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Jordan-Hölder Theorem for Multigroups

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Abstract. Multigroup theory is the application of multisets to the theory of groups. Many group's theoretic notions have been studied in multigroup theory, however, the ideas of maximal normal subgroup, simple group, normal series, composition series, and the Jordan-Hölder Theorem are yet to be investigated in multiset context. In this article, we define simple multigroup, maximal normal submultigroup, normal series for multigroup, and composition series for multigroup with examples. With these concepts, we establish the Jordan-Hölder Theorem in multigroup theory. It is shown that every finite multigroup defined over a finite group has a composition series. In addition, it is established that every finite multigroup defined over a finite group has at least two composition series which are equivalent.

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One constraint of set theory is the refusal to allow repeated elements in a collection, which is admissible in real-world applications. The word "multiset" refers to an extensional set where an element can be repeated in a collection [1]. According to DeBruijin [2], the concept of multiset was introduced to D. E. Knuth by N. G. de Bruijn in a private message, and since then, the word has been used to depict a set with repeated elements/members. The relevance of multiset has led to many studies and applications in a number of fields [3–9]. By relaxing the condition of definite collection in set, Zadeh [10] introduced fuzzy sets, which was applied to group theory by proposing fuzzy group theory [11]. Some properties of the fuzzy group theory were discussed [12–16]. Nazmul et al. [17] utilized multisets in group theory to introduce the theory of multigroups.

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A comprehensive research on multigroup theory was conducted in [18] and some results in multigroup theory were discussed [19, 20]. The concepts of highly invariant submultigroups, characteristic submultigroups, normal submultigroups, and Frattini submultigroups were investigated in multigroup settings, yielding some relevant results [21–24]. In addition, the order of multigroups, cyclic multigroups, comultisets, and factor multigroups were established [25–28]. The studies in [29–34] examined various notions in multigroup contexts, such as direct products and actions. Some algebraic systems have been examined using the idea of multisets [35–38]. Finally, the idea of soluble multigroups was introduced and many of its properties were studied in [39].

Although many group theoretic concepts have been addressed under multiset context, the notions of maximal normal subgroup, simple group, normal series, composition series, and the Jordan-Hölder Theorem are yet to be studied in multiset domain. Hence, it is appropriate to investigate simple multigroup, maximal normal submultigroup, normal series for multigroup, composition series for multigroup, and the Jordan-Hölder Theorem for multigroups because the necessary concepts needed for the establishment of these concepts have been studied in multigroup theory. Thus, this article establishes simple multigroup, maximal normal submultigroup, normal series for multigroup, composition series for multigroup, and the Jordan-Hölder Theorem for multigroups, respectively. The remainder of the article is organized as follows: Section 2 presents the preliminaries for the study, Section 3 covers the main results of the articles, and Section 4 concludes and makes suggestions for further research.

1. Preliminaries

Let S and G represent a non-empty set and a group, respectively.

Definition 1 ([10])**.** A fuzzy subset $\mathfrak{F}$ of S is presented as:

$$\mathfrak{F} = \{\langle s, \mathfrak{F}_m(s)\rangle \mid s \in S\}, \tag{1}$$

where $\mathfrak{F}_m\colon S \to [0,1]$ is the membership degree of $s \in S$.

Definition 2 ([11])**.** A fuzzy subset $\mathfrak{F}$ of G is a fuzzy subgroup of G if

(i) $\mathfrak{F}_m(xy) \geq \min\left\{\mathfrak{F}_m(x), \mathfrak{F}_m(y)\right\}\ \forall\ x, y \in G$,

(ii) $\mathfrak{F}_m(x^{-1}) = \mathfrak{F}_m(x)\ \forall\ x \in G$.

In addition, $\mathfrak{F}_m(e) = \mathfrak{F}_m(xx^{-1}) \geq \min\left\{\mathfrak{F}_m(x), \mathfrak{F}_m(x)\right\} = \mathfrak{F}_m(x)\ \forall\ x \in G$, where e is the unit element of G.

Definition 3 ([6])**.** A multiset $\mathfrak{D}$ of S is a pair $\langle S, C_{\mathfrak{D}}\rangle$, where

$$C_{\mathfrak{D}} : S \to \mathcal{N} = \{1, 2, ...\} \tag{2}$$

is a function, such that for $s \in S$ implies $\mathfrak{D}(s) = C_{\mathfrak{D}}(s) > 0$ and $C_{\mathfrak{D}}(s)$ is the multiplicity of s in $\mathfrak{D}$. If $C_{\mathfrak{D}}(s) = 0$, then $s \notin S$.

Definition 4 ([8]). Suppose $\mathfrak{D}$ and $\mathfrak{E}$ are multisets of S, then

(i) $\mathfrak{D} = \mathfrak{E} \Longleftrightarrow C_{\mathfrak{D}}(s) = C_{\mathfrak{E}}(s)\ \forall\ s \in S$,

(ii) $\mathfrak{D} \subseteq \mathfrak{E} \Longleftrightarrow C_{\mathfrak{D}}(s) \leq C_{\mathfrak{E}}(s)\ \forall\ s \in S$,

(iii) $\mathfrak{D} \cap \mathfrak{E} \Longrightarrow C_{\mathfrak{D}\cap\mathfrak{E}}(s) = \min\{C_{\mathfrak{D}}(s), C_{\mathfrak{E}}(s)\}\ \forall\ s \in S$,

(iv) $\mathfrak{D} \cup \mathfrak{E} \Longrightarrow C_{\mathfrak{D}\cup\mathfrak{E}}(s) = \max\{C_{\mathfrak{D}}(s), C_{\mathfrak{E}}(s)\}\ \forall\ s \in S$,

(v) $\mathfrak{D} \oplus \mathfrak{E} \Longrightarrow C_{\mathfrak{D}\oplus\mathfrak{E}}(s) = C_{\mathfrak{D}}(s) \oplus C_{\mathfrak{E}}(s)\ \forall\ s \in S$.

