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Certain Properties of Δ_h Legendre Polynomials and Applications in Computer Modelling

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Abstract. This study investigates the development of polynomials, with a particular focus on the unique Δ_h Legendre polynomials. Explicit formulas for these polynomials are derived, along with summation formulae that provide further structural insights. Additionally, the monomiality principle is established, reinforcing the algebraic framework of these polynomials. Symmetric identities are also formulated, highlighting their fundamental properties. The study concludes with remarks summarizing the key findings and potential directions for future research.

2020 Mathematics Subject Classifications: 33E20, 33B10, 33E30, 11T23

Key Words and Phrases: Δ_h sequences, Monomiality principle, explicit forms, Symmetric identities

1. Introduction and preliminaries

Many fields, such as statistical mechanics and quantum mechanics, have used specific polynomials to represent and characterize the behavior of complex systems. There are several other fields, including statistics and quantum mechanics, where complex systems have been described and analyzed using these special polynomials. In a number of mathematical fields, including combinatorics, entropy, and algebraic combinatorics, polynomial sequences are essential. In approximation theory and physics, Legendre polynomials,

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named after the French mathematician Edmond Legendre, were introduced in the 19th century. These polynomials arise as solutions to the second-order Legendre differential equation and are defined on the interval $[0, +\infty)$, commonly denoted by $\mathbb{S}_n(u)$, where n represents the degree.

Legendre polynomials exhibit orthogonality with respect to the weight function e^{-u} on $[0, +\infty)$, ensuring that the weighted integral of two polynomials with different degrees is zero. This orthogonality and their recurrence relation enable efficient computation of higher-degree polynomials from lower-degree ones. Additionally, they allow functions to be expressed as series expansions using their generating function, simplifying the derivation of closed-form solutions for certain differential equations. Widely applied in mathematics, physics, and engineering, Legendre polynomials are integral to solving the Schrödinger equation for spherically symmetric quantum systems, including the hydrogen atom. They also play a crucial role in problems involving diffusion, wave propagation, and heat conduction.

Much recent work has focused on two-variable special polynomials in mathematical physics. There are characteristics of a class of polynomials called two-variable special polynomials, such as [1–10]. They are widely studied in the subject of algebraic geometry and have many applications in mathematics and other domains. Notable instances of two-variable special polynomials include bivariate Chebyshev, Hermite, Laguerre, and Legendre polynomials, etcetra. Approximation theory, numerical analysis, and signal processing all make extensive use of them. Bivariate Legendre polynomials are Legendre polynomials extended to two variables. They are useful in quantum mechanics, potential theory, and random matrix theory, and they satisfy a bivariate counterpart of the Legendre differential equation. The behavior of two-degree-of-freedom systems can be studied with the help of bivariate Legendre polynomials. These polynomials are widely studied in mathematical physics, probability theory, and approximation theory because they meet a specific orthogonality condition concerning a weight function. These two-variable special polynomials are important because they can be used to solve problems in a variety of mathematical and scientific fields, offer a rich framework for expressing and analyzing multivariate functions, and have particular characteristics that make them appropriate for particular applications.

Two-variable Legendre polynomials, denoted as $\mathbb{S}_n(u, v)$ [11], hold significant mathematical value and serve as essential tools in physics due to their broad applications. These polynomials provide a powerful framework for analyzing solutions to various partial differential equations commonly arising in physical contexts.

The 2-variable Legendre polynomials (2VLeP) $\mathbb{S}_n(u, v)$ are defined through the following generating function:

$$e^{vt} J_0(2t\sqrt{-u}) = \sum_{n=0}^{\infty} \mathbb{S}_n(u, v) \frac{t^n}{n!}, \tag{1}$$

where $J_0(ut)$ is the 0^{th} order ordinary Bessel function of first kind [12] defined by

$$J_n(2\sqrt{u}) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (\sqrt{u})^{n+2k}}{k! \ (n+k)!}. \tag{2}$$

We also note that

$$\exp(-\alpha D_u^{-1}) = J_0(2\sqrt{\alpha u}), \qquad D_u^{-n}\{1\} := \frac{u^n}{n!} \tag{3}$$

is the inverse derivative operator.

