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An Iterative Fractional Operational Matrix Method for Solving Fractional Undamped Duffing Equation with High Nonlinearity

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Abstract. This study presents a novel iterative fractional operational matrix method for solving the highly nonlinear fractional undamped Duffing equation. The proposed method efficiently approximates the solution by transforming the original fractional differential equation into a system of algebraic equations using modified operational matrices. The accuracy and effectiveness of this approach are validated through comparisons with established numerical methods and alternative analytical techniques from the literature. The results demonstrate that the proposed method provides a highly accurate approximation with rapid convergence. Furthermore, a rigorous convergence analysis is conducted to establish the existence and uniqueness of the solution. Notably, the study explores the impact of varying the fractional order revealing its significant influence on the system's dynamic behavior. As the fractional order approaches unity, the fractional model converges to its classical counterpart, highlighting the role of fractional derivatives in capturing memory effects and hereditary properties in physical systems.

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1. Introduction

The domain of fractional calculus broadens the conventional concepts of differentiation and integration to encompass non-integer orders, thereby offering a more comprehensive mathematical framework for these operations. The origins of fractional differentiation and integration date back to 1695, when Leibniz and Euler first introduced these concepts [1]. A distinguishing characteristic of fractional operators is their non-local behavior, which provides valuable insights into the historical and dynamic significance of fractional models.

In recent years, fractional differential equations have found widespread applications in modeling diverse real-world phenomena. For instance, it is examined their utility in analyzing and designing control systems, see [2]. Magin [3] identified three bioengineering domains where fractional calculus principles have been instrumental in developing innovative mathematical models. Similarly, Matlob and Jamali [4] employed fractional differential equations to study the dynamics of viscoelastic systems. In image processing, fractional orders have been utilized to improve denoising techniques [5], while other notable applications include finance [6], solid mechanics [7], and mathematical biology [8–12].

A significant recent development in fractional calculus is the Atangana-Baleanu fractional derivative [8]. Proposed by Abdon Atangana and Dumitru Baleanu in 2016, this derivative incorporates non-singular kernels and the Mittag-Leffler function. The application of fractional differentiation in [8] led to the formulation of a mathematical model for heat conduction in materials. This methodology has proven effective in addressing a wide range of real-world problems, as evidenced in [13–23].

In 1918, George Duffing introduced the Duffing equation in his publication [24]. Within his work, Duffing proposed a simplified mathematical model

$$x''(t) + a^2 x(t) - \beta x^2(t) - \gamma x^3(t) = k \sin(\omega t), \tag{1}$$

where he calculated the first term, $H \sin(\omega t)$, of the periodic solution. Duffing also proposed simplified forms of this equation to describe the motion of pendulums. For a symmetric pendulum, the equation takes the form

$$x''(t) + \alpha x(t) - \gamma x^3(t) = 0, \tag{2}$$

while for an asymmetric pendulum, it becomes:

$$x''(t) + \alpha x(t) - \gamma x^2(t) = 0. \tag{3}$$

Since its introduction, differential equations with polynomial nonlinearities have been referred to as Duffing's equations [25]. Ganji et al. [26] formulated a nonlinear differential equation for the cubic free undamped Duffing oscillator

$$x''(t) + \alpha x(t) + \beta x^3(t) = 0, \tag{4}$$

subject to the initial conditions:

$$x(0) = A, \quad x'(0) = 0. \tag{5}$$

Various researchers have addressed this problem numerically. For instance, He [27] applied the homotopy perturbation method to solve the Duffing equation, while Beléndez et al. [28] employed a modified version of the homotopy perturbation method. Additionally, Ramos, Syam, and Wazwaz utilized the variational iteration method [29–34], and Ghosh et al. [35] used the Adomian decomposition method. Other techniques include the artificial parameter decomposition method by Ramos [36] and He's parameter expanding method by Syam [37]. Several analytical methods have been developed for solving Duffing equations. For weak nonlinearity, approaches such as the method of multiple scales [38], the Krylov-Bogolubov method [14], straightforward expansions [39], and the Lindstedt-Poincaré method [39] have been employed. Strong cubic nonlinearity has also been studied, as discussed in [40, 41].

In this paper, we extend the problem defined in equations (4)-(5) to the fractional domain. The generalized form is given by

$$D^{2\alpha}x(t) + \beta x(t) + \gamma x^3(t) = 0, \quad \frac{1}{2} < \alpha \leq 1, \quad 0 < t < T, \tag{6}$$

subject to the initial conditions

$$x(0) = A, \quad D^{\alpha}x(0) = 0. \tag{7}$$

Here, the derivative D^{α} is interpreted in the Caputo sense.

