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Analyzing Approximate and Exact Wave Solutions of Optical Fibres in the Time-Fractional Estevez-Mansfield-Clarkson Equation

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Abstract. The study aims to employ the dynamics of hyperbolic and solitons wave solutions to the time-fractional Estevez-Mansfield-Clarkson equation, which is utilize to stimulate the optical fibre waves in communication system and image processing. The proposed method entails transforming nonlinear fractional partial differential equations into nonlinear ordinary differential equations. Various solutions for the current model are derived in the form of exponential and trigonometric functions. The 2D and 3D under some suitable values of parameters are also plotted. The accomplished solutions show that these are reliable, applicable, efficient and intricate dynamics inherent in the communication system, image processing and might be used in further works to found novel solutions for other types of nonlinear evolution equations ascending in physical science and engineering. The novelty of our result is to offer new and critical insights into the intricate wave behaviors of the time-fractional Estevez-Mansfield-Clarkson equation by illustrating the basic mechanics of the importance of waves interaction and propagation in optical fibres in deepening our knowledge about the basic physical model.

2020 Mathematics Subject Classifications: 35Qxx, 26A33
Key Words and Phrases: Nonlinear differential equations, stability analysis, rescaling method, fractional calculus, traveling wave transformation

1. Introduction

In applied science, engineering, and mathematics, as well as in the modeling of a variety of physical nonlinear phenomena, nonlinear fractional partial differential equations (NFPDEs) are essential [1], [2] . The numerous fractional models [3] created in [4] have attracted a lot of interest in the domains of fluid dynamics, fluid mechanics, neurons, optical fibers, electric circuits, water waves, plasma waves, capillary-gravity waves, chemical

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physics, and plasma physics. It is crucial to comprehend how different elements behave in different scientific domains, which emphasizes the need to investigate methods for solving non-functional partial differential equations (PDEs) [5], [6] and [7]. Investigating techniques for solving fractional nonlinear partial differential equations (PDEs) is necessary to better understand the dynamics of these real-world components. This investigation is necessary to learn more about and comprehend the intricate behaviors that are present in the systems indicated above. The greater richness and generality that fractional nonlinear PDE solutions offer, which surpasses classical solutions in terms of descriptive power, makes them of scholarly interest [3].
To investigate new findings, mathematicians have developed and implemented innovative and reliable techniques. Some of these methods are, generalized Khater method, modified Khater method, simple equation method, Poincare-Lighthill-Kuo method, generalized Kudryashov method, fractional sub-equation method, modified extended direct algebraic method and (G'/G)-expansion method. Recent studies have used a variety of techniques to investigate solutions for solitary waves, revealing a wide range of approaches to comprehending intricate physical phenomena [8]. Many nonlinear models have been thoroughly examined for their qualitative responses in addition to their analytical solutions, including stability evaluations by changing parameters using bifurcation theory. Moreover, chaos theory has been used to examine their unexpected character under different initial conditions. Even in the most simple systems, chaos can appear as erratic patterns. A thorough review of the literature suggests that chaos seems to be a recurring theme in a number of events. Smoke rising from cigarettes, for example, becomes turbulent swirls. Additionally, weather patterns and the movement of oil through underground pipelines are examples of disorder. This study focuses on 3-D and 2-D plots, which are the most efficient ways to identify irregular patterns [9].
In [10], the Estevez-Mansfield-Clarkson Equation, introduced by Pilar Estevez, Elizabeth Mansfield, and Peter Clarkson in 1997, is a nonlinear partial differential equation [11], which is a nonlinear evolution equation of the fourth order. This equation has proven advantageous in examining wave dynamics in optical fibre, as it was initially employed in their investigation of pattern dispersion in wave. In [4], [12] the model under study is the nonlinear (1+1)-dimensional Estevez-Mansfield-Clarkson (EMC) equation given as

$$u_{txxx} + \beta(u_x u_{tx} + u_{xx} u_t) + u_{tt} = 0 \tag{1}$$

where $u = u(x,t)$ is a wave function. The parameter β represents a non-zero constant. In the literature, different methods are applied to solve Equation (1). The time-fractional EMC equation is expressed in its formal form as follows:

$$D_t^{\alpha} u_{xxx} + \beta(u_x D_t^{\alpha} u_x + u_{xx} D_t^{\alpha} u) + D_{2t}^{2\alpha} u = 0 \tag{2}$$

Here $t > 0$, $0 < \alpha \leq 1$. In the broad field of nonlinear science, the EMC equation is a crucial formula. It can be used in image processing, fluid dynamics, optics, and plasma physics and engineering. Recent research [13] has brought attention to the significance of the EMC equation and its critical role in understanding complex physical processes. Wave behavior in shallow water and optical fiber is the main applications for the EMC equation, which poses several difficulties. Researchers have used a variety of research approaches in their investigations. In [13], the authors used the sine-Gordon expansion technique to investigate soliton solutions of Eq. (2) and the Riemann model. Using a simplified equation technique, [14] and [15] investigated the EMC model to determine traveling wave patterns.