Or, Alternatively by

$$e^{vt} C_0(-ut^2) = \sum_{n=0}^{\infty} \mathbb{S}_n(u,v) \frac{t^n}{n!}, \tag{4}$$

where $C_0(ut)$ is the 0^{th} order Tricommi function of first kind [12] with

$$C_0(-ut^2) = e^{D_u^{-1} t^2}. \tag{5}$$

Thus, in view of equation (3) or (5), the generating expression for Legendre polynomials can be casted as:

$$e^{vt} \ e^{D_u^{-1} t^2} = \sum_{n=0}^{\infty} \mathbb{S}_n(u,v) \frac{t^n}{n!}. \tag{6}$$

In recent years, mathematicians have shown a growing interest in the study of Δ_h forms of special polynomials, motivated by their analytical significance and computational utility. Several generalizations of these polynomials have been explored in works such as [13–18]. Expanding upon these studies, the classical finite difference operator Δ_h has been employed to introduce a novel class known as Δ_h special polynomials, as discussed in [19–23].

These polynomials are not only significant in pure mathematical theory but also hold substantial applications in computational modeling. In numerical analysis, Δ_h special polynomials serve as essential tools for approximating solutions to differential and integral equations, particularly in discretized frameworks. In physics and engineering, they provide efficient computational schemes for modeling wave propagation, heat conduction, and fluid dynamics. Their structured recurrence relations and explicit forms allow for the development of stable numerical algorithms, making them valuable in computer simulations.

Additionally, in statistical computing, these polynomials facilitate the design of discrete probability distributions and contribute to data interpolation techniques, enhancing predictive modeling. Their role in symbolic computation further extends to software implementations for solving partial difference equations in image processing and digital signal analysis. Given their computational efficiency and adaptability, Δ_h special polynomials are emerging as powerful mathematical tools with extensive applications in computer-aided modeling and simulations. In symbolic computation and computer algebra systems (CAS), these polynomials aid in the efficient manipulation of large-scale algebraic expressions. Their structured operational rules allow for automated simplifications and symbolic differentiation, making them valuable in software implementations for solving discrete versions of differential equations, such as in image processing, digital signal analysis, and network modelling. Furthermore, in graph theory and combinatorial optimization, Δ_h polynomials provide analytical tools for studying discrete structures and optimizing network algorithms. Their application extends to shortest path problems, spanning tree calculations, and network flow optimizations, which are crucial in computer science and logistics.

"These Δ_h-Appell polynomial are represented as:

$$\mathbb{A}_n^{[h]}(u) := \mathbb{A}_n(u), \quad n \in \mathbb{N}_0 \tag{7}$$

and defined by

$$\mathbb{A}_n^{[h]}(u) = nh\mathcal{A}_{n-1}(u), \quad n \in \mathbb{N}, \tag{8}$$

where Δ_h is the finite difference operator:

$$_{\Delta_h}\mathbb{H}^{[h]}(u) = \mathbb{H}(u+h) - \mathbb{H}(u)". \tag{9}$$

The Δ_h-Appell polynomials $\mathbb{A}_n(u)$ are specified by the following generating function [19]:

$$\gamma(t)(1+ht)^{\frac{u}{h}} = \sum_{n=0}^{\infty} \mathbb{A}_n^{[h]}(u)\frac{t^n}{n!}, \tag{10}$$

where

$$\gamma(t) = \sum_{n=0}^{\infty} \gamma_{n,h}\frac{t^n}{n!}, \qquad \gamma_{0,h} \neq 0. \tag{11}$$

Motivated by Costabile [19], here we introduced the two variable Δ_h Legendre polynomials:

$$(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} = \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} \tag{12}$$

through the generating function concept.

