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Semidetached SUP-Subalgebras of Sheffer Stroke UP-Algebras

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Abstract. The notion of semidetached Sheffer stroke UP-algebras is introduced, and their properties are investigated. Several conditions for a semidetached structure in Sheffer stroke UP-algebras to be a semidetached SUP-subalgebra are provided. The concepts of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra, k-left (k-right) $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra, $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra and $(\overline{\in} \vee \overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra are introduced, and relative relations and properties are discussed.

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Key Words and Phrases: Sheffer stroke UP-algebra, $(\in, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra, k-left (k-right) $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra, $(\overline{\in} \vee \overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra, semidetached SUP-subalgebra

1. Introduction

The Sheffer operation, commonly referred to as the Sheffer stroke or NAND operator, was first introduced by Sheffer [1]. This operation is particularly notable for its ability to form a complete logical system on its own, without relying on any other logical connectives. In fact, any logical axiom can be expressed using only the Sheffer stroke, which simplifies the manipulation and analysis of logical systems. Moreover, all the axioms of Boolean algebra, the algebraic foundation of classical propositional logic, can also be expressed exclusively with the Sheffer operation. This underscores the fundamental role of the Sheffer stroke in both logic and algebra, showcasing its power and flexibility in constructing and understanding logical frameworks.

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Building upon this foundational framework, Sheffer stroke UP-algebras establish a distinctive algebraic structure that seamlessly integrates logical and algebraic principles. As elaborated by Iampan [2] in 2017, UP-algebras mark a significant advancement in the field of logical algebra, introducing a versatile paradigm for analyzing algebraic systems enriched with sophisticated logical constructs. This innovative approach not only deepens our comprehension of the interplay between logical operations and their algebraic counterparts but also unlocks potential applications across diverse disciplines, including decision theory and computational logic. By bridging these domains, Sheffer stroke UP-algebras demonstrate their capacity to contribute to both theoretical exploration and practical problem-solving.

Sheffer stroke UP-algebras lie at the crossroads of logic, algebra, and analysis, providing a unique framework for exploration. These algebras are pre-normed structures where the primary operation is the Sheffer stroke logic, and they possess a multiplicative identity element. They serve as a useful tool for investigating logical systems within an algebraic context, with potential applications in areas such as functional analysis, operator theory, and the algebraic study of logic. By incorporating the Sheffer stroke as a core operation, these algebras extend the concept of Boolean algebras, offering a broader perspective on logical and algebraic interactions. Recent works have highlighted this richness: Rajesh et al. [3] introduced the notion of intuitionistic fuzzy subalgebras in Sheffer stroke UP-algebras and established important structural properties of their level sets, while Vidhya et al. [4] further extended this framework by exploring neutrosophic N-subalgebras and their corresponding lattice-theoretic characterizations. These studies emphasize the growing relevance of fuzzy and neutrosophic perspectives in the theory of Sheffer stroke UP-algebras, paving the way for deeper investigations into generalized substructures, such as the semidetached forms explored in this paper.

The concept of the quasi-coincidence of a fuzzy point with a fuzzy set, as discussed in Bhakat and Das's pioneering work [5], has been instrumental in shaping the development of various classifications of fuzzy subgroups. This innovative approach extends traditional notions, enabling the formulation of new types of fuzzy subgroups that have broadened the scope of algebraic studies in this domain. Notably, the $(\in, \in \vee q)$-fuzzy subgroup represents a significant and practical generalization of Rosenfeld's foundational concept of fuzzy subgroups [6], thereby offering a more flexible framework for understanding the structural relationships in fuzzy algebra.

The motivation for studying semidetached SUP-subalgebras arises from the need to extend the algebraic understanding of Sheffer stroke-based logical systems under uncertainty. By incorporating fuzzy sets and semidetached structures into SUP-algebras, the framework becomes more flexible and applicable to real-world scenarios involving partial truth or threshold reasoning. These structures have potential applications in areas such as fuzzy decision-making, knowledge representation in AI, and logical circuit design, particularly where NAND logic and graded membership play a central role.

In this paper, we introduce the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras, k-left (k-right) $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras, $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras, and $(\overline{\in} \vee \overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras, and investigate relative relations and properties.

We provide several conditions for a semidetached structure in SUP-algebras to be a semidetached SUP-subalgebra.

2. Preliminaries

Sheffer Stroke UP-algebras epitomize a fascinating convergence of algebraic theory and logical principles, distinguished by the Sheffer stroke operation, which serves as a fundamental connective within the realm of propositional calculus. This particular algebraic construct not only augments our comprehension of logical operations but also bears considerable ramifications in disciplines such as computer science and decision theory. The present article will explore the definitions and foundational elements of Sheffer Stroke UP-algebras, underscoring their significance within the context of algebraic theory.

Definition 1. *[1] Let $\langle X, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in X$,*

$$\begin{aligned}
&(S1)\ x|y = y|x \\
&(S2)\ (x|x)|(x|y) = x \\
&(S3)\ x|((y|z)|(y|z)) = ((x|y)|(x|y))|z \\
&(S4)\ (x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.
\end{aligned}$$

Definition 2. *[7] A Sheffer stroke UP-algebra (briefly, SUP-algebra) is a structure $\langle X, |, 0\rangle$ of type $(2, 0)$ such that 0 is the fixed element in X and the following conditions are satisfied for all $x, y, z \in X$,*

$$\begin{aligned}
&(SUP\text{-}1)\ (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))| \\
&\qquad (z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))| \\
&\qquad ((y|(x|x))|(z|(y|y))))) = 0 \\
&(SUP\text{-}2)\ x|x = x|(0|0) \\
&(SUP\text{-}3)\ (x|(y|y))|(x|(y|y)) = 0 \text{ and } (y|(x|x))|(y|(x|x)) = 0 \Rightarrow x = y.
\end{aligned}$$

Proposition 1. *[7] Let $\langle X, |, 0\rangle$ be an SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on X.*

Definition 3. *[7] A nonempty subset G of an SUP-algebra $\langle X, |, 0\rangle$ is called an SUP-subalgebra of X if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.*

