(s) = \gamma'(s) = (1, y'(s), z'(s)),$$

$$N(s) = \frac{1}{\kappa(s)}\gamma''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)),$$

$$B(s) = T(s) \times N(s) = \frac{1}{\kappa(s)}(0, -z''(s), y''(s)).$$

Finally, the Frenet derivative formulas can be represented in matrix form as follows:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.1}$$

3. Quasi-Frame and Quasi Equations

This section delves into the concept of the quasi-frame (Q-frame) and its relationship with the classical Frenet frame, as well as an examination of the quasi equations within the context of Galilean three-dimensional space. Let $\alpha(s)$ represent a curve in $\mathbb{G}_3$. The Q-frame is established using three orthonormal vectors: $T(s)$, the unit tangent vector; $N_q(s)$, the unit Q-normal vector; and $B_q(s)$, the unit Q-binormal vector. The Q-frame $\{T(s), N_q(s), B_q(s)\}$ is defined as follows:

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N_q = \frac{T \times \mathbf{z}}{\|T \times \mathbf{z}\|}, \quad B_q = T \times N_q, \tag{3.1}$$

where $\mathbf{z}$ is a projection vector that can be chosen from $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$.

Denote the standard Frenet frame by $\{T, N, B\}$, and let $\theta(s)$ denote the angle between the vectors N and N_q. We can express N_q and B_q in terms of N and B as follows:

$$N_q = \cos(\theta)N + \sin(\theta)B, \tag{3.2}$$

$$B_q = -\sin(\theta)N + \cos(\theta)B. \tag{3.3}$$

From these relationships, it is also possible to rewrite N and B in terms of the Q-frame:

$$N = \cos(\theta)N_q - \sin(\theta)B_q, \tag{3.4}$$

$$B = \sin(\theta)N_q + \cos(\theta)B_q. \tag{3.5}$$

Utilizing the Frenet formulas, we have:

$$T' = \kappa N = \kappa(\cos(\theta)N_q - \sin(\theta)B_q).$$

By defining $K_1 = \kappa \cos(\theta)$ and $K_2 = \kappa \sin(\theta)$, we can express the derivative of T as:

$$T' = K_1 N_q - K_2 B_q. \tag{3.6}$$

Using the established relationships, the derivatives N_q' and B_q' can be expressed as:

$$N_q' = K_3 B_q, \quad B_q' = -K_3 N_q, \tag{3.7}$$

where $K_3 = \theta' + \tau$. Consequently, the quasi equations can be presented in a matrix form:

$$\begin{pmatrix} T' \\ N_q' \\ B_q' \end{pmatrix} = \begin{pmatrix} 0 & K_1 & -K_2 \\ 0 & 0 & K_3 \\ 0 & -K_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_q \\ B_q \end{pmatrix}. \tag{3.8}$$

The quasi curvatures K_1, K_2, and K_3 are associated with the Q-frame as follows:

Corollary 1. *If $\alpha(s)$ is a curve in $\mathbb{G}_3$, then the quasi curvatures K_1, K_2, and K_3 in terms of the Q-frame are given by*

$$K_1 = g(T', N_q), \quad K_2 = -g(T', B_q), \quad K_3 = g(N_q', B_q) = -g(B_q', N_q).$$

Corollary 2. *If $K_2 = 0$, the Q-frame coincides with the Frenet frame, demonstrating that the Q-frame generalizes the Frenet frame in $\mathbb{G}_3$.*

4. Generalized position vector

This section examines the spatial coordinates of position vectors within the Galilean three-dimensional manifold. We derive the components associated with the tangential, Q-normal, and Q-binormal vectors under specific conditions. Additionally, we investigate the properties of Q-rectifying and Q-osculating curves in $\mathbb{G}_3$ with respect to the Q-frame of reference. Furthermore, we demonstrate the absence of normal curves within the Galilean spatial framework.

