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Edge k-Product Cordial Labeling of Graphs

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Abstract. In this paper, we introduce a new labeling namely 'edge k-product cordial labeling' as follows: For a graph $G = (V(G), E(G))$ having no isolated vertex, an edge labeling $f : E(G) \to \{0, 1, ..., k-1\}$, where $k > 1$ is an integer, is said to be an edge k-product cordial labeling if it induces a vertex labeling $f^\star : V(G) \to \{0, 1, ..., k-1\}$ defined by $f^\star(v) = \prod_{uv \in E(G)} f(uv)(mod\ k)$ satisfies $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^\star}(i) - v_{f^\star}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$, where $e_f(i)$ and $v_{f^\star}(i)$ denote the number of edges and vertices respectively having a label i ($i = 0, 1, ..., k-1$). Further, we study the edge k-product cordial behavior of star, bistar, shadow and splitting graph of star, path union of star, bistar and cycle graphs.

2020 Mathematics Subject Classifications: 05C78

Key Words and Phrases: Product cordial labeling, k-product cordial labeling, edge k-product cordial labeling, shadow graph, splitting graph, path union of graph

1. Introduction

In mathematics, the field of graph theory revolves around the examination of graphs, which are the fundamental objects within discrete mathematics. Over the past six decades,

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one particular aspect of graph theory called graph labeling, has gained significant popularity due to its diverse applications. Labeling involves assigning real numbers, typically positive integers, to the elements of a graph. In 1967, Rosa [10] published an influential paper that laid the groundwork for various graph labeling problems. Since then, numerous authors have delved into researching various graph labeling techniques and a detailed survey is available in [4]. Among these labeling techniques, 'cordial labeling' by Cahit [3] stands out as a less stringent version compared to graceful and harmonious labeling. Of these, graceful labeling is more popular since it has various practical applications. Cordial labeling has potential applications in areas such as network design, error-correcting codes, and cryptography. In particular, cordial labeling could be useful in designing efficient communication protocols where balancing two types of nodes (positive and negative) is essential. In the subsequent years, several variants of cordial labeling, such as, 'product cordial labeling', 'k-product cordial labeling', 'edge product cordial labeling' and more are introduced. In 'edge product cordial labeling' [11], the roles of vertices and edges in product cordial labeling [6] are swapped. Building on this notion, several results have been established. See [2, 5, 9, 12-19]. Researchers have also explored the applications of specific graph labeling techniques, for instance, use of 'mean cordial labeling' in digraph representations of blood circulation in the human body [1] and the 3-total edge product cordial labeling (another variant of cordial labeling) in carbon nanotube network [7].

Motivated by the concept of 'edge product cordial labeling', and the several results established on this concept, we take a step further and introduce a new labeling namely 'Edge k-product cordial labeling' as follows: For a graph $G = (V(G), E(G))$ having no isolated vertex, an edge labeling $f : E(G) \to \{0, 1, ..., k-1\}$, where $k > 1$ is an integer, is said to be an edge k-product cordial labeling if it induces a vertex labeling $f^\star : V(G) \to \{0, 1, ..., k-1\}$ defined by $f^\star(v) = \prod_{uv \in E(G)} f(uv)(mod\, k)$ satisfies $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^\star}(i) - v_{f^\star}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$, where $e_f(i)$ and $v_{f^\star}(i)$ denote the number of edges and vertices respectively having a label i $(i = 0, 1, ..., k-1)$. A graph that admits an edge k-product cordial labeling is called edge k-product cordial graph. In this study, we explore the edge k-product cordial behavior of some standard graphs. We present our study as follows: Followed by the introduction, the edge k-product cordial behavior of star, bistar, complete graph and complete bipartite graph are investigated in the second section. In the third section, we focus on the edge k-product cordial behavior of the shadow and splitting graph of star. In the fourth section, we investigate the edge k-product cordial behavior of the path union of graphs. The definitions of the following graph structures are also useful for the present study.

Definition 1 [4]. Let G be a graph and G' be a copy of G. Let v' be the vertex in G' corresponding to the vertex v of G. The shadow graph of a graph G, denoted as $D_2(G)$ is a graph obtained by the following operation: Join each vertex v in G to the neighbors of the vertex v' in G' which corresponds to v.

Definition 2 [4]. The splitting graph of a graph G, denoted as $S'(G)$ is the graph obtained from G by taking a new vertex u' for each $u \in V(G)$ and joining u' to all vertices of G adjacent to u.

Definition 3 [8]. Let $G_1, G_2, ..., G_n,\ n \geq 2$, be n copies of a graph G. Let $v_i \in$

$V(G_i), i = 1, 2, ..., n$ be the vertex corresponding to the vertex $v \in V(G)$ in the i^{th} copy of G_i. We denoted by $P(n.G^v)$ the graph obtained by adding the edge v_iv_{i+1} to G_i and G_{i+1} , $1 \leq i \leq n-1$, and we call $P(n.G^v)$ the path union of n copies of the graph G.

2. Edge k-Product Cordial Labeling of Star, Bistar, Complete Graph and Complete Bipartite Graph

In this section, first we establish that the star graph $K_{1,n}$ and the bistar graph $B_{n,n}$ admit an edge k-product cordial labeling for $n \geq k$. In the next two theorems, we give the necessary condition for the complete graph K_n and the complete bipartite graph $K_{m,n}$ to admit an edge k-product cordial labeling.

Theorem 1. *For $n \geq k$, the star $K_{1,n}$ admits an edge k-product cordial labeling.*

Proof. Let the vertex set and edge set of $K_{1,n}$ be $V(K_{1,n}) = \{u, u_i; 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i; 1 \leq i \leq n\}$ respectively.
Let $n \equiv r \ (mod\ k)$; $0 \leq r \leq k-1$.
Define $f : E(K_{1,n}) \to \{0, 1, 2, ..., k-1\}$ for $n \geq k$ as follows:

$f(uu_i) \quad = 0 \ ; \ 1 \leq i \leq \lfloor \frac{n}{k} \rfloor$,

$$f(uu_{\lfloor \frac{n}{k} \rfloor + i}) = \begin{cases} q & ; i \equiv q \ (mod\ (k-1)), 1 \leq q \leq k-2 \\ k-1 & ; i \equiv 0 \ (mod\ (k-1)) \end{cases} \quad ; 1 \leq i \leq n - \lfloor \frac{n}{k} \rfloor.$$

From this labeling we get,

$$e_f(i) = \begin{cases} \lfloor \frac{n}{k} \rfloor & ; i = 0 \ ; \ r < i \leq k-1 \\ \lfloor \frac{n}{k} \rfloor + 1 & ; 1 \leq i \leq r, \end{cases}$$

$$v_{f^*}(i) = \begin{cases} \lfloor \frac{n}{k} \rfloor & ; r < i \leq k-1 \\ \lfloor \frac{n}{k} \rfloor + 1 & ; 0 \leq i \leq r. \end{cases}$$

Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, 2, ..., k-1\}$. Hence, $K_{1,n}$ is an edge k-product cordial graph for $n \geq k$.

Example 1. An edge 4-product cordial labeling of $K_{1,9}$ is given in Figure 1.

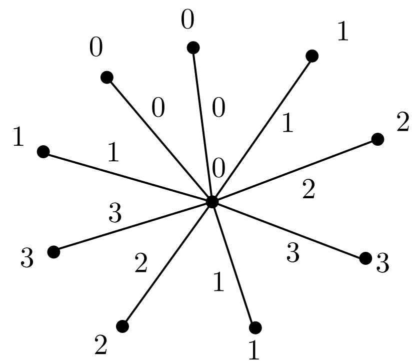

Figure 1: Edge 4-product cordial labeling of $K_{1,9}$

Theorem 2. *For $n \geq k$, the bistar $B_{n,n}$ admits an edge k-product cordial labeling.*

Proof. Let the vertex set and edge set of $B_{n,n}$ be $V(B_{n,n}) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i; 1 \leq i \leq n\}$ respectively.
Let $n \equiv r \ (mod\ k)\ ;\ 0 \leq r \leq k-1$.
Define $f : E(B_{n,n}) \to \{0, 1, 2, ..., k-1\}$ for $n \geq k$ as follows:

$$f(uv) = 0,$$
$$f(uu_i) = 0\ ;\ 1 \leq i \leq \lfloor \tfrac{n}{k} \rfloor,$$
$$f(uu_{\lfloor \frac{n}{k} \rfloor + i}) = \begin{cases} q & ; i \equiv q \ (mod\ (k-1)), 1 \leq q \leq k-2 \\ k-1 & ; i \equiv 0 \ (mod\ (k-1)) \end{cases} ; 1 \leq i \leq n - \lfloor \tfrac{n}{k} \rfloor,$$
$$f(vv_i) = 0\ ;\ 1 \leq i \leq \lfloor \tfrac{n}{k} \rfloor - 1,$$
$$f(vv_{\lfloor \frac{n}{k} \rfloor}) = \begin{cases} 0 & ; n \equiv 1,2 \ (mod\ 3)\ ,\ k = 3\ ;\ n \not\equiv 0,1 \ (mod\ k)\ ,\ k > 3 \\ 1 & ; n \equiv 0 \ (mod\ k)\ ,\ k \geq 3 \\ k-2 & ; k = 2\ ;\ n \equiv 1 \ (mod\ k)\ ,\ k > 3, \end{cases}$$
$$f(vv_{\lfloor \frac{n}{k} \rfloor + i}) = \begin{cases} k-q & ; i \equiv q \ (mod\ (k-1)), 1 \leq q \leq k-2 \\ 1 & ; i \equiv 0 \ (mod\ (k-1)) \end{cases} ; 1 \leq i \leq n - \lfloor \tfrac{n}{k} \rfloor.$$

From this labeling we obtain,

$$e_f(i) = \begin{cases} \frac{2n}{k} & ; i \neq 1 \\ \frac{2n}{k} + 1 & ; i = 1 \end{cases} ; n \equiv 0 \ (mod\ k),$$
$$e_f(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor & ; i = 0\ ,\ k > 3\ ;\ 2 \leq i \leq k-3 \\ 2\lfloor \frac{n}{k} \rfloor + 1 & ; i = 1, k-1, k-2\ ;\ i = 0\ ,\ k = 3 \end{cases} ; n \equiv 1 \ (mod\ k)\ ,\ k \geq 3,$$
$$e_f(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor + 1 & ; i = 0 \\ 2\lfloor \frac{n}{k} \rfloor + 2 & ; 1 \leq i \leq k-1 \end{cases} ; n \equiv k-1 \ (mod\ k),$$
$$e_f(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor + 1 & ; i = 0, 1\ ;\ 2 \leq i \leq r < k-i \\ & ;\ k-i \leq r < i \\ 2\lfloor \frac{n}{k} \rfloor + 2 & ; i \leq r\ ,\ k-i \leq r \end{cases} ; n \not\equiv 0, 1, k-1 \ (mod\ k)\ ,\ k \geq 3,$$
$$v_{f^*}(i) = \begin{cases} e_f(i) + 1 & ; i = 0 \\ e_f(i) & ; 1 \leq i \leq k-1. \end{cases}$$

Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^\star}(i) - v_{f^\star}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$. Therefore, $B_{n,n}$ is an edge k-product cordial graph if $n \geq k$.