The article is organized to give readers a thorough grasp of the Δ_h Legendre polynomials, with Section 2 outlining how they are generated, how they recur, and how they are evaluated. Effective calculation is made possible by the formulas provided in Section 3 for evaluating or adding up these polynomials under particular circumstances. By analyzing the behavior of Δ_h Legendre polynomials under different operations and determining their determinant form, Section 4 explores the Momomiality principle. In Section 5, symmetric identities are obtained for these polynomials. By summarizing the results and going over applications, consequences, and possible future research areas related to Δ_h Legendre polynomials, the conclusion enhances understanding of their behavior and usefulness in a variety of mathematical contexts.

2. Two variable Δ_h Legendre polynomials

This section explores a new class of two-variable Δ_h Legendre polynomials, establishing their fundamental properties and expanding the scope of polynomial theory. The development of their generating function, $\mathbb{S}_\omega^{[h]}(u,v)$, provides deeper insight into their structure and behavior, which is crucial for applications in combinatorics, analysis, and mathematical physics. By linking these polynomials to their generating function, this study enhances the understanding of polynomial families and their applications. The findings offer valuable perspectives on their special properties, paving the way for further research in mathematical and scientific domains.

First, we derive the generating function for these Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$ by proving the following result:

Theorem 1. *For the two variable Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$, the succeeding generating relation holds true:*

$$(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} = \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}, \tag{13}$$

or equivalently

$$(1+ht)^{\frac{v}{h}} C_0\left(\frac{-u}{h}\log(1+ht^2)\right) = \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}. \tag{14}$$

Proof. Expanding $(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}$ around $u=v=0$ using a Newton series for finite differences and analyzing the product's development with respect to t's powers, we identify the polynomials $\mathbb{S}_n^{[h]}(u,v)$ as coefficients of $\frac{t^n}{n!}$. These coefficients, given in equation (13), represent the generating function of the two-variable Δ_h-Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$.

Theorem 2. *For the two variable Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$, the succeeding relations hold true:*

$$\begin{aligned} \frac{{}_v\Delta_h}{h}\mathbb{S}_n^{[h]}(u,v) &= n\ \mathbb{S}_{n-1}^{[h]}(u,v) \\ \frac{{}_u\Delta_h}{h}\mathbb{S}_n^{[h]}(u,v) &= n(n-1)\ \mathbb{S}_{n-2}^{[h]}(u,v), \quad D_u^{-1}\to u. \end{aligned} \tag{15}$$

Proof. By differentiating (13) w.r.t. v by taking into consideration of expression (5), we have

$$\begin{aligned} {}_v\Delta_h(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} &= (1+ht)^{\frac{v+h}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} - (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} \\ &= (1+ht-1)(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} \end{aligned}$$

$$= ht\ (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}. \tag{16}$$

By substituting r.h.s. of expression (13) in (16), we find

$$ {}_v\Delta_h \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} = h\sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^{n+1}}{n!}. \tag{17}$$

Substituting $n \to n-1$ into the right-hand side of (16) and equating the coefficients of identical powers of t in the resulting expression leads to the derivation of (15).

We now derive the explicit form of the two-variable Δ_h Legendre polynomials, $\mathbb{S}_n^{[h]}(u,v)$, as stated in the following theorem:

Theorem 3. *The two-variable Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$ satisfy the following relations:*

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{d=0}^{\frac{v}{h}} \binom{n}{d}\binom{\frac{v}{h}}{d}\ h^d\ \mathbb{S}_{n-d}^{[h]}(u). \tag{18}$$

Proof. Expanding generating relation (13) in the given manner:

$$(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} = \sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d}\frac{(ht)^d}{d!}\ \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,0)\frac{t^n}{n!} \tag{19}$$

which can further be written as

$$\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d}\ h^d\ \mathbb{S}_n^{[h]}(u,0)\frac{t^{n+d}}{n!\ d!}. \tag{20}$$

By replacing $n \to n-d$ in the r.h.s. of previous expression, it follows that

$$\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d}\ h^d\ \mathbb{S}_n^{[h]}(u,0)\frac{t^{n}}{(n-d)!\ d!}. \tag{21}$$

On multiplying and dividing by $n!$ in the r.h.s. of previous expression (21) and comparing the coeffecients of same exponents of t on both sides, assertion (18) is deduced.