Definition 5 ([17]). A multiset $\mathfrak{D}$ of G is called a multigroup of G if:

(i) $C_{\mathfrak{D}}(xy) \geq \min\{C_{\mathfrak{D}}(x), C_{\mathfrak{D}}(y)\}\ \forall\ x, y \in G$,

(ii) $C_{\mathfrak{D}}(x^{-1}) = C_{\mathfrak{D}}(x)\ \forall\ x \in G$.

It is worthy to note that, $C_{\mathfrak{D}}(e) \geq C_{\mathfrak{D}}(x)\ \forall\ x \in X$ since

$$C_{\mathfrak{D}}(e) = C_{\mathfrak{D}}(xx^{-1}) \geq \min\{C_{\mathfrak{D}}(x), C_{\mathfrak{D}}(x)\} = C_{\mathfrak{D}}(x)\ \forall x \in G.$$

In addition, $\mathfrak{D}_*$ defined by $\mathfrak{D}_* = \{x \in G \mid C_{\mathfrak{D}}(x) > 0\}$ is a subgroup of G.

Definition 6 ([26]). If $\mathfrak{D}$ is a multigroup of G, then the order of $\mathfrak{D}$ is the sum of the multiplicities for each of the elements in $\mathfrak{D}$. It is mathematically presented as:

$$|\mathfrak{D}| = \sum_{i=1}^{n} C_{\mathfrak{D}}(x_i)\ \forall x_i \in G. \tag{3}$$

Definition 7 ([31]). A multigroup $\mathfrak{D}$ of G is commutative if $C_{\mathfrak{D}}(xy) = C_{\mathfrak{D}}(yx)\ \forall\ x, y \in G$. If G is commutative, then a multigroup $\mathfrak{D}$ of G is a commutative multigroup.

Definition 8 ([31]). Suppose $\mathfrak{D}$ and $\mathfrak{E}$ are multigroups of G, then $\mathfrak{D}$ is a submultigroup of $\mathfrak{E}$ if $\mathfrak{D} \subseteq \mathfrak{E}$. Again, $\mathfrak{D}$ is a proper submultigroup of $\mathfrak{E}$ if $\mathfrak{D} \subseteq \mathfrak{E}$ and $\mathfrak{D} \neq \mathfrak{E}$.

Definition 9 ([22]). Suppose $\mathfrak{D}$ is a submultigroup of a multigroup $\mathfrak{E}$ of G, then $\mathfrak{D}$ is normal in $\mathfrak{E}$ denoted by $\mathfrak{D} \triangleleft \mathfrak{E}$ if $C_{\mathfrak{D}}(xy) = C_{\mathfrak{D}}(yx) \Longleftrightarrow C_{\mathfrak{D}}(y) = C_{\mathfrak{D}}(x^{-1}yx)\ \forall\ x, y \in G$. Certainly, any normal submultigroup is commutative and self-normal.

Definition 10 ([28]). Let $\mathfrak{D}$ be a submultigroup of a multigroup $\mathfrak{E}$ of G. Then, the submultiset $y\mathfrak{D}$ of $\mathfrak{E}$ for $y \in G$ defined by $C_{y\mathfrak{D}}(x) = C_{\mathfrak{D}}(y^{-1}x)\ \forall\ x \in G$ is a left comultiset of $\mathfrak{D}$. Similarly, $\mathfrak{D}y$ of $\mathfrak{E}$ such that $C_{\mathfrak{D}y}(x) = C_{\mathfrak{D}}(xy^{-1})\ \forall\ x \in G$ is a right comultiset of $\mathfrak{D}$.

Definition 11 ([18]). Suppose $\mathfrak{D}$ and $\mathfrak{E}$ are multigroups of G. Then, the product $\mathfrak{D} \circ \mathfrak{E}$ is a multiset of G defined as follows:

$$C_{\mathfrak{D}\circ\mathfrak{E}}(x) = \begin{cases} \bigvee_{x=yz} \min\{C_{\mathfrak{D}}(y), C_{\mathfrak{E}}(z)\}, & \text{if } \exists\, y, z \in G \text{ where } x = yz \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Definition 12 ([28]). Suppose $\mathfrak{E}$ is a multigroup of G and $\mathfrak{D}$ a normal submultigroup in $\mathfrak{E}$. Then, the set of right/left comultisets of $\mathfrak{D}$ such that $C_{x\mathfrak{D}\circ y\mathfrak{D}}(z) = C_{xy\mathfrak{D}}(z)\ \forall\ x, y, z \in G$ is a factor/quotient multigroup of $\mathfrak{E}$ by $\mathfrak{D}$, represented as $\mathfrak{E}/\mathfrak{D}$.

2. Main Results

Before the introduction of normal series, composition series, and Jordan-Hölder therorem under multigroups, we first reiterate solvable multigroups as established in [39] as follows:

Definition 13. For every finite multigroup $\mathfrak{D}$ of a finite G, there is a chain of consecutive submultigroups of $\mathfrak{D}$:

$$\mathfrak{D}_0 \subseteq \mathfrak{D}_1 \subseteq \cdots \subseteq \mathfrak{D}_n = \mathfrak{D}, \tag{5}$$

where $(\mathfrak{D}_0)_* = (\mathfrak{D}_1)_* = \cdots = (\mathfrak{D}_n)_* = \mathfrak{D}_*$.