The Operational Matrix Method (OMM) is a robust numerical approach widely employed for solving differential equations, particularly those arising in engineering and applied mathematics [42–49]. By converting differential operators into operational matrices, OMM effectively transforms differential equations into algebraic matrix equations [43, 44]. This transformation eliminates the necessity for explicit analytical integration or symbolic computation, streamlining the process of finding solutions. OMM is particularly advantageous in scenarios where the solutions can be approximated using polynomials or linear combinations of polynomial functions. It has found extensive applications in fields such as control system analysis, structural dynamics, heat transfer, and fluid mechanics. The method's versatility and computational efficiency have made it an essential tool in addressing a diverse array of problems across scientific and engineering disciplines. The effectiveness of the Operational Matrix Method lies in its ability to accurately approximate solutions by converting differential or integral equations into a system of algebraic equations. This is achieved through the use of operational matrices, which efficiently handle the underlying mathematical operations. The method is especially well-suited for problems where solutions can either be directly expressed in terms of polynomial functions or approximated as linear combinations of such functions. The OMM approach simplifies the computational process, reduces computational overhead, and enhances the accuracy of the results. These advantages have made it a preferred technique in numerous scientific and engineering domains. Applications of OMM include, but are not limited to, control system design, modeling of structural dynamics, heat conduction problems, and fluid flow analysis. The extensive use of OMM in these areas highlights its adaptability and robustness in solving a broad spectrum of mathematical and physical problems. The

Operational Matrix Method stands out as a highly efficient numerical technique for solving both linear and nonlinear differential equations, particularly those involving polynomial forms, [45, 50–56]. Its ability to transform complex mathematical problems into simpler algebraic systems has solidified its position as a valuable tool in applied mathematics and engineering. The references [50–53] provide further insights into the diverse applications of OMM, showcasing its reliability and effectiveness in tackling real-world challenges.

The organization of this paper is as follows. Section 2 outlines essential definitions and lemmas that serve as the foundation for the subsequent analysis. In Section 3, we present the modified iterative formulation of the Operational Matrix Method (OMM), detailing its theoretical framework and implementation. Section 4 is devoted to establishing key theoretical findings, including proofs for the existence and uniqueness of solutions, error estimation, and a comprehensive convergence analysis. In Section 5, we illustrate the practical applicability of the proposed method through three numerical examples, demonstrating its convergence to the unique solution of Problem (6)-(7). The paper concludes with Section 6, where we provide a summary of our findings and offer remarks on potential future research directions.

2. Preliminarily

This section presents several essential definitions and results that are integral to the analysis within this paper.

Definition 1. *[1] For $\alpha \in (0,1)$ and $t > 0$, the Caputo derivative (CD) of the function $x(t)$ is defined as*

$$D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s)\, ds, \tag{8}$$

while the fractional integral (FI) operator is expressed as:

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds. \tag{9}$$

The subsequent lemma outlines key properties of the Caputo derivative and the fractional integral.

Lemma 1. *[1] For $\alpha \in (0,1)$ and $x(t) \in C[0,T]$, the following relations hold:*

$$I^{\alpha}D^{\alpha}x(t) = x(t) - x(0), \tag{10}$$

$$D^{\alpha}I^{\alpha}x(t) = x(t). \tag{11}$$

In this paper, we utilize the concept of the sequential Caputo derivative, defined as follows.

Definition 2. *[43] Let $\alpha \in (0,1)$, and for a given function $x(t)$, the sequential Caputo derivative of order α is defined recursively as*

$${}^{c}D^{L\alpha}x(t) = {}^{c}D^{\alpha}\left({}^{c}D^{(L-1)\alpha}x(s)\right), \quad L \in \mathbb{N}, \tag{12}$$

where ${}^{sc}D^{L\alpha}x(t)$ *denotes the sequential Caputo derivative of order* α*. For ease of notation, we will refer to it simply as* $D^{L\alpha}x(t)$ *when discussing the sequential derivative.*

Next, we introduce the Operational Matrix Method (OMM) with the following definition.

Definition 3. *[42, 49] Let* $t_s = s\Delta$ *for* $s \in \{0, 1, 2, \dots, M-1\}$ *and* $\Delta = \frac{T}{M}$*, where* $M \in \mathbb{N}$*. The s-block pulse function (BPF) is defined as:*

$$\mu_s(t) = \begin{cases} 1, & t_s \le t < t_{s+1}, \\ 0, & \textit{otherwise}, \end{cases} \quad 0 \le s < M.$$

The product and orthogonality properties of BPFs follow directly from the above definition, as given in the next theorem.