Example 2. An edge 4-product cordial labeling of $B_{9,9}$ is given in Figure 2.

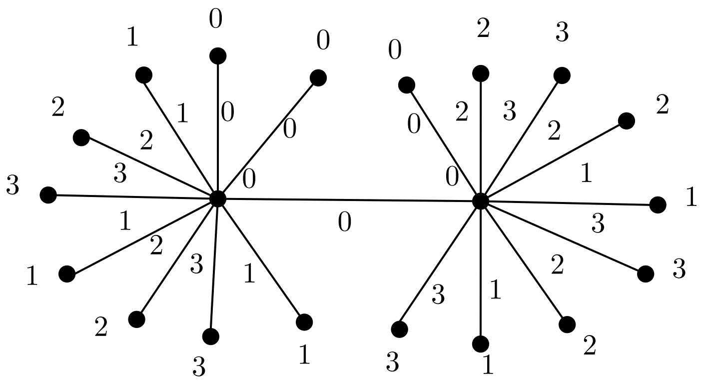

Figure 2: Edge 4-product cordial labeling of $B_{9,9}$

Theorem 3. *A complete graph K_n does not admit edge k-product cordial labeling if* $3 \leq k \leq \frac{n(n-1)}{2}$.

Proof. Let $3 \leq k \leq \frac{n(n-1)}{2}$, then $\lfloor \frac{n(n-1)}{2k} \rfloor \geq 1$.
Let f be an edge k-product cordial labeling of K_n, then $e_f(i) = \lfloor \frac{n(n-1)}{2k} \rfloor$ or $\lfloor \frac{n(n-1)}{2k} \rfloor + 1$. We have the following two cases.

Case (i): For $n \leq k$, we have $v_{f^*}(i) = 0$ or 1. If $e_f(0) = \lfloor \frac{n(n-1)}{2k} \rfloor$, then $v_{f^*}(0) \geq \lfloor \frac{n(n-1)}{2k} \rfloor + 1 > 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k-1$, which is a contradiction.

Case (ii): For $n > k$, we have $v_{f^*}(i) = \lfloor \frac{n}{k} \rfloor$ or $\lfloor \frac{n}{k} \rfloor + 1$. If $e_f(0) = \lfloor \frac{n(n-1)}{2k} \rfloor$, then $v_{f^*}(0) \geq \lfloor \frac{n(n-1)}{2k} \rfloor + 1 > \lfloor \frac{n}{k} \rfloor + 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k-1$, which is a contradiction.

Hence, K_n is not an edge k-product cordial graph if $3 \leq k \leq \frac{n(n-1)}{2}$.

Theorem 4. *A complete bipartite graph $K_{m,n}$ with $m \equiv r_1$ (mod k) and $n \equiv r_2$ (mod k) does not admit edge k-product cordial labeling if* $r_1 + r_2 < k \leq r_1 r_2$.

Proof. Let $m \equiv r_1$ (mod k) and $n \equiv r_2$ (mod k). Then $|V(K_{m,n})| = k(\lfloor \frac{m}{k} \rfloor + \lfloor \frac{n}{k} \rfloor) + r_1 + r_2$ and $|E(K_{m,n})| = k(k\lfloor \frac{m}{k} \rfloor \lfloor \frac{n}{k} \rfloor + r_2 \lfloor \frac{m}{k} \rfloor + r_1 \lfloor \frac{n}{k} \rfloor) + r_1 r_2$.
Let f be an edge k-product cordial labeling of $K_{m,n}$. Then $e_f(i) = k\lfloor \frac{m}{k} \rfloor \lfloor \frac{n}{k} \rfloor + r_2 \lfloor \frac{m}{k} \rfloor + r_1 \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 r_2}{k} \rfloor$ or $k\lfloor \frac{m}{k} \rfloor \lfloor \frac{n}{k} \rfloor + r_2 \lfloor \frac{m}{k} \rfloor + r_1 \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 r_2}{k} \rfloor + 1$ and $v_{f^*}(i) = \lfloor \frac{m}{k} \rfloor + \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 + r_2}{k} \rfloor$ or $\lfloor \frac{m}{k} \rfloor + \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 + r_2}{k} \rfloor + 1$.
Since $r_1 + r_2 < k \leq r_1 r_2$, we have $\lfloor \frac{r_1 + r_2}{k} \rfloor < \lfloor \frac{r_1 r_2}{k} \rfloor$. Now, $e_f(0) = k\lfloor \frac{m}{k} \rfloor \lfloor \frac{n}{k} \rfloor + r_2 \lfloor \frac{m}{k} \rfloor + r_1 \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 r_2}{k} \rfloor$ implies $v_{f^*}(0) \geq k\lfloor \frac{m}{k} \rfloor \lfloor \frac{n}{k} \rfloor + r_2 \lfloor \frac{m}{k} \rfloor + r_1 \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 r_2}{k} \rfloor + 1 > \lfloor \frac{m}{k} \rfloor + \lfloor \frac{n}{k} \rfloor + \lfloor \frac{r_1 + r_2}{k} \rfloor + 1$, which is a contradiction. Hence, $K_{m,n}$ is not an edge k-product cordial graph if $r_1 + r_2 < k \leq r_1 r_2$.

3. Edge k-Product Cordial Behavior of Shadow and Splitting Graph of Star

In order to prove the edge k-product cordial behavior of shadow and splitting graph of star, we prove the following general result.

Theorem 5. *A graph G with $k \le |V| \le |E|$ does not admit an edge k-product cordial labeling if $|V| \equiv 0 (mod\ k)$.*

Proof. Let G be a graph with $k \le |V| \le |E|$ and $|V| = tk$ $(t \ge 1)$. Then $|E| = tk + j$, where $0 \le j \le \lfloor \frac{tk(tk-3)}{2} \rfloor$. Let f be an edge k-product cordial labeling of G. Then, $e_f(i)$ is either $t + \lfloor \frac{j}{k} \rfloor$ or $t + \lfloor \frac{j}{k} \rfloor + 1$ and $v_{f^*}(i) = t$ $(i = 0, 1, ..., k-1)$. If $e_f(0) = t + \lfloor \frac{j}{k} \rfloor$, then $v_{f^*}(0) \ge t + \lfloor \frac{j}{k} \rfloor + 1 > t$, which is not possible. Hence, $e_f(0) = t + \lfloor \frac{j}{k} \rfloor + 1$, which implies $v_{f^*}(0) \ge t + \lfloor \frac{j}{k} \rfloor + 2 > t$. Then we get, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k-1$, that is a contradiction. Therefore, G is not an edge k-product cordial graph.

3.1. Shadow Graph of Star

In this subsection, we establish that the shadow graph of a star graph $D_2(K_{1,n})$ does not admit the edge k-product cordial labeling for $k \le n$. In addition, we investigate the edge k-product cordial behavior $D_2(K_{1,n})$ for $k = 3, 4, 5$.

Theorem 6. *The graph $D_2(K_{1,n})$ does not admit an edge k-product cordial labeling for $k \le n$.*

Proof. Let $k \le n$. We consider the following two cases.
Case (i): For $n = tk + k - 1$, we have $|V| = 2tk + 2k$ and $|E| = 4tk + 4(k-1)$. Clearly, $|V| \equiv 0$ $(mod\ k)$ and $k < |V| < |E|$. By Theorem 5, $D_2(K_{1,n})$ is not an edge k-product cordial graph.

Case (ii): For $n = tk + r$; $t \ge 1, 0 \le r \le k-2$, we have $|V| = 2tk + 2(r+1)$ and $|E| = 4tk + 4r$. If f is an edge k-product cordial labeling of $D_2(K_{1,n})$, then

$$e_f(i) = \begin{cases} 4t & ; r = 0 \\ 4t + \lfloor \frac{4r}{k} \rfloor \ or\ 4t + \lfloor \frac{4r}{k} \rfloor + 1 & ; 1 \le r \le k-2, \end{cases}$$

$$v_{f^*}(i) = \begin{cases} 2t + 1 & ; r = 0\ ,\ k = 2 \\ 2t \ or\ 2t + 1 & ; r = 0\ ,\ k \ge 3 \\ 2t + \lfloor \frac{2r+2}{k} \rfloor \ or\ 2t + \lfloor \frac{2r+2}{k} \rfloor + 1 & ; 1 \le r \le k-2. \end{cases}$$

For the case where $r = 0$, $e_f(0) = 4t$ implies $v_{f^*}(0) \ge 4t + 1 > 2t + 1$ for $t \ge 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k-1$, which is a contradiction. In the other cases, if $e_f(0) = 4t + \lfloor \frac{4r}{k} \rfloor$, then $v_{f^*}(0) \ge 4t + \lfloor \frac{4r}{k} \rfloor + 1$. Since $\lfloor \frac{4r}{k} \rfloor \ge \lfloor \frac{2r+2}{k} \rfloor$ for $1 \le r \le k-2$, we get $v_{f^*}(0) > 2t + \lfloor \frac{2r+2}{k} \rfloor + 1$ for $t \ge 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k-1$, which is a contradiction. Hence, $D_2(K_{1,n})$ is not an edge k-product cordial graph for $k \le n$.