Let $\alpha = \alpha(s)$ be a unit speed curve in $\mathbb{G}_3$. We can write the position vector concerning the Q-frame in $\mathbb{G}_3$ as

$$\alpha = \alpha(s) = m_1(s)T + m_2(s)N_q + m_3(s)B_q, \tag{4.1}$$

for some differentiable functions $m_1(s), m_2(s)$ and $m_3(s)$. By differentiating Equation (4.1) and using Equation (3.8), we have

$$m_1' = 1, \tag{4.2}$$

$$m_1 K_1 + m_2' - m_3 K_3 = 0, \tag{4.3}$$

$$-m_1 K_2 + m_2 K_3 + m_3' = 0. \tag{4.4}$$

From Equation (4.2), we get:

$$m_1 = C + s, \tag{4.5}$$

where C is constant.

By substitution from Equation (4.5) into Equation (4.3), we obtain:

$$m_3 = \frac{(C+s)K_1 + m_2'}{K_3}. \tag{4.6}$$

By differentiate Equation (4.6), we have

$$m_3' = \frac{K_3[sK_1' + K_1 + CK_1' + m''_2] - K_3'[sK_1 + CK_1 + m_2']}{(K_3)^2}. \tag{4.7}$$

No general solution has been found for this system. Because of this, we give the solution in some special cases.

Case 1: By substituting from Equation (4.7) into Equation (4.4), and substituting $K_1 = K_2 = 0, K_3 = constant = a \neq 0$, we obtain a non-homogeneous second linear differential equation

$$K_3 m_2'' + K_3^3 m_2 = 0.$$

So,

$$m_2 = C_1 \cos as + C_2 \sin as, \tag{4.8}$$

By taking the derivative of (4.8) and substituting $K_1 = 0, K_3 = a$, into Equation (4.6), we obtain:

$$m_3 = -C_1 \sin as + C_2 \cos as. \tag{4.9}$$

Therefore, we can write the position vector as

$$\alpha(s) = (C+s)T + (C_1 \cos as + C_2 \sin as)N_q + (-C_1 \cos as + C_2 \sin as)B_q,$$

where C, C_1, C_2, A and a are constants.

Case 2: Let $m_2 = C_3 \neq 0$, from Equation (4.3), we obtain:

$$m_3 = (C+s)\frac{K_1}{K_3}.$$

Therefore, in this case, we can write the position vector as

$$\alpha(s) = (C+s)T + C_3 N_q + (C+s)\frac{K_1}{K_3}B_q.$$

From Equation (4.4), we have a linear first-order differential equation

$$(\frac{K_1}{K_3})' + \frac{1}{C+s}(\frac{K_1}{K_3}) = K_2 - \frac{C_3}{C+s}K_3. \tag{4.10}$$

The integrating factor is given by $\mu = e^{\int \frac{1}{c+s}ds} = (c+s)$, therefore the solution of Equation(4.10) is given by

$$\frac{K_1}{K_3} = \frac{1}{C+s}\int [(C+s)K_2 - C_3 K_3]ds + C_4. \tag{4.11}$$

Case 3: Let $m_3 = C_5 \neq 0$ from Equation (4.4), we obtain:

$$m_2 = (C+s)\frac{K_2}{K_3}. \tag{4.12}$$

Therefore, in this case, we can write the position vector as

$$\alpha(s) = (C+s)T + (C+s)\frac{K_2}{K_3}N_q + C_5 B_q.$$

Substituting Equation (4.12) into Equation (4.3) yields a linear first-order differential equation:

$$(\frac{K_2}{K_3})' + \frac{1}{C+s}(\frac{K_2}{K_3}) = \frac{C_5}{C+s}K_3 + K_1.$$

So,

$$\frac{K_2}{K_3} = \frac{1}{C+s}\int (C_5K_3 + (C+s)K_1)ds + C_6. \tag{4.13}$$

Corollary 3. *In the case of the Frenet curve, we can put $K_2 = 0, K_3 = \tau$, and $K_1 = \kappa$. Then, Equations (4.2), (4.3), and (4.4) become*

$$m_1' = 1, \tag{4.14}$$

$$m_1\kappa + m_2' - m_3\tau = 0, \tag{4.15}$$

$$m_2\tau + m_3' = 0. \tag{4.16}$$

Therefore, $m_1 = C + s$.
*In **case 2**, if $m_2 = C_3$, Thus $m_3 = (C+s)\frac{\kappa}{\tau}$.*
Also, Equation (4.11), becomes

$$\frac{-C_3}{C+s}\int [\tau ds + C_4] = \frac{\kappa}{\tau}$$

.
*In **case 3**, From Equation (4.16), we obtain $\tau = 0$. From Equation (4.15), we obtain*

$$m_2 = -\int (c+s)\kappa ds.$$

Also, Equation (4.13) becomes

$$C_6 = -\int ((C+s)\kappa)ds.$$

These results are consistent with those in [34].