NFPDEs are converted into nonlinear ordinary differential equations (NODEs) using the suggested procedure [16], [17], [18], [19], [20], [21] and [22]. Solitons with a variety of mathematical structures, such as trigonometric, exponential, rational, and hyperbolic functions, have been found as a result of this research. Their true wave characteristics may be thoroughly examined thanks to these structures. In particular, the motivation of this paper is to show that these behaviors are different kinds of solitons. This is supported by showing how they look in 3D, and 2D graphs. Interestingly, our results produce a wider variety of results than previous solutions, thus providing new and significant insights into the underlying phenomena of the model under study. In short, Our obtained results are newer than the existing results of the model under study. Additionally, graphs representing the physical solutions of various fractional orders can be compared.

The rest of this investigation is organized in the following sections: Section 2 describes the main points of characteristics for the nonlinear-time fractional EMC equation; Section 3 shows the findings solutions for the time-fractional EMC equation; Section 4 includes the corresponding graphical representation; Section 5 includes discussion and Section 6, conclusion are summarized in brief points.

2. Characteristics

In quantum mechanics and field theory, this equation is applicable in quantum systems where nonlinearity and fractional dynamics play a role, such as fractional quantum mechanics or Bose-Einstein condensates. Is signal processing, this equation is used in signal and image processing for systems exhibiting fractal or self-similar behavior. In engineering, used in modeling and control of nonlinear dynamic systems where fractional-order dynamics are observed, such as robotics or materials with memory effects. In nonlinear optics, the time-fractional EMC equation is used to describe light propagation in nonlinear and dispersive media, particularly in fiber optics where fractional effects are significant. In fluid dynamics, it models the behavior of nonlinear waves and instabilities in fluid systems, especially those with nonlocal interactions or anomalous flow properties.

The mathematical importance of nonlinear time-fractional EMC is to finding exact or approximate solutions to the nonlinear time-fractional EMC equation is a key area of research, which helps in understanding the underlying physics of complex systems. This equation serves as a platform for analyzing fractional solitons and their interactions. By incorporating fractional calculus, the EMC equation becomes a powerful tool to model a wide range of phenomena that traditional integer-order models cannot handle, making it valuable across multiple scientific and engineering disciplines. The robustness of the propose method refers to its ability to understand numerical stability, handling of singu-

larities and computational efficiency.
The role of the nonlinear time-fractional EMC equation is to extends the classical EMC equation to accommodate fractional-order derivatives, making it applicable to systems with nonlocal and memory-dependent behavior. It plays a critical role in modeling nonlinear wave propagation in complex media. This equation bridges linear fractional dynamics with nonlinear effects, enabling the modeling of phenomena such as solitons, rogue waves, and bifurcations.
Physical Interpretation of the nonlinear time-fractional EMC equation lies in its ability to describe phenomena governed by fractional dynamics and nonlinear interactions. The time-fractional derivative reflects a nonlocal memory effect, meaning the system's present state is influenced by its past states in a weighted manner. The fractional nature implies that energy dissipation or storage is distributed over time in a non-exponential manner, aligning with real-world observations in complex systems. This is seen in materials with viscoelastic properties, where the stress-strain relationship depends on the entire deformation history.

3. Methodology

In this paragraph, we have applied stability analysis and rescaling method for finding exact and approximate solutions to the time-fractional EMC equation.
Step 1: First, a general form of a time-fractional differential equation is examined

$$P(D_t^{\alpha}u, u_x, D_{2t}^{2\alpha}u, u_{xx}, ...) = 0 \tag{3}$$

where P is the polynomial in the function $u(x,t)$ and its various partial derivatives and $D_{2t}^{2\alpha}u$ indicates two times sequential conformable fractional derivatives of the function $u(x,t)$.
Step 2: Applying the transformation

$$u(x,t) = f(\xi), \quad \xi = \sqrt{k}\left(x - \frac{wt^{\alpha}}{\Gamma(1+\alpha)}\right) \tag{4}$$