Theorem 7. *The graph $D_2(K_{1,n})$ admits an edge 3-product cordial labeling if and only if $n = 1$.*

Proof. Let the vertex set and edge set of $D_2(K_{1,n})$ be $V(D_2(K_{1,n})) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(D_2(K_{1,n})) = \{uu_i, vv_i, uv_i, vu_i; 1 \leq i \leq n\}$ respectively.
Define the edge labeling $f : E(D_2(K_{1,1})) \to \{0, 1, 2\}$ as follows:
$f(uu_1) = 0, f(vv_1) = f(uv_1) = 1, f(vu_1) = 2$.
From this labeling we get, $e_f(0) = e_f(1) - 1 = e_f(2) = 1$ and $v_{f^*}(0) - 1 = v_{f^*}(1) = v_{f^*}(2) = 1$. Hence, $D_2(K_{1,1})$ is an edge 3-product cordial graph.

For $n = 2$, $|V| = 6$ and $|E| = 8$. By Theorem 5, $D_2(K_{1,2})$ is not an edge 3-product cordial graph. Also, by Theorem 6, $D_2(K_{1,n})$; $n \geq 3$ is not an edge 3-product cordial graph.

Theorem 8. *The graph $D_2(K_{1,n})$ does not admit an edge 4-product cordial labeling.*

Proof. Let the vertex set and edge set of $D_2(K_{1,n})$ be $V(D_2(K_{1,n})) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(D_2(K_{1,n})) = \{uu_i, vv_i, uv_i, vu_i; 1 \leq i \leq n\}$ respectively.
For $n = 1$, $|V| = 4$ and $|E| = 4$. For $n = 3$, $|V| = 8$ and $|E| = 12$. By Theorem 5, $D_2(K_{1,1})$ and $D_2(K_{1,3})$ are not edge 4-product cordial graphs.
Let f be an edge 4-product cordial labeling of $D_2(K_{1,2})$. Then, $e_f(i) = 2$ and $v_{f^*}(i) = 1$ or 2. If $e_f(0) = 2$, then $v_{f^*}(0) \geq 3$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, 3$, which is a contradiction. Hence, $D_2(K_{1,2})$ is not an edge 4-product cordial graph. Clearly, by Theorem 6, $D_2(K_{1,n})$ is not an edge 4-product cordial graph if $n \geq 4$.

Theorem 9. *The graph $D_2(K_{1,n})$ admits an edge 5-product cordial labeling if and only if $n = 2$.*

Proof. Let the vertex set and edge set of $D_2(K_{1,n})$ be $V(D_2(K_{1,n})) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(D_2(K_{1,n})) = \{uu_i, vv_i, uv_i, vu_i; 1 \leq i \leq n\}$ respectively.
Define the edge labeling $f : E(D_2(K_{1,2})) \to \{0, 1, 2, 3, 4\}$ as $f(uu_1) = 4, f(uu_2) = 1, f(uv_1) = 3, f(uv_2) = 0, f(vu_1) = 2, f(vu_2) = 1, f(vv_1) = 4, f(vv_2) = 3$.
From this labeling we get, $e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) = e_f(4) = 2$ and $v_{f^*}(0) - 1 = v_{f^*}(1) = v_{f^*}(2) = v_*(3) = v_{f^*}(4) = 1$.
Hence, $D_2(K_{1,2})$ is an edge 5-product cordial graph.

For $n = 1$, $|V| = |E| = 4$. If f is an edge 5-product cordial labeling of $D_2(K_{1,1})$, then $e_f(i)$ and $v_{f^*}(i)$ are either 0 or 1 for $i = 0, 1, 2, 3, 4$. Clearly, $e_f(0) = 0$ otherwise $v_{f^*}(0) = 2$. So, $e_f(i) = v_{f^*}(i) = 1$ $(i = 1, 2, 3, 4)$. In order to get the vertex label 1, there must be two adjacent edges, say uu_1 and uv_1 with labels 2 and 3 respectively. To get the vertex label 2, we must have $f(vu_1) = 1$ and $f(vv_1) = 4$, which results in $v_{f^*}(2) = 2$, which is a contradiction. Hence, $D_2(K_{1,1})$ is not an edge 5-product cordial graph.
For $n = 3$, $|V| = 8$ and $|E| = 12$. Let f be an edge 3-product cordial labeling of $D_2(K_{1,3})$. Then $e_f(i) = 2$ or 3 $(i = 0, 1, 2, 3, 4)$ and $v_{f^*}(i) = 1$ or 2 $(i = 0, 1, 2, 3, 4)$. Now, $e_f(0) = 2$

implies $v_{f^*}(0) \geq 3 > 2$. Therefore $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, 3, 4$, which is a contradiction. Hence, $D_2(K_{1,3})$ is not an edge 5-product cordial graph.
For $n = 4$, $|V| = 10$ and $|E| = 16$. By Theorem 5, $D_2(K_{1,4})$ is not an edge 5-product cordial graph. Also, by Theorem 6, $D_2(K_{1,n})$; $n \geq 5$ is not an edge 5-product cordial graph.

3.2. Splitting Graph of Star

In this subsection, we show that the splitting graph of a star graph $S'(K_{1,n})$ does not admit the edge k-product cordial labeling for $k \leq n$. Also, we study the edge k-product cordial behavior of $S'(K_{1,n})$ for $k = 3, 4, 5$.

Theorem 10. *The graph $S'(K_{1,n})$ dos not admit an edge k-product cordial labeling for $k \leq n$.*

Proof. Let $k \leq n$. We consider the following two cases.
Case (i): For $n = tk + k - 1$, we have $|V| = 2tk + 2k$ and $|E| = 3tk + 3(k - 1)$. Clearly, $|V| \equiv 0 \ (mod\ k)$ and $k < |V| < |E|$. By Theorem 5, $S'(K_n)$ is not an edge k-product cordial graph.
Case (ii): For $n = tk + r$; $t \geq 1$ and $0 \leq r \leq k - 2$, we have $|V| = 2tk + 2(r + 1)$ and $|E| = 3tk + 3r$. If f is an edge k-product cordial labeling of $S'(K_{1,n})$, then

$$e_f(i) = \begin{cases} 3t & ; r = 0 \\ 3t + 1 & ; r = 1\ ,\ k = 3 \\ 3t \ or\ 3t + 1 & ; r = 1\ ,\ k \geq 4 \\ 3t + \lfloor \frac{3r}{k} \rfloor \ or\ 3t + \lfloor \frac{3r}{k} \rfloor + 1 & ; 2 \leq r \leq k - 2, \end{cases} ; i \in \{0, 1, ..., k - 1\},$$

$$v_{f^*}(i) = \begin{cases} 2t + 1 & ; r = 0\ ,\ k = 2 \\ 2t + 1 \ or\ 2t + 2 & ; r = 1\ ,\ k = 3 \\ 2t + 1 & ; r = 1\ ,\ k = 4 \\ 2t \ or\ 2t + 1 & ; r = 0\ ,\ k \geq 3\ ;\ r = 1\ ,\ k \geq 5 \\ 2t + \lfloor \frac{2r+2}{k} \rfloor \ or\ 2t + \lfloor \frac{2r+2}{k} \rfloor + 1 & ; 2 \leq r \leq k - 2. \end{cases} ; i \in \{0, 1, .., k - 1\}.$$

For the case where $0 \leq r \leq 1$, $e_f(0) = 3t$ implies $v_{f^*}(0) \geq 3t + 1 > 2t + 1$ for $t \geq 1$. Also, $e_f(0) = 3t + 1$ implies $v_{f^*}(0) \geq 3t + 2 > 2t + 2$ for $t \geq 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, ..., k - 1$, which is a contradiction. In the other cases, if $e_f(0) = 3t + \lfloor \frac{3r}{k} \rfloor$, then $v_{f^*}(0) \geq 3t + \lfloor \frac{3r}{k} \rfloor + 1$. Since $\lfloor \frac{3r}{k} \rfloor \geq \lfloor \frac{2r+2}{k} \rfloor$ for $2 \leq r \leq k - 2$, we have $v_{f^*}(0) > 2t + \lfloor \frac{2r+2}{k} \rfloor + 1$ for $t \geq 1$. Therefore, $|v_{f^*}(0) - v_{f^*}(j)| > 1$ for some $j = 1, 2, .., k - 1$, which is a contradiction. Hence, $S'(K_{1,n})$ is not an edge k-product cordial graph if $k \leq n$.

Theorem 11. *The graph $S'(K_{1,n})$ admits an edge 3-product cordial labeling if and only if $n = 1$.*

Proof. Let Let the vertex set and edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, u_i, v, v_i; 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{uu_i, vu_i, uv_i; 1 \leq i \leq n\}$ respectively.
Define an edge labeling $f : E(S'(K_{1,1})) \to \{0, 1, 2\}$ as follows:

$f(uu_1) = 0, f(vu_1) = 1, f(uv_1) = 2.$
From this labeling we get, $e_f(0) = e_f(1) = e_f(2) = 1$ and $v_{f^*}(0)-1 = v_{f^*}(1) = v_{f^*}(2) = 1$. Hence, $S'(K_{1,1})$ is an edge 3-product cordial graph.

For $n = 2$, $|V| = |E| = 6$. By Theorem 5, $S'(K_{1,2})$ is not an edge 3-product cordial graph. Also, by Theorem 10, $S'(K_{1,n})$; $n \geq 3$ is not an edge 3-product cordial graph.

Theorem 12. *The graph $S'(K_{1,n})$ admits an edge 4-product cordial labeling if and only if $n = 2$.*

Proof. Let the vertex set and edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, u_i, v, v_i; 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{uu_i, vu_i, uv_i; 1 \leq i \leq n\}$ respectively.
Define an edge labeling $f : E(S'(K_{1,2})) \to \{0, 1, 2, 3\}$ as follows:
$f(uu_1) = 0, f(uu_2) = 1, f(vu_1) = 3, f(vu_2) = 1, f(uv_1) = 2, f(uv_2) = 2.$
From this labeling we get, $e_f(0) = e_f(1) - 1 = e_f(2) - 1 = e_f(3) = 1$ and $v_{f^*}(0) - 1 = v_{f^*}(1) = v_{f^*}(2) - 1 = v_{f^*}(3) = 1$. Hence, $S'(K_{1,2})$ is an edge 4-product cordial graph.