3. Summation formulae

This section presents summation formulae, essential tools in mathematical analysis for understanding polynomial structures in two variables. These formulas systematically compute sums of specific polynomials, revealing hidden symmetries and interrelationships between variables. Their applications extend to mathematical physics, probability theory, and combinatorics, aiding in the development of efficient computational techniques. Summation equations serve as fundamental building blocks for advancing mathematical theory and its real-world applications. Below, we establish these summation formulae by proving the following key results:

Theorem 4. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(v+1,u) = \sum_{m=0}^{n} \binom{n}{m} \left(-\frac{1}{h}\right)_m (-h)^m \mathbb{S}_{n-m}^{[h]}(u,v). \tag{22}$$

Proof. From (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(v+1,u)\frac{t^n}{n!} - \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} &= (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} \left((1+ht)^{\frac{1}{h}} - 1\right) \\ &= \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} \left(\sum_{m=0}^{\infty} \left(-\frac{1}{h}\right)_m (-h)^m \frac{t^m}{m!} - 1\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \left(-\frac{1}{h}\right)_m (-h)^m \mathbb{S}_{n-m}^{[h]}(u,v)\right) \frac{t^n}{n!} - \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}. \end{aligned} \tag{23}$$

Comparing the coefficients of t, we obtain (22).

Theorem 5. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{j=0}^{[\frac{n}{2}]} \left(-\frac{v}{h}\right)_{n-2j} (-h)^{n-j} \left(-\frac{u}{h}\right)_j (-1)^j \frac{n!}{(n-2j)!(j!)^2}. \tag{24}$$

Proof. From (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} &= (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} \\ &= \sum_{n=0}^{\infty} \left(-\frac{v}{h}\right)_n (-h)^n \frac{t^n}{n!} \sum_{j=0}^{\infty} \left(-\frac{u}{h}\right)_j (-1)^j (-h)^j \frac{t^{2j}}{j!j!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{[\frac{n}{2}]} \left(-\frac{v}{h}\right)_{n-2j} (-h)^{n-j} \left(-\frac{u}{h}\right)_j (-1)^j \frac{t^n}{(n-2j)!(j!)^2}. \end{aligned} \tag{25}$$

Comparing the coefficients of t, we obtain (24).

Now, we investigate the connection between the Stirling numbers of the first kind and 2-variable Δ_h Legendre polynomials.

$$\frac{[\log(1+t)]^k}{k!} = \sum_{i=k}^{\infty} S_1(i,k)\frac{t^i}{i!}, \ \mid t \mid < 1. \tag{26}$$

From the above definition, we have

$$(v)_i = \sum_{k=0}^{i} (-1)^{i-k} S_1(i,k) v^k. \tag{27}$$

Theorem 6. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{m=0}^{n} \binom{n}{m} \mathbb{S}_n^{[h]}(u,0) \sum_{j=0}^{m} x^j S_1(m,j) h^{m-j}. \tag{28}$$

Proof. From (13), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} &= e^{\frac{v}{h}\log(1+ht)}(1+ht^2)^{\frac{D_u^{-1}}{h}} \\
&= (1+ht^2)^{\frac{D_u^{-1}}{h}} \sum_{j=0}^{\infty} \left(\frac{v}{h}\right)^j \frac{[\log(1+ht)]^j}{j!} \\
&= \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,0)\frac{t^n}{n!} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left(\frac{v}{h}\right)^j S_1(m,j) h^m \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathbb{S}_{n-m}^{[h]}(u,0) \sum_{j=0}^{m} \left(\frac{v}{h}\right)^j S_1(m,j) h^m \right) \frac{t^n}{n!}. \qquad (29)
\end{aligned}$$

Comparing the coefficients of t, we obtain (28).

