EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5897
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Stability of Hyper 3-Homomorphisms and Hyper 3-Derivations in Ternary Algebras

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Abstract. In this paper, we introduce hyper 3-homomorphisms and hyper 3-derivations in complex ternary algebras and we prove the Hyers-Ulam stability of hyper 3-homomorphisms and hyper 3-derivations in complex ternary algebras for the following 3-additive functional equation

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k). \tag{1}$$

Further, we investigate isomorphisms between complex ternary algebras, associated with the 3-additive functional equation.

2020 Mathematics Subject Classifications: 11E20, 39B52, 39B82

Key Words and Phrases: Hyers-Ulam stability, 3-additive functional equation, ternary algebra, hyper 3-homomorphism, hyper 3-derivation

1. Introduction and Preliminaries

The first stability proplem was raised by Ulam [1] during his talk at University of Wisconsin in 1940. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ and for some $\delta > 0$. Then, there exists a unique additive mapping $l : E \to E'$ such that

$$\|f(x) - l(x)\| \leq \delta$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5897

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for all $x \in E$. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation $g(x+y) = g(x) + g(y)$. In 1978, Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. Moreover if $f(\mu x)$ is continuous in $\mu \in \mathbb{R}$ for each fixed $x \in E$, then l is $\mathbb{R}$-linear. Găvruta [4] obtained a generalized result of the Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [5–14]).

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics (see [15–17]). A general ternary algebra is defined as internal ternary multiplication in a vector space. Let $\mathcal{A}$ be a linear space over a complex number field equipped with a mapping $[\cdot,\cdot,\cdot] : \mathcal{A}^3 = \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with $(x, y, z) \mapsto [x, y, z]$, which is $\mathbb{C}$-inear in each outer variable and conjugate $\mathbb{C}$-linear in the middle variable, and satisfies the following associative identity condition

$$[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$$

for all $x, y, z, u, v \in \mathcal{A}$. Then the pair $(\mathcal{A}, [\cdot,\cdot,\cdot])$ is called a complex ternary algebra. Assume that $\mathcal{A}$ is a complex ternary algebra. Then we say that $\mathcal{A}$ has a unit if there exist an element $e \in \mathcal{A}$ such that $[e, e, a] = [e, a, e] = [a, e, e] = a$ for all $a \in \mathcal{A}$. Park [18] and Moslehian [19] contributed works on the stability problem of ternary homomorphisms and ternary derivations and Bavand Savadkouhi [20] investigated the stability problem of ternary Jordan homomorphisms and ternary Jordan derivations. The stability probelms of several functional equations have been extensively investigated by a number of authors and there are many interesting results, containing ternary homomorphisms and ternary derivations, concerning this problem (see [21–26]).

Let $\mathcal{A}$ and $\mathcal{A}'$ be complex ternary algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{A}'$ is called a ternary algebra homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in \mathcal{A}$. If, in addition, the $\mathbb{C}$-linear mapping H is bijective, then the $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{A}'$ is called a ternary algebra isomorphism. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a ternary algebra derivation if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in \mathcal{A}$ (see [27–30]).

Let X be a complex ternary algebra. A mapping $f : X^3 \to X$ is *3-additive* if

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k)$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$. A mapping $f : X^3 \to X$ is called *3-linear* if f is 3-additive and $\mathbb{C}$-linear for each variable.

Throughout the paper, assume that X is a complex ternary algebra, Y is a complex ternary Banach algebra and t is a fixed nonzero real number with $|t| < 1$.

2. Stability of hyper 3-homomorphisms in ternary algebras

In this section, we prove the Hyers-Ulam stability of hyper 3-homomorphisms in complex ternary algebras and we investigate ternary algebra isomorphisms between complex ternary algebras, associated with the 3-additive functional equation (1).

Definition 1. *Let X and Y be complex ternary algebras. A 3-linear mapping $h : X^3 \to Y$ is called a hyper 3-additivec mapping if h satisfies*

$$8h(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} h(x_i + (-1)^i x_2, y_j + (-1)^j y_2, z_1 + (-1)^k z_2) \tag{2}$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$.