For $n = 1$, $|V| = 4$ and $|E| = 3$. If f is an edge 4-product cordial labeling of $S'(K_{1,1})$, then $e_f(i)$ is either 0 or 1 for $i = 0, 1, 2, 3$ and $v_{f^*}(i) = 1$ for all $i = 0, 1, 2, 3$. Clearly, $e_f(0) = 0$ otherwise $v_{f^*}(0) = 2$. Thus, $e_f(i) = 1$ for all $i = 1, 2, 3$. But $e_f(2) = 1$ implies $v_{f^*}(0) = 0$ and $v_{f^*}(2) = 2$. Therefore, $|v_{f^*}(0) - v_{f^*}(2)| > 1$ which is a contradiction. Hence, $S'(K_{1,1})$ is not an edge 4-product cordial graph.
For $n = 3$, $|V| = 8$ and $|E| = 9$. By Theorem 5, $S'(K_{1,3})$ is not an edge 4-product cordial graph. Also, by Theorem 10, $S'(K_{1,n})$ is not an edge 4-product cordial graph if $n \geq 4$.

Theorem 13. *The graph $S'(K_{1,n})$ admits an edge 5-product cordial labeling if and only if $n \leq 3$.*

Proof. Let the vertex set and edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, u_i, v, v_i; 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{uu_i, vu_i, uv_i; 1 \leq i \leq n\}$ respectively. Define an edge labeling $f : S'(K_{1,n}) \to \{0, 1, ..., k-1\}$ for $n \leq 3$ as follows:

$$f(uu_i) = \begin{cases} 0 & ; i = 1 \ , \ n = 2, 3 \\ 1 & ; i = 2 \ , \ n = 2, 3 \\ 2 & ; i = n \ , \ n = 1, 3, \end{cases}$$

$$f(vu_i) = \begin{cases} 1 & ; i = 1 \ , \ n = 3 \ ; \ i = n = 2 \\ 3 & ; i = n = 3 \\ 4 & ; i = 1 \ , \ n = 1, 2 \ ; \ i = 2 \ , \ n = 3, \end{cases}$$

$$f(uv_i) = \begin{cases} 1 & ; i = n = 1 \\ 2 & ; i = 1 \ , \ n = 2, 3 \\ 3 & ; i = 2 \ , \ n = 2, 3 \\ 4 & ; i = n = 3. \end{cases}$$

Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, 2, 3, 4\}$. Hence, $S'(K_{1,n})$ is an edge 5-product cordial graph if $n \leq 3$.

For $n = 4$, $|V| = 10$ and $|E| = 12$. By Theorem 5, $S'(K_{1,4})$ is not an edge 5-product cordial graph. Also, by Theorem 10, $S'(K_{1,n})$ is not an edge 5-product cordial graph if $n \geq 5$.

4. Edge k-Product Cordial Labeling of Path Union of Graphs

In this section, we explore the edge k-product cordial behavior of the path union of star, bistar and cycle graphs. In the following general result, we show that the path union of an edge k-product cordial graph with multiple of k edges also admits an edge k-product cordial labeling.

Theorem 14. *Let G be an edge k-product cordial graph with multiple of k edges. Then $P(n.G^v)$, where v is a vertex of G such that at least one of its incident edges is labeled with 0 admits an edge k-product cordial labeling.*

Proof. Let the vertex and edge set of $P(n.G^v)$ be $V(P(n.G^v)) = \bigcup_{1\leq i\leq n} V(G_i)$ and $E(P(n.G^v)) = \bigcup_{1\leq i\leq n} E(G_i) \cup \{e_i \ : \ e_i = v_i v_{i+1} \ , \ v_i \in V(G_i) \ , \ 1 \leq i \leq n-1\}$ respectively. Let g be an edge k-product cordial labeling of G. Since G has kt edges, $e_g(i) = t$ for all $i = 0, 1, ..., k-1$ and $|v_{g^*}(i) - v_{g^*}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$.
Define an edge labeling $f : E(P(n.G^v)) \to \{0, 1, ..., k-1\}$ for $n \equiv r \ (mod \ k) \ ; \ 0 \leq r \leq k-1$ as follows:
$f(e) = g(e) \ ; \ e \in E(G_i) \ , \ 1 \leq i \leq n,$

$$f(e_i) = \begin{cases} 0 & ; 1 \leq i \leq \lfloor \frac{n-1}{k} \rfloor \\ 1 & ; \lfloor \frac{n-1}{k} \rfloor + 1 \leq i \leq 2\lfloor \frac{n-1}{k} \rfloor \\ 2 & ; 2\lfloor \frac{n-1}{k} \rfloor + 1 \leq i \leq 3\lfloor \frac{n-1}{k} \rfloor \\ : & \\ : & \\ k-1 & ; (k-1)\lfloor \frac{n-1}{k} \rfloor + 1 \leq i \leq k\lfloor \frac{n-1}{k} \rfloor, \end{cases}$$

$f(e_{k\lfloor \frac{n-1}{k} \rfloor + i}) = j \ ; \ i \equiv j \ (mod \ k) \ , \ 1 \leq i \leq n - 1 - k\lfloor \frac{n-1}{k} \rfloor.$
From this labeling we obtain,

$$e_f(i) = \begin{cases} ne_g(i) + \lfloor \frac{n-1}{k} \rfloor & ; i \geq r \\ ne_g(i) + \lfloor \frac{n-1}{k} \rfloor + 1 & ; i < r, \end{cases}$$

$v_{f^*}(i) = v_{g^*}(i)$.
Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, 2.., k-1\}$. Hence, $P(n.G^v)$ is an edge k-product cordial graph.

4.1. Path Union of Star

In this subsection, we prove that the path union of a star graph $P(n.K^v_{1,m})$, where v is a root vertex of $K_{1,m}$ admits an edge k-product cordial labeling for $n \equiv 0, 1 \ (mod \ k)$. Also, we show that the graph $P(n.K^v_{1,m})$ admits an edge k-product cordial labeling for $n \equiv k-1 \ (mod \ k)$ if $m \equiv 0, k-1 \ (mod \ k)$.

Theorem 15. *The path union of star graph $P(n.K^v_{1,m})$, where v is a root vertex of $K_{1,m}$ admits an edge k-product cordial labeling if $n \equiv 0, 1 \ (mod\ k)$.*

Proof. Let the vertex and edge set of $P(n.K^v_{1,m})$ be $V(P(n.K^v_{1,m})) = \{v_i, v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P(n.K^v_{1,m})) = \{v_iv_{i+1}, v_iv_i^j, v_nv_n^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$ respectively.
If $m \equiv 0 \ (mod\ k)$, by Theorems 1 and 14, $P(n.K^v_{1,m})$ is an edge k-product cordial graph.
Define $f : E(P(n.K^v_{1,m})) \rightarrow \{0, 1, 2, ..., k-1\}$ for $n \equiv 0, 1 \ (mod\ k)$ and $m \equiv r \ (mod\ k)$; $1 \leq r \leq k-1$ as follows:
$f(v_iv_{i+1}) = 0$; $1 \leq i \leq n-1$,
We consider the following two cases.

Case(i): If $n \equiv 0 \ (mod\ k)$, then

$$f(v_iv_i^j) = \begin{cases} 0 & ; 1 \leq i \leq \frac{n}{k} \ , \ 1 \leq j \leq k\lfloor \frac{m}{k} \rfloor - (k-1) + r \\ 1 & ; 1 \leq i \leq \frac{n}{k} \ , \ j = k\lfloor \frac{m}{k} \rfloor - (k-1) + r + 1 \ ; \ \frac{n}{k} + 1 \leq i \leq \frac{2n}{k} \\ 2 & ; 1 \leq i \leq \frac{n}{k} \ , \ j = k\lfloor \frac{m}{k} \rfloor - (k-1) + r + 2 \ ; \ \frac{2n}{k} + 1 \leq i \leq \frac{3n}{k} \\ : & \\ : & \\ k-1 & ; 1 \leq i \leq \frac{n}{k} \ , \ j = k\lfloor \frac{m}{k} \rfloor + r \ ; \ \frac{(k-1)n}{k} + 1 \leq i \leq n. \end{cases}$$

From this labeling we get,

$$e_f(i) = \begin{cases} n\lfloor \frac{m}{k} \rfloor + \frac{n}{k}(1+r) - 1 & ; i = 0 \\ n\lfloor \frac{m}{k} \rfloor + \frac{n}{k}(1+r) & ; 1 \leq i \leq k-1, \end{cases}$$

$v_{f^*}(i) = n\lfloor \frac{m}{k} \rfloor + \frac{n}{k}(1+r)$; $0 \leq i \leq k-1$.

Case (ii): If $n \equiv 1 \ (mod\ k)$, then
$f(v_iv_i^j)$; $1 \leq i \leq n-1$, $1 \leq j \leq m$ as in Case (i),

$$f(v_nv_n^j) = \begin{cases} 0 & ; 1 \leq j \leq \lfloor \frac{m}{k} \rfloor \\ 1 & ; \lfloor \frac{m}{k} \rfloor + 1 \leq j \leq 2\lfloor \frac{m}{k} \rfloor \\ 2 & ; 2\lfloor \frac{m}{k} \rfloor + 1 \leq j \leq 3\lfloor \frac{m}{k} \rfloor \\ : & \\ : & \\ k-1 & ; (k-1)\lfloor \frac{m}{k} \rfloor + 1 \leq j \leq k\lfloor \frac{m}{k} \rfloor, \end{cases}$$

$f(v_nv_n^{k\lfloor \frac{m}{k} \rfloor + j}) = j$; $1 \leq j \leq r$.
From this labeling we have,

$$e_f(i) = \begin{cases} \lfloor \frac{n}{k} \rfloor + n\lfloor \frac{m}{k} \rfloor + r\lfloor \frac{n}{k} \rfloor & ; i = 0 \ ; \ i > r \\ \lfloor \frac{n}{k} \rfloor + n\lfloor \frac{m}{k} \rfloor + r\lfloor \frac{n}{k} \rfloor + 1 & ; 1 \leq i \leq r, \end{cases}$$

$$v_{f^*}(i) = \begin{cases} \lfloor \frac{n}{k} \rfloor + n\lfloor \frac{m}{k} \rfloor + r\lfloor \frac{n}{k} \rfloor & ; i > r \\ \lfloor \frac{n}{k} \rfloor + n\lfloor \frac{m}{k} \rfloor + r\lfloor \frac{n}{k} \rfloor + 1 & ; 0 \leq i \leq r. \end{cases}$$

Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$. Hence, $P(n.K^v_{1,m})$ is an edge k-product cordial graph if $n \equiv 0, 1 \ (mod\ k)$.