Using (4), we can write (3) in the following integer order nonlinear ordinary differential equation (ODE):

$$Q(f', f'', f''', ...) = 0 \tag{5}$$

where $f' = \frac{df}{d\xi}$, $f'' = \frac{d^2f}{d\xi^2}$, etc and Q is the polynomial in f and its derivatives.
Step 3: Now, we take a new independent variable $X = X(\xi)$, $Y = Y(\xi)$ such that

$$X(\xi) = f(\xi), \quad Y(\xi) = \frac{df}{d\xi}$$

Thus, (5) reduces to a new system of ODEs for stability analysis

$$X'(\xi) = f(X,Y) = Y$$

$$Y'(\xi) = g(X,Y) = X''$$

3.1. Determination of approximate solution by using Stability Analysis

In this section, to study the stability of the system, we first found the fixed points by solving the equations where all changes in the system (time derivatives) become zero. After identifying the fixed points, we calculated the Jacobian matrix, which is made up of partial derivatives of the system's equations.

$$J(X,Y) = \begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{bmatrix}$$

To check the stability of these fixed points, we calculated the eigenvalues and eigenvectors of the Jacobian matrix. If all eigenvalues had negative real parts, the fixed point was stable, meaning the system would return to it after small disturbances. If any eigenvalue had a positive real part, the fixed point was unstable. Complex eigenvalues with imaginary parts indicated oscillatory behavior. Eigenvectors helped us understand the directions in which the system grows or shrinks around the fixed points, providing a clearer picture of stability or phase space. The phase space is a graph where we plot the variables of the system against each other to see how the system changes over time.
Analyze the trajectory to understand how the system behaves overall. We used numerical simulations to plot trajectories starting from different initial conditions. These trajectories showed how the system moves toward fixed points, cycles, or more complex behaviors like chaos. The eigenvectors helped us identify stable and unstable directions near the fixed points, which guided how the trajectories flowed. This analysis gave us a complete picture of both the local and global behavior of the system.

3.2. Determination of exact solution by using Rescaling

The determination of an exact solution by rescaling into its simplest form and then integrating involves a structured methodology. First, the problem is analyzed to identify the governing equation, variables, and parameters. The equation is examined for scaling laws to guide the rescaling process. Next, dimensionless variables are introduced to reduce complexity, typically by substituting scaled variables like

$$z = h_1(w,k), \quad V = h_2(w,k,\beta)$$

where α, β are scaling factors. These substitutions simplify the original equation by reducing the number of parameters or their interaction. After rescaling, the equation is rewritten in terms of the new variables, often resulting in a simpler or standard form, such as a separable equation or a known ordinary differential equation (ODE).
Once simplified, the equation is solved analytically. If the equation is separable, direct integration is performed. Finally, the solution is rescaled back to the original variables to ensure dimensional consistency and physical applicability. The obtained solution is verified by substituting it into the original equation and ensuring consistency with the problem's constraints. This systematic approach simplifies complex equations into solvable forms and provides exact solutions with clarity and precision. Sometimes to handle

ODE by using elementary techniques and our system not give exact answers that's why we use rescaling method.

4. Applications

The purpose of this section is to provide approximate for case 1 by applying Stability analysis and case 2 is for exact solutions by using rescaling to time-fractional EMC equation with nonlinearity.

Case 1

We take traveling wave transformation for our purpose.

$$u(x,t) = U(\xi), \quad \xi = \sqrt{k}\left(x - \frac{wt^{\alpha}}{\Gamma(1+\alpha)}\right) \tag{6}$$

where k is a wavelength, $\frac{w}{\Gamma(1+\alpha)}$ is angular frequency and α is a fractional order. By means of the wave transformation (6), Equation (2) converts into the under mentioned NODE

$$-k^2wU'''' + \beta(-wk\sqrt{k}U'U'' - wk\sqrt{k}U'U'') + kw^2U'' = 0 \tag{7}$$

Further simplifying (7) and canceling w and k from (7)

$$kU'''' + 2\beta\sqrt{k}U'U'' - wU'' = 0 \tag{8}$$

By once integrating (8) and ignoring the integral constant, we can determine

$$kU''' + 2\beta\sqrt{k}(U')^2 - wU' = 0 \tag{9}$$

If we set $U' = V$, then (9) becomes

$$kV'' + 2\beta\sqrt{k}V^2 - wV = 0 \tag{10}$$

where $V' = \frac{dV}{d\xi}$. Let's take $r = 2\beta\sqrt{k}$ and $s = \frac{w}{k}$