Theorem 1. *[43–45] Let* $\{t_0 = 0, t_1, \dots, t_M = T\}$ *be a uniform partition of* $[0, T]$*, and let* $\{\mu_0(t), \mu_1(t), \dots, \mu_{M-1}(t)\}$ *be the corresponding BPFs. Then, for any* $0 \le i, j \le M-1$*, the following relations hold:*

$$\mu_i(t)\mu_j(t) = \begin{cases} \mu_i(t), & \textit{if } i = j, \\ 0, & \textit{otherwise}. \end{cases} \tag{13}$$

and

$$\int_0^T \mu_i(t)\mu_j(t)\,dt = \begin{cases} \Delta, & \textit{if } i = j, \\ 0, & \textit{otherwise}. \end{cases} \tag{14}$$

We conclude this section by stating the completeness property in the following lemma.

Lemma 2. *[45–47] If* $x \in L^2[0, T)$*, then we have the expansion:*

$$x(t) = \sum_{s=0}^{\infty} x_s \mu_s(t), \tag{15}$$

where

$$x_s = \frac{1}{\Delta} \int_{s\Delta}^{(s+1)\Delta} x(r)\,dr. \tag{16}$$

For numerical purposes, we approximate $x(t)$ by:

$$x(t) \approx \sum_{s=0}^{M-1} x_s \mu_s(t), \quad M \gg 1. \tag{17}$$

3. Iterative Operational Matrix Method

Syam et al. [42–44, 46–49], introduced an inefficient iterative method which reduce the computational time and cost, and increase the effecinet of the OMM. In this section, we will follow this approach to solve our problem (6)-(7). First, by applying the fractional integral operator (9) to both sides of (6), we obtain the following equation:

$$D^{\alpha}x(t) - D^{\alpha}x(0) + \frac{\beta}{\Gamma(\alpha)}\int_0^t x(s)(t-s)^{\alpha-1}ds + \frac{\gamma}{\Gamma(\alpha)}\int_0^t x^3(s)(t-s)^{\alpha-1}ds = 0.$$

Since $D^{\alpha}x(0) = 0$, this simplifies to:

$$D^{\alpha}x(t) = -\frac{\beta}{\Gamma(\alpha)}\int_0^t x(s)(t-s)^{\alpha-1}ds - \frac{\gamma}{\Gamma(\alpha)}\int_0^t x^3(s)(t-s)^{\alpha-1}ds. \tag{18}$$

Next, applying the fractional integral operator (9) again to both sides of equation (18), we obtain

$$x(t) - x(0) = -\frac{\beta}{\Gamma^2(\alpha)}\int_0^t\int_0^s x(w)(t-s)^{\alpha-1}(s-w)^{\alpha-1}dwds$$
$$-\frac{\gamma}{\Gamma^2(\alpha)}\int_0^t\int_0^s x^3(w)(t-s)^{\alpha-1}(s-w)^{\alpha-1}dw\,ds.$$

Since $x(0) = A$, we can rewrite this as:

$$x(t) = A - \frac{\beta}{\Gamma^2(\alpha)}\int_0^t\int_0^s x(w)(t-s)^{\alpha-1}(s-w)^{\alpha-1}dw\,ds$$
$$-\frac{\gamma}{\Gamma^2(\alpha)}\int_0^t\int_0^s x^3(w)(t-s)^{\alpha-1}(s-w)^{\alpha-1}dwds. \tag{19}$$

Let $\{t_0 = 0, t_1, \ldots, t_M = T\}$ represent a uniform partition of $[0, T]$, and let $\{\mu_0(t), \mu_1(t), \ldots, \mu_{M-1}(t)\}$ be the corresponding block pulse functions (BPFs). We approximate $x(t)$ using these BPFs as follows

$$x(t) = \sum_{i=0}^{M-1} x_i\mu_i(t). \tag{20}$$

By applying the collocation method to equation (20) with collocation points at t_j, where $1 \le j < M$, we get

$$x(t_j) = \sum_{i=0}^{M-1} x_i\mu_i(t_j) = x_j, \quad 0 \le j \le M-1, \tag{21}$$

which leads to the following equation:

$$x_j = A - \frac{\beta}{\Gamma^2(\alpha)}\int_0^{t_j}\int_0^s \left(\sum_{i=0}^{M-1} x_i\mu_i(w)\right)(t_j-s)^{\alpha-1}(s-w)^{\alpha-1}dw\,ds$$

$$- \frac{\gamma}{\Gamma^2(\alpha)} \int_0^{t_j} \int_0^s \left(\sum_{i=0}^{M-1} x_i \mu_i(w) \right)^3 (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds. \tag{22}$$