Definition 2. *Let X and Y be complex ternary algebras. A 3-linear mapping $h : X^3 \to Y$ is called a hyper 3-homomorphism if h satisfies*

$$h([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) = [h(x_1, x_2, x_3), h(y_1, y_2, y_3), h(z_1, z_2, z_3)]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$.

Lemma 1. *Let X and Y be complex ternary algebras. Let $h : X^3 \to Y$ be a hyper 3-additive mapping and satisfy $h(2x, 2y, 2z) = 8h(x, y, z)$ for all $x, y, z \in X^3$, then h is 3-additive.*

Proof. For $x_1, x_2, y_1, y_2, z_1, z_2 \in X$, we define

$$p_1 := \frac{x_1 + x_2}{2}, p_2 := \frac{x_1 - x_2}{2}, q_1 := \frac{y_1 + y_2}{2},$$
$$q_2 := \frac{y_1 - y_2}{2}, r_1 := \frac{z_1 + z_2}{2} \quad and \quad r_2 := \frac{z_1 - z_2}{2}.$$

It follows from (2) that

$$\begin{aligned}
&h(x_1 + x_2, y_1 + y_2, z_1 + z_2) = h(2p_1, 2q_1, 2r_1) \\
&= 8h(p_1, q_1, r_1) \\
&= \sum_{i,j,k=1}^{2} h(p_1 + (-1)^i p_2, q_1 + (-1)^j q_2, r_1 + (-1)^k r_2) \\
&= \sum_{i,j,k=1}^{2} h(x_i, y_j, z_k).
\end{aligned}$$

This completes the proof.

Lemma 2. [31] *Let X and Y be complex vector spaces and $f : X^3 \to Y$ be a* 3*-additive mapping such that*

$$f(\lambda x, \mu y, \nu z) = \lambda\mu\nu f(x, y, z)$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1 := \{\kappa \in \mathbb{R} \mid |\kappa| = 1\}$ and $x, y, z \in X$. Then f is 3*-linear.*

Theorem 1. *Let X and Y be complex ternary algebras and t be a real number satisfying $|t| < 1$. Assume that a mapping $h : X^3 \to Y$ satisfies*

$$h(0, a, b) = h(a, 0, b) = h(a, b, 0) = 0$$

and

$$\left\| 8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right\| \tag{3}$$
$$\leq \left\| t \left(8 \sum_{i,j,k=1}^{2} h\left(\frac{x_1 + (-1)^i x_2}{2}, \frac{y_1 + (-1)^j y_2}{2}, \frac{z_1 + (-1)^k z_2}{2} \right) - 8h(x_1, y_1, z_1) \right) \right\|$$

for all $a, b \in X$ and all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. Then h is hyper 3-additive.

Proof. Letting $x_1 = x_2 := x, y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (3), we get

$$\|h(2x, 2y, 2z) - 8h(x, y, z)\| \leq 0$$

for all $x, y, z \in X$. So $h(2x, 2y, 2z) = 8h(x, y, z)$ for all $x, y, z \in X$. It follows from (3) that

$$\left\| 8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right\|$$
$$\leq \left\| t \left(8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right) \right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. Thus

$$8h(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, since $|t| < 1$. Thus, the mapping h is hyper 3-additive.

Theorem 2. *Let X be a complex ternary algebra, Y be a complex ternary Banach algebra and t be a real number satisfying $|t| < 1$. Let $\varphi : X^6 \to [0, \infty)$ and $\psi : X^9 \to [0, \infty)$ be functions such that*

$$\sum_{j=1}^{+\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty$$

and

$$\sum_{j=1}^{+\infty} 8^{3j} \psi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}, \frac{z}{2^j} \right) < \infty$$

for all $x, y, z \in X$. Assume that a mapping $h : X^3 \to Y$ satisfies

$$h(0, a, b) = h(a, 0, b) = h(a, b, 0) = 0$$

and

$$\left\| 8\mu h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2)) \right\| \tag{4}$$
$$\leq \left\| t\left(8\mu h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h\left(\mu\frac{x_1 + (-1)^i x_2}{2}, \mu\frac{y_1 + (-1)^j y_2}{2}, \mu\frac{z_1 + (-1)^k z_2}{2}\right)\right)\right\|$$
$$+\varphi(x_1, y_1, z_1, x_2, y_2, z_2)$$