After simplification, we get

$$V'' = sV - rV^2 \tag{11}$$

Introducing $X = V(\xi)$ and $Y = V'(\xi)$ as new variables, then equation (11) is equivalent to 2-D autonomous system

$$\frac{dX}{d\xi} = Y, \quad \frac{dY}{d\xi} = Y' = sV - rV^2 \tag{12}$$

As a result, we obtain this form

$$X' = f(X,Y) = Y \tag{13}$$

$$Y' = g(X,Y) = sV - rV^2 \tag{14}$$

To find the fixed points of the system, the smooth functions can be written in the following equations

Fixed Points:

$$f(X,Y)=0, \quad \implies Y=0$$

$$g(X,Y)=0, \quad \implies sX-rX^2=0, \quad \implies X=0,\frac{s}{r}$$

The fixed points for the system are

$$(X,Y)=(0,0), \quad (\frac{s}{r},0)$$

Jacobian Matrix:

To find the Jacobian matrix of the system, we have

$$J(X,Y)=\begin{bmatrix}\frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y}\\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y}\end{bmatrix}=\begin{bmatrix}0 & 1\\ s-2rX & 0\end{bmatrix}$$

$$J(0,0)=\begin{bmatrix}0 & 1\\ s & 0\end{bmatrix}$$

$$J(\frac{s}{r},0)=\begin{bmatrix}0 & 1\\ -s & 0\end{bmatrix}$$

Eigenvalues at Fixed Points:

Eigenvalues of Jacobian matrix J, can be found by following characteristics equation

$$det(J-\lambda I)=0$$

- **Case 1(a)** At fixed point $J(0,0)$

$$|J(0,0)-\lambda I|=\begin{vmatrix}-\lambda & 1\\ s & -\lambda\end{vmatrix}=0 \quad where \quad s=\frac{w}{k}$$

The eigenvalue can be written as

$$\lambda=\pm\sqrt{\frac{w}{k}} \implies \lambda_{11}=\sqrt{\frac{w}{k}}, \quad \lambda_{12}=-\sqrt{\frac{w}{k}}$$

If we choose the values of parameters $\beta=1$, $w=16$, $k=4$, $r=1$
we get

$$\lambda_{11}=2, \quad \lambda_{12}=-2$$

The eigenvalues $\lambda_{11}=2$ and $\lambda_{12}=-2$ indicates that the fixed points (0,0) is saddle. With these parameteric values, the system of ODEs simplifies to

$$X'=Y, \quad Y'=4X-X^2 \tag{15}$$

Eigenvector at $\lambda = \pm\sqrt{\frac{w}{k}}$ **:**
For each eigenvalue λ_{11} and λ_{12}, the eigenvectors are

$$v_{11} = \begin{bmatrix} 1 \\ \lambda_{11} \end{bmatrix}, \quad v_{12} = \begin{bmatrix} 1 \\ \lambda_{12} \end{bmatrix}$$

$$\implies \quad v_{11} = \begin{bmatrix} 1 \\ \sqrt{\frac{w}{k}} \end{bmatrix}, \quad v_{12} = \begin{bmatrix} 1 \\ -\sqrt{\frac{w}{k}} \end{bmatrix}$$

Trajectory
For the system, the general form is

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = c_1 v_{11} e^{\lambda_{11}\xi} + c_2 v_{12} e^{\lambda_{12}\xi} = c_1 \begin{bmatrix} 1 \\ \sqrt{\frac{w}{k}} \end{bmatrix} e^{\lambda_{11}\xi} + c_2 \begin{bmatrix} 1 \\ -\sqrt{\frac{w}{k}} \end{bmatrix} e^{\lambda_{12}\xi}$$

$$U'(\xi) = V(\xi) = X(\xi) = c_1 exp\left(\sqrt{\frac{w}{k}}\xi\right) + c_2 exp\left(-\sqrt{\frac{w}{k}}\xi\right)$$

After integration

$$U(\xi) = \sqrt{\frac{w}{k}} c_1 exp\left(\sqrt{\frac{w}{k}}\xi\right) - \sqrt{\frac{w}{k}} c_2 exp\left(-\sqrt{\frac{w}{k}}\xi\right) + c_3 \tag{16}$$

$$u_{11}(x,t) = \sqrt{\frac{w}{k}} c_1 exp\left[\sqrt{\frac{w}{k}}\sqrt{k}\left(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\right)\right] - \sqrt{\frac{w}{k}} c_2 exp\left[-\sqrt{\frac{w}{k}}\sqrt{k}\left(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\right)\right] + c_3$$

$$u_{11}(x,t) = \sqrt{\frac{w}{k}} c_1 exp\left[\sqrt{w}\left(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\right)\right] - \sqrt{\frac{w}{k}} c_2 exp\left[-\sqrt{w}\left(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\right)\right] + c_3 \tag{17}$$