This equation can be solved due to the fact that

$$\mu_i(t_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

By utilizing the properties of the Riemann integral, equation (22) simplifies to

$$x_j = A - \frac{\beta}{\Gamma^2(\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \int_0^s \left(\sum_{i=0}^{M-1} x_i \mu_i(w) \right) (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds$$
$$- \frac{\gamma}{\Gamma^2(\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \int_0^s \left(\sum_{i=0}^{M-1} x_i \mu_i(w) \right)^3 (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds. \tag{23}$$

For any $1 \leq k, j < M$, we know that

$$\mu_j(t) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}, \quad t \in [t_k, t_{k+1}).$$

This leads to the following simplifications

$$\int_{t_k}^{t_{k+1}} \int_0^s \left(\sum_{i=0}^{M-1} x_i \mu_i(w) \right) (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds$$
$$= \int_{t_k}^{t_{k+1}} \int_0^s x_k (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds,$$
$$\int_{t_k}^{t_{k+1}} \int_0^s \left(\sum_{i=0}^{M-1} x_i \mu_i(w) \right)^3 (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds$$
$$= \int_{t_k}^{t_{k+1}} \int_0^s x_k^3 (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds.$$

Thus, we can express the equation for x_j as

$$x_j = A - \frac{\beta}{\Gamma^2(\alpha)} \sum_{k=0}^{j-1} \eta_{jk} x_k - \frac{\gamma}{\Gamma^2(\alpha)} \sum_{k=0}^{j-1} \eta_{jk} x_k^3, \tag{24}$$

where the coefficients η_{jk} are defined as

$$\eta_{jk} = \int_{t_k}^{t_{k+1}} \int_0^s (t_j - s)^{\alpha-1} (s-w)^{\alpha-1} dw\, ds. \tag{25}$$

4. Error Analysis

Let $x \in L^2([0,T))$ be a function, and its norm is defined by the following formula:

$$||x|| = \sqrt{\int_0^T |x(t)|^2 \, dt}. \tag{26}$$

According to Lemma (2), the function $x(t)$ can be approximated as

$$x_M(t) = \sum_{i=0}^{M-1} x_i \mu_i(t). \tag{27}$$

The aim of Theorem (2) is to demonstrate that the mean square error is minimized when the coefficients x_i are chosen according to Equation (16).

Theorem 2. *Let $x \in L^2([0,T))$ and define $x_M(t)$ as given in Equation (27). Then, the mean square error term*

$$\mathcal{A}(x_0, x_1, \ldots, x_{M-1}) = \int_0^T (x(t) - x_M(t))^2 \, dt$$

is minimized when x_i is computed according to Equation (16) for all $i = 0, 1, \ldots, M-1$.

Proof. For $0 \le i \le M-1$, by applying Theorem (1), we compute

$$\frac{\partial \mathcal{A}}{\partial x_i} = 2 \int_0^T (x(t) - x_M(t)) \, \mu_i(t) \, dt = 2 \left(\int_0^T x(t)\mu_i(t) \, dt - \Delta x_i \right) = 0.$$

Therefore, we obtain

$$x_i = \frac{1}{\Delta} \int_0^T x(t)\mu_i(t) \, dt.$$

Additionally, Theorem (1) provides the second-order derivatives as:

$$\frac{\partial^2 \mathcal{A}}{\partial x_i \partial x_j} = \begin{cases} 2\Delta, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases} \quad 0 \le i, j \le M-1.$$

For any $0 \le i \le M-1$, we compute the determinant of the Hessian matrix:

$$\begin{vmatrix} \frac{\partial^2 \mathcal{A}}{\partial x_0^2} & \frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_i} \\ \frac{\partial^2 \mathcal{A}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{A}}{\partial x_1^2} & \cdots & \frac{\partial^2 \mathcal{A}}{\partial x_1 \partial x_i} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{A}}{\partial x_i \partial x_0} & \frac{\partial^2 \mathcal{A}}{\partial x_i \partial x_1} & \cdots & \frac{\partial^2 \mathcal{A}}{\partial x_i^2} \end{vmatrix} = \begin{vmatrix} 2\Delta & 0 & \ldots & 0 \\ 0 & 2\Delta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2\Delta \end{vmatrix} = (2\Delta)^{i+1} > 0.$$

Thus, the error $\mathcal{A}$ is minimized when x_i is computed according to Equation (16) for all $i = 0, 1, \ldots, M-1$.