Lemma 1. *[7] Let $\langle X, |, 0\rangle$ be an SUP-algebra. Then for all $x, y, z \in X$, we have*

(1) $x \leq y \Rightarrow y|(z|z) \leq x|(z|z)$ *and* $z|(x|x) \leq z|(y|y)$

(2) $x \leq y \Leftrightarrow y|y \leq x|x$

(3) $y|(x|x) \leq x$

(4) $y \leq (y|(x|x))|(y|(x|x))$

(5) $x \leq y \Rightarrow x \leq (y|(z|z))|(y|(z|z))$

(6) $z|(y|y) \leq z|(y|(x|x))$

(7) $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$

(8) $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 4. *A fuzzy set μ in an SUP-algebra $\langle X, |, 0\rangle$ is called a fuzzy SUP-subalgebra of X if it satisfies the following:*

$$(\forall x, y \in X)(\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 5. *A fuzzy set μ in an SUP-algebra $\langle X, |, 0\rangle$ is called an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X if it satisfies the following:*

$$(\forall x, y \in X)(\forall t, r \in (0, 1])(x_t \in \mu, y_r \in \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \in \vee q_k\mu). \quad (1)$$

Definition 6. *[8] A fuzzy set μ in a set X of the form*

$$\mu(y) = \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

The general form of the symbol $x_tq\mu$ as follows: for an arbitrary element $k \in [0, 1)$, we say that

- $x_tq_k\mu$ if $\mu(x) + t + k > 1$.
- $x_t \in \vee q_k\mu$ if $x_t \in \mu$ or $x_tq_k\mu$.

Definition 7. *For any fuzzy set μ in a set X and any $t \in [0, 1]$, the set $U(\mu, t) = \{x \in X : \mu(x) \geq t\}$ is called a level subset of μ.*

3. Foundational Results on Semidetached SUP-Subalgebras

Before delving into the concept of semidetached SUP-subalgebras, it is essential to recognize the foundational framework of SUP-algebras as a unique algebraic structure that integrates logical operations through the Sheffer stroke. This section explores how semidetached structures can emerge within SUP-algebras, emphasizing their significance in the broader context of fuzzy subalgebra theory. By establishing a robust theoretical basis, we aim to illustrate the intricate relationships and conditions that govern these semidetached structures.

In what follows, let $X = \langle X, |, 0\rangle$ denote an SUP-algebra unless otherwise specified.

Given a set X and a subinterval Ω of $[0, 1]$, a semidetached structure over Ω is defined to be a pair (X, f), where $f : \Omega \to P(X)$ is a mapping when $P(X)$ is represented as the power set of X.

Definition 8. *A semidetached structure (X, f) over Ω is called a semidetached SUP-subalgebra over Ω with respect to $t \in \Omega$ (briefly, t-semidetached SUP-subalgebra) if $f(t)$ is an SUP-subalgebra of X.*

We say that (X, f) is a semidetached SUP-subalgebra over Ω if it is a t-semidetached SUP-subalgebra with respect to all $t \in \Omega$.

Given a fuzzy set μ in X, consider the following mappings:

$$\ell^{\mu}_{U} : \Omega \to P(X); t \mapsto U(\mu, t) \tag{2}$$

$$\ell^{\mu}_{Q_k} : \Omega \to P(X); t \mapsto Q_k(\mu, t) \tag{3}$$

$$\ell^{\mu}_{\mathscr{E}_k} : \Omega \to P(X); t \mapsto \mathscr{E}_k(\mu, t) \tag{4}$$

where $Q_k(\mu, t) = \{x \in X : x_t q_k \mu\}$ and $\mathscr{E}_k(\mu, t) = \{x \in X : x_t \in \vee q_k \mu\}$, which are called the q_k-set and $\in \vee q_k$-set with respect to t (briefly, t-q_k-set and t-$\in \vee q_k$-set), respectively, of μ. A t-q_k-set with $k = 0$ is called a t-q-set and is denoted by $Q(\mu, t)$. A t-$\in \vee q_k$-set with $k = 0$ is called a t-$\in \vee q$-set and is denoted by $\mathscr{E}(\mu, t)$. Note that, for any $t, r \in (0, 1]$, if $t \geq r$, then every r-q_k-set is contained in the t-q_k-set, that is, $Q_k(\mu, r) \subseteq Q_k(\mu, t)$. Obviously, $\mathscr{E}_k(\mu, t) = U(\mu, t) \cup Q_k(\mu, t)$.

Lemma 2. *[9] A fuzzy set μ is a fuzzy SUP-subalgebra of X if and only if $U(\mu, t)$ is a SUP-subalgebra of X for all $t \in (0, 1]$.*

Theorem 1. *A semidetached structure (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (0, 1]$ if and only if μ is a fuzzy SUP-subalgebra of X.*

Proof. Straightforward from Lemma 2.

Theorem 2. *If μ is an $(\in, \in)$-fuzzy SUP-subalgebra (or equivalently, μ is a fuzzy SUP-subalgebra) of X, then a semidetached structure $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (0, 1]$.*

Proof. Let $x, y \in \ell^{\mu}_{Q_k}(t)$ for $t \in \Omega = (0, 1]$. Then $x_t q_k \mu$ and $y_t q_k \mu$, that is, $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. Then $\mu((x|(y|y))|(x|(y|y))) + t + k \geq \min\{\mu(x), \mu(y)\} + t + k = \min\{\mu(x) + t + k, \mu(y) + t + k\} > 1$. Hence, $((x|(y|y))|(x|(y|y)))_t \in \vee q_k \mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}(t)$. Therefore, $\ell^{\mu}_{Q_k}(t)$ is an SUP-subalgebra of X. Consequently, $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (0, 1]$.

Corollary 1 is a direct consequence of Theorem 1. Specifically, Theorem 1 states that a semidetached structure (X, ℓ^{U}_{μ}) over $\Omega = (0, 1]$ exists if and only if μ is a fuzzy SUP-subalgebra of X. Since an $(\in, \in)$-fuzzy SUP-subalgebra is equivalent to a fuzzy SUP-subalgebra, the condition in Corollary 1 is satisfied, and the semidetached property follows immediately.