5. Q-Normal Curves

In this section, we demonstrate the absence of normal curves within the context of Galilean 3-space, as defined by both the Frenet frame and the Q-frame. A curve $\alpha(s)$ is classified as a Q-normal curve in $\mathbb{G}_3$ if it resides entirely within its Q-normal plane. Formally, this means that the curve α satisfies the equation

$$\alpha(s) = \lambda(s)N_q(s) + \pi(s)B_q(s),$$

where $N_q(s)$ and $B_q(s)$ represent the Q-normal and Q-binormal vectors, respectively.

Theorem 1. *For any admissible differentiable curve parameterized by s, there do not exist quasi-normal curves in $\mathbb{G}_3$.*

Proof. Let $\beta(s)$ denote an admissible differentiable curve parameterized by the Galilean invariant arc length s in $\mathbb{G}_3$. We can express $\beta(s)$ in the following form:

$$\beta(s) = (s, y(s), z(s)).$$

Differentiating with respect to s, we obtain the tangent vector:

$$T = (1, y', z').$$

From the Galilean metric, we find that $g(\beta, T) = s \neq 0$, which implies that β cannot be classified as a Q-normal curve.

Corollary 4. *If $\beta(s)$ is an admissible differentiable curve parameterized by the Galilean invariant arc length s, then $\beta(s)$ does not constitute a normal curve in $\mathbb{G}_3$ or $\mathbb{G}_n$. Consequently, the results presented in [23] are invalid.*

6. Quasi-rectifying curves

In this subsection, we establish the fundamental criteria for characterizing a Q-curve with position vector $\Gamma(s)$ as a Q-rectifying curve within $\mathbb{G}_3$. The definition of a Q-rectifying curve is predicated on its geometric relationship to the Q-rectifying plane. Specifically, a curve Γ is classified as Q-rectifying if and only if its position vector can be expressed as a linear combination of its tangent vector $T(s)$ and Q-binormal vector $B_q(s)$, such that $\Gamma(s) = \gamma(s)T(s) + \epsilon(s)B_q(s)$, where $\gamma(s)$ and $\epsilon(s)$ are scalar functions.

Theorem 2. *Let $\Gamma(s)$ be a Q-rectifying curve in $\mathbb{G}_3$. Then the tangential and the binormal components of the position vector $\Gamma(s)$ are given, respectively, by*

$$g(\Gamma, T) = C + s, \qquad g(\Gamma, B_q) = (C + s)\frac{K_1}{K_3}.$$

Proof. Suppose that $\Gamma(s)$ is a Q-rectifying curve, then

$$\Gamma(s) = \gamma(s)T(s) + \epsilon(s)B_q(s), \tag{6.1}$$

for some differentiable functions $\gamma(s)$ and $\epsilon(s)$. We can deduce that

$$\gamma(s) = C + s, \quad \epsilon(s) = (C+s)\frac{K_1}{K_3}.$$

Therefore,

$$g(\Gamma, T) = C + s, \; g(\Gamma, B_q) = (C+s)\frac{K_1}{K_3}.$$

Thus,

$$\Gamma(s) = (C+s)T + (C+s)\frac{K_1}{K_3}B_q.$$

Corollary 5. *Let $\Gamma(s)$ be a Frenet rectifying curve in $\mathbb{G}_3$. Then the Frenet tangential and the binormal components of the position vector are given, respectively, by*

$$g(\Gamma, T) = C + s, \qquad g(\Gamma, B) = (C+s)\frac{\kappa}{\tau}.$$

Theorem 3. *Let $\Gamma(s)$ be a quasi curve with position vector in $\mathbb{G}_3$. The curve is Q-rectifying if and only if*

$$\frac{K_1}{K_3} = \frac{1}{C+s}\int (C+s)K_2 ds + C_7,$$

where C_7 is constant.