Next, we aim to establish the order of the mean square error in approximating $x(t)$ over the interval $[0, T)$.

Theorem 3. *Let $x(t)$ be a differentiable function on $[0, T)$ such that*

$$|x'(t)| \leq \mathcal{F} \tag{28}$$

for all $t \in [0, T)$, where $\mathcal{F}$ is a positive constant. Then,

$$||\mathcal{A}||^2 \leq C\Delta^2 \tag{29}$$

where $\mathcal{A}(x) = x(t) - x_M(t)$, $t \in [0, T)$, and C is a positive constant, with $\Delta = \frac{T}{M}$.

Proof. Let $t_i = i\Delta$ and $\mathcal{A}_i = [t_i, t_{i+1})$, where $\Delta = \frac{T}{M}$ and $i = 0, 1, \ldots, M-1$. By applying the mean value theorem for integrals and Equation (16), we obtain

$$\begin{aligned} x_M(t) &= x_i, \quad t \in [t_i, t_{i+1}), \\ &= \frac{1}{\Delta}\int_{t_i}^{t_{i+1}} x(t)\, dt, \\ &= x(\nu_i), \quad \nu_i \in [t_i, t_{i+1}), \quad i = 0, 1, \ldots, M-1. \end{aligned}$$

Using the mean value theorem for integrals, we have:

$$\begin{aligned} ||\mathcal{A}||^2 &= \int_0^T (x(t) - x_M(t))^2\, dx \\ &= \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (x(t) - x_M(t))^2\, dx \\ &= \Delta \sum_{i=0}^{M-1} (x(\omega_i) - x(\nu_i))^2 \end{aligned}$$

where $\omega_i, \nu_i \in [t_i, t_{i+1})$. By the mean value theorem and Equation (28), we get:

$$\begin{aligned} ||\mathcal{A}||^2 &\leq \Delta\mathcal{F}^2 \sum_{i=0}^{M-1} (\omega_i - \nu_i)^2 \\ &\leq \Delta\mathcal{F}^2 \sum_{i=0}^{M-1} \Delta^2 \\ &= M\mathcal{F}^2\Delta^3 = (T\mathcal{F}^2)\Delta^2 = C\Delta^2. \end{aligned}$$

Thus, $C = \mathcal{F}^2 T$.

The above result confirms that the mean square error in the approximation is of order Δ^2.

t	HPM	MHPM	SHPM	Present	Mathematica
0.5	0.762476	0.768902	0.768766	0.768802	0.768802
1.0	0.176929	0.233741	0.233680	0.233692	0.233692
2.0	−1.055110	−0.891260	−0.859323	−0.859349	−0.859349
3.5	−0.461650	−0.079433	−0.093034	−0.093013	−0.093013
5.0	2.049041	0.996472	0.947107	0.947130	0.947130

Table 1: Comparison of approximate solutions for $A = 1, \alpha = 1, \beta = 1$, and $\gamma = 1$.

t	HPM	MHPM	SHPM	Present	Mathematica
1	0.056288	0.080176	0.080519	0.080527	0.080527
2	−0.808192	−0.739174	−0.729000	−0.729018	−0.729018
3	−0.339208	−0.239413	−0.238620	−0.238626	−0.238626
4	0.891267	0.706827	0.667953	0.668022	0.668022
5	0.893003	0.395315	0.387550	0.387551	0.387551

Table 2: Comparison of approximate solutions for $A = 0.75, \alpha = 1, \beta = 1.5$, and $\gamma = 1.5$.

5. Numerical Results

We first analyze Problem (6)-(7) for the case where $\alpha = 1$. Since the exact solution for this problem is unavailable, numerical solutions were computed using the built-in Wolfram Mathematica routine, employing the fully explicit Runge-Kutta method. These solutions serve as the standard reference for comparison. In Tables 1 and 2, we present a comparison of our results with those obtained using HPM [54], MHPM [26], and SHPM [54], alongside the numerical solution. Table 1 corresponds to the parameters $A = 1, \alpha = 1, \beta = 1$, and $\gamma = 1$, while Table 2 pertains to $A = 0.75, \alpha = 1, \beta = 1.5$, and $\gamma = 1.5$.