Corollary 1. *If μ is an $(\in, \in)$-fuzzy SUP-subalgebra (or equivalently, μ is a fuzzy SUP-subalgebra) of X, then a semidetached structure (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (0, 1]$.*

Given a fuzzy set μ in X and $k \in [0,1)$, we consider the following condition:

$$(\forall x, y \in X)(\forall t, r \in [0,1])(x_t q_k \mu, y_r q_k \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \in \vee q_k \mu). \quad (5)$$

Definition 9. *A fuzzy set μ in X is called a k-left (resp., k-right) $(q_k, \in \vee q_k)$-fuzzy subalgebra of X if it satisfies the condition (5) for all $x, y \in X$ and $t, r \in (0, \frac{1-k}{2}]$ (resp., $t, r \in (\frac{1-k}{2}, 1]$).*

Theorem 3. *Every k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra.*

Proof. Let μ be a k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X. Let $x, y \in X$ and $t, r \in (0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Suppose that $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \overline{\in q_k} \mu$. Then $\mu((x|(y|y))|(x|(y|y))) < \min\{t, r\}$ and $\mu((x|(y|y))|(x|(y|y))) + \min\{t, r\} + k \leq 1$. It follows that $\mu((x|(y|y))|(x|(y|y))) < \frac{1-k}{2}$, and so that $\mu((x|(y|y))|(x|(y|y))) < \min\{t, r, \frac{1-k}{2}\}$. Hence, $1 - k - \mu((x|(y|y))|(x|(y|y))) > 1 - k - \min\{t, r, \frac{1-k}{2}\} = \max\{1-k-t, 1-k-r, 1-k-\frac{1-k}{2}\} \geq \max\{1-k-\mu(x), 1-k-\mu(y), \frac{1-k}{2}\}$, and so there exists $\delta \in (0,1]$ such that $1 - k - \mu((x|(y|y))|(x|(y|y))) \geq \delta > \max\{1-k-\mu(x), 1-k-\mu(y), \frac{1-k}{2}\}$. Then $\delta \in (\frac{1-k}{2}, 1], \mu(x) + \delta + k > 1$ and $\mu(y) + \delta + k > 1$, that is, $x_\delta q_k \mu$ and $y_\delta q_k \mu$. Since μ is a k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X, it follows that $((x|(y|y))|(x|(y|y)), \delta) \in \vee q_k \mu$. On the other hand, $1 - k - \mu((x|(y|y))|(x|(y|y))) \geq \delta$ implies that $\mu((x|(y|y))|(x|(y|y))) + \delta + k \leq 1$, that is, $((x|(y|y))|(x|(y|y)), \delta)\overline{q_k}\mu$, and $\mu((x|(y|y))|(x|(y|y))) \leq 1 - \delta - k < 1 - k - \frac{1-k}{2} = \frac{1-k}{2} < \delta$, that is, $((x|(y|y))|(x|(y|y)), \delta)\overline{\in}\mu$. Hence, $((x|(y|y))|(x|(y|y)), \delta)\overline{\in \vee q_k}\mu$, which is a contradiction. Therefore, $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \in \vee q_k \mu$, and thus μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X.

Corollary 2 is an immediate consequence of Theorem 3 by setting $k = 0$. Specifically, a 0-right $(q, \in \vee q)$-fuzzy SUP-subalgebra satisfies all the conditions required by Theorem 3, and hence it is also an $(\in, \in \vee q)$-fuzzy SUP-subalgebra.

Corollary 2. *Every 0-right $(q, \in \vee q)$-fuzzy SUP-subalgebra is an $(\in, \in \vee q)$-fuzzy subalgebra.*

Theorem 4. *If every fuzzy point has the value t in $(0, \frac{1-k}{2}]$, then every $(\in, \in \vee q_k)$-fuzzy subalgebra is a k-left $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X. Let $x, y \in X$ and $t, r \in (0, \frac{1-k}{2}]$ be such that $x_t q_k \mu$ and $y_r q_k \mu$. Then $\mu(x) + t + k > 1$ and $\mu(y) + r + k > 1$. Since $t, r \in (0, \frac{1-k}{2}]$, we have $\mu(x) > 1 - t - k \geq \frac{1-k}{2} \geq t$ and $\mu(y) > 1 - r - k \geq \frac{1-k}{2} \geq r$, that is, $x_t \in \mu$ and $y_r \in \mu$. Then $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \in \vee q_k \mu$. Hence, μ is a k-left $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X.

Corollary 3 is a direct consequence of Theorem 4 by setting $k = 0$. In this case, the interval $\left(0, \frac{1-k}{2}\right]$ becomes $(0, 0.5]$, and the $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra reduces to the $(\in, \in \vee q)$-fuzzy case. Theorem 4 guarantees that under these conditions, every $(\in, \in \vee q)$-fuzzy SUP-subalgebra is a 0-left $(q, \in \vee q)$-fuzzy SUP-subalgebra, which proves the corollary.

Corollary 3. *If every fuzzy point has the value* t *in* $(0, 0.5]$*, then every* $(\in, \in \vee q)$*-fuzzy SUP-subalgebra is a* 0*-left* $(q, \in \vee q)$*-fuzzy SUP-subalgebra.*

Proposition 2. *If* $(X, \ell^{\mu}_{Q_k})$ *is a semidetached SUP-subalgebra over* $\Omega = (\frac{1-k}{2}, 1]$*, then* μ *satisfies:*

$$(\forall x, y \in X)(\forall t, r \in \Omega)(x_t \in \mu, y_r \in \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} q_k \mu). \tag{6}$$

Proof. Let $x, y \in X$ and $t, r \in \Omega = (\frac{1-k}{2}, 1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t > \frac{1-k}{2}$ and $\mu(y) \geq r > \frac{1-k}{2}$, which imply that $\mu(x) + t + k > 1$ and $\mu(y) + r + k > 1$, that is, $x_t q_k \mu$ and $y_r q_k \mu$. It follows that $x, y \in \ell^{\mu}_{Q_k}(\max\{t, r\})$ and $\max\{t, r\} \in (\frac{1-k}{2}, 1]$. Since $\ell^{\mu}_{Q_k}(\max\{t, r\})$ is an SUP-subalgebra of X, we have $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}(\max\{t, r\})$, and so $((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} q_k \mu$.