for all $a, b \in X$ and all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and all $\mu \in \mathbb{T}^1$. Let $h : X^3 \to X$ satisfy

$$\|h([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [h(x_1, x_2, x_3), h(y_1, y_2, y_3), h(z_1, z_2, x_3)]\| \tag{5}$$
$$\leq \psi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Then there exists a unique hyper 3-homomorphism $H : X^3 \to Y$ such that

$$\|h(x, y, z) - H(x, y, z)\| \leq \sum_{j=0}^{+\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \tag{6}$$

for all $x, y, z \in X$.

Proof. Letting $\mu = 1, x_1 = x_2 := x, y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (4), we get

$$\|h(2x, 2y, 2z) - 8h(x, y, z)\| \leq \varphi(x, y, z, x, y, z)$$

and so

$$\left\| h(x, y, z) - 8h\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Hence

$$\left\| 8^l h\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right) - 8^{l+k} h\left(\frac{x}{2^{l+k}}, \frac{y}{2^{l+k}}, \frac{z}{2^{l+k}}\right) \right\| \tag{7}$$
$$\leq \sum_{j=0}^{k-1} \left\| 8^{l+j} h\left(\frac{x}{2^{l+j}}, \frac{y}{2^{l+j}}, \frac{z}{2^{l+j}}\right) - 8^{l+(j+1)} h\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right) \right\|$$
$$= \sum_{j=0}^{k-1} 8^{l+j} \left\| h\left(\frac{x}{2^{l+j}}, \frac{y}{2^{l+j}}, \frac{z}{2^{l+j}}\right) - 8h\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right) \right\|$$

$$\leq \sum_{j=0}^{k-1} 8^{l+j}\varphi\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}, \frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right)$$

for all nonnegative integers l, k and all $x, y, z \in X$. It follows that $\left\{8^j h\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right)\right\}$ is a Cauchy sequence for each $(x, y, z) \in X^3$. Since Y is complete, $\left\{8^j h\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right)\right\}$ converges. Thus one can define the mapping $H : X^3 \to Y$ by

$$H(x, y, z) := \lim_{n\to+\infty} 8^n h\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)$$

for all $(x, y, z) \in X^3$. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (7), we get (6). It follows from (4) that

$$\begin{aligned}
&\left\|8\mu H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2))\right\| \\
&= \lim_{n\to+\infty} 8^n \left\| 8\mu h\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}\right)\right. \\
&\qquad \left.- \sum_{i,j,k=1}^{2} h\left(\mu\frac{x_1 + (-1)^i x_2}{2^n}, \mu\frac{y_1 + (-1)^j y_2}{2^n}, \mu\frac{z_1 + (-1)^k z_2}{2^n}\right)\right\| \\
&\leq \lim_{n\to+\infty} 8^n \left\| t\left(8\mu h\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}\right)\right.\right. \\
&\qquad \left.\left.- \sum_{i,j,k=1}^{2} h\left(\mu\frac{x_1 + (-1)^i x_2}{2^n}, \mu\frac{y_1 + (-1)^j y_2}{2^n}, \mu\frac{z_1 + (-1)^k z_2}{2^n}\right)\right)\right\| \\
&+ \lim_{n\to+\infty} 8^n \varphi\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}, \frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}\right) \\
&= \| t\,(8\mu H(x_1, y_1, z_1) \\
&\qquad \left.- \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2))\right)\Bigg\|
\end{aligned}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Thus

$$\begin{aligned}
&\left\|8\mu H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2))\right\| \\
&\leq \| t\,(8\mu H(x_1, y_1, z_1) \\
&\qquad \left.- \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2))\right)\Bigg\|
\end{aligned} \tag{8}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Let $\mu = 1$ in (8). By Theorem 1, the mapping $H : X^3 \to X$ is 3-additive. It follows from (8) and the 3-additivity of H that

$$\begin{aligned}
&\left\| 8\mu H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2)) \right\| \\
&\leq \Big\| t \Big(8\mu H(x_1, y_1, z_1) \\
&\quad - \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2)) \Big) \Big\|
\end{aligned}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Since $|t| < 1$,