- **Case 1(b)** At fixed point $J(\frac{s}{r}, 0)$

$$|J(\frac{s}{r}, 0) - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -s & -\lambda \end{vmatrix} = 0$$

The eigenvalue can be written as

$$\lambda = \pm i\sqrt{\frac{w}{k}} \implies \lambda_{13} = i\sqrt{\frac{w}{k}}, \quad \lambda_{14} = -i\sqrt{\frac{w}{k}}$$

If we choose the values of parameters $\beta = \frac{1}{2}$, $w = 32$, $k = 2$, $r = \frac{1}{\sqrt{2}}$
we get

$$\lambda_{13} = 4i, \quad \lambda_{14} = -4i$$

The eigenvalues $\lambda_{13} = 4i$ and $\lambda_{14} = -4i$ show that the fixed points $(\frac{s}{r}, 0)$ is centre and the system of ODEs simplifies under these parameters.

$$X' = Y, \quad Y' = 16X - \frac{1}{\sqrt{2}}X^2 \tag{18}$$

Eigenvector at $\lambda = \pm i\sqrt{\frac{w}{k}}$:
For each eigenvalue λ_{13} and λ_{14}, the eigenvectors are

$$v_{13} = \begin{bmatrix} 1 \\ \lambda_{13} \end{bmatrix}, \quad v_{14} = \begin{bmatrix} 1 \\ \lambda_{14} \end{bmatrix}$$

$$\implies \quad v_{13} = \begin{bmatrix} 1 \\ i\sqrt{\frac{w}{k}} \end{bmatrix}, \quad v_{14} = \begin{bmatrix} 1 \\ -i\sqrt{\frac{w}{k}} \end{bmatrix}$$

Trajectory

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = c_4 v_{13} e^{\lambda_{13}\xi} + c_5 v_{14} e^{\lambda_{14}\xi} = c_4 \begin{bmatrix} 1 \\ i\sqrt{\frac{w}{k}} \end{bmatrix} e^{\lambda_{13}\xi} + c_5 \begin{bmatrix} 1 \\ -i\sqrt{\frac{w}{k}} \end{bmatrix} e^{\lambda_{14}\xi}$$

$$U'(\xi) = V(\xi) = X(\xi) = c_4 exp\Big(i\sqrt{\frac{w}{k}}\xi\Big) + c_5 exp\Big(-i\sqrt{\frac{w}{k}}\xi\Big)$$

After integration

$$U(\xi) = c_4\Big(cos\sqrt{\frac{w}{k}}\xi + isin\sqrt{\frac{w}{k}}\xi\Big) + c_5\Big(cos\sqrt{\frac{w}{k}}\xi - isin\sqrt{\frac{w}{k}}\xi\Big) + c_6 \tag{19}$$

$$U(\xi) = c_4 + c_5\Big(cos\sqrt{\frac{w}{k}}\xi\Big) + i(c_4 - c_5)\Big(sin\sqrt{\frac{w}{k}}\xi\Big) + c_6$$

If we take $c_4 + c_5 = A$ and $i(c_4 - c_5) = B$

$$u_{12}(x,t) = Acos\Big[\sqrt{w}\Big(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\Big)\Big] + Bsin\Big[\sqrt{w}\Big(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\Big)\Big] + c_6 \tag{20}$$

Case 2
To find the exact solution of ODE (10) using traveling wave transformation (6), first rescaling it by using new variables

$$z = \sqrt{\frac{w}{k}}\xi, \quad V = \frac{w}{\beta\sqrt{k}}W(z) \tag{21}$$

$$\frac{dV}{dz} = \frac{w}{\beta\sqrt{k}}W'(z), \quad \frac{d^2V}{dz^2} = \frac{w}{\beta\sqrt{k}}W''(z)$$

Substitute these relations into ODE (10), we get

$$k\left(\frac{w}{\beta\sqrt{k}}W''(z)\right) + 2\beta\sqrt{k}\left(\frac{w}{\beta\sqrt{k}}W(z)\right)^2 - w\left(\frac{w}{\beta\sqrt{k}}W(z)\right) = 0$$

$$\frac{w\sqrt{k}}{\beta}W''(z) + 2\frac{w^2}{\beta\sqrt{k}}W^2(z) - \frac{w^2}{\beta\sqrt{k}}W(z) = 0 \tag{22}$$