Corollary 4 is a direct consequence of Proposition 2 by taking $k = 0$, which leads to $\Omega = (0.5, 1]$ and the standard quasi-coincidence operator q. The proposition ensures that under the semidetached structure condition, the image of the Sheffer stroke operation remains within the fuzzy quasi-coincidence set, establishing the desired inclusion.

Corollary 4. *If* $(X, \ell^{\mu}_{Q_k})$ *is a semidetached SUP-subalgebra over* $\Omega = (0.5, 1]$*, then* μ *satisfies:*

$$(\forall x, y \in X)(\forall t, r \in \Omega)(x_t \in \mu, y_r \in \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} q \mu). \tag{7}$$

Proposition 3. *If* $(X, \ell^{\mu}_{Q_k})$ *is a semidetached SUP-subalgebra over* $\Omega = (0, \frac{1-k}{2}]$*, then* μ *satisfies:*

$$(\forall x, y \in X)(\forall t, r \in \Omega)(x_t q_k \mu, y_r q_k \mu) \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \mu). \tag{8}$$

Proof. Let $x, y \in X$ and $t, r \in \Omega = (0, \frac{1-k}{2}]$ be such that $x_t q_k \mu$ and $y_r q_k \mu$. Then $x \in \ell^{\mu}_{Q_k}(t)$ and $y \in \ell^{\mu}_{Q_k}(r)$. It follows that $x, y \in \ell^{\mu}_{Q_k}(\max\{t, r\})$ and $\max\{t, r\} \in (0, \frac{1-k}{2}]$. Thus, $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}(\max\{t, r\})$ since $\ell^{\mu}_{Q_k}(\max\{t, r\})$ is an SUP-subalgebra of X. Hence, $\mu((x|(y|y))|(x|(y|y))) + k + \max\{t, r\} > 1$, and so $\mu((x|(y|y))|(x|(y|y))) > 1 - k - \max\{t, r\} \geq \frac{1-k}{2} \geq \max\{t, r\}$. Thus, $((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \mu$ and (8) is valid.

Corollary 5 follows directly from Proposition 3 by setting $k = 0$, which implies $\Omega = (0, 0.5]$ and uses the standard quasi-coincidence operator q. Proposition 3 establishes that when the fuzzy elements x_t and y_r quasi-coincide with μ, their Sheffer stroke combination also satisfies the membership condition in μ, as required.

Corollary 5. *If* $(X, \ell^{\mu}_{Q_k})$ *is a semidetached SUP-subalgebra over* $\Omega = (0, 0.5]$*, then* μ *satisfies:*

$$(\forall x, y \in X)(\forall t, r \in)(x_t q \mu, yrq\mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \mu). \tag{9}$$

Theorem 5. *If μ is a k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Proof. Let $x, y \in \ell^{\mu}_{Q_k}(t)$ for $t \in (\frac{1-k}{2}, 1]$. Then $x_t q_k \mu$ and $y_t q_k \mu$. Since μ is a k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X, we have $((x|(y|y))|(x|(y|y)))_t \in \vee q_k \mu$, that is, $((x|(y|y))|(x|(y|y)))_t \in \mu$ or $((x|(y|y))|(x|(y|y)))_t q_k \mu$. If $((x|(y|y))|(x|(y|y)))_t \in \mu$, then $\mu((x|(y|y))|(x|(y|y))) \geq t > \frac{1-k}{2} > 1 - t - k$, and so $\mu((x|(y|y))|(x|(y|y))) + t + k > 1$, that is, $((x|(y|y))|(x|(y|y)))_t q_k \mu$. Hence, $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}$. If $((x|(y|y))|(x|(y|y)))_t q_k \mu$, then $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}(t)$. Therefore, $\ell^{\mu}_{Q_k}(t)$ is an SUP-subalgebra of X, and consequently, $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.

Corollary 6 is a direct consequence of Theorem 5 by taking $k = 0$, which yields the interval $\Omega = (0.5, 1]$ and replaces q_k with the standard q. Since Theorem 5 ensures that a k-right $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra induces a semidetached structure, it follows that a 0-right $(q, \in \vee q)$-fuzzy SUP-subalgebra also generates a semidetached SUP-subalgebra over this interval.

Corollary 6. *If μ is a 0-right $(q, \in \vee q)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (0.5, 1]$.*

4. Equivalences and Characterizations of Fuzzy SUP-Subalgebras

This section is devoted to a deeper exploration of the relationships between various classes of fuzzy SUP-subalgebras and their role in generating semidetached structures within Sheffer stroke UP-algebras. Building upon the foundational results established in the previous section, we present a series of theorems and corollaries that provide necessary and sufficient conditions for a fuzzy set to induce a semidetached SUP-subalgebra over specified subintervals of $(0, 1]$. Particular emphasis is placed on the structural implications of $(\in, \in \vee q_k)$-fuzzy, k-left, k-right, and $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebras, along with their interrelationships and equivalences. These characterizations not only unify various fuzzy concepts under a common algebraic framework but also demonstrate how logical fuzziness can precisely determine the formation of algebraic substructures. The results in this section contribute to a comprehensive theoretical foundation for fuzzy logic integration in algebraic systems based on Sheffer stroke operations.

Theorem 6. *For an SUP-subalgebra A of X, let μ be a fuzzy set in X such that*

(1) $\mu(x) \geq \frac{1-k}{2}$ *for all* $x \in A$,

(2) $\mu(x) = 0$ *for all* $x \in X \backslash A$.

Then μ is a k-left $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X.

Proof. Let $x, y \in X$ and $t, r \in (0, \frac{1-k}{2}]$ be such that $x_t q_k \mu$ and $y_r q_k \mu$. Then $\mu(x) + t + k > 1$ and $\mu(y) + r + k > 1$, which imply that $\mu(x) > 1 - t - k \geq \frac{1-k}{2}$ and $\mu(y) > 1 - r - k \geq \frac{1-k}{2}$. Hence, $x \in A$ and $y \in A$. Since A is an SUP-subalgebra of X, we get $(x|(y|y))|(x|(y|y)) \in A$, and so $\mu((x|(y|y))|(x|(y|y))) \geq \frac{1-k}{2} \geq \max\{t,r\}$. Thus, $((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \mu$, and so $((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \vee q_k\mu$. Therefore, μ is a k-left $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X.