The chain of the consecutive submultigroups is also presented as:

$$C_{\mathfrak{D}_0}(x) \leq C_{\mathfrak{D}_1}(x) \leq \cdots \leq C_{\mathfrak{D}_n}(x) = C_{\mathfrak{D}}(x),\ \forall x \in G \tag{6}$$

such that $(\mathfrak{D}_0)_* = (\mathfrak{D}_1)_* = \cdots = (\mathfrak{D}_n)_* = \mathfrak{D}_*$.

Definition 14. If $\mathfrak{D}$ is a multigroup of G, then $\mathfrak{D}$ is solvable if it has a chain of consecutive submultigroups:

$$\mathfrak{D}_0 \subseteq \mathfrak{D}_1 \subseteq \cdots \subseteq \mathfrak{D}_n = \mathfrak{D} \tag{7}$$

such that $(\mathfrak{D}_0)_* = (\mathfrak{D}_1)_* = \cdots = (\mathfrak{D}_n)_* = \mathfrak{D}_*$, where $\mathfrak{D}_{i-1} \triangleleft \mathfrak{D}_i$ and $\mathfrak{D}_i/\mathfrak{D}_{i-1}$ is commutative $\forall\ 1 \leq i \leq n$.

The finite chain of consecutive submultigroups of $\mathfrak{D}$ is a solvable series for $\mathfrak{D}$ denoted by $\mathfrak{D}_i$. In fact, the solvable series for $\mathfrak{D}$ is presented as:

$$\mathfrak{D}_0 \triangleleft \mathfrak{D}_1 \triangleleft \cdots \triangleleft \mathfrak{D}_n = \mathfrak{D}. \tag{8}$$

Next, we shall define the concepts of maximal normal submultigroup of a multigroup and simple multigroup. From the concept of normal submultigroup in Definition 9, we define a maximal normal submultigroup of a multigroup as follows:

Definition 15. Let $\mathfrak{C}$ and $\mathfrak{D}$ be multigroups of G such that $\mathfrak{C} \triangleleft \mathfrak{D}$. Then

(i) $\mathfrak{C}$ is a maximal non-trivial normal submultigroup if it is the largest proper non-trivial normal submultigroup of $\mathfrak{D}$.

(ii) $\mathfrak{D}$ is simple if it has no proper non-trivial normal submultigroup.

Remark 1. A submultigroup $\mathfrak{B}$ of a multigroup $\mathfrak{D}$ of G is trivial if:

(i) $\mathfrak{B}$ is the identity element or the identity element with multiplicity,

(ii) $\mathfrak{B}$ is a subgroup of G or G itself,

(iii) $\mathfrak{B}$ is the same as $\mathfrak{D}$.

Example 1. Suppose $G = \{1, d, d^2, d^3\}$ where $d^4 = 1$, $d^{-1} = d^3$, $(d^2)^{-1} = d^2$, and $(d^3)^{-1} = d$. Then, a multigroup of G is:

$$\mathfrak{D} = \{1, 1, 1, 1, d, d, d^2, d^2, d^2, d^3, d^3\}.$$

Certainly, $\mathfrak{D}$ is commutative. The submultigroups of $\mathfrak{D}$ can be presented in terms of:

(i) the properties of G without multiplicity (since every subgroup is a submultigroup of trivial multiplicity),

(ii) $\mathfrak{D}_*$ with multiplicity,

(iii) the properties of G and multiplicity.

By Case (i), the submultigroups of $\mathfrak{D}$ are as follows:

$$\mathfrak{D}_0 = \{1\}, \mathfrak{D}_1 = \{1, d^2\}, \mathfrak{D}_2 = \mathfrak{D}_* = \{1, d, d^2, d^3\}.$$

In this case, $\mathfrak{D}$ is not included as a submultigroup of itself because it contains multiplicity. Clearly, $\{1, d, d^3\}$ is not a submultigroup of $\mathfrak{D}$ because $d.d$, $d^3.d^3 \notin \{1, d, d^3\}$. Since $\mathfrak{D}_0$, $\mathfrak{D}_1$, and $\mathfrak{D}_2$ are trivial submultigroups of $\mathfrak{D}$ (because their multiplicity is 1), $\mathfrak{D}$ has neither proper non-trivial normal submultigroup nor a maximal non-trivial normal submultigroup and hence, $\mathfrak{D}$ is a simple multigroup.

Using Case (ii), the submultigroups of $\mathfrak{D}$ are in Table 1:

Table 1: Submultigroups of $\mathfrak{D}$ based on Case (ii)

Submultigroups and their structures
$\hat{\mathfrak{D}}_1 = \mathfrak{D}_2 = \{1, d, d^2, d^3\}$, $\hat{\mathfrak{D}}_2 = \{1, 1, d, d^2, d^3\}$,
$\hat{\mathfrak{D}}_3 = \{1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_4 = \{1, 1, 1, d, d^2, d^3\}$
$\hat{\mathfrak{D}}_5 = \{1, 1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_6 = \{1, 1, 1, d, d, d^2, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_7 = \{1, 1, 1, 1, d, d^2, d^3\}$,
$\hat{\mathfrak{D}}_8 = \{1, 1, 1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_9 = \mathfrak{D} = \{1, 1, 1, 1, d, d, d^2, d^2, d^2, d^3, d^3\}$

Among the list, $\hat{\mathfrak{D}}_1$ and $\hat{\mathfrak{D}}_9$ are trivial submultigroups because $\hat{\mathfrak{D}}_1 = \mathfrak{D}_*$ and $\hat{\mathfrak{D}}_9$ is a submultigroup of itself, $\hat{\mathfrak{D}}_2$–$\hat{\mathfrak{D}}_8$ are proper non-trivial normal submultigroups of $\mathfrak{D}$, and $\hat{\mathfrak{D}}_8$ is the maximal non-trivial normal submultigroup of $\mathfrak{D}$. Hence, $\mathfrak{D}$ is not simple.