Multiply β and divide by w and $\sqrt{k}$ through (22), we have

$$W''(z) + 2\frac{w}{k}W^2(z) - \frac{w}{k}W(z) = 0 \tag{23}$$

Let $m = \frac{w}{k}$. Then (22) becomes

$$W''(z) + 2mW^2(z) - mW(z) = 0 \tag{24}$$

Reduce the Order

Multiply through by $W'(z)$, which allows us to integrate directly

$$W'(z)W''(z) + 2mW^2(z)W'(z) - mW(z)W'(z) = 0 \tag{25}$$

For simplicity

$$W'(z)W''(z) = \frac{1}{2}\frac{d}{dz}(W'(z))^2$$

$$\frac{1}{2}\frac{d}{dz}(W'(z))^2 + m\frac{d}{dz}\left(W^3(z) - \frac{1}{2}W^2(z)\right) = 0$$

Integrate with respect to "z" and setting constant of integration is zero.

$$\frac{1}{2}(W'(z))^2 + m\left(W^3(z) - \frac{1}{2}W^2(z)\right) = 0$$

$$(W'(z))^2 + 2m\left(W^3(z) - \frac{1}{2}W^2(z)\right) = 0$$

$$(W'(z))^2 = mW^2(z) - 2mW^3(z)$$

Taking square root of both sides

$$W'(z) = \pm\sqrt{mW^2(z) - 2mW^3(z)}$$

After Integration,

$$W(z) = \frac{1}{2}tan^2\left(\frac{c_7}{2}\sqrt{m} - \frac{1}{2}z\sqrt{m}\right) + \frac{1}{2} \tag{26}$$

where c_7 is constant of integration.

$$W(z) = \frac{1}{2}tan^2\left(\frac{c_7}{2}\sqrt{\frac{w}{k}} - \frac{1}{2}\sqrt{\frac{w}{k}}\xi\sqrt{\frac{w}{k}}\right) + \frac{1}{2}$$

$$W(z) = \frac{1}{2}tan^2\left(\frac{c_7}{2}\sqrt{\frac{w}{k}} - \frac{1}{2}\frac{w}{k}\xi\right) + \frac{1}{2} \tag{27}$$

As above $V = \frac{w}{\beta\sqrt{k}}W(z) \implies W(z) = \frac{\beta\sqrt{k}}{w}V$, then (27) becomes

$$V = \frac{w}{2\beta\sqrt{k}}tan^2\left(\frac{c_7}{2}\sqrt{\frac{w}{k}} - \frac{1}{2}\frac{w}{k}\xi\right) + \frac{1}{2}$$

From $U' = V$ and after integration, we get

$$U(\xi) = -\frac{\sqrt{k}tan\left(\frac{c_7\sqrt{\frac{w}{k}}k-2w\xi}{2k}\right)}{2\beta} + \frac{\sqrt{k}arctan\left(tan\left(\frac{c_7\sqrt{\frac{w}{k}}k-2w\xi}{2k}\right)\right)}{2\beta} + \frac{1}{2}\xi + c_8$$

$$u(x,t) = -\frac{\sqrt{k}tan(\theta)}{2\beta} + \frac{\sqrt{k}arctan[tan(\theta)]}{2\beta} + \frac{1}{2}\sqrt{k}\left(x - \frac{wt^\alpha}{\Gamma(1+\alpha)}\right) + c_8 \tag{28}$$

where $\theta = \frac{c_7\sqrt{\frac{w}{k}}k-2w\sqrt{k}\left(x-\frac{wt^\alpha}{\Gamma(1+\alpha)}\right)}{2k}$ and c_8 is constant of integration. The exact solution $u(x,t)$ in (28) shown in Figure (7) and Figure (8) whereas $\frac{w}{\Gamma(1+\alpha)}$ indicates sudden change in the angular frequency.

5. Graphical Representation

The physical structure of applied phenomena in the real world can only be depicted through the use of graphic representations. The evaluated soliton solutions in Equations (15), (18), given the supplied values for the free parameters, give a range of geometrical shapes that represent the hyperbolic phase space in Figure (1) and Figure (4). The results in (17), (20) and (28) can understand the waves propagating to the optical fibres when we set $\alpha = 0.5$ in Figure (3) and Figure (5), when we set $\alpha = 0.9$ in Figure (8) and when we set $\alpha = 1$ in Figure (2), Figure (6) and Figure (7).