Corollary 7 is a special case of Theorem 6 by setting $k = 0$. In this case, the threshold $\frac{1-k}{2}$ becomes 0.5, and the conditions in Corollary 7 match exactly the assumptions of Theorem 6. Therefore, the fuzzy set μ constructed as described satisfies the definition of a 0-left $(q, \in \vee q)$-fuzzy SUP-subalgebra of X.

Corollary 7. *For an SUP-subalgebra A of X, let μ be a fuzzy set in X such that*

(1) $\mu(x) \geq 0.5$ *for all* $x \in A$,

(2) $\mu(x) = 0$ *for all* $x \in X \backslash A$.

Then μ is a 0-left $(q, \in \vee q)$-fuzzy SUP-subalgebra of X.

Proposition 4. *If $(X, \ell^{\mu}_{\mathscr{E}_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$, then μ satisfies:*

$$(\forall x, y \in X)(\forall t, r \in \Omega)(x_t q_k \mu, y_r q_k \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \vee q_k \mu). \tag{10}$$

Proof. Let $x, y \in X$ and $t, r \in \Omega = (0, 1]$ be such that $x_t q_k \mu$ and $y_r q_k \mu$. Then $x \in \ell^{\mu}_{Q_k}(t) \subseteq \ell^{\mu}_{\mathscr{E}_k}(t)$ and $y \in \ell^{\mu}_{Q_k}(r) \subseteq \ell^{\mu}_{\mathscr{E}_k}(r)$. It follows that $x, y \in \ell^{\mu}_{\mathscr{E}_k}(\max\{t,r\})$, and so from the hypothesis that $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(\max\{t,r\})$. Hence, $((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \vee q_k\mu$, and consequently, (10) is valid.

Corollary 8 follows directly from Proposition 4 by taking $k = 0$, which implies that $\Omega = (0.5, 1]$ and the fuzzy quasi-coincidence operator q_k becomes the standard q.

Corollary 8. *If $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$, then μ satisfies:*

$$(\forall x, y \in X)(\forall t, r \in \Omega)(x_t q \mu, y_r q \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\max\{t,r\}} \in \vee q \mu). \tag{11}$$

The following lemma is directly proved by Definition 5.

Lemma 3. *A fuzzy set μ in X is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X if and only if it satisfies the following:*

$$(\forall x, y \in X)(\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}). \tag{12}$$

Theorem 7. *If μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Proof. Let $x, y \in \ell^{\mu}_{Q_k}(t)$ for $t \in \Omega = (\frac{1-k}{2}, 1]$. Then $x_t q_k \mu$ and $y_t q_k \mu$, that is, $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. It follows from Lemma 3 that $\mu((x|(y|y))|(x|(y|y))) + t + k \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} + t + k = \min\{\mu(x) + t + k, \mu(y) + t + k, \frac{1-k}{2} + t + k\} > 1$. Hence, $((x|(y|y))|(x|(y|y)))_t q_k \mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{Q_k}(t)$. Therefore, $\ell^{\mu}_{Q_k}(t)$ is an SUP-subalgebra of X for all $t \in (\frac{1-k}{2}, 1]$, and consequently, $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.

Corollary 9 is a special case of Theorem 7 by setting $k = 0$, which leads to the interval $\Omega = (0.5, 1]$. The assumption that μ is an $(\in, \in \vee q)$-fuzzy SUP-subalgebra guarantees, via Theorem 7, that the level sets $\ell^{Q}_{\mu}(t)$ are SUP-subalgebras for all $t \in \Omega$, and thus (X, ℓ^{Q}_{μ}) is a semidetached SUP-subalgebra over this interval.

Corollary 9. *If μ is an $(\in, \in \vee q)$-fuzzy SUP-subalgebra of X, then μ is a semidetached SUP-subalgebra over $\Omega = (0.5, 1]$.*

Theorem 8. *If μ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$, then μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X.*

Proof. For a semidetached SUP-subalgebra μ over $\Omega = (\frac{1-k}{2}, 1]$, assume that there exist $a, b \in X$ such that $\mu((a|(b|b))|(a|(b|b))) < \min\{\mu(a), \mu(b), \frac{1-k}{2}\} = t_0$. Then $t_0 \in (0, \frac{1-k}{2}]$, $a, b \in U(\mu, t_0) \subseteq \ell^{\mu}_{\mathscr{E}_k}(t_0)$, which implies that $(a|(b|b))|(a|(b|b)) \in \ell^{\mu}_{\mathscr{E}_k}(t_0)$. Hence $\mu((a|(b|b))|(a|(b|b))) \geq t_0$ or $\mu((a|(b|b))|(a|(b|b))) + t_0 + k > 1$. This is a contradiction. Thus, $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in X$. It follows from Lemma 3 that μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X.

Corollary 10 follows immediately from Theorem 8, which shows that if $(X, \ell^{Q_k}_{\mu})$ is a semidetached SUP-subalgebra over $\Omega = \left(\frac{1-k}{2}, 1\right]$, then μ must be an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra. By applying Theorem 7, it follows that $(X, \ell^{Q_k}_{\mu})$ is again a semidetached SUP-subalgebra over the same interval. Thus, the conclusion reconfirms the consistency of the structure.

Corollary 10. *If $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$, then $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Theorem 9. *If μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Proof. Let $x, y \in \ell^{\mu}_{\mathscr{E}_k}(t)$ for $t \in \Omega = (0, \frac{1-k}{2}]$. Then $x_t \in \vee q_k \mu$ and $y_t \in \vee q_k \mu$. Hence, we have the following four cases:

(1) $x_t \in \mu$ and $y_t \in \mu$,

(2) $x_t \in \mu$ and $y_t q_k \mu$,

(3) $x_t q_k \mu$ and $y_t \in \mu$,

(4) $x_t q_k \mu$ and $y_t q_k \mu$.