Example 3. An edge 4-product cordial labeling of $P(4.K^v_{1,5})$ is shown in Figure 3.

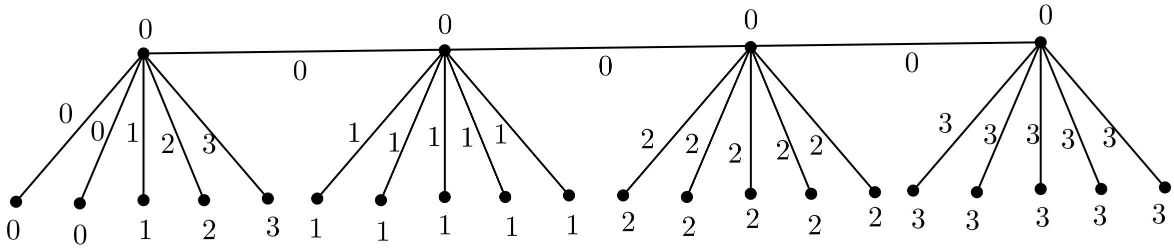

Figure 3: Edge 4-product cordial labeling of $P(4.K^v_{1,5})$

Theorem 16. *The path union star graph $P(n.K^v_{1,m})$, where v is a root vertex of $K_{1,m}$ admits an edge k-product cordial labeling if $n \equiv k-1$ $(mod\ k)$ and $m \equiv 0, k-1$ $(mod\ k)$.*

Proof. Let the vertex and edge set of $P(n.K^v_{1,m})$ be $V(P(n.K^v_{1,m})) = \{v_i, v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P(n.K^v_{1,m})) = \{v_iv_{i+1}, v_iv_i^j, v_nv_n^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$ respectively.
If $m \equiv 0$ $(mod\ k)$, then by Theorems 1 and 14, $P(n.K^v_{1,m})$ is an edge k-product cordial graph.
Define $f : E(P(n.K^v_{1,m})) \to \{0, 1, 2, ..., k-1\}$ for $n \equiv k-1$ $(mod\ k)$ and $m \equiv k-1$ $(mod\ k)$ as follows:
$f(v_iv_i^j)$ for $1 \leq i \leq n-k+1$, $1 \leq j \leq m$ as in Case (i) of Theorem 15,
$f(v_iv_{i+1}) = 0$; $1 \leq i \leq n-1$,

$$f(v_iv_i^j) = \begin{cases} 0 & ; n-k+2 \leq i \leq n\ ,\ 1 \leq j \leq \lfloor \frac{m}{k} \rfloor \\ 1 & ; n-k+2 \leq i \leq n\ ,\ \lfloor \frac{m}{k} \rfloor + 1 \leq j \leq 2\lfloor \frac{m}{k} \rfloor \\ & ; i = n-k+2\ ,\ m-k+2 \leq j \leq m \\ 2 & ; n-k+2 \leq i \leq n\ ,\ 2\lfloor \frac{m}{k} \rfloor + 1 \leq j \leq 3\lfloor \frac{m}{k} \rfloor \\ & ; i = n-k+3\ ,\ m-k+2 \leq j \leq m \\ : & \\ : & \\ k-1 & ; n-k+2 \leq i \leq n\ ,\ (k-1)\lfloor \frac{m}{k} \rfloor + 1 \leq j \leq k\lfloor \frac{m}{k} \rfloor \\ & ; i = n\ ,\ m-k+2 \leq j \leq m. \end{cases}$$

From this labeling we have,

$$e_f(i) = \begin{cases} k\lfloor \frac{n}{k} \rfloor \lfloor \frac{m}{k} \rfloor + k\lfloor \frac{n}{k} \rfloor + (k-1)(1 + \lfloor \frac{m}{k} \rfloor) - 1 & ; i = 0 \\ k\lfloor \frac{n}{k} \rfloor \lfloor \frac{m}{k} \rfloor + k\lfloor \frac{n}{k} \rfloor + (k-1)(1 + \lfloor \frac{m}{k} \rfloor) & ; 1 \leq i \leq k-1, \end{cases}$$

$v_{f^*}(i) = k\lfloor \frac{n}{k} \rfloor \lfloor \frac{m}{k} \rfloor + k\lfloor \frac{n}{k} \rfloor + (k-1)(1 + \lfloor \frac{m}{k} \rfloor)$; $0 \leq i \leq k-1$.
Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$. Hence, $P(n.K^v_{1,m})$ is an edge k-product cordial graph if $n \equiv k-1$ $(mod\ k)$ and $m \equiv 0, k-1$ $(mod\ k)$.

Example 4. An edge 5-product cordial labeling of $P(4.K^v_{1,5})$ is shown in Figure 4.

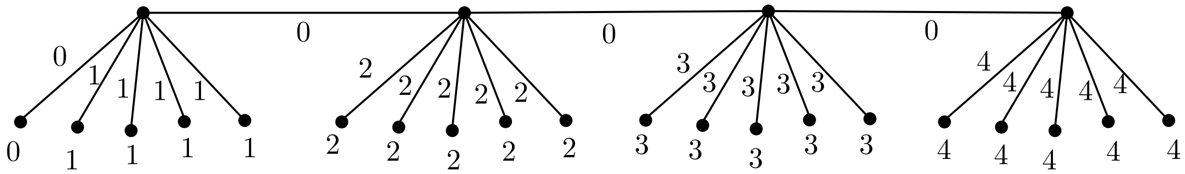

Figure 4: Edge 5-product cordial labeling of $P(4.K_{1,5}^v)$

4.2. Path Union of Bistar

In this subsection, we prove that the path union of a bistar graph $P(n.B_{m,m}^v)$, where v is a root vertex of $B_{m,m}$ admits an edge k-product cordial labeling for $n \equiv 0, 1\ (mod\ k)$. Also, we show that the graph $P(n.B_{m,m}^v)$ admits an edge k-product cordial labeling for $n \equiv k-1\ (mod\ k)$ if $m \equiv 0, k-1\ (mod\ k)$.

Theorem 17. *The path union of bistar graph $P(n.B_{m,m}^v)$, where v is a root vertex of $B_{m,m}$ admits an edge k-product cordial labeling if $n \equiv 0, 1\ (mod\ k)$.*

Proof. Let the vertex and edge set of $P(n.B_{m,m}^v)$ be $V(P(n.B_{m,m}^v)) = \{v_i, u_i, v_i^j, u_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(P(n.B_{m,m}^v)) = \{v_iv_{i+1}, v_iu_i, v_iv_i^j, u_iu_i^j, v_nu_n, v_nv_n^j, u_nu_n^j : 1 \le i \le n-1, 1 \le j \le m\}$ respectively.
Define $f : E(P(n.B_{m,m}^v)) \to \{0, 1, 2, ..., k-1\}$ for $n \equiv 0, 1\ (mod\ k)$ and $m \equiv r\ (mod\ k)\ ;\ 0 \le r \le k-1$ as follows:
$f(v_iv_{i+1}) = 0\ ;\ 1 \le i \le n-1,$
$f(v_iu_i) \ = 0\ ;\ 1 \le i \le n,$
We have the following two cases.
Case (i): If $n \equiv 0\ (mod\ k)$, then
$f(v_iv_i^j) = 0\ ;\ 1 \le i \le n\ ,\ 1 \le j \le \lfloor \frac{m}{k} \rfloor - 1,$

$$f(v_iv_i^{\lfloor \frac{m}{k} \rfloor}) = \begin{cases} 0 & ;\ 1 \le i \le \frac{(r+1)n}{k} \\ 1 & ;\ \frac{(r+1)n}{k} + 1 \le i \le \frac{(r+2)n}{k} \\ 2 & ;\ \frac{(r+2)n}{k} + 1 \le i \le \frac{(r+3)n}{k} \\ : & \\ : & \\ k-r-1 & ;\ \frac{(k-1)n}{k} \le i \le n \end{cases} \quad ;\ m \not\equiv k-1\ (mod\ k),$$

$$f(v_iv_i^{\lfloor \frac{m}{k} \rfloor + j}) = \begin{cases} q & ;\ j \equiv q\ (mod\ k-1)\ ,\ 1 \le q \le k-2 \\ k-1 & ;\ j \equiv 0\ (mod\ k-1) \end{cases} \quad ;\ 1 \le i \le n\ ,\ 1 \le j \le (k-1)\lfloor \frac{m}{k} \rfloor,$$

$$f(v_iv_i^{k\lfloor \frac{m}{k} \rfloor + j}) = \begin{cases} k-q & ;\ j \equiv q\ (mod\ k-1)\ ,\ 1 \le q \le k-2 \\ 1 & ;\ j \equiv 0\ (mod\ k-1) \end{cases} \quad ;\ 1 \le i \le \frac{n}{k}\ ,\ 1 \le j \le r,$$

$$f(v_{\frac{n}{k}+i}v_{\frac{n}{k}+i}^{k\lfloor \frac{m}{k} \rfloor + j}) = \begin{cases} q & ;\ i \equiv q\ (mod\ k-1)\ ,\ 1 \le q \le k-2 \\ k-1 & ;\ i \equiv 0\ (mod\ k-1) \end{cases} \quad ;\ 1 \le i \le \frac{(k-1)n}{k}\ ,\ 1 \le j \le r,$$

$f(u_iu_i^j) = f(v_iv_i^j)$; $1 \leq i \leq n$, $1 \leq j \leq m$.
From this labeling we have,

$$e_f(i) = \begin{cases} \frac{2n}{k} + 2n\lfloor\frac{m}{k}\rfloor + 2r\frac{n}{k} - 1 & ; i = 0 \\ \frac{2n}{k} + 2n\lfloor\frac{m}{k}\rfloor + 2r\frac{n}{k} & ; 1 \leq i \leq k-1, \end{cases}$$

$v_{f^*}(i) = \frac{2n}{k} + 2n\lfloor\frac{m}{k}\rfloor + 2r\frac{n}{k}$; $0 \leq i \leq k-1$.