Figures 1–5 illustrate the comparisons between the current method and numerical solutions under varying parameters. Specifically, Figure 1 evaluates $A = 1, \alpha = 1, \beta = 0.5$, and $\gamma = 2$; Figure 2 examines $A = 1.5, \alpha = 1, \beta = 1$, and $\gamma = 0.5$; Figure 3 considers $A = 1.5, \alpha = 1, \beta = 0.5$, and $\gamma = 1.5$; Figure 4 studies $A = 2, \alpha = 1, \beta = 1.5$, and $\gamma = 1$; and Figure 5 assesses $A = 1.5, \alpha = 1, \beta = 1.5$, and $\gamma = 1.5$.

Finally, the behavior of Problem (6)-(7) is investigated for $A = 1, \beta = 0.5$, and $\gamma = 2$ with varying values of α ($\alpha = 0.75, 0.85, 0.9, 0.95, 0.995, 1$). Results are depicted in Figures 6 and 7. Similarly, for $A = 1.5, \beta = 1.5$, and $\gamma = 1.5$, outcomes for $\alpha = 0.75, 0.85, 0.9, 0.95, 0.995, 1$ are illustrated in Figures 8 and 9. As α approaches 1, the results converge toward the solution of the classical second-order differential equation.

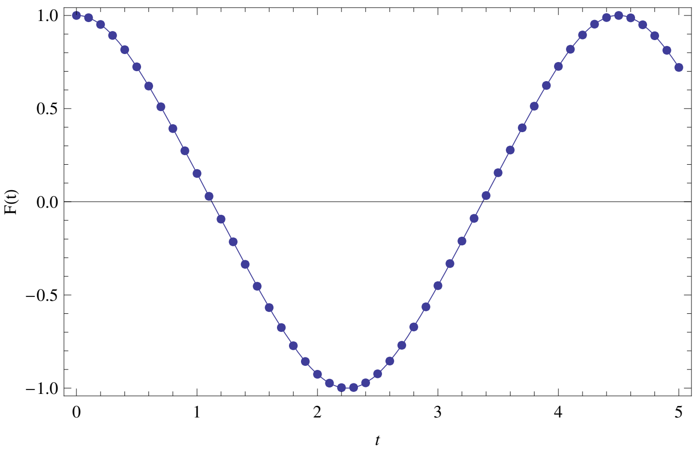

Figure 1: $A = 1, \alpha = 1, \beta = 0.5$, and $\gamma = 2$. Numerical (solid) vs. present method (dashed).

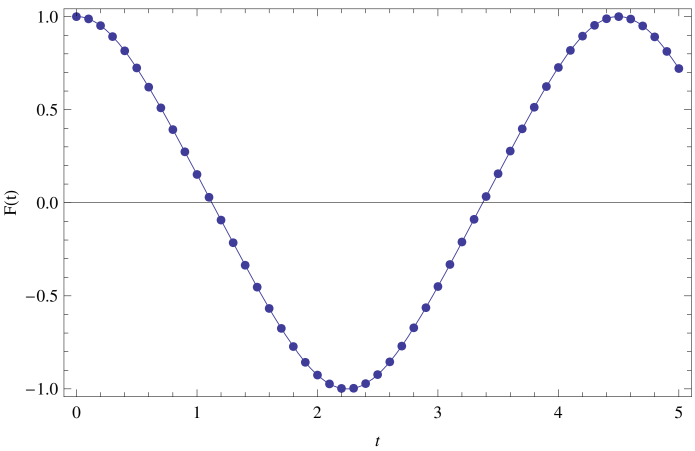

Figure 2: $A = 1.5, \alpha = 1, \beta = 1$, and $\gamma = 0.5$. Numerical (solid) vs. present method (dashed).

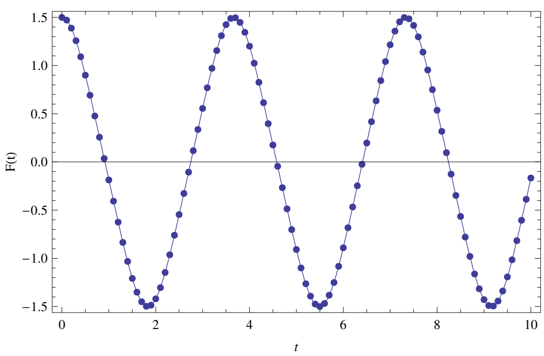

Figure 3: $A = 1.5, \alpha = 1, \beta = 0.5$, and $\gamma = 1.5$. Numerical (solid) vs. present method (dashed).