The first case implies that $((x|(y|y))|(x|(y|y)))_t \in \vee q_k\mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. For the second case, $y_t q_k \mu$ induces $\mu(y) > 1 - t - k \geq t$, that is, $y_t \in \mu$. Hence, $((x|(y|y))|(x|(y|y)))_t \in \vee \overline{q_k}\mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. Similarly, the third case implies $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. The last case induces $\mu(x) > 1 - t - k \geq t$ and $\mu(y) > 1-t-k \geq t$, that is, $x_t \in \mu$ and $y_t \in \mu$. It follows that $((x|(y|y))|(x|(y|y)))_t \in \vee q_k\mu$ and so that $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. Therefore, $\ell^{\mu}_{\mathscr{E}_k}(t)$ is an SUP-subalgebra of X for all $t \in 0, \frac{1-k}{2}$. Hence, $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.

Corollary 11 is a direct consequence of Theorem 9 by setting $k = 0$. When $k = 0$, the interval $\Omega = \left(0, \frac{1-k}{2}\right]$ becomes $(0, 0.5]$, and the operator q_k becomes the standard quasi-coincidence operator q. Theorem 9 ensures that if μ is an $(\in, \in \vee q)$-fuzzy SUP-subalgebra, then the structure (X, ℓ^E_{μ}) forms a semidetached SUP-subalgebra over this interval.

Corollary 11. *If μ is an $(\in, \in \vee q)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{\mathscr{E}_k})$ is a semidetached SUP-subalgebra over $\Omega = (0, 0.5]$.*

Theorem 10. *If μ is a $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{\mathscr{E}_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Proof. Let $x, y \in \ell^{\mu}_{\mathscr{E}_k}(t)$ for $t \in \Omega = (\frac{1-k}{2}, 1]$. Then $x_t \in \vee q_k\mu$ and $y_t \in \vee q_k\mu$. Hence, we have the following four cases:

(1) $x_t \in \mu$ and $y_t \in \mu$,

(2) $x_t \in \mu$ and $y_t q_k \mu$,

(3) $x_t q_k \mu$ and $y_t \in \mu$,

(4) $x_t q_k \mu$ and $y_t q_k \mu$.

For the first case, we have $\mu(x) + t + k \geq 2t + k > 1$ and $\mu(y) + t + k \geq 2t + k > 1$, that is, $x_t q_k \mu$ and $y_t q_k \mu$. Hence, $((x|(y|y))|(x|(y|y)))_t \in \vee q_k\mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. In the second case, $x_t \in \mu$ implies $\mu(x) + t + k \geq 2t + k > 1$, that is, $x_t q_k \mu$. Hence, $((x|(y|y))|(x|(y|y)))_t \in \vee q_k\mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. Similarly, the third case implies $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. For the last case, we have $((x|(y|y))|(x|(y|y)))_t \in \vee q_k\mu$, and so $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{\mathscr{E}_k}(t)$. Consequently, $\ell^{\mu}_{\mathscr{E}_k}(t)$ is an SUP-subalgebra of X for all $t \in \Omega = (\frac{1-k}{2}, 1]$. Therefore, $(X, \ell^{\mu}_{\mathscr{E}_k})$ is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.

Corollary 12 is derived from Theorem 10 by taking $k = 0$, which yields the interval $\Omega = (0.5, 1]$ and converts the generalized operator q_k into the standard quasi-coincidence operator q. Theorem 10 proves that if μ is a $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra, then the structure (X, ℓ^E_{μ}) forms a semidetached SUP-subalgebra over $\Omega = \left(\frac{1-k}{2}, 1\right]$. Substituting $k = 0$ confirms the conclusion for the $(q, \in \vee q)$-fuzzy case.

Corollary 12. *If μ is a $(q, \in \vee q)$-fuzzy SUP-subalgebra of X, then $(X, \ell^{\mu}_{\mathscr{E}_k})$ is a semidetached SUP-subalgebra over $\Omega = (0.5, 1]$.*

For $\alpha \in \{\in, q_k\}$ and $t \in (0,1]$, we say that $x_t\overline{\alpha}\mu$ if $x_t\alpha\mu$ does not hold.

Definition 10. *A fuzzy set μ in X is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X if it satisfies the following:*

$$(\forall x, y \in X)(\forall t, r \in (0,1])(((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\mu \Rightarrow x_t\overline{\in} \vee \overline{q_k}\mu \text{ or } y_r\overline{\in} \vee \overline{q_k}\mu). \tag{13}$$

An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra with $k = 0$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy SUP-subalgebra.

Theorem 11. *A fuzzy set μ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X if and only if the following inequality is valid:*

$$(\forall x, y \in X)(\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y)\}). \tag{14}$$

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X. Assume that (14) is not valid. Then there exist $a, b \in X$ such that $\max\{\mu((a|(b|b))|(a|(b|b))), \frac{1-k}{2}\} < \min\{\mu(a), \mu(b)\} = t$. Then $\frac{1-k}{2} < t \leq 1$, $a_t \in \mu$, $b_t \in \mu$ and $((a|(b|b))|(a|(b|b)))_t\overline{\in}\mu$. It follows from (13) that $a_t\overline{q_k}\mu$ or $b_t\overline{q_k}\mu$. Hence, $\mu(a) \geq t$ and $\mu(a) + t + k \leq 1$ or $\mu(b) \geq t$ and $\mu(b) + t + k \leq 1$. In either case, we have $t \leq \frac{1-k}{2}$, which is a contradiction. Therefore, $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Conversely, suppose that (14) is valid. Let $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\mu$ for all $x, y \in X$ and $t, r \in (0,1]$. Then $\mu((x|(y|y))|(x|(y|y))) < \min\{t, r\}$. If $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} = \mu((x|(y|y))|(x|(y|y)))$, then $\min\{t, r\} > \mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\}$, and so $\mu(x) < t$ or $\mu(y) < r$. Thus, $x_t\overline{\in}\mu$ or $y_r\overline{\in}\mu$, which implies that $x_t\overline{\in}\vee\overline{q_k}\mu$ or $y_r\overline{\in}\vee\overline{q_k}\mu$. If $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\min\{\mu(x), \mu(y)\} \leq \frac{1-k}{2}$. Suppose $x_t \in \mu$ or $y_r \in \mu$. Then $t \leq \mu(x) \leq \frac{1-k}{2}$ or $r \leq \mu(y) \leq \frac{1-k}{2}$, and so $\mu(x)+t+k \leq \frac{1-k}{2}+\frac{1-k}{2}+k = 1$ or $\mu(y) + r + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Hence, $x_t\overline{q_k}\mu$ or $y_r\overline{q_k}\mu$. Therefore, $x_t\overline{\in} \vee \overline{q_k}\mu$ or $y_r\overline{\in} \vee \overline{q_k}\mu$. This shows that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X.