Case (ii): If $n \equiv 1\ (mod\ k)$, then
$f(v_iv_i^j)$; $1 \leq i \leq n-1$, $1 \leq j \leq m$ as in Case (i),
$f(u_iu_i^j)$; $1 \leq i \leq n-1$, $1 \leq j \leq m$ as in Case (i),
$f(v_nv_n^j) = 0$; $1 \leq j \leq \lfloor\frac{m}{k}\rfloor$,
$f(u_nu_n^j) = 0$; $1 \leq j \leq \lfloor\frac{m}{k}\rfloor - 1$,

$$f(v_nv_n^{\lfloor\frac{m}{k}\rfloor+j}) = \begin{cases} q & ; j \equiv q\ (mod\ k-1)\ ,\ 1 \leq q \leq k-2 \\ k-1 & ; j \equiv 0\ (mod\ k-1) \end{cases} ; 1 \leq j \leq m - \lfloor\frac{m}{k}\rfloor,$$

$$f(u_nu_n^{\lfloor\frac{m}{k}\rfloor}) = \begin{cases} 0 & ; m \equiv 1,2\ (mod\ 3)\ ,\ k = 3\ ;\ m \not\equiv 0,1\ (mod\ k)\ ,\ k > 3 \\ 1 & ; m \equiv 0\ (mod\ k)\ ,\ k \geq 3 \\ k-2 & ; k = 2\ ;\ m \equiv 1\ (mod\ k)\ ,\ k > 3, \end{cases}$$

$$f(u_nu_n^{\lfloor\frac{m}{k}\rfloor+j}) = \begin{cases} k-q & ; j \equiv q\ (mod\ k-1)\ ,\ 1 \leq q \leq k-2 \\ 1 & ; j \equiv 0\ (mod\ k-1) \end{cases} ; 1 \leq j \leq m - \lfloor\frac{m}{k}\rfloor.$$

From this labeling we have,

$$e_f(i) = \begin{cases} 2\lfloor\frac{n}{k}\rfloor + \frac{2mn}{k} & ; i = 0\ ;\ 2 \leq i \leq k-1 \\ 2\lfloor\frac{n}{k}\rfloor + \frac{2mn}{k} + 1 & ; i = 1 \end{cases} ;\ m \equiv 0\ (mod\ k),$$

$$e_f(i) = \begin{cases} 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2\lfloor\frac{n}{k}\rfloor & ; i = 0\ ,\ k > 3\ ;\ 2 \leq i \leq k-3 \\ & ; m \equiv 1\ (mod\ k)\ ,\ k \geq 3, \\ 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2\lfloor\frac{n}{k}\rfloor + 1 & ; i = 1, k-1, k-2\ ;\ i = 0\ ,\ k = 3 \end{cases}$$

$$e_f(i) = \begin{cases} 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2(k-1)\lfloor\frac{n}{k}\rfloor + 1 & ; i = 0 \\ 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2(k-1)\lfloor\frac{n}{k}\rfloor + 2 & ; 1 \leq i \leq k-1 \end{cases} ;\ m \equiv k-1\ (mod\ k),$$

$$e_f(i) = \begin{cases} 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2r\lfloor\frac{n}{k}\rfloor + 1 & ; i = 0,1\ ;\ 2 \leq i \leq r < k-i\ ;\ k-i \leq r < i \\ & ; r \neq 0, 1, k-1,\ k \geq 3, \\ 2\lfloor\frac{n}{k}\rfloor + 2n\lfloor\frac{m}{k}\rfloor + 2r\lfloor\frac{n}{k}\rfloor + 2 & ; i \leq r\ ,\ k-i \leq r \end{cases}$$

$$v_{f^*}(i) = \begin{cases} e_f(i) + 1 & ; i = 0 \\ e_f(i) & ; 1 \leq i \leq k-1. \end{cases}$$

Clearly, $|e_f(i) - e_f(j)| \leq 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \leq 1$ for $i, j \in \{0, 1, ..., k-1\}$. Hence, $P(n.B^v_{m,m})$ is an edge k-product cordial graph if $n \equiv 0, 1\ (mod\ k)$.

Example 5. An edge 4-product cordial labeling of $P(4.B^v_{5,5})$ is shown in Figure 5.

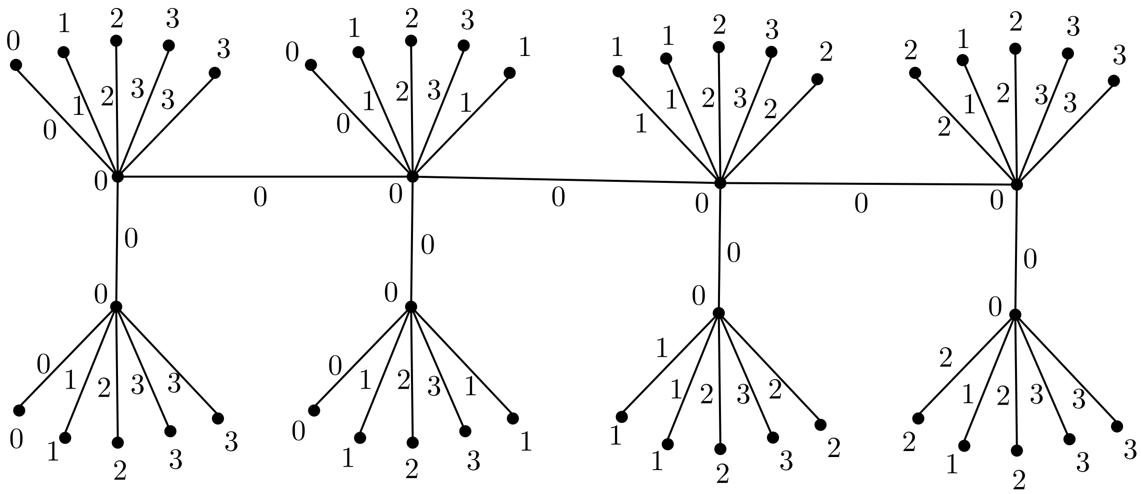

Figure 5: Edge 4-product cordial labeling of $P(4.B^v_{5,5})$

Theorem 18. *The path union of bistar graph $P(n.B^v_{m,m})$, where v is a root vertex of $B_{m,m}$ admits an edge k-product cordial labeling if $n \equiv k-1$ $(mod\ k)$ and $m \equiv 0, k-1$ $(mod\ k)$.*

Proof. Let the vertex and edge set of $P(n.B^v_{m,m})$ be $V(P(n.B^v_{m,m})) = \{v_i, u_i, v_i^j, u_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(P(n.B^v_{m,m})) = \{v_iv_{i+1}, v_iu_i, v_iv_i^j, u_iu_i^j, v_nu_n, v_nv_n^j, u_nu_n^j : 1 \le i \le n-1, 1 \le j \le m\}$ respectively.
Define $f : E(P(n.B^v_{m,m})) \to \{0, 1, 2, ..., k-1\}$ for $n \equiv k-1$ $(mod\ k)$ and $m \equiv 0, k-1$ $(mod\ k)$ as follows:
$f(v_iv_i^j)$; $1 \le i \le n-k+1$, $1 \le j \le m$ as in Case (i) of Theorem 17,
$f(u_iu_i^j)$; $1 \le i \le n-k+1$, $1 \le j \le m$ as in Case (i) of Theorem 17,
$f(v_iv_{i+1}) = 0$; $1 \le i \le n-1$,
$f(v_iu_i) \quad = 0$; $1 \le i \le n$,
We have the following two cases.
Case (i): If $m \equiv 0$ $(mod\ k)$, then
$f(v_iv_i^j) = f(u_iu_i^j) = 0$; $n-k+2 \le i \le n$, $1 \le j \le \frac{m}{k} - 1$,
$f(v_{n-k+1+i}v_{n-k+1+i}^{\frac{m}{k}}) = i$; $1 \le i \le k-2$,
$f(v_nv_n^{\frac{m}{k}}) = f(u_nu_n^{\frac{m}{k}}) = 0$,
$f(u_{n-k+1+i}u_{n-k+1+i}^{\frac{m}{k}}) = k-i$; $1 \le i \le k-2$,

$$f(v_iv_i^j) = f(u_iu_i^j) = \begin{cases} 1 & ; \frac{m}{k}+1 \le j \le \frac{2m}{k} \\ 2 & ; \frac{2m}{k}+1 \le j \le \frac{3m}{k} \\ : & \\ : & \\ k-1 & ; \frac{(k-1)m}{k}+1 \le j \le m \end{cases} \quad ; n-k+2 \le i \le n.$$

From this labeling we have,

$$e_f(i) = \begin{cases} 2\lfloor\frac{n}{k}\rfloor + \frac{2mn}{k} + 1 & ; i = 0, 1, k-1 \\ 2\lfloor\frac{n}{k}\rfloor + \frac{2mn}{k} + 2 & ; 2 \le i \le k-2, \end{cases}$$

$v_{f^*}(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor + \frac{2mn}{k} + 1 & ; i = 1, k-1 \\ 2\lfloor \frac{n}{k} \rfloor + \frac{2mn}{k} + 2 & ; i = 0 \ ; \ 2 \le i \le k-2. \end{cases}$

Case (ii): If $m \equiv k-1 \ (mod \ k)$, then

$f(v_i v_i^j) \ ; \ n-k+2 \le i \le n \ , \ 1 \le j \le m-k+1$ as in Case (i),

$f(u_i u_i^j) \ ; \ n-k+2 \le i \le n \ , \ 1 \le j \le m-k+1$ as in Case (i),

$f(v_i v_{i+1}) = 0 \ ; \ 1 \le i \le n-1,$

$f(v_i u_i) \ = 0 \ ; \ 1 \le i \le n,$

$f(v_i v_i^j) \ = 0 \ ; \ n-k+2 \le i \le n \ , \ m-k+2 \le j \le m-k+\lfloor \frac{m}{k} \rfloor + 1,$

$f(v_{n-k+1+i} v_{n-k+1+i}^j) = 0 \ ; \ 2 \le i \le k-2 \ , \ m-k+\lfloor \frac{m}{k} \rfloor + 2 \le j \le m-k+2\lfloor \frac{m}{k} \rfloor + 1,$

$f(v_{n-k+1+i} v_{n-k+1+i}^j) = i \ ; \ 1 \le i \le k-1 \ , \ m-k+2\lfloor \frac{m}{k} \rfloor + 2 \le j \le m,$

$f(v_{n-k+1+i} v_{n-k+1+i}^j) = i \ ; \ i = 1, k-1 \ , \ m-k+2 \le j \le m-k+\lfloor \frac{m}{k} \rfloor + 1,$

$f(u_{n-k+1+i} u_{n-k+1+i}^j) = i \ ; \ 1 \le i, j \le k-1.$

From this labeling we have,

$e_f(i) \ = \begin{cases} 2n\lfloor \frac{m}{k} \rfloor + 2n - 1 & ; i = 0 \\ 2n\lfloor \frac{m}{k} \rfloor + 2n & ; 1 \le i \le k-1, \end{cases}$

$v_{f^*}(i) = 2n\lfloor \frac{m}{k} \rfloor + 2n \ ; \ 0 \le i \le k-1.$

Clearly, $|e_f(i) - e_f(j)| \le 1$ and $|v_{f^*}(i) - v_{f^*}(j)| \le 1$ for $i, j \in \{0, 1, ..., k-1\}$. Hence, $P(n.B^v_{m,m})$ is an edge k-product cordial graph if $n \equiv k-1 \ (mod \ k)$ and $m \equiv 0, k-1 \ (mod \ k)$.