$$8\mu H(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} H(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2)),$$

and $H(\mu(x_1, y_1, z_1)) = \mu H(x_1, y_1, z_1)$ for all $(x_1, y_1, z_1) \in X^3$ and $\mu \in \mathbb{T}^1$. By Lemma 2, the mapping $H : X^3 \to X$ is 3-linear. It follows from (5) and the 3-additivity of H that

$$\begin{aligned}
&\| H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)] \| \\
&= \lim_{n \to +\infty} 8^{3n} \Big\| h\left(\frac{[x_1, y_1, z_1]}{8^n}, \frac{[x_2, y_2, z_2]}{8^n}, \frac{[x_3, y_3, z_3]}{8^n} \right) \\
&- \left[h\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n} \right), h\left(\frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n} \right), h\left(\frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n} \right) \right] \Big\| \\
&\leq \lim_{n \to +\infty} 8^{3n} \psi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}, \frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n} \right) = 0.
\end{aligned}$$

So

$$H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) = [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Therefore, the mapping H is a unique hyper 3-homomor-phism satisfying (6).

Theorem 3. *Let X be a complex ternary algebra, Y be a complex ternary Banach algebra and t be a real number satisfying $|t| < 1$. Let $h : X^3 \to Y$ be a bijective mapping satisfying (4) such that*

$$h([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) = [h(x_1, x_2, x_3), h(y_1, y_2, y_3), h(z_1, z_2, z_3)] \tag{9}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. If $h(\alpha x_0, \beta y_0, \gamma z_0)$ is continuous in $\alpha, \beta, \gamma \in \mathbb{R}$ for each fixed $(x_0, y_0, z_0) \in X^3$ and $\lim_{n \to +\infty} 8^n h\left(\frac{e}{2^n}, \frac{e}{2^n}, \frac{e}{2^n}\right) = e'$, then the mapping $h : X^3 \to Y$ is a hyper 3-isomorphism.

Proof. Since h satisfies (9), the mapping $h : X^3 \to Y$ satisfies (4) by Theorem 1, there exists a hyper 3-homomorphism $H : X^3 \to Y$ satisfying (6). The mapping $H : X^3 \to Y$ is defined by

$$H(x, y, z) := \lim_{n\to+\infty} 8^n h\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)$$

for all $x, y, z \in X$. It follows from (9) that

$$\begin{aligned}
&\|[H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)] - [H(x_1, x_2, x_3), H(y_1, y_2, y_3), h(z_1, z_2, z_3)]\| \\
&= \|H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [H(x_1, x_2, x_3), H(y_1, y_2, y_3), h(z_1, z_2, z_3)]\| \\
&= \lim_{n\to+\infty} 8^{2n} \left\| h\left(\left[\frac{x_1}{2^n}, \frac{y_1}{2^n}, z_1\right], \left[\frac{x_2}{2^n}, \frac{y_2}{2^n}, z_2\right], \left[\frac{x_3}{2^n}, \frac{y_3}{2^n}, z_3\right]\right)\right. \\
&\left. - \left[h\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}\right), h\left(\frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}\right), h(z_1, z_2, z_3)\right] \right\| \\
&\le \lim_{n\to+\infty} 8^{2n} \psi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}, z_1, z_2, z_3\right) = 0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. So

$$[H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)] = [H(x_1, x_2, x_3), H(y_1, y_2, y_3), h(z_1, z_2, z_3)]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Letting $x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = e$ in the last equality, we get $h(z_1, z_2, z_3) = H(z_1, z_2, z_3)$ for all $z_1, z_2, z_3 \in X$. Therefore, the bijective mapping $h : X^3 \to Y$ is a hyper 3-isomorphism.

3. Stability of hyper 3-derivations in ternary algebras

In this section, we prove the Hyers-Ulam stability of hyper 3-derivations in complex ternary algebras.

Definition 3. *Let X be a ternary algebra. A 3-linear mapping $f : X^3 \to X$ is called a hyper 3-derivation if f satisfies*

$$\begin{aligned}
f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) &= [f(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]] \\
&+ [[x_1, x_2, x_3], f(y_1, y_2, y_3), [z_1, z_2, z_3]] \\
&+ [[x_1, x_2, x_3], [y_1, y_2, y_3], f(z_1, z_2, z_3)]
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$.