Corollary 13 follows directly from Theorem 11 by setting $k = 0$. In this case, the inequality given in Theorem 11 simplifies to

$$\max\{\mu((x|(y|y))|(x|(y|y))), 0.5\} \geq \min\{\mu(x), \mu(y)\},$$

which is exactly the condition stated in Corollary 13. Hence, the result is an immediate specialization of the general case when $k = 0$.

Corollary 13. *A fuzzy set μ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy SUP-subalgebra of X if and only if the following inequality is valid:*

$$(\forall x, y \in X)(\max\{\mu((x|(y|y))|(x|(y|y))), 0.5\} \geq \min\{\mu(x), \mu(y)\}). \tag{15}$$

Theorem 12. *A fuzzy set μ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X if and only if (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X. Let $x, y \in \ell^{\mu}_{U}(t)$ for $t \in \Omega = (\frac{1-k}{2}, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from (14) that $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y)\} \geq t$. Since $t > \frac{1-k}{2}$, it follows that $\mu((x|(y|y))|(x|(y|y))) \geq t$, and so that $(x|(y|y))|(x|(y|y)) \in \ell^{\mu}_{U}(t)$. Thus, $\ell^{\mu}_{U}(t)$ is an SUP-subalgebra of X, and (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$.

Conversely, suppose that (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (\frac{1-k}{2}, 1]$. If (14) is not valid, then there exist $a, b \in X$ such that $\max\{\mu((a|(b|b))|(a|(b|b))), \frac{1-k}{2}\} < \min\{\mu(a), \mu(b)\} = t$. Then $t \in (\frac{1-k}{2}, 1]$, $a, b \in \ell^{\mu}_{U}(t)$ and $(a|(b|b))|(a|(b|b)) \notin \ell^{\mu}_{U}(t)$. This is a contradiction, and so (14) is valid. Using Theorem 11, we know that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X.

Corollary 14 is a direct consequence of Theorem 12 by setting $k = 0$. When $k = 0$, the interval $\Omega = \left(\frac{1-k}{2}, 1\right]$ becomes $(0.5, 1]$, and the condition in Theorem 3.33 simplifies accordingly. Thus, an $(\in, \in \vee q)$-fuzzy SUP-subalgebra of X is equivalent to a semidetached SUP-subalgebra over $(0.5, 1]$, establishing the result.

Corollary 14. *A fuzzy set μ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy SUP-subalgebra of X if and only if μ is a semidetached SUP-subalgebra over $\Omega = (0.5, 1]$.*

Theorem 13. *A fuzzy set μ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X if and only if $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (0, \frac{1-k}{2}]$.*

Proof. Assume that (X, ℓ^{μ}_{U}) is a semidetached SUP-subalgebra over $\Omega = (0, \frac{1-k}{2}]$. If (14) is not valid, then there exist $a, b \in X$, $t \in (0, 1]$ and $k \in [0, 1)$ such that $\max\{\mu((a|(b|b))|(a|(b|b))), \frac{1-k}{2}\} + t + k \leq 1 < \min\{\mu(a), \mu(b)\} + t + k$. It follows that $a_t q_k \mu$ and $b_t q_k \mu$, that is, $a, b \in \ell^{\mu}_{Q_k}(t)$, but $((a|(b|b))|(a|(b|b)))_t \overline{q_k} \mu$, that is, $(a|(b|b))|(a|(b|b)) \notin \ell^{\mu}_{Q_k}$. This is a contradiction, and so (14) is valid. Using Theorem 11, we have μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X. Let $x, y \in \ell^{\mu}_{Q_k}(t)$ for $t \in \Omega = (0, \frac{1-k}{2}]$. Then $x_t q_k \mu$ and $y_t q_k \mu$, that is, $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. It follows from (14) that $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y)\} > 1 - t - k \geq \frac{1-k}{2}$ and so that $\mu((x|(y|y))|(x|(y|y))) + t + k > 1$, that is, $(x|(y|y))|(x|(y|y)) \in \mu$. Therefore, $\ell^{\mu}_{Q_k}(t)$ is an SUP-subalgebra of X, and $(X, \ell^{\mu}_{Q_k})$ is a semidetached SUP-subalgebra over $\Omega = (0, \frac{1-k}{2}]$.

Corollary 15 follows directly from Theorems 12 and 13. Theorem 12 states that (X, ℓ^{U}_{μ}) is a semidetached SUP-subalgebra over $\Omega = \left(\frac{1-k}{2}, 1\right]$ if and only if μ is an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra. Similarly, Theorem 13 establishes that $(X, \ell^{Q_k}_{\mu})$ is a semidetached SUP-subalgebra over $\Omega = \left(0, \frac{1-k}{2}\right]$ under the same condition. Therefore, the equivalence between the two semidetached structures directly follows.

Corollary 15. *For a fuzzy set μ in X, the following are equivalent.*

(1) (X, ℓ^{μ}_{U}) *is a semidetached SUP-subalgebra over* $\Omega = (\frac{1-k}{2}, 1]$,

(2) $(X, \ell^{\mu}_{Q_k})$ *is a semidetached SUP-subalgebra over* $\Omega = (0, \frac{1-k}{2}]$.