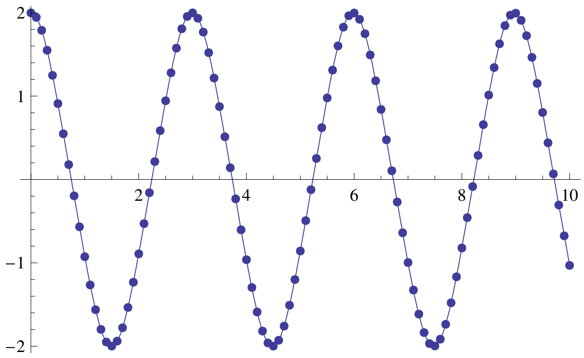

Figure 4: $A = 2, \alpha = 1, \beta = 1.5$, and $\gamma = 1$. Numerical (solid) vs. present method (dashed).

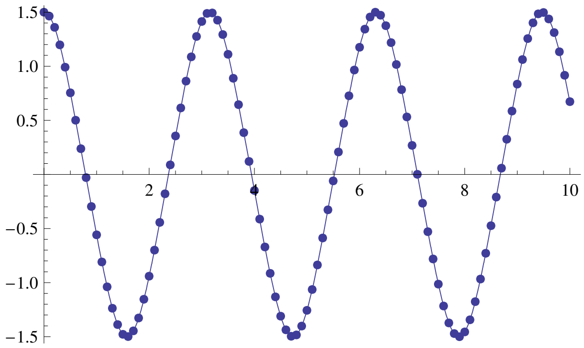

Figure 5: $A = 1.5, \alpha = 1, \beta = 1.5$, and $\gamma = 1.5$. Numerical (solid) vs. present method (dashed).

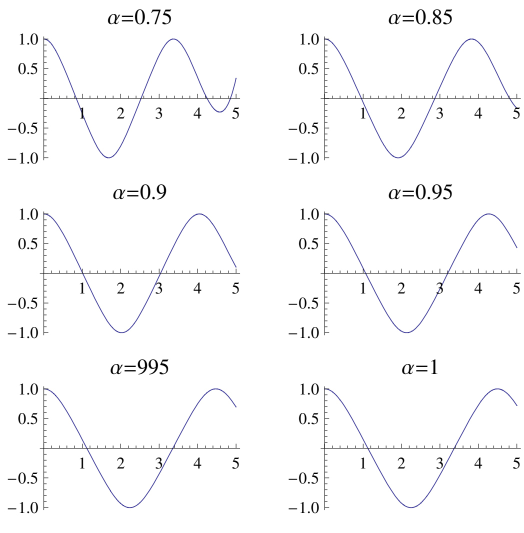

Figure 6: $A = 1, \beta = 0.5$, and $\gamma = 2$ for $\alpha = 0.75, 0.85, 0.9, 0.95, 0.995, 1$.

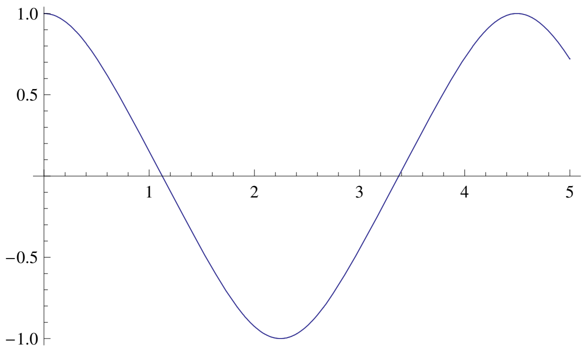

Figure 7: Results for $A = 1, \beta = 0.5, \gamma = 2, \alpha = 0.995, 1$.