Example 6. An edge 3-product cordial labeling of $P(5.B^v_{3,3})$ is given in Figure 6.

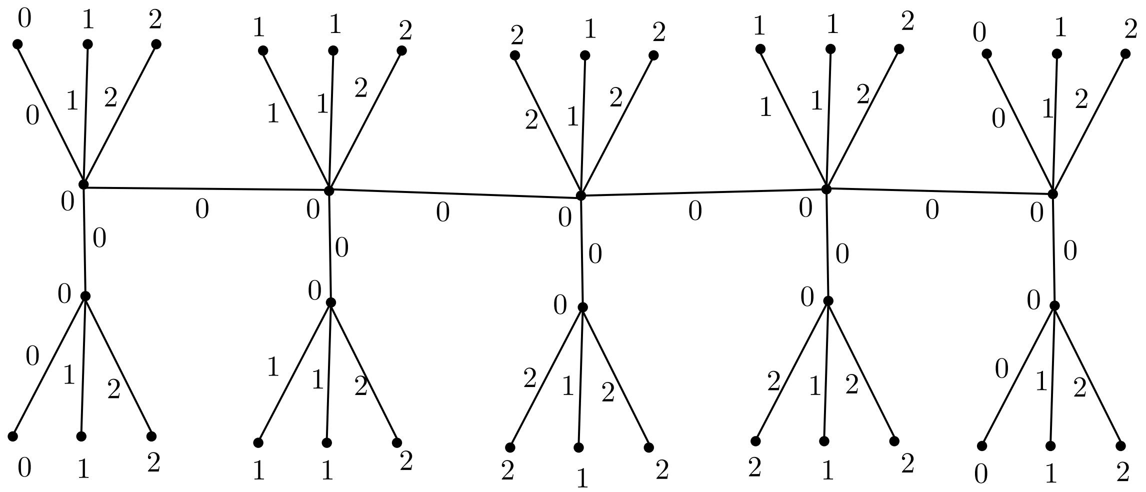

Figure 6: Edge 3-product cordial labeling of $P(5.B^v_{3,3})$

4.3. Path Union of Cycle

In this subsection, we establish the necessary conditions for the path union of a cycle graph $P(n.C^v_m)$ to admit an edge k-product cordial labeling. Also, we investigate the edge 3-product and 4-product cordial behavior of $P(n.C^v_m)$.

Let v be a vertex of a cycle C_m ; $m \ge 3$. According to the symmetry, all $P(n.C^v_m)$ are isomorphic. Hence, we use the notation $P(n.C_m)$.

In order to establish the necessary condition for the path union of a cycle graph $P(n.C_m)$ to admit an edge k-product cordial labeling, we prove the following general result.

Theorem 19. *Any graph G with $\lfloor \frac{|V|}{k} \rfloor < \lfloor \frac{|E|}{k} \rfloor$ does not an admit edge k-product cordial labeling.*

Proof. Let f be an edge k-product cordial labeling of a graph G with $\lfloor \frac{|V|}{k} \rfloor < \lfloor \frac{|E|}{k} \rfloor$. Then $e_f(i)$ is either $\lfloor \frac{|E|}{k} \rfloor$ or $\lfloor \frac{|E|}{k} \rfloor + 1$ and $v_{f^*}(i)$ is either $\lfloor \frac{|V|}{k} \rfloor$ or $\lfloor \frac{|V|}{k} \rfloor + 1$ $(i = 0, 1, ..., k-1)$. If $e_f(0) = \lfloor \frac{|E|}{k} \rfloor$, then $v_{f^*}(0) \geq \lfloor \frac{|E|}{k} \rfloor + 1 > \lfloor \frac{|V|}{k} \rfloor + 1$, which is a contradiction. Therefore, $e_f(0) = \lfloor \frac{|E|}{k} \rfloor + 1$, which results $v_{f^*}(0) \geq \lfloor \frac{|E|}{k} \rfloor + 2 > \lfloor \frac{|V|}{k} \rfloor + 1$, a contradiction again. Hence, G is not an edge k-product cordial graph.

Theorem 20. *The path union of cycle graph $P(n.C_m)$ does not admit an edge k-product cordial labeling if $n \geq k$.*

Proof. For the path union of cycle graph $P(n.C_m)$, we have $|V| = nm$ and $|E| = nm + n - 1$. Let f be an edge k-product cordial labeling of $P(n.C_m)$; $n \geq k$. We have the following two cases.

Case(i): For $n = k$, we have $e_f(i) = m$ or $m+1$ $(i = 0, 1, 2, ..., k-1)$ and $v_{f^*}(i) = m$ $(i = 0, 1, 2, ..., k-1)$. If $e_f(0) = m$, then $v_{f^*}(0) \geq m + 1$, which is a contradiction.

Case (ii): For $n \geq k + 1$, we have $\lfloor \frac{nm+n-1}{k} \rfloor > \lfloor \frac{nm}{k} \rfloor$. By Theorem 19, $P(n.C_m)$ is not an edge k-product cordial graph.

Hence, $P(n.C_m)$ is not an edge k-product cordial graph if $n \geq k$.

Theorem 21. *The path union of cycle graph $P(n.C_m)$ does not admit an edge k-product cordial labeling if m is a multiple of k.*

Proof. Let $m = kt$; $t \geq 1$. For the path union of cycle graph $P(n.C_{kt})$, we have $|V| = ktn$ and $|E| = ktn+n-1$. Let f be an edge k-product cordial labeling of $P(n.C_{kt})$; $n < k$. since $n-1 < k$, we have $e_f(i) = tn$ or $tn+1$ and $v_{f^*}(i) = tn$ $(i = 0, 1, 2, ..., k-1)$. If $e_f(0) = tn$, then $v_{f^*}(0) \geq tn+1$. Therefore, $|v_{f^*}(0) - v_{f^*}(i)| \geq 2$ for some $i \in \{1, 2, ..., k-1\}$, which is a contradiction. Thus, f is not an edge k-product cordial labeling of $P(n.C_{kt})$; $n < k$. By Theorem 20, $P(n.C_m)$; $n \geq k$ is not an edge k-product cordial graph. Hence, $P(n.C_{kt})$ is not an edge k-product cordial graph.

Theorem 22. *The path union of cycle graph $P(n.C_m)$ admits an edge 3-product cordial labeling if and only if $n = 2$ and $m \equiv 2$ $(mod\ 3)$.*

Proof. Let the vertex and edge set of $P(n.C_m)$ be $V(P(n.C_m)) = \{v_i^j \ ; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P(n.C_m)) = \{v_i^1 v_{i+1}^1, v_i^j v_i^{j+1}, v_n^j v_n^{j+1}, v_i^m v_i^1, v_n^m v_n^1 \ ; 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$ respectively.

Define an edge labeling $f : E(P(2.C_{3t+2})) \to \{0, 1, 2\}$ for $t \geq 1$ as follows:

$f(v_1^1v_2^1) \quad = 0,$
$f(v_i^{3t+2}v_i^1) = 1\ ;\ 1 \leq i \leq 2,$
$f(v_1^jv_1^{j+1}) \ = 0\ ;\ 1 \leq j \leq 2t,$

$$f(v_1^{2t+j}v_1^{2t+j+1}) = \begin{cases} 1 & ; j \equiv 0,3\ (mod\ 4) \\ 2 & ;\ j \equiv 1,2\ (mod\ 4) \end{cases} ; 1 \leq j \leq t+1,$$

$$f(v_2^jv_2^{j+1}) \qquad = \begin{cases} 1 & ; j \equiv 0,1\ (mod\ 4) \\ 2 & ;\ j \equiv 2,3\ (mod\ 4) \end{cases} ; 1 \leq j \leq 3t+1.$$

From this labeling we have,

$$e_f(i) \ = \begin{cases} 2t+1 & ; i = 0 \\ 2t+2 & ; i = 1,2, \end{cases}$$

$$v_{f^*}(i) = \begin{cases} 2t+1 & ; i = 1,2 \\ 2t+2 & ; i = 0. \end{cases}$$

Conversely, let f be an edge 3-product cordial labeling of $P(2.C_{3t+1})\ ;\ t \geq 1$. Then, $e_f(i) = 2t+1\ (i = 0,1,2)$ and $v_{f^*}(i) = 2t$ or $2t+1\ (i = 0,1,2)$. Clearly, $e_f(0) = 2t+1$ implies $v_{f^*}(0) \geq 2t+2 > 2t+1$, which is a contradiction. Hence, $P(2.C_{3t+1})\ ;\ t \geq 1$ is not an edge 3-product cordial graph.
By Theorem 21, $P(2.C_{3t})\ ;\ t \geq 1$ is not an edge 3-product cordial graph. Also, by Theorem 20, $P(n.C_m)\ ;\ n \geq 3$ is not an edge 3-product cordial graph.

Example 7. An edge 3-product cordial labeling of $P(n.C_8)$ is shown in Figure 7.