Proof. Suppose that $\Gamma(s)$ is a Q-rectifying curve. So, we can put $m_2 = 0$ in Equation (4.3) and Equation (4.4), we obtain:

$$(C+s)K_1 - \epsilon K_3 = 0,$$
$$-(C+s)K_2 + \epsilon' = 0.$$

By solving these equations we get a linear first-order differential equation

$$(\frac{K_1}{K_3})' + \frac{1}{C+s}\frac{K_1}{K_3} = K_2.$$

So,

$$\frac{K_1}{K_3} = \frac{1}{C+s}\int (C+s)K_2 ds + C_7. \tag{6.2}$$

Conversely, Let the condition (6.2) be satisfied. Let a vector r be as follows

$$r = \Gamma(s) - (C+s)T_q - (C+s)\frac{K_1}{K_3}B_q. \tag{6.3}$$

By differentiating Equation (6.3) with respect to s and using Equation (6.2), we deduce $r' = 0$. Thus

$$\Gamma(s) - r = (C+s)T_q + (C+s)\frac{K_1}{K_3}B_q.$$

Up to a transformation with r, we find Γ is a Q-rectifying curve in $\mathbb{G}_3$.

Corollary 6. *let $\Gamma(s)$ be curve with position vector in $\mathbb{G}_3$. The curve is a Frenet rectifying curve if and only if*

$$\frac{\kappa}{\tau} = \frac{C_7^*}{C+s},$$

where C_7^* is constant.

7. Quasi-Osculating Curves

In this section, we establish the necessary and sufficient conditions for a curve with position vector $\eta(s)$ to be classified as a Q-osculating curve within the framework of $\mathbb{G}_3$. A curve $\eta(s)$ is deemed a Q-osculating curve if it remains contained within its Q-osculating plane. In mathematical terms, the curve η satisfies the following representation:

$$\eta(s) = \varrho(s)T(s) + \varepsilon(s)N_q(s),$$

where $T(s)$ and $N_q(s)$ denote the tangent and Q-normal vectors, respectively.

Theorem 4. *Let $\eta(s)$ be a Q-osculating curve in $\mathbb{G}_3$. Then the tangential and Q-normal components of the position vector η can be expressed as follows:*

$$g(\eta, T) = C + s, \quad g(\eta, N_q) = -\int (C+s)K_1\, ds + C_8,$$

where C and C_8 are constants.

Proof. Assuming that $\eta(s)$ is a Q-osculating curve, we can write:

$$\eta(s) = \varrho(s)T(s) + \varepsilon(s)N_q(s).$$

From this representation, we derive:

$$\varrho(s) = C + s, \quad \varepsilon(s) = -\int (C+s)K_1\, ds + C_8.$$

Thus, we find that:

$$g(\eta, T) = C + s, \quad g(\eta, N_q) = -\int (C+s)K_1\, ds + C_8.$$

Consequently, we can express $\eta(s)$ as:

$$\eta(s) = (C+s)T + \left[-\int (C+s)K_1\, ds + C_8\right] N_q,$$

where C and C_8 are constants.

Corollary 7. *If the curve $\eta(s)$ is a Frenet osculating curve, then its tangential and normal components are given by:*

$$g(\eta, T) = C + s, \quad g(\eta, N) = -\int (C+s)\kappa\, ds + C^*,$$

where C^* is a constant.

Theorem 5. *Consider a quasi curve with position vector $\eta(s)$ in $\mathbb{G}_3$. The curve is classified as Q-osculating if and only if the following condition holds:*

$$\frac{K_2}{K_3} = \frac{1}{C+s} \int [-(C+s)K_1]\, ds + C_9,$$

where C_9 is a constant.

Proof.