By Case (iii), the submultigroups of $\mathfrak{D}$ are in Table 2:

Table 2: Submultigroups of $\mathfrak{D}$ based on Case (iii)

Submultigroups and their structures
$\mathfrak{D}_0 = \{1\}$, $\mathfrak{D}_1 = \{1, d^2\}$, $\mathfrak{D}_2 = \hat{\mathfrak{D}}_1 = \{1, d, d^2, d^3\}$,
$\hat{\mathfrak{D}}_2 = \{1, 1, d, d^2, d^3\}$, $\hat{\mathfrak{D}}_3 = \{1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_4 = \{1, 1, 1, d, d^2, d^3\}$, $\hat{\mathfrak{D}}_5 = \{1, 1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_6 = \{1, 1, 1, d, d, d^2, d^2, d^2, d^3, d^3\}$, $\hat{\mathfrak{D}}_7 = \{1, 1, 1, 1, d, d^2, d^3\}$,
$\hat{\mathfrak{D}}_8 = \{1, 1, 1, 1, d, d, d^2, d^2, d^3, d^3\}$,
$\hat{\mathfrak{D}}_9 = \mathfrak{D} = \{1, 1, 1, 1, d, d, d^2, d^2, d^2, d^3, d^3\}$,
$\dot{\mathfrak{D}}_1 = \{1, 1\}$, $\dot{\mathfrak{D}}_2 = \{1, 1, 1\}$, $\dot{\mathfrak{D}}_3 = \{1, 1, 1, 1\}$, $\dot{\mathfrak{D}}_4 = \{1, 1, d^2\}$,
$\dot{\mathfrak{D}}_5 = \{1, 1, d^2, d^2\}$, $\dot{\mathfrak{D}}_6 = \{1, 1, 1, d^2\}$, $\dot{\mathfrak{D}}_7 = \{1, 1, 1, d^2, d^2\}$,
$\dot{\mathfrak{D}}_8 = \{1, 1, 1, d^2, d^2, d^2\}$, $\dot{\mathfrak{D}}_9 = \{1, 1, 1, 1, d^2\}$,
$\dot{\mathfrak{D}}_{10} = \{1, 1, 1, 1, d^2, d^2\}$, $\dot{\mathfrak{D}}_{11} = \{1, 1, 1, 1, d^2, d^2, d^2\}$

Among the list, $\mathfrak{D}_0$–$\mathfrak{D}_2$, $\dot{\mathfrak{D}}_1$–$\dot{\mathfrak{D}}_3$, and $\hat{\mathfrak{D}}_9$ are trivial submultigroups of $\mathfrak{D}$. Submultigroups $\hat{\mathfrak{D}}_2$–$\hat{\mathfrak{D}}_8$ and $\dot{\mathfrak{D}}_4$–$\dot{\mathfrak{D}}_{11}$ are proper non-trivial normal submultigroups of $\mathfrak{D}$, and $\hat{\mathfrak{D}}_8$ is the maximal non-trivial normal submultigroups of $\mathfrak{D}$. Again, $\mathfrak{D}$ is not a simple multigroup.

Example 2. In a symmetry group S_n for $n = 3$ (i.e., $S = \{1, 2, 3\}$), $A_3 = \{\rho_0, \rho_1, \rho_2\} \subseteq S_3$ is a simple group ($\rho_1^{-1} = \rho_2$, $\rho_2^{-1} = \rho_1$, and ρ_0 is the identity element). Using A_3, a multigroup defined over A_3 is:

$$\mathfrak{E} = \{\rho_0, \rho_0, \rho_0, \rho_1, \rho_1, \rho_2, \rho_2\}.$$

Certainly, A_3 is a trivial multigroup. Since $\rho_1.\rho_2 = \rho_2.\rho_1 = \rho_0$, $\mathfrak{E}$ is commutative.

By Case (i), the submultigroups of $\mathfrak{E}$ are as follows:

$$\mathfrak{E}_0 = \{\rho_0\},\ \mathfrak{E}_1 = A_3 = \mathfrak{E}_* = \{\rho_0, \rho_1, \rho_2\}.$$

Because $\mathfrak{E}_0$ and $\mathfrak{E}_1$ are trivial, then $\mathfrak{E}$ is a simple multigroup (because it has no proper non-trivial normal submultigroup and hence, no maximal non-trivial normal submultigroup).

By Case (ii), the submultigroups of $\mathfrak{E}$ are in Table 3:

Table 3: Submultigroups of $\mathfrak{E}$ based on Case (ii)

Submultigroups and their Structures
$\hat{\mathfrak{E}}_1 = \mathfrak{E}_1 = \{\rho_0, \rho_1, \rho_2\}$, $\hat{\mathfrak{E}}_2 = \{\rho_0, \rho_0, \rho_1, \rho_2\}$,
$\hat{\mathfrak{E}}_3 = \{\rho_0, \rho_0, \rho_1, \rho_1, \rho_2, \rho_2\}$, $\hat{\mathfrak{E}}_4 = \{\rho_0, \rho_0, \rho_0, \rho_1, \rho_2\}$,
$\hat{\mathfrak{E}}_5 = \mathfrak{E} = \{\rho_0, \rho_0, \rho_0, \rho_1, \rho_1, \rho_2, \rho_2\}$

Here, it is observed that $\hat{\mathfrak{E}}_1$ and $\hat{\mathfrak{E}}_5$ are trivial, and $\hat{\mathfrak{E}}_2$–$\hat{\mathfrak{E}}_4$ are proper non-trivial normal submultigroups of $\mathfrak{E}$ without a maximal non-trivial normal submultigroup of $\mathfrak{D}$ since either $\hat{\mathfrak{E}}_3 \nsubseteq \hat{\mathfrak{E}}_4$ or $\hat{\mathfrak{E}}_4 \nsubseteq \hat{\mathfrak{E}}_3$. Hence, $\mathfrak{E}$ is not simple.