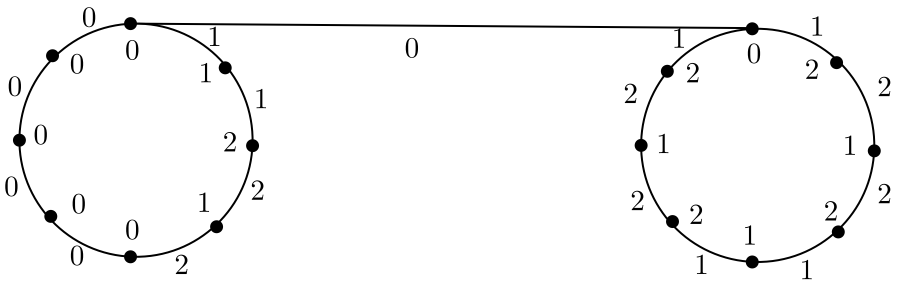

Figure 7: Edge 3-product cordial labeling of $P(n.C_8)$

Theorem 23. *The path union of cycle graph $P(n.C_m)$ admits an edge 4-product cordial labeling if and only if $n = 2$ and $m = 3, 5$.*

Proof. Let the vertex and edge set of $P(n.C_m)$ be $V(P(n.C_m)) = \{v_i^j\ ; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P(n.C_m)) = \{v_i^1v_{i+1}^1, v_i^jv_i^{j+1}, v_n^jv_n^{j+1}, v_i^mv_i^1, v_n^mv_n^1\ ; 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$ respectively. Define $f : E(P(2.C_m) \to \{0,1,2,3\}$ for $m = 3,5$ as follows:
$f(v_1^1v_2^1) \quad = 2,$

$$f(v_i^jv_i^{j+1}) = \begin{cases} 0 & ; i = 1\ ,\ j = 1 \\ 1 & ; i = 1\ ,\ j = 2 \\ 3 & ; i = 2\ ,\ 1 \leq j \leq 2 \end{cases} ;\ m = 3,$$

$$f(v_i^jv_i^{j+1}) = \begin{cases} 0 & ; i = 1\ ,\ 1 \leq j \leq 2 \\ 1 & ; i = 1\ ,\ j = 4\ ;\ i = 2\ ,\ 2 \leq j \leq 3 \\ 2 & ; i = 1\ ,\ j = 3 \\ 3 & ; i = 2\ ,\ j = 1,4 \end{cases} ;\ m = 5,$$

$$f(v_i^m v_i^1) \;=\begin{cases} 1 & ; i=2\ ,\ m=3 \\ 2 & ; i=1\ ,\ m=3,5 \\ 3 & ; i=2\ ,\ m=5. \end{cases}$$

From this labeling we have,

$$e_f(i) \;=\begin{cases} 2\lfloor \frac{m}{4} \rfloor & ; i=0 \\ 2\lfloor \frac{m}{4} \rfloor+1 & ; 1 \le i \le 3, \end{cases}$$

$$v_{f^*}(i) =\begin{cases} 2\lfloor \frac{m}{4} \rfloor & ; i=1,3 \\ 2\lfloor \frac{m}{4} \rfloor+1 & ; i=0,2. \end{cases}$$

Hence, $P(2.C_3)$ and $P(2.C_5)$ are edge 4-product cordial graphs.

To prove the converse part, we consider the following two cases.

Case(i): If $n=2$ and $m=4t+r$, where $1 \le r \le 3$, $t \ge 2$ for $r=1$ and $t \ge 1$ for $2 \le r \le 3$, then $|V|=8t+2r$ and $|E|=8t+2r+1$.

Let f be an edge 4-product cordial labeling of the graph $P(2.C_m)$. Then we have,

$$e_f(i) \;=\begin{cases} 2t \ or\ 2t+1 & ; r=1 \\ 2t+1 \ or\ 2t+2 & ; r=2,3, \end{cases}$$

$$v_{f^*}(i) \;=\begin{cases} 2t \ or\ 2t+1 & ; r=1 \\ 2t+1 & ; r=2 \\ 2t+1 \ or\ 2t+2 & ; r=3. \end{cases}$$

For $r=1$, we must have the following conditions.

(i) $e_f(0)=2t$,

(ii) two adjacent edges cannot be labeled with 2,

(iii) 0 must be assigned consecutively

otherwise $v_{f^*}(0) \ge 2t+2$. Hence, $e_f(i)=2t+1$ $(i=1,2,3)$ and $2t+1$ non adjacent edges must be labeled with 2. This implies $v_{f^*}(2) \ge 2t+2$. By similar argument, for $r=3$, we get $e_f(0)=2t+1$, $e_f(i)=2t+2$ $(i=1,2,3)$ and $v_{f^*}(2) \ge 2t+3$. For $r=2$, $e_f(0)=2t+1$ implies $v_{f^*}(0) \ge 2t+2$. Therefore in all the cases, we obtain $|v_{f^*}(0)-v_{f^*}(2)| \ge 2$, which is a contradiction.

Case(ii): If $n=3$ and $m=4t+r$, where $1 \le r \le 3$, $t \ge 1$ for $1 \le r \le 2$ and $t \ge 0$ for $r=3$, then $|V|=12t+3r$ and $|E|=12t+3r+2$.

Let g be an edge 4-product cordial labeling of the graph $P(3.C_m)$. Then we have,

$$e_g(i) \;=\begin{cases} 3t+1 \ or\ 3t+2 & ; r=1 \\ 3t+2 & ; r=2 \\ 3t+2 \ or\ 3t+3 & ; r=3, \end{cases}$$

$$v_{g^*}(i) =\begin{cases} 3t \ or\ 3t+1 & ; r=1 \\ 3t+1 \ or\ 3t+2 & ; r=2 \\ 3t+2 \ or\ 3t+3 & ; r=3. \end{cases}$$

For $1 \le r \le 2$, $e_g(0)=3t+r$ implies $v_{g^*}(0) \ge 3t+r+1$. For $r=3$, as in case(i) we get, $e_g(0)=3t+2$, $e_g(i)=3t+3$ $(i=1,2,3)$ and $v_{g^*}(i)=3t+2$ $(i=1,2,3)$, which result $v_{g^*}(2) \ge 3t+3$. Therefore, in all the cases, we obtain $|v_{g^*}(0)-v_{g^*}(2)| \ge 2$, which is a contradiction.

By Theorem 21, $P(n.C_{4t})\ ;\ t \ge 1$ is not an edge 4-product cordial graph. Also, by Theorem 20, $P(n.C_m)\ ;\ n \ge 4$ is not an edge 4-product cordial graph.

References

[1] R. Manoharan A. Anto Cathrin Aanisha. Mean Cordial Labeling in Graph Representations of Human Anatomy and Circular Systems. *Communications on Applied Nonlinear Analysis*, 31(7):453–458, 2024.

[2] C. M. Barasara. Edge and Total Edge Product Cordial Labeling of Some New Graphs. *International Journal of Engineering, Science and Mathematics*, 7(2):263–273, 2018.

[3] I. Cahit. Cordial graphs: A Weaker Version of Graceful and Harmonious Graphs. *Ars Combin.*, 23:201–207, 1987.

[4] J. A. Gallian. A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, # DS6, 2023.

[5] E. Roshdy M. Aboshady, R. Elbarkouky and M. Abdel-Azim Seoud. Further Results on Edge Product Cordial Labeling. *Proceedings of the Pakistan Academy of Sciences*, 57(4):23–32, 2020.

[6] S. Somasundaram M. Sundaram, R. Ponraj. Product Cordial Labeling of Graphs. *Bulletin of Pure and Applied Sciences*, 23E(1):155–163, 2004.

[7] Andrea Semaničová-Feňovčíková Martin Bača, Muhammad Irfan. On 3-Total Edge Product Cordial Labeling of a Carbon Nanotube Network. *AKCE International Journal of Graphs and Combinatorics*, 16:310–318, 2019.

[8] Andrea Semaničová-Feňovčíková P. Jeyanthi, K. Jeya Daisy. Z_k-Magic Labeling of Path Union of Graphs. *CUBO A Mathematical Journal*, 21:15–35, 2019.

[9] R. Rajeswari R. Thamizharasi. Edge Product Cordial Labeling and Total Magic Cordial Labeling of Regular Digraphs. *International Journal on Information Sciences and Computing*, 9(2):1–4, 2015.

[10] A. Rosa. On Certain Valuations of the Vertices of a Graph, Theory of Graphs. *Internat. Sympos.*, pages 349–355, 1966.

[11] C. M. Barasara S. K. Vaidya. Edge Product Cordial Labeling of Graphs. *Journal of Mathematical and Computational Science*, 2(5):1436–1450, 2012.

[12] C. M. Barasara S. K. Vaidya. Edge Product Cordial Labeling in the Context of Some Graph Operations. *International Journal of Mathematics and Scientific Computing*, 3(1):4–7, 2013.

[13] C. M. Barasara S. K. Vaidya. Some Edge Product Cordial Graphs. *International Journal of Mathematics and Soft Computing*, 3(3):49–53, 2013.

[14] C. M. Barasara S. K. Vaidya. Some New Families of Edge Product Cordial Graphs. *Advanced Modeling and Optimization*, 15(1):103–111, 2013.

[15] C. M. Barasara S. K. Vaidya. On Edge Product Cordial Labeling of Some Product Related Graphs. *International Journal of Mathematics and its Applications*, 2(2):15–22, 2014.

[16] C. M. Barasara S. K. Vaidya. Product and Edge Product Cordial Labeling of Degree Splitting Graph of Some Graphs. *Advances and Applications in Discrete Mathematics*, 15(1):61–74, 2015.

[17] N. B. Patel U. M. Prajapati. Edge Product Cordial Labeling of Some Cycle Related Graphs. *Open Journal of Discrete Mathematics*, 6:268–278, 2016.

[18] N. B. Patel U. M. Prajapati. Edge Product Cordial Labeling of Some Graphs. *Journal of Applied Mathematics and Computational Mechanics*, 18(1):69–76, 2019.

[19] N. B. Patel U. M. Prajapati. Edge Product Cordial Labeling of Switching Operations on Some Graphs. *TWMS J. App. and Eng. Math.*, 12(1):191–199, 2022.

- [1–19] are journal articles.