Suppose that $\eta(s)$ is a Q-osculating curve. Then, setting $m_3 = 0$ in Equations (4.3) and (4.4), we obtain:

$$(C+s)K_1 + \epsilon' = 0, \tag{7.1}$$

$$-(C+s)K_2 + \epsilon K_3 = 0. \tag{7.2}$$

From equation (7.1), we deduce $m_2 = -\int(C+s)K_1 ds + C_9$, and by solving these equations we get a linear first-order differential equation

$$(\frac{K_2}{K_3})' + \frac{1}{C+s}(\frac{K_2}{K_3}) = -K_1.$$

So,

$$\frac{K_2}{K_3} = \frac{1}{C+s} \int -(C+s)K_1 ds + C_{10}. \tag{7.3}$$

Conversely, Let the condition (7.3) be satisfied. Let a vector p be as follows

$$p = \eta(s) - (C+s)T_q - (C+s)\frac{K_2}{K_3}N_q. \tag{7.4}$$

By differentiating Equation(7.4), we deduce $p' = 0$, therefore p is constant. Thus

$$\eta(s) - p = (C+s)T_q + (C+s)\frac{K_2}{K_3}Nq.$$

Up to a transformation with p, we find η is a Q-osculating curve in $\mathbb{G}_3$.

Corollary 8. *let $\eta(s)$ be curve with position vector in $\mathbb{G}_3$. The curve is a Frenet osculating curve if and only if*

$$\int (C+s)\kappa ds = C^{**},$$

where C and C^{**} are constants.

8. Conclusion

In this paper, we have investigated the theoretical foundations of the quasi-frame (Q-frame) in three-dimensional Galilean space ($\mathbb{G}_3$) and explored its applications to the study of curves in this geometric framework. Our work has provided a comprehensive analysis of the quasi-frame, its relationship with the classical Frenet frame, and the geometric properties of curves defined in relation to this frame. The quasi-frame serves as a generalization of the Frenet frame, particularly useful in scenarios where the curvature vanishes and the Frenet frame becomes undefined. By establishing the quasi equations in matrix form, we derived the quasi curvatures K_1, K_2, and K_3, which govern the behavior of curves in $\mathbb{G}_3$. This framework not only extends the applicability of the Frenet frame but also provides a more robust tool for analyzing curves in Galilean space.

We derived explicit expressions for the position vectors of curves in $\mathbb{G}_3$ with respect to the Q-frame. By examining specific cases, we obtained solutions for the components of the position vector under various conditions. This analysis allowed us to classify curves into quasi-rectifying and quasi-osculating categories based on their geometric properties. A curve is quasi-rectifying if its position vector lies in the plane spanned by its tangent and quasi-binormal vectors, while a curve is quasi-osculating if it remains entirely within its quasi-osculating plane, defined by the tangent and quasi-normal vectors. These classifications provide a deeper understanding of the geometric behavior of curves in $\mathbb{G}_3$.

One of the most significant findings of this study is the demonstration that normal curves do not exist in $\mathbb{G}_3$. This result challenges existing theories and provides new insights into the geometric structure of Galilean space. By proving the absence of normal curves, we have clarified the limitations of certain geometric classifications in this context. Furthermore, our results generalize several well-known properties of curves in Euclidean and Minkowski spaces to the Galilean setting. For instance, the conditions for quasi-rectifying and quasi-osculating curves reduce to their Frenet counterparts when the quasi curvatures are appropriately specialized. This demonstrates the versatility of the Q-frame and its ability to unify various geometric frameworks.

The findings of this study have several important implications for both theoretical and applied mathematics. The quasi-frame provides a powerful tool for analyzing curves in Galilean space, particularly in cases where the Frenet frame fails. This has potential applications in physics, engineering, and computer graphics, where Galilean geometry is often used to model motion and spatial relationships. Future research could explore extending the quasi-frame to higher-dimensional Galilean spaces ($\mathbb{G}_n$), providing new insights into the geometry of curves and surfaces in these settings. Additionally, the quasi-frame could be applied to problems in classical and relativistic mechanics, where Galilean transformations play a central role. Investigating the properties of surfaces generated by quasi-curves, such as quasi-ruled surfaces, could also lead to new geometric constructions and classifications.

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