By Case (iii), the submultigroups of $\mathfrak{E}$ are in Table 4:

Table 4: Submultigroups of $\mathfrak{E}$ based on Case (iii)

Submultigroups and their structures
$\mathfrak{E}_0 = \{\rho_0\}$, $\hat{\mathfrak{E}}_1 = \mathfrak{E}_1 = \{\rho_0, \rho_1, \rho_2\}$, $\hat{\mathfrak{E}}_2 = \{\rho_0, \rho_0, \rho_1, \rho_2\}$,
$\hat{\mathfrak{E}}_3 = \{\rho_0, \rho_0, \rho_1, \rho_1, \rho_2, \rho_2\}$, $\hat{\mathfrak{E}}_4 = \{\rho_0, \rho_0, \rho_0, \rho_1, \rho_2\}$,
$\hat{\mathfrak{E}}_5 = \mathfrak{E} = \{\rho_0, \rho_0, \rho_0, \rho_1, \rho_1, \rho_2, \rho_2\}$, $\dot{\mathfrak{E}}_1 = \{\rho_0, \rho_0\}$,
$\dot{\mathfrak{E}}_2 = \{\rho_0, \rho_0, \rho_0\}$

Here, $\mathfrak{E}_0$, $\hat{\mathfrak{E}}_1$, $\dot{\mathfrak{E}}_1$, $\dot{\mathfrak{E}}_2$ and $\hat{\mathfrak{E}}_5$ are trivial, and $\hat{\mathfrak{E}}_2$–$\hat{\mathfrak{E}}_4$ are proper non-trivial normal submultigroups of $\mathfrak{E}$ without a maximal non-trivial normal submultigroup of $\mathfrak{E}$ since either $\hat{\mathfrak{E}}_3 \nsubseteq \hat{\mathfrak{E}}_4$ or $\hat{\mathfrak{E}}_4 \nsubseteq \hat{\mathfrak{E}}_3$. Hence, $\mathfrak{E}$ is not a simple multigroup.

Example 3. Let $G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be a group of order 8 with two generators a and b, which satisfy the relations:

(i) $a^4 = b^2 = 1$ and $ba = a^3b = a^{-1}b$, which is a group recognize as D_4.

(ii) $a^4 = 1$, $a^2 = b^2$, and $ba = a^3b$, which is a group of unit quaternions.

Indeed, $a^{-1} = a^3$, $(a^3)^{-1} = a$, $(a^2)^{-1} = a^2$, $b^{-1} = b$, $(ab)^{-1} = a^3b$, $(a^2b)^{-1} = a^2b$, and $(a^3b)^{-1} = ab$. Certainly, G is non-abelian since $b(ab) \neq (ab)b$ because $b(ab) = (ba)b = a^3b^2 = a^3$ and $(ab)b = ab^2 = a$.

Using G, a multigroup defined over G is:

$$\mathfrak{F} = \{1, 1, 1, a, a, a^2, a^2, a^2, a^3, a^3, b, b, ab, ab, a^2b, a^2b, a^3b, a^3b\},$$

which is also non-commutative.

The following structures in Table 5 are the submultigroups of $\mathfrak{F}$:

Table 5: Submultigroups of $\mathfrak{F}$

Submultigroups and their structures
$\mathfrak{F}_0 = \{1\}$, $\mathfrak{F}_1 = \{1, 1\}$, $\mathfrak{F}_2 = \{1, 1, 1\}$, $\mathfrak{F}_3 = \{1, a, a^2, a^3\}$,
$\mathfrak{F}_4 = \{1, 1, a, a^2, a^3\}$, $\mathfrak{F}_5 = \{1, 1, a, a, a^2, a^2, a^3, a^3\}$,
$\mathfrak{F}_6 = \{1, 1, 1, a, a^2, a^3\}$, $\mathfrak{F}_7 = \{1, 1, 1, a, a, a^2, a^2, a^3, a^3\}$,
$\mathfrak{F}_8 = \{1, 1, 1, a, a, a^2, a^2, a^2, a^3, a^3\}$, $\mathfrak{F}_9 = \{1, a^2\}$,
$\mathfrak{F}_{10} = \{1, 1, a^2, a^2\}$, $\mathfrak{F}_{11} = \{1, 1, 1, a^2, a^2, a^2\}$, $\mathfrak{F}_{12} = \{1, b\}$,
$\mathfrak{F}_{13} = \{1, 1, b\}$, $\mathfrak{F}_{14} = \{1, 1, b, b\}$, $\mathfrak{F}_{15} = \{1, 1, 1, b, b\}$,
$\mathfrak{F}_{16} = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$,
$\mathfrak{F}_{17} = \{1, 1, a, a^2, a^3, b, ab, a^2b, a^3b\}$,
$\mathfrak{F}_{18} = \{1, 1, a, a, a^2, a^2, a^3, a^3, b, b, ab, ab, a^2b, a^2b, a^3b, a^3b\}$,
$\mathfrak{F}_{19} = \{1, 1, 1, a, a^2, a^3, b, ab, a^2b, a^3b\}$,
$\mathfrak{F}_{20} = \{1, 1, 1, a, a, a^2, a^2, a^3, a^3, b, b, ab, ab, a^2b, a^2b, a^3b, a^3b\}$,
$\mathfrak{F} = \{1, 1, 1, a, a, a^2, a^2, a^2, a^3, a^3, b, b, ab, ab, a^2b, a^2b, a^3b, a^3b\}$

Here, $\mathfrak{F}_0$–$\mathfrak{F}_3$, $\mathfrak{F}_9$, $\mathfrak{F}_{12}$, and $\mathfrak{F}$ are trivial. The submultigroups $\mathfrak{F}_4$–$\mathfrak{F}_8$, $\mathfrak{F}_{10}$, $\mathfrak{F}_{11}$, and $\mathfrak{F}_{13}$–$\mathfrak{F}_{20}$ are proper non-trivial normal submultigroups of $\mathfrak{F}$. Hence, $\mathfrak{F}$ is not a simple multigroup. Again, the maximal non-trivial normal submultigroup of $\mathfrak{F}$ is $\mathfrak{F}_{20}$.