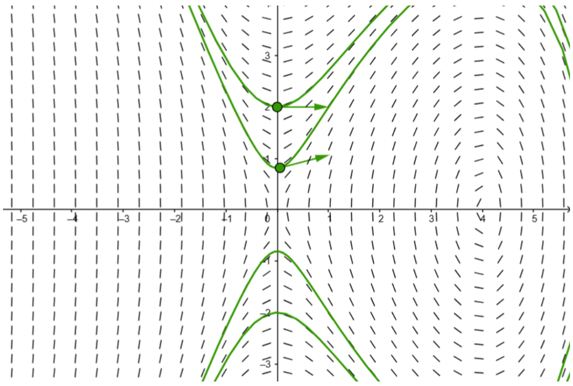

Figure 1: Hyperbolic phase space for solution of (15) when $\beta = 1$, $w = 16$, $k = 4$, $r = 1$

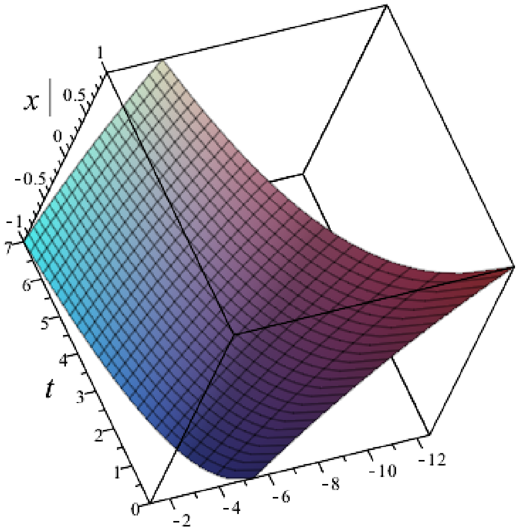

Figure 2: 3D for solution of (17) when $\alpha = 1$

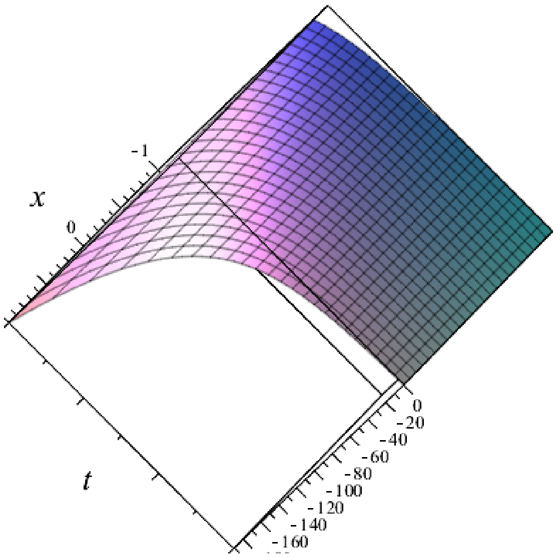

Figure 3: 3D for solution of (17) when $\alpha = \frac{1}{2}$

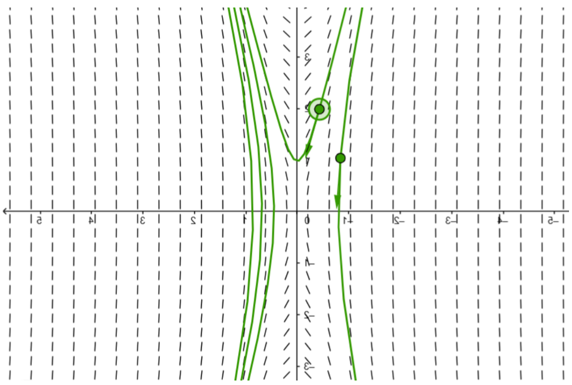

Figure 4: Hyperbolic phase space for solution of (18) when $\beta = \frac{1}{2}$, $w = 32$, $k = 2$, $r = \frac{1}{\sqrt{2}}$