Theorem 4. *Let X be a ternary algebra and t be a real number satisfying $|t| < 1$. If a mapping $f : X^3 \to X$ satisfies*

$$\left\| f(x_1 + x_2, y_1 + y_2, z_1 + z_2) - \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right\| \tag{10}$$

$$\leq \left\| t \left(8f\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) - \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right) \right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. *Then* f *is 3-additive.*

Proof. Letting $x_1 = x_2 := x, y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (10), we get

$$\|f(2x, 2y, 2z) - 8f(x, y, z)\| \leq 0$$

for all $x, y, z \in X$. So $f(2x, 2y, 2z) = 8f(x, y, z)$ for all $x, y, z \in X$. It follows from (10) that

$$\left\| f(x_1+x_2, y_1+y_2, z_1+z_2) - \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right\|$$
$$\leq \left\| t \left(f(x_1+x_2, y_1+y_2, z_1+z_2) - \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right) \right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. Thus

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, since $|t| < 1$. Thus the mapping f is 3-additive.

Theorem 5. *Let* X *be a ternary Banach algebra and* t *be a real number satisfying* $|t| < 1$. *Let* $\varphi : X^6 \to [0, \infty)$ *and* $\psi : X^9 \to [0, \infty)$ *be functions such that*

$$\sum_{j=1}^{+\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

and

$$\sum_{j=1}^{+\infty} 8^{3j} \psi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in X$. *Let* $f : X^3 \to X$ *be a mapping satisfying*

$$\left\| f(\mu(x_1+x_2, y_1+y_2, z_1+z_2)) - \mu \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right\| \tag{11}$$
$$\leq \left\| t \left(8f\left(\mu\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)\right) - \mu \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k) \right) \right\|$$

$$+\varphi(x_1, y_1, z_1, x_2, y_2, z_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ *and* $\mu \in \mathbb{T}^1$, *and*

$$\begin{aligned}&\|f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [f(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]] \\ &-[[x_1, x_2, x_3], f(y_1, y_2, y_3), [z_1, z_2, z_3]] - [[x_1, x_2, x_3], [y_1, y_2, y_3], f(z_1, z_2, z_3)]\| \\ &\leq \psi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)\end{aligned} \tag{12}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. *Then there exists a unique hyper 3-derivation* $D : X^3 \to X$ *such that*

$$\|f(x, y, z) - D(x, y, z)\| \leq \sum_{j=0}^{+\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \tag{13}$$

for all $x, y, z \in X$.

Proof. Letting $\mu = 1, x_1 = x_2 := x, y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (11), we get

$$\|f(2x, 2y, 2z) - 8f(x, y, z)\| \leq \varphi(x, y, z, x, y, z)$$

for all $x, y, z \in X$. By induction, we have

$$\left\|f(x, y, z) - 8^n f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)\right\| \leq \sum_{j=0}^{n-1} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right)$$

for all $x, y, z \in X$. Hence

$$\begin{aligned}&\left\|8^l f\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right) - 8^k f\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right)\right\| \\ &\leq \sum_{j=l}^{k-1} \left\|8^j f\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{k-1} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\end{aligned} \tag{14}$$

for all nonnegative integers $l, k (k > l)$ and all $x, y, z \in X$. It follows that the sequence $\left\{8^k f\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right)\right\}$ is a Cauchy sequence for each $(x, y, z) \in X^3$. Since X is complete, the sequence $\left\{8^k f\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right)\right\}$ converges. Thus one can define the mapping $D : X^3 \to X$ by

$$D(x, y, z) := \lim_{n \to +\infty} 8^n f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)$$

for all $(x, y, z) \in X^3$. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (14), we get (13). It follows from (11) that

$$\|D(\mu(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \mu \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\|$$