Remark 2. Every multigroup whose submultigroups are trivial is a simple multigroup.

To see this, given a multigroup $\mathfrak{E} = \{\rho_0, \rho_0, \rho_1, \rho_2\}$ defined over $A_3 = \{\rho_0, \rho_1, \rho_2\}$. Then, the submultigroups of $\mathfrak{E}$ are:

$$\mathfrak{E}_0 = \{\rho_0\},$$
$$\mathfrak{E}_1 = \{\rho_0, \rho_1, \rho_2\},$$
$$\mathfrak{E}_2 = \{\rho_0, \rho_0, \rho_1, \rho_2\} = \mathfrak{E}.$$

Among these submultigroups of $\mathfrak{E}$, we notice that all of them are trivial. Thus, $\mathfrak{E}$ is simple.

Definition 16. Let G be a finite group and $\mathfrak{D}$ be a multigroup of G with a finite multiplicity. Then, $\mathfrak{D}$ has a normal series if there exist:

$$C_{\mathfrak{D}_0}(x) \leq C_{\mathfrak{D}_1}(x) \leq \cdots \leq C_{\mathfrak{D}_n}(x) = C_{\mathfrak{D}}(x)\ \forall x \in G, \tag{9}$$

such that $(\mathfrak{D}_0)_* = (\mathfrak{D}_1)_* = \cdots = (\mathfrak{D}_n)_* = \mathfrak{D}_*$ and $\mathfrak{D}_i \triangleleft \mathfrak{D}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Example 4. Using the submultigroups of $\mathfrak{D}$ in Example 1, we observe that normal series only exists for Case (ii) and Case (iii). It does not exist for Case (i) because $(\mathfrak{D}_0)_* \neq \mathfrak{D}_*$, $(\mathfrak{D}_1)_* \neq \mathfrak{D}_*$, and $(\mathfrak{D}_2)_* \neq \mathfrak{D}_*$. The normal series for Case (ii) is identical to Case (iii) because all the submultigroups in Case (ii) are in Case (iii), and the rest of the submultigroups in Case (iii) do not share the same elements as G. Thus, the normal series for $\mathfrak{D}$ are:

$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_4 \subseteq \hat{\mathfrak{D}}_7 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D},$$
$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_3 \subseteq \hat{\mathfrak{D}}_5 \subseteq \hat{\mathfrak{D}}_6 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D},$$
$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_4 \subseteq \hat{\mathfrak{D}}_5 \subseteq \hat{\mathfrak{D}}_6 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D}.$$

Certainly, $\hat{\mathfrak{D}}_i \triangleleft \hat{\mathfrak{D}}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Example 5. Using the submultigroups of $\mathfrak{E}$ in Example 2, we have the following normal series:

$$\hat{\mathfrak{E}}_1 \subseteq \hat{\mathfrak{E}}_2 \subseteq \hat{\mathfrak{E}}_3 \subseteq \hat{\mathfrak{E}}_5 = \mathfrak{E},$$
$$\hat{\mathfrak{E}}_1 \subseteq \hat{\mathfrak{E}}_2 \subseteq \hat{\mathfrak{E}}_4 \subseteq \hat{\mathfrak{E}}_5 = \mathfrak{E},$$

where $\hat{\mathfrak{E}}_i \triangleleft \hat{\mathfrak{E}}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Closely related to normal series is the concept of composition series.

Definition 17. Let G be a finite group and $\mathfrak{D}$ be a multigroup of G with a finite multiplicity. Then, $\mathfrak{D}$ has a composition series if there exist a chain of consecutive submultigroups:

$$C_{\mathfrak{D}_0}(x) \leq C_{\mathfrak{D}_1}(x) \leq \cdots \leq C_{\mathfrak{D}_n}(x) = C_{\mathfrak{D}}(x) \; \forall x \in G, \tag{10}$$

such that $(\mathfrak{D}_0)_* = (\mathfrak{D}_1)_* = \cdots = (\mathfrak{D}_n)_* = \mathfrak{D}_*$ with the properties

(i) $\mathfrak{D}_i \lhd \mathfrak{D}_{i+1} \; \forall \; 0 \leq i \leq n-1$,

(ii) $\mathfrak{D}_{i+1}/\mathfrak{D}_i$ is simple $\forall \; 0 \leq i \leq n-1$.

For Example 1, the composition series for $\mathfrak{D}$ are:

$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_4 \subseteq \hat{\mathfrak{D}}_7 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D},$$
$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_3 \subseteq \hat{\mathfrak{D}}_5 \subseteq \hat{\mathfrak{D}}_6 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D},$$
$$\hat{\mathfrak{D}}_1 \subseteq \hat{\mathfrak{D}}_2 \subseteq \hat{\mathfrak{D}}_4 \subseteq \hat{\mathfrak{D}}_5 \subseteq \hat{\mathfrak{D}}_6 \subseteq \hat{\mathfrak{D}}_8 \subseteq \hat{\mathfrak{D}}_9 = \mathfrak{D}.$$

Certainly, $\hat{\mathfrak{D}}_i \lhd \hat{\mathfrak{D}}_{i+1} \; \forall \; 0 \leq i \leq n-1$ and $\hat{\mathfrak{D}}_{i+1}/\hat{\mathfrak{D}}_i$ is simple $\forall \; 0 \leq i \leq n-1$.