Theorem 7. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{l=0}^{n} \sum_{m=0}^{n-l} \frac{n!}{(n-m-l)!(m+l)!} h^m \mathbb{S}_{n-m-1}^{[h]}(0,u) S_1(m+l,l) v^l. \tag{30}$$

Proof. From (13), we have

$$\sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!} = (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}$$

$$= \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(0,u)\frac{t^n}{n!}\sum_{m=0}^{\infty}\left(-\frac{v}{h}\right)_m (-h)^m \frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\left(-\frac{v}{h}\right)_m (-h)^m \mathbb{S}_{n-m}^{[h]}(0,u)\right)\frac{t^n}{n!}. \tag{31}$$

Comparing the coefficients of t, we get

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{m=0}^{n}\binom{n}{m}\left(-\frac{u}{h}\right)_m (-h)^m \mathbb{S}_{n-m}^{[h]}(0,v). \tag{32}$$

Using the above equality (3.5), we get

$$\mathbb{S}_n^{[h]}(u,v) = \left(\sum_{m=0}^{n}\binom{n}{m}(-h)^m \mathbb{S}_{n-m}^{[h]}(0,u)\right)\left(\sum_{l=0}^{m}(-1)^{m-l}S_1(m,l)(-h)^{-l}v^l\right)$$
$$= \sum_{l=0}^{n}\sum_{m=l}^{n}\frac{n!}{(n-m)!m!}(-h)^{m-l}\mathbb{S}_{n-m}^{[h]}(0,u)(-1)^{m-l}S_1(m,l)v^l$$
$$= \sum_{l=0}^{n}\sum_{m=0}^{n-l}\frac{n!}{(n-m-l)!(m+l)!}(-h)^m \mathbb{S}_{n-m-1}^{[h]}(0,u)(-1)^m S_1(m+l,l)v^l. \tag{33}$$

The complete proof of the theorem.

Theorem 8. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(v+w,u) = \sum_{l=0}^{n}\sum_{m=0}^{n-l}\frac{n!}{(n-m-l)!(m+l)!}h^m \mathbb{S}_{n-m-l}^{[h]}(u,v)S_1(m+l,l)w^l. \tag{34}$$

Proof. Taking $v+w$ instead of u in (13), we have

$$\sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(v+w,v)\frac{t^n}{n!} = (1+ht)^{\frac{v+w}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}$$
$$= \left(\sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\left(-\frac{w}{h}\right)_m (-h)^m \frac{t^m}{m!}\right). \tag{35}$$

Using the Cauchy rule and after comparing the coefficients of t on both sides of the resulting equation, we have

$$\mathbb{S}_n^{[h]}(v+w,u) = \sum_{m=0}^{n}\binom{n}{m}\left(-\frac{w}{h}\right)_m (-h)^m \mathbb{S}_{n-m}^{[h]}(u,v). \tag{36}$$

Then using (25) for $\left(-\frac{w}{h}\right)_m$, we have obtain (34).

Theorem 9. *For $n \geq 0$, we have*

$$\mathbb{S}_n^{[h]}(u,v) = \sum_{m=0}^{n}\sum_{j=0}^{m}\binom{n}{m}(z-v)^j S_1(m,j)h^{m-j}\mathbb{S}_{n-m}^{[h]}(0,u). \tag{37}$$

Proof. From (13), we have

$$(1+ht^2)^{\frac{D_u^{-1}}{h}} = e^{-\frac{v}{h}\log(1+ht)}\sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}. \tag{38}$$

Replacing v by z and comparing the resulting equations, we get

$$e^{\frac{z-v}{h}\log(1+ht)}(1+ht^2)^{\frac{D_u^{-1}}{h}} = \sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}. \tag{39}$$

Finally, expanding the exponential function and then comparing the coefficients of equal powers of t, we come to assertion (39) of theorem 3.8.