$$= \lim_{n\to+\infty} 8^n \left\| f\left(\mu\left(\frac{x_1+x_2}{2^n}, \frac{y_1+y_2}{2^n}, \frac{z_1+z_2}{2^n}\right) - \mu\sum_{i,j,k=1}^{2} f\left(\frac{x_i}{2^n}, \frac{y_j}{2^n}, \frac{z_k}{2^n}\right)\right)\right\|$$
$$\leq \lim_{n\to+\infty} 8^n \left\| t\left(8f\left(\mu\left(\frac{x_1+x_2}{2^{n+1}}, \frac{y_1+y_2}{2^{n+1}}, \frac{z_1+z_2}{2^{n+1}}\right)\right) - \mu\sum_{i,j,k=1}^{2} f\left(\frac{x_i}{2^n}, \frac{y_j}{2^n}, \frac{z_k}{2^n}\right)\right)\right\|$$
$$+ \lim_{n\to+\infty} 8^n\varphi\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}, \frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}\right)$$
$$= \left\| t\left(8D\left(\mu\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)\right) - \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\right)\right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Thus

$$\|D(\mu(x_1+x_2, y_1+y_2, z_1+z_2)) - \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\| \tag{15}$$
$$\leq \left\| t\left(8D\left(\mu\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)\right) - \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\right)\right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Let $\mu = 1$ in (15). By Theorem 4, the mapping $D : X^3 \to X$ is 3-additive. It follows from (15) and the 3-additivity of D that

$$\|D(\mu(x_1+x_2, y_1+y_2, z_1+z_2)) - \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\|$$
$$\leq \left\| t\left(D(\mu(x_1+x_2, y_1+y_2, z_1+z_2)) - \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\right)\right\|$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\mu \in \mathbb{T}^1$. Since $|t| < 1$,

$$D(\mu(x_1+x_2, y_1+y_2, z_1+z_2)) = \mu\sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)$$

and $D(\mu(x_1, y_1, z_1)) = \mu D(x_1, y_1, z_1)$ for all $(x_1, y_1, z_1) \in X^3$ and $\mu \in \mathbb{T}^1$. By Lemma 2, the mapping $D : X^3 \to X$ is 3-linear. It follows from (12) and the 3-additivity of D that

$$\|D([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [D(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]]$$
$$-[[x_1, x_2, x_3], D(y_1, y_2, y_3), [z_1, z_2, z_3]] - [[x_1, x_2, x_3], [y_1, y_2, y_3], D(z_1, z_2, z_3)]\|$$
$$= \lim_{n\to+\infty} 8^{3n} \left\| f\left(\frac{[x_1, y_1, z_1]}{8^n}, \frac{[x_2, y_2, z_2]}{8^n}, \frac{[x_3, y_3, z_3]}{8^n}\right)\right.$$
$$- \left[f\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}\right), \frac{[y_1, y_2, y_3]}{8^n}, \frac{[z_1, z_2, z_3]}{8^n}\right]$$

$$
\begin{aligned}
&- \left[\frac{[x_1, x_2, x_3]}{8^n}, f\left(\frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}\right), \frac{[z_1, z_2, z_3]}{8^n} \right] \\
&- \left[\frac{[x_1, x_2, x_3]}{8^n}, \frac{[y_1, y_2, y_3]}{8^n}, f\left(\frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}\right) \right] \Big\| \\
\leq & \lim_{n\to+\infty} 8^{3n} \psi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}, \frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}\right) = 0
\end{aligned}
$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. So

$$
\begin{aligned}
&D([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) = [D(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]] \\
&-[[x_1, x_2, x_3], D(y_1, y_2, y_3), [z_1, z_2, z_3]] - [[x_1, x_2, x_3], [y_1, y_2, y_3], D(z_1, z_2, z_3)]
\end{aligned}
$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Therefore, the mapping H is a unique hyper 3-derivation satisfying (13).

4. Conclusion and future work

In this paper, we introduced hyper 3-homomorphisms and hyper 3-derivations in ternary algebras and we proved the Hyers-Ulam stability of hyper 3-homomorphisms and hyper 3-derivations in ternary Banach algebras, associated with the 3-additive functional equation (1). We will provide suitable examples and useful applications in next work.

Acknowledgements

The authors are thankful to the editors and the anonymous reviewers for many valuable suggestions to improve this paper.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

Fundings
S. Donganont was supported by the University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5020/2567).

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