For Example 2, we have the following normal series:

$$\hat{\mathfrak{E}}_1 \subseteq \hat{\mathfrak{E}}_2 \subseteq \hat{\mathfrak{E}}_3 \subseteq \hat{\mathfrak{E}}_5 = \mathfrak{E},$$
$$\hat{\mathfrak{E}}_1 \subseteq \hat{\mathfrak{E}}_2 \subseteq \hat{\mathfrak{E}}_4 \subseteq \hat{\mathfrak{E}}_5 = \mathfrak{E}.$$

Since $\hat{\mathfrak{E}}_i$ for $i = 1, 2, 3, 4, 5$ are normal, then $\hat{\mathfrak{E}}_i \lhd \hat{\mathfrak{E}}_{i+1} \; \forall \; 0 \leq i \leq n-1$ and $\hat{\mathfrak{E}}_{i+1}/\hat{\mathfrak{E}}_i$ is simple $\forall \; 0 \leq i \leq n-1$.

Example 6. Let $\mathfrak{C} = \{0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5\}$ be a multigroup of $\mathbb{Z}_6$. Then, the non-trivial submultigroups of $\mathfrak{C}$ which share the same elements as G are:

$$\hat{\mathfrak{C}}_1 = \{0, 1, 2, 3, 4, 5\},$$
$$\hat{\mathfrak{C}}_2 = \{0, 0, 1, 2, 3, 4, 5\},$$
$$\hat{\mathfrak{C}}_3 = \{0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5\},$$
$$\hat{\mathfrak{C}}_4 = \{0, 0, 0, 1, 2, 3, 4, 5\},$$
$$\hat{\mathfrak{C}}_5 = \{0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5\} = \mathfrak{C}.$$

Then, the composition series are:

$$\hat{\mathfrak{C}}_1 \subseteq \hat{\mathfrak{C}}_2 \subseteq \hat{\mathfrak{C}}_3 \subseteq \hat{\mathfrak{C}}_5 = \mathfrak{C},$$
$$\hat{\mathfrak{C}}_1 \subseteq \hat{\mathfrak{C}}_2 \subseteq \hat{\mathfrak{C}}_4 \subseteq \hat{\mathfrak{C}}_5 = \mathfrak{C}.$$

Because $\hat{\mathfrak{C}}_i \lhd \mathfrak{C}$ for $i = 1, 2, 3, 4$, then $\hat{\mathfrak{C}}_i \lhd \hat{\mathfrak{C}}_{i+1} \; \forall \; 0 \leq i \leq n-1$ and $\hat{\mathfrak{C}}_{i+1}/\hat{\mathfrak{C}}_i$ is simple $\forall$ $0 \leq i \leq n-1$.

Theorem 1. *Every finite multigroup defined over a finite group has a composition series.*

Proof. Let $\mathfrak{D}$ be a finite multigroup over a finite G. We establish the proof by the principle of induction. Suppose every finite multigroup of order less than $|\mathfrak{D}|$ has a composition series. Now, if $\mathfrak{D}$ is simple, then there is no composition series since no non-trivial proper normal submultigroup exist. On the other hand, if $\mathfrak{D}$ is not simple, then there must be a non-trivial proper normal submultigroup. Since $\mathfrak{D}$ is finite, there is a maximal non-trivial normal submultigroup in $\mathfrak{D}$, which we denote as $\mathfrak{B}$. Certainly, $|\mathfrak{B}| < |\mathfrak{D}|$.

By induction, $|\mathfrak{B}|$ has a composition series:

$$C_{\mathfrak{B}_0}(x) \leq C_{\mathfrak{B}_1}(x) \leq \cdots \leq C_{\mathfrak{B}_n}(x) = C_{\mathfrak{B}}(x) \;\forall x \in G.$$

But then, $\mathfrak{B} \lhd \mathfrak{D}$ since $\mathfrak{B}$ is maximal in $\mathfrak{D}$, and so

$$C_{\mathfrak{B}_0}(x) \leq C_{\mathfrak{B}_1}(x) \leq \cdots \leq C_{\mathfrak{B}_n}(x) = C_{\mathfrak{B}}(x) \leq C_{\mathfrak{D}}(x) \;\forall x \in G,$$

which is the composition series for $\mathfrak{D}$.

Theorem 2 (The Jordan-Hölder Theorem)**.** *Every finite multigroup defined over a finite group has at least two composition series which are equivalent.*

Proof. Let $\mathfrak{D}$ be a finite multigroup over a finite group G. Suppose we have two composition series for $\mathfrak{D}$:

$$\begin{gathered}C_{\mathfrak{D}_0}(x) \leq C_{\mathfrak{D}_1}(x) \leq \cdots \leq C_{\mathfrak{D}_n}(x) = C_{\mathfrak{D}}(x) \;\forall x \in G,\\ C_{\mathfrak{B}_0}(x) \leq C_{\mathfrak{B}_1}(x) \leq \cdots \leq C_{\mathfrak{B}_m}(x) = C_{\mathfrak{D}}(x) \;\forall x \in G,\end{gathered}$$

such that $\mathfrak{D}_i \lhd \mathfrak{D}_{i+1}$ with $\mathfrak{D}_{i+1}/\mathfrak{D}_i$ simple $\forall\; 0 \leq i \leq n-1$ and $\mathfrak{B}_j \lhd \mathfrak{B}_{j+1}$ with $\mathfrak{B}_{j+1}/\mathfrak{B}_j$ simple $\forall\; 0 \leq j \leq m-1$. We need to prove that $n = m$ and $(\mathfrak{D}_1/\mathfrak{D}_0, \mathfrak{D}_2/\mathfrak{D}_1, \cdots, \mathfrak{D}_n/\mathfrak{D}_{n-1})$ is a rearrangement (denoted as $\sim$) of $(\mathfrak{B}_1/\mathfrak{B}_0, \mathfrak{B}_2/\mathfrak{B}_1, \cdots, \mathfrak{B}_m/\mathfrak{B}_{m-1})$.

