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Solving Partial Differential Equations Via The Double Sumudu-Shehu Transform

Monther Al-Momani[1], Ali Jaradat[2], Baha' Abughazaleh[3,*], Abdulkarim Farah[3]

[1] *Department of Basic Sciences, Al-Ahliyya Amman University, Amman, Jordan*
[2] *Department of Mathematics, Amman Arab University, Amman, Jordan*
[3] *Department of Mathematics, Isra University, Amman, Jordan*

Abstract. This paper introduces a new double hybrid transform yielding single integral transforms and their generalizations. The main purpose of this study is to propose the most common form for generalized transformations in terms of Hybrid Sumudu and Shehu transforms. In this paper, we introduce a just invented transform and research its basic characteristics such as existence, inversion, along with related theorems. The study also introduces novel results with respect to partials and generalizes the double convolution theorem. Furthermore, it uses the developed properties and theorems to solve specific kinds of differential equations that have very important applications in physics and science. The purpose of this research is to show the applicability and efficiency of a novel transform in solving differential equations with multiple variable to solve.

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Key Words and Phrases: Sumudu transform, Shehu transform, The Double Sumudu-Shehu transform.

1. Introduction

Integral transforms are a class of mathematical operators that map functions from one space to another through the process of integration. These transforms simplify the manipulation of certain properties of the original functions by moving them to a new functional space. After transformation the function can be changed back to its original space using inverse of integral transformation.

They are an integral part of physics, chemistry, engineering and economy since they help in modeling real world phenomena. Thus, mathematicians keep coming up with new techniques to solve a more and more wider group of differential equations.

*Corresponding author.
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Email addresses: montheralmomani72@gmail.com (M. Al-Momani), a.jaradat@aau.edu.jo (A. Jaradat), baha.abughazaleh@iu.edu.jo (B. Abughazaleh), karim.farah@iu.edu.jo (A. Farah)

Integral transformations are known for their effectiveness and simplicity, particularly when applied to differential equations with initial or boundary conditions. They simplify the process by converting differential equations, reducing the complexity from derivative operations to algebraic ones. By carefully selecting the appropriate integral transformation, it becomes easier to manage not only the derivatives in intricate differential equations but also the boundary conditions, leading to a form of the equation that is more straightforward to solve.

One of the most well-known transforms is the Laplace transform, which was introduced in 1780. It is used in various fields such as science and engineering. In 1993, the Sumudu transform was defined by [1]. More recently, the Shehu transform was introduced by [2] in 2019, which represents a generalization of the Laplace and Sumudu transforms. For additional details on the Shehu transform.

Additionally, Double transforms have been defined for solving differential equations involving more than one variable. Examples of Double transforms include the Double Laplace transform [3], the Double Sumudu transform [4], Double Mellin-ARA Transform [5], for more details about integral transform see [6], [7], [8], [9], [10] and [11]. In this research, we define the Double Sumudu-Shehu transform(DSHT). We explore its properties, including the conditions for its existence, linearity. The study employs this hybrid transform across various fundamental functions, revealing its potential in the realms of convolution theory and derivative operations. We also apply the Double Sumudu-Shehu transform to solve partial differential equations.

2. Sumudu and Shehu transforms

This section provides a brief overview and fundamental properties of the single transforms: Sumudu, and Shehu transforms.

2.1. Sumudu transform

Definition 1. *For a continuous function $r(\tau)$ defined on $(0,\infty)$, the Sumudu is defined as follows:*

$$R(\kappa) = S(r(\tau)) = \frac{1}{\kappa}\int_0^\infty e^{-\frac{\tau}{\kappa}} r(\tau) d\tau, \kappa \in \mathbb{C}.$$

Here, we present some fundamental properties of the Sumudu transform.

Let $R(\kappa) = S(r(\tau))$, then for nonzero constants β and γ, we have

$$S(\beta r_1(\tau) + \gamma r_2(\tau)) = \beta S(r_1(\tau)) + \gamma S(r_2(\tau)), \tag{1}$$

where $r_1(\tau)$ and $r_2(\tau)$ are continuous functions on $(0,\infty)$.

$$S(\tau^{\beta}) = \Gamma(\beta+1)\kappa^{\beta} \tag{2}$$

$$S(e^{\beta\tau}) = \frac{1}{1-\kappa\beta},\ \beta \in \mathbb{R} \tag{3}$$

$$S(r'(\tau)) = \frac{R(\kappa)}{\kappa} - \frac{r(0)}{\kappa} \tag{4}$$

$$S(r''(\tau)) = \frac{R(\kappa)}{\kappa^2} - \frac{r(0)}{\kappa^2} - \frac{r'(0)}{\kappa}. \tag{5}$$