Definition 11. *A fuzzy set μ in X is called an $(\overline{\in}\vee\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X if it satisfies the following:*

$$(\forall x,y\in X)(\forall t,r\in(0,1])(((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\vee\overline{q_k}\mu\Rightarrow x_t\overline{\in}\vee\overline{q_k}\mu \text{ or } y_r\overline{\in}\vee\overline{q_k}\mu). \tag{16}$$

Theorem 14. *Every $(\overline{\in}\vee\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra.*

Proof. Let $x,y\in X$ and $t,r\in(0,1]$ be such that $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\mu$. Then $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\vee\overline{q_k}\mu$, and so $x_t\overline{\in}\vee\overline{q_k}\mu$ or $y_r\overline{\in}\vee\overline{q_k}\mu$ by (16). Therefore, μ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X.

Corollary 16 is an immediate consequence of Theorem 14, which states that every $(\in\vee q_k,\in\vee q_k)$-fuzzy SUP-subalgebra is also an $(\in,\in\vee q_k)$-fuzzy SUP-subalgebra. From this, properties (2), (3), and (4) follow directly by invoking Theorems 11, 12, and 13, respectively. Thus, the corollary summarizes the logical implications of Theorem 14 and the previously established equivalences.

Corollary 16. *If μ is an $(\overline{\in}\vee\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X, then*

(1) *μ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra,*

(2) *μ satisfies the condition (14),*

(3) *(X,ℓ_U^μ) is a semidetached SUP-subalgebra over $\Omega=(\frac{1-k}{2},1]$,*

(4) *$(X,\ell_{Q_k}^\mu)$ is a semidetached SUP-subalgebra over $\Omega=(0,\frac{1-k}{2}]$.*

Definition 12. *A fuzzy set μ in X is called a $(\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X if if it satisfies the following:*

$$(\forall x,y\in X)(\forall t,r\in(0,1])(((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{q_k}\mu\Rightarrow x_t\overline{\in}\vee\overline{q_k}\mu \text{ or } y_r\overline{\in}\vee\overline{q_k}\mu). \tag{17}$$

Theorem 15. *Assume that $\min\{t,r\}\leq\frac{1-k}{2}$ for any $t,r\in(0,1]$. Then every $(\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra.*

Proof. Let μ be an $(\overline{q_k},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X. Assume that

$$((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\mu$$

for $x,y\in X$ and $t,r\in(0,1]$ with $\min\{t,r\}\leq\frac{1-k}{2}$. Then $\mu((x|(y|y))|(x|(y|y)))<\min\{t,r\}\leq\frac{1-k}{2}$, and so $\mu((x|(y|y))|(x|(y|y)))+k+\min\{t,r\}<\frac{1-k}{2}+\frac{1-k}{2}+k=1$, that is, $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}q_k\mu$. It follows from (17) that $x_t\overline{\in}\vee\overline{q_k}\mu$ or $y_r\overline{\in}\vee\overline{q_k}\mu$. Therefore, μ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy SUP-subalgebra of X.

Corollary 17 directly follows from Theorem 15 by setting $k=0$. In this case, the condition $\min\{t,r\}\leq\frac{1-k}{2}$ becomes $\min\{t,r\}\leq 0.5$. Theorem 15 ensures that under this condition, every $(q_k, \in \vee q_k)$-fuzzy SUP-subalgebra is also an $(\in, \in \vee q_k)$-fuzzy SUP-subalgebra. Therefore, for $k = 0$, every $(q, \in \vee q)$-fuzzy SUP-subalgebra becomes an $(\in, \in \vee q)$-fuzzy SUP-subalgebra.

Corollary 17. *Assume that* $\min\{t, r\} \leq 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{q}, \overline{\in} \vee \overline{q})$*-fuzzy SUP-subalgebra is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy SUP-subalgebra.*

Theorem 16. *Assume that* $\min\{t, r\} > \frac{1-k}{2}$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy SUP-subalgebra is a* $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$*-fuzzy SUP-subalgebra.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X. Assume that

$$((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{q_k}\mu$$

for $x, y \in X$ and $t, r \in (0, 1]$ with $\min\{t, r\} > \frac{1-k}{2}$. If $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}} \in \mu$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{t, r\}$, and so $\mu((x|(y|y))|(x|(y|y))) + k + \min\{t, r\} > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Hence, $((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}q_k\mu$, a contradiction. Thus,

$$((x|(y|y))|(x|(y|y)))_{\min\{t,r\}}\overline{\in}\mu,$$

which implies from (13) that $x_t \overline{\in} \vee \overline{q_k}\mu$ or $y_r \overline{\in} \vee \overline{q_k}\mu$. Therefore, μ is a $(\overline{q_k}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebra of X.

Corollary 18. *Assume that* $\min\{t, r\} > 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy SUP-subalgebra is a* $(\overline{q}, \overline{\in} \vee \overline{q})$*-fuzzy SUP-subalgebra.*

5. Conclusion

In this paper, we have introduced the concept of semidetached SUP-algebras and explored their fundamental properties. The investigation into these algebraic structures has revealed several significant findings. The notion of semidetached SUP-subalgebras has been clearly defined, providing a new perspective on the relationships within SUP-algebras. This contributes to a deeper understanding of their algebraic properties and potential applications in various fields of study, including logic and functional analysis. Additionally, we have discussed various types of fuzzy SUP-subalgebras, including $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras and k-left (k-right) $(q_k, \overline{\in} \vee \overline{q_k})$-fuzzy SUP-subalgebras. These classifications are crucial for establishing the framework necessary for further research and applications of these algebraic structures.

The paper has provided several conditions under which a semidetached structure can be classified as a semidetached SUP-subalgebra. This is essential for validating the theoretical framework and ensuring that the properties discussed are applicable in practical scenarios. The findings of this study open avenues for future research, particularly in exploring the applications of semidetached SUP-algebras in logical systems and operator theory. The flexibility and completeness of the Sheffer stroke as a logical operator suggest that further investigations could yield valuable insights into both algebraic and logical frameworks. In conclusion, the exploration of semidetached SUP-subalgebras in SUP-algebras not only enhances our understanding of these structures but also sets the stage for future research that could bridge the gap between algebra, logic, and analysis. The potential applications of these findings are vast, and we encourage further exploration in this promising area of study.

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