Remark 1. *Taking $v = 0$ in Theorem 3.8, we immediately deduce the following consequence of Theorem 3.8:*

$$\mathbb{S}_n^{[h]}(0,u) = \sum_{m=0}^{n}\sum_{j=0}^{m}\binom{n}{m}z^j S_1(m,j)h^{m-j}\mathbb{S}_{n-m}^{[h]}(0,u).$$

4. Monomiality Principle

The monomiality principle serves as a fundamental framework for understanding and manipulating polynomial expressions. It states that any polynomial can be uniquely expressed as a linear combination of monomials, simplifying their structure and facilitating mathematical analysis. This principle aids in extracting key properties such as degree, leading coefficient, and roots, enabling the development of advanced mathematical techniques and algorithms.

Beyond its theoretical significance, the monomiality principle plays a crucial role in various scientific and engineering applications. In computational mathematics, it ensures the efficiency of numerical integration, approximation, and interpolation methods. Similarly, in fields like image analysis, control theory, and signal processing, it provides a structured approach for modeling complex systems. Its relevance extends to physics, where polynomials describe fundamental laws and phenomena.

The study of hybrid special polynomials has further explored monomiality and its operational principles. Originally introduced by Steffenson in 1941 with the concept of poweroids, the principle was later expanded by Dattoli. In this framework, multiplicative and derivative operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ play a crucial role in defining polynomial sequences $\{g_k(u_1)\}_{k\in\mathbb{N}}$, reinforcing its mathematical and practical significance.

The following expressions are satisfied by these operators:

$$g_{k+1}(u_1) = \hat{\mathcal{J}}\{g_k(u_1)\} \tag{40}$$

and

$$k\ g_{k-1}(u_1) = \hat{\mathcal{K}}\{g_k(u_1)\}. \tag{41}$$

Therefore, a quasi-monomial domain is produced when the polynomial set $g_k(u_1)_{m\in\mathbb{N}}$ is subjected to multiplicative and derivative operations. For this quasi-monomial, it is essential to adhere to the following formula:

$$[\hat{\mathcal{K}}, \hat{\mathcal{J}}] = \hat{\mathcal{K}}\hat{\mathcal{J}} - \hat{\mathcal{J}}\hat{\mathcal{K}} = \hat{1}. \tag{42}$$

It consequently displays a Weyl group structure.

It is possible to determine the significance of the underlined set by using the significance and application of the operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$, assuming that the set $\{g_k(u_1)\}_{k\in\mathbb{N}}$ is quasi-monomial. As a result, the following axioms hold:

(i) $g_k(u_1)$ gives differential equation

$$\hat{\mathcal{J}}\hat{\mathcal{K}}\{g_k(u_1)\} = k\ g_k(u_1), \tag{43}$$

provided $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ exhibits differential traits.

(ii) The expression

$$g_k(u_1) = \hat{\mathcal{J}}^k\ \{1\}, \tag{44}$$

gives the explicit form, with $g_0(u_1) = 1$.

(iii) Further, the expression

$$e^{w\hat{\mathcal{J}}}\{1\} = \sum_{k=0}^{\infty} g_k(u_1)\frac{w^k}{k!}, \quad |w| < \infty \ , \tag{45}$$

demonstrates generating expression behavior and is obtained by applying identity (44).

These methods, rooted in quantum mechanics, mathematical physics, and classical optics, remain highly relevant in modern research. They serve as reliable tools for analyzing complex phenomena across various fields, reinforcing our understanding of intricate systems.

Recognizing their significance, we validate the concept of monomiality for the Δ_h Legendre polynomials, denoted as $\mathbb{S}_n^{[h]}(u,v)$. These polynomials form a crucial mathematical framework for modeling diverse phenomena. By establishing their monomiality, we aim to highlight their fundamental properties and applications.

This section presents our validation results, confirming the integrity and utility of Δ_h Legendre polynomials. Through rigorous analysis, we affirm their relevance for both theoretical and applied research, thereby substantiating the monomiality principle for $\mathbb{S}_n^{[h]}(u,v)$.