2.2. The Shehu transform

Definition 2. *For a continuous function $t(\upsilon)$ defined on $(0,\infty)$, the Shehu is defined as follows:*

$$T(\lambda,\mu) = H(t(\upsilon)) = \int_0^{\infty} e^{-\frac{\lambda\upsilon}{\mu}} t(\upsilon)d\upsilon.$$

We now outline the fundamental properties of the Shehu transform.

Suppose that $T_1(\lambda,\mu) = H(t_1(\upsilon))$ and $T_2(\lambda,\mu) = H(t_2(\upsilon))$, and β and γ are nonzero real numbers, then the following properties hold:

$$H(\beta t_1(\upsilon) + \gamma t_2(\upsilon)) = \beta H(t_1(\upsilon)) + \gamma H(t_2(\upsilon)) \tag{6}$$

$$H(\upsilon^{\beta}) = \Gamma(\beta+1)\left(\frac{\mu}{\lambda}\right)^{\beta+1} \tag{7}$$

$$H(e^{\gamma\upsilon}) = \frac{\mu}{\lambda-\gamma\mu} \tag{8}$$

$$H(t'(\upsilon)) = \frac{\lambda}{\mu}T(\lambda,\mu) - t(0) \tag{9}$$

$$H(t''(\upsilon)) = \frac{\lambda^2}{\mu^2}T(\lambda,\mu) - \frac{\lambda}{\mu}t(0) - t'(0). \tag{10}$$

3. The Double Sumudu-Shehu transform

This section introduces DSHT, a novel mathematical tool combining the Sumudu and Shehu transforms. It outlines its core properties linearity, invertibility, and behavior with partial derivatives and establishes a dedicated convolution theorem. Practical examples demonstrate its application to fundamental functions.

The DSHT transform is defined as follows:

$$Q(\kappa,\lambda,\mu)=S_\tau H_\upsilon(q(\tau,\upsilon))=\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}q(\tau,\upsilon)\;d\tau d\upsilon, \tag{11}$$

where $q(\tau,\upsilon)$ is a continuous function on $(0,\infty)\times(0,\infty)$.

If $q(\tau,\upsilon)$ can be written as $q(\tau,\upsilon)=w(\tau)z(\upsilon)$ for some continuous functions w and z, then $S_\tau H_\upsilon(q(\tau,\upsilon))=S(w(\tau))H(z(\upsilon))$. In fact

$$\begin{aligned}
S_\tau H_\upsilon(q(\tau,\upsilon)) &= S_\tau H_\upsilon(w(\tau)z(\upsilon))\\
&= \frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}w(\tau)z(\upsilon)d\tau d\upsilon\\
&= \left(\frac{1}{\kappa}\int_0^\infty e^{-\frac{\tau}{\kappa}}w(\tau)d\tau\right)\left(\int_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}z(\upsilon)d\upsilon\right)\\
&= S(w(\tau))H(z(\upsilon)).
\end{aligned}$$

3.1. The DSHT for some basic functions

(i)

$$S_\tau H_\upsilon(1)=\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}d\tau d\upsilon$$

$$=\left(\frac{1}{\kappa}\int_0^\infty e^{-\frac{\tau}{\kappa}}\right)\left(\int_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}d\upsilon\right)=1\times\frac{\mu}{\lambda}=\frac{\mu}{\lambda},\ \mathrm{Re}(\kappa)>0.$$

(ii)

$$S_\tau H_\upsilon(\tau^\beta\upsilon^\gamma)=\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}\tau^\beta\upsilon^\gamma d\tau d\upsilon$$

$$=\left(\frac{1}{\kappa}\int_0^\infty \tau^\beta e^{-\frac{\tau}{\kappa}}d\tau\right)\left(\int_0^\infty \upsilon^\gamma e^{-\frac{\lambda\upsilon}{\mu}}d\upsilon\right)$$

$$=\Gamma(\beta+1)\kappa^\beta\times\Gamma(\gamma+1)\left(\frac{\mu}{\lambda}\right)^{\gamma+1}$$

$$=\frac{\kappa^\beta\mu^{\gamma+1}}{\lambda^{\gamma+