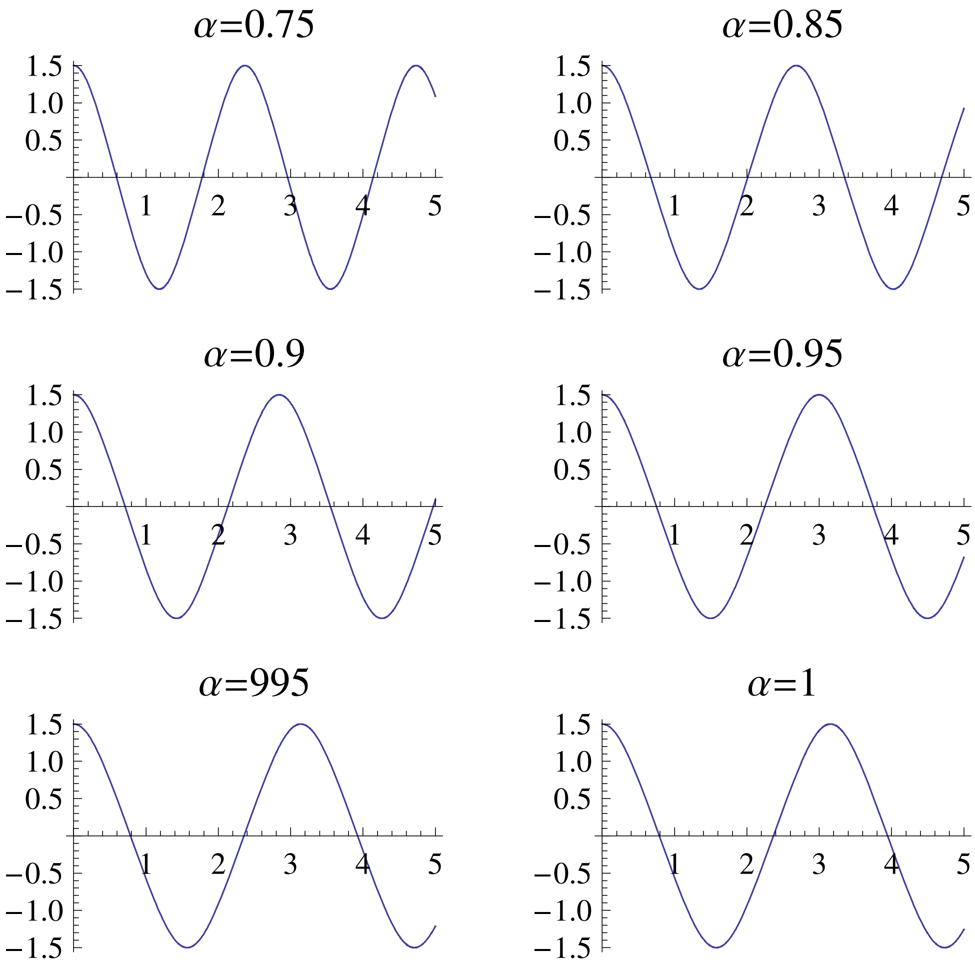

Figure 8: Results for $A = 1.5, \beta = 1.5, \gamma = 1.5, \alpha = 0.75, 0.85, 0.9, 0.95, 0.995, 1$.

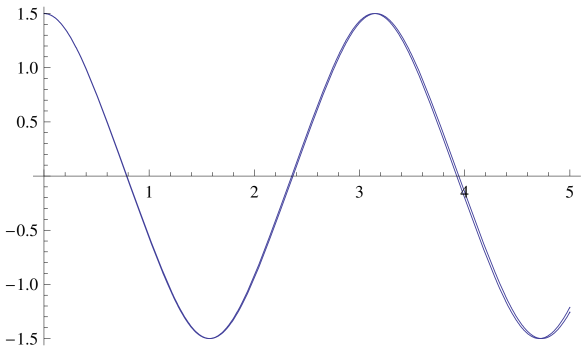

Figure 9: Results for $A = 1.5, \beta = 1.5, \gamma = 1.5, \alpha = 0.995, 1.$

6. Conclusion

In this work, we developed and analyzed an iterative fractional operational matrix method, based on the method described in [13], for solving the nonlinear fractional undamped Duffing equation. The proposed method efficiently reduces computational complexity while maintaining high accuracy, as demonstrated through comparisons with existing analytical and numerical techniques. The results confirm that the method provides an excellent approximation, outperforming classical perturbation-based approaches such as the Homotopy Perturbation Method (HPM), Modified HPM (MHPM), and Simplified HPM (SHPM).

A key contribution of this study is the investigation of the effect of varying the fractional order α beyond the provided range. Our findings reveal that decreasing α leads to a significant deviation from the classical Duffing oscillator, introducing memory-dependent effects and altering the system's response dynamics. As α approaches unity, the system gradually transitions to the standard integer-order Duffing equation, confirming the robustness of fractional modeling in capturing real-world physical behaviors. This insight is crucial for applications in nonlinear dynamics, where fractional-order models provide a more generalized framework for analyzing oscillatory systems with memory effects.

Future research directions include extending the method to more complex nonlinear oscillatory systems, exploring higher-dimensional fractional models, and investigating real-world applications such as mechanical vibrations and bioengineering systems. Additionally, further refinement of the numerical scheme may improve computational efficiency for large-scale problems.

Author Contributions:

All authors have the same contribution.

Data Availability Statement:

Not applicable.

Conflicts of Interest:

The authors declare no conflict of interest.

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