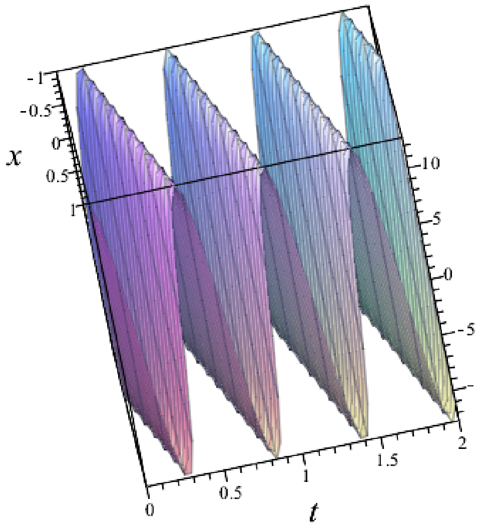

Figure 5: 3D for solution of (20) when $\alpha = \frac{1}{2}$

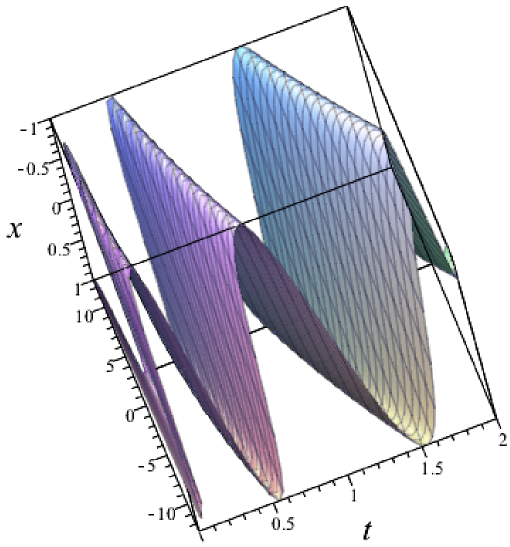

Figure 6: 3D for solution of (20) when $\alpha = 1$

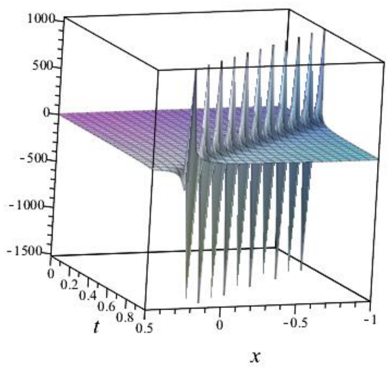

Figure 7: 3D for solution of (28) when $\alpha = 1$

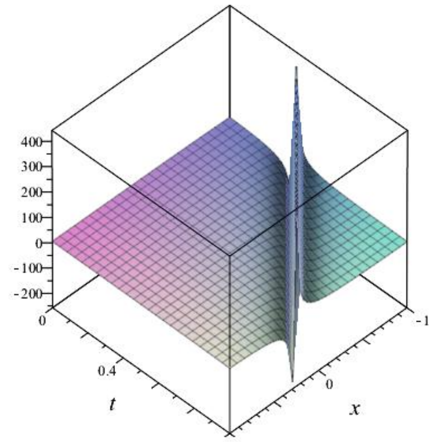

Figure 8: 3D for solution of (28) when $\alpha = 0.9$

6. Discussion

The stability analysis method has been utilized to study and solve the nonlinear time-fractional EMC equation. This approach used methodology to transform NFPDES into NODEs. Different answers in the form of exponential and trigonometric functions were obtained by solving the resulting problem. Using appropriate values for the associated parameters, we successfully examined a few generated solutions of exact and approximate. Our analysis revealed that the solution of case 1 shows the behavior i-e saddle and centre and case 2 shows exact solution. Solutions derived from the soliton simulations illustrated the fundamental ideas that controlled the wave's interaction and propagation. The complex dynamics of the EMC equation can be better understood thanks to these answers. Researchers may better grasp how the model behaves in different contexts by looking at these solutions, which will help them comprehend problems like wave dynamics in optical fiber. As a result, the findings of our research are crucial for tackling the real-world difficulties associated with examining various nonlinear structures that arise in shallow water. In addition to helping us understand the dynamics in these environments, they may be used to forecast and control fundamental fluid mechanics events.

7. Conclusion and future work

We successfully obtained two cases of wave solutions to solved fractional optical fiber equation which are the nonlinear space-time fractional EMC equation. We obtained different kinds of hyperbolic and soliton wave solutions, using stability analysis method to describe stable, unstable or saddle point and rescaling method to reduce the equation to get exact solution. Using Mathematica software, the derived solutions are validated by substituting into the equation. Both cases of the new analytical and approximate the nonlinear space-time fractional EMC equation appeared in Equations (17), (20) and (28). These solutions might also be helpful for optical fibre in engineering and scientific domains. Both two and three-dimensional graphs are used to illustrate the results, about how sudden change in angular frequency. Stability analysis is performed to ensure that the solutions are approximate while rescaling ensure that the solution are exact and accurate. The findings will be useful for further research on the related system. The techniques employed are readily applicable to other fractional partial differential equations that are nonlinear. Future studies could concentrate on examining different approaches to solving the problem, like numerical simulations or other analytical methods like the inverse scattering transform, as well as how the EMC equation behaves under various perturbations and boundary conditions to better understand its physical implications and possible uses.

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