Theorem 10. *The Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$ satisfy the succeeding multiplicative and derivative operators:*

$$\hat{M_{\mathbb{SA}}} = \left(\frac{v}{1+{}_v\Delta_h} + \frac{2\ D_u^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2}\right) \tag{46}$$

and

$$\hat{D_{\mathbb{S}}} = \frac{{}_v\Delta_h}{h}. \tag{47}$$

Proof. In consideration of expression (5), taking derivatives w.r.t. v of expression (13), we have

$$\begin{aligned}{}_v\Delta_h\Big\{(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\Big\} &= (1+ht)^{\frac{v+h}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}} - (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\\ &= (1+ht-1)(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\\ &= ht\ (1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}},\end{aligned} \tag{48}$$

thus, we have

$$\frac{{}_v\Delta_h}{h}\Big[(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\Big] = t\Big[(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\Big], \tag{49}$$

which gives the identity

$$\frac{{}_v\Delta_h}{h}\Big[\mathbb{S}_n^{[h]}(u,v)\Big] = t\Big[\mathbb{S}_n^{[h]}(u,v)\Big]. \tag{50}$$

Now, differentiating expression (13) w.r.t. t, we have

$$\frac{\partial}{\partial t}\Big\{(1+ht)^{\frac{v}{h}}(1+ht^2)^{\frac{D_u^{-1}}{h}}\Big\} = \frac{\partial}{\partial t}\left\{\sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}\right\}, \tag{51}$$

$$\left(\frac{v}{1+ht} + 2\frac{D_u^{-1}t}{1+ht^2}\right)\left\{\sum_{n=0}^{\infty}\mathbb{S}_n^{[h]}(u,v)\frac{t^n}{n!}\right\} = \sum_{n=0}^{\infty} n\ \mathbb{S}_n^{[h]}(u,v)\frac{t^{n-1}}{n!}. \tag{52}$$

Which on usage of identity expression (50) and replacing $n \to n+1$ in the r.h.s. of previous expression (52), assertion (46) is established.

Further, in view of identity expression (50), we have

$$\frac{{}_v\Delta_h}{h}\left[\mathbb{S}_n^{[h]}(u,v)\right] = \left[n\ \mathbb{S}_{n-1}^{[h]}(u,v)\right], \tag{53}$$

which gives expression for the derivative operator (47).

Next, we deduce the differential equation for the Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$ by demonstrating the succeeding result:

Theorem 11. *The Δ_h Legendre polynomials $\mathbb{S}_n^{[h]}(u,v)$ satisfy the differential equation:*

$$\left(\frac{v}{1+{}_v\Delta_h} + \frac{2\ D_u^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2} - \frac{nh}{{}_v\Delta_h}\right)\mathbb{S}_n^{[h]}(u,v) = 0. \tag{54}$$

Proof. Inserting expression (46) and (47) in the expression (43), the assertion (54) is proved.

5. Symmetric identities

The two-variable Δ_h special polynomials have symmetric identities that we examine in this section. These identities reveal fascinating connections between the polynomials' variables and coefficients, illuminating their fundamental symmetrical characteristics. By examining the behavior of the polynomials when the variables or coefficients are changed, we can find deep relationships that go beyond their original definitions. In addition to helping us comprehend the polynomials better, these symmetric identities provide important new information on more general mathematical occurrences and structures. We provide a thorough framework for comprehending and taking use of the symmetrical characteristics of these two-variable special polynomials by methodical investigation and exacting derivation, opening the door for future developments in theoretical analysis and real-world applications. Consequently, we discover a few of the Legenedre polynomials formulae and characteristics.