We prove by induction on $|\mathfrak{D}|$. For $|\mathfrak{D}| = 1$, the result is trivial. Assume $\mathfrak{D}_{n-1} = \mathfrak{B}_{m-1}$, the result follows by induction. Then, assume $\mathfrak{D}_{n-1} \neq \mathfrak{B}_{m-1}$. Set $\Phi = \mathfrak{D}_{n-1}$, $\Psi = \mathfrak{B}_{m-1}$, and $\Omega = \Phi \cap \Psi$, where Ω is a maximal submultigroup of $\mathfrak{D}_{n-1}$ and $\mathfrak{B}_{m-1}$.

Now, Ω has a composition series, $C_{\Omega_0}(x) \leq C_{\Omega_1}(x) \leq \cdots \leq C_{\Omega_t}(x) = C_{\Omega}(x) \;\forall x \in G$. Then,

$$\begin{gathered}C_{\mathfrak{D}_0}(x) \leq C_{\mathfrak{D}_1}(x) \leq \cdots \leq C_{\mathfrak{D}_{n-1}}(x) = C_{\Phi}(x) \;\forall x \in G \text{ and}\\ C_{\Omega_0}(x) \leq C_{\Omega_1}(x) \leq \cdots \leq C_{\Omega_t}(x) = C_{\Omega}(x) \leq C_{\Phi}(x) \;\forall x \in G\end{gathered}$$

are both composition series for Φ. By induction, we have $n-1 = t+1 \Rightarrow n-2 = t$, and

$$(\mathfrak{D}_1/\mathfrak{D}_0, \mathfrak{D}_2/\mathfrak{D}_1, \cdots, \mathfrak{D}_{n-1}/\mathfrak{D}_{n-2}) \sim (\Omega_1/\Omega_0, \Omega_2/\Omega_1, \cdots, \Omega_t/\Omega_{t-1}, \Phi/\Omega). \tag{11}$$

Similarly,

$$\begin{gathered}C_{\mathfrak{B}_0}(x) \leq C_{\mathfrak{B}_1}(x) \leq \cdots \leq C_{\mathfrak{B}_{m-1}}(x) = C_{\Psi}(x) \;\forall x \in G \text{ and}\\ C_{\Omega_0}(x) \leq C_{\Omega_1}(x) \leq \cdots \leq C_{\Omega_t}(x) = C_{\Omega}(x) \leq C_{\Psi}(x) \;\forall x \in G\end{gathered}$$

are both composition series for Ψ. Thus, $m-1=t+1 \Rightarrow m-2=t$, and

$$(\mathfrak{B}_1/\mathfrak{B}_0, \mathfrak{B}_2/\mathfrak{B}_1, \cdots, \mathfrak{B}_{m-1}/\mathfrak{B}_{m-2}) \sim (\Omega_1/\Omega_0, \Omega_2/\Omega_1, \cdots, \Omega_t/\Omega_{t-1}, \Psi/\Omega). \tag{12}$$

From $m-1=t+1$ and $n-1=t+1$, we have $n=m$. By appending $\mathfrak{D}/\Phi$ to both sides of (11), we have

$$(\mathfrak{D}_1/\mathfrak{D}_0, \cdots, \mathfrak{D}_{n-1}/\mathfrak{D}_{n-2}, \mathfrak{D}/\mathfrak{D}_{n-1}) \sim (\Omega_1/\Omega_0, \cdots, \Omega_t/\Omega_{t-1}, \Phi/\Omega, \mathfrak{D}/\Phi). \tag{13}$$

Similarly, appending $\mathfrak{D}/\Psi$ to both sides of (12), we have

$$(\mathfrak{B}_1/\mathfrak{B}_0, \cdots, \mathfrak{B}_{m-1}/\mathfrak{B}_{m-2}, \mathfrak{D}/\mathfrak{B}_{m-1}) \sim (\Omega_1/\Omega_0, \cdots, \Omega_t/\Omega_{t-1}, \Psi/\Omega, \mathfrak{D}/\Psi). \tag{14}$$

The right hand side of (13) and (14) are identical except $(\Phi/\Omega, \mathfrak{D}/\Phi)$ and $(\Psi/\Omega, \mathfrak{D}/\Psi)$. Hence, $(\Phi/\Omega, \mathfrak{D}/\Phi) \sim (\Psi/\Omega, \mathfrak{D}/\Psi)$ and so

$$(\mathfrak{D}_1/\mathfrak{D}_0, \cdots, \mathfrak{D}_n/\mathfrak{D}_{n-1}) \sim (\mathfrak{B}_1/\mathfrak{B}_0, \cdots, \mathfrak{B}_m/\mathfrak{B}_{m-1}).$$

3. Conclusion

In this paper, the notions of simple multigroup, maximal normal submultigroup, normal series for multigroup, and composition series for multigroup were defined as algebraic structures in multiset context and characterized with examples and some results. In addition, it was proven that every finite multigroup defined over a finite group has a composition series. Finally, the Jordan-Hölder Theorem was presented in multiset context, and it was established that every finite multigroup defined over a finite group has at least two composition series which are equivalent. The multigroup structures can be applicable in cryptography for the security of data, molecular symmetry and molecular behavior during chemical reactions, quantum mechanics and particle physics, algorithms in areas related to coding theory and data structure, robotics and computer graphics, etc. For further investigation, the concept of nilpotency in multigroup setting could be considered.

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