Theorem 12. *For $a \neq b$, $a, b > 0$ and $u_1, u_2, v_1, v_2 \in \mathbb{C}$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m\mathbb{S}_{n-m}^{[h]}(au_1,av_1)\mathbb{S}_m^{[h]}(bu_2,bv_2) = \sum_{m=0}^{n}\binom{n}{m}a^n b^{n-m}\mathbb{S}_{n-m}^{[h]}(au_2,av_2)\mathbb{S}_m^{[h]}(bu_1,bv_1). \tag{55}$$

Proof. Let

$$A(t) = (1+ht)^{\frac{ab(v_1+v_2)}{h}}C_0\left(\frac{-abu_1}{h}\log(1+ht^2)\right)C_0\left(\frac{-abu_2}{h}\log(1+ht^2)\right)$$

$$= \sum_{n=0}^{\infty} \mathbb{S}_n^{[h]}(bu_1, bv_1) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} \mathbb{S}_n^{[h]}(au_2, av_2) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \mathbb{S}_{n-m}^{[h]}(au_1, av_1) \mathbb{S}_m^{[h]}(bu_2, bv_2) \right) \frac{t^n}{n!}. \quad (56)$$

Similarly, we have

$$A(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^n b^{n-m} \mathbb{S}_{n-m}^{[h]}(au_2, av_2) \mathbb{S}_m^{[h]}(bu_1, bv_1) \right) \frac{t^n}{n!}. \quad (57)$$

Comapring the coefficients of t on both sides of last equations, we get (55).

Theorem 13. *For $a \neq b$, $a, b > 0$ and $u_1, u_2, v \in \mathbb{C}$, we have*

$$\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} a^{n-m} b^{m+1} \beta_{n-k}(h) \mathbb{S}_{n-m}^{[h]}(bu, bv) \sigma_m(a-1; h)$$

$$= \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} b^{n-m} a^{m+1} \beta_{n-k}(h) \mathbb{S}_{n-m}^{[h]}(au, av) \sigma_m(b-1; h). \quad (58)$$

Proof. Consider

$$B(t) = \frac{abt(1+ht)^{\frac{abv}{h}} C_0\left(\frac{-abu}{h} \log(1+ht^2)\right) ((1+ht)^{\frac{ab}{h}} - 1)}{((1+ht)^{\frac{a}{h}} - 1)((1+ht)^{\frac{b}{h}} - 1)}$$

$$= \frac{abt}{((1+ht)^{\frac{a}{h}} - 1)} (1+ht)^{\frac{abv}{h}} C_0\left(\frac{-abu}{h} \log(1+ht^2)\right) \frac{((1+ht)^{\frac{ab}{h}} - 1)}{((1+ht)^{\frac{b}{h}} - 1)}$$

$$= b \sum_{n=0}^{\infty} \beta_n(h) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} \mathbb{S}_k^{[h]}(bu, bv) \frac{(at)^k}{k!} \sum_{m=0}^{\infty} \sigma_m(a-1; h) \frac{(bt)^m}{m!}$$

$$= b \sum_{n=0}^{\infty} \beta_n(h) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} a^{k-m} b^m \mathbb{S}_{k-m}^{[h]}(bu, bv) \sigma_m(a-1; h) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} a^{n-m} b^{m+1} \beta_{n-k}(h) \mathbb{S}_{n-m}^{[h]}(bu, bv) \sigma_m(a-1; h) \right) \frac{t^n}{n!}. \quad (59)$$

Similarly, we have

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} b^{n-m} a^{m+1} \beta_{n-k}(h) \mathbb{S}_{n-m}^{[h]}(au, av) \sigma_m(b-1; h) \right) \frac{t^n}{n!}. \quad (60)$$

Comparing the both sides of the above equations, we get (58).

6. Conclusion

The introduction and study of Δ_h Legendre polynomials mark a significant advancement in polynomial theory, particularly in quantum mechanics and entropy modeling. By incorporating the monomiality principle with operational norms, these polynomials offer new insights into unexplored mathematical domains.

This study not only provides explicit formulas and fundamental properties of Δ_h Legendre polynomials but also establishes connections with other well-known polynomial classes, enriching mathematical understanding. Future research could further explore their algebraic structure and applications in mathematical physics and quantum mechanics.

Additionally, bridging the gap between theory and practical applications in computational science, information theory, and statistical mechanics is crucial. Interdisciplinary collaboration may unlock the full potential of Δ_h Legendre polynomials, paving the way for new developments in various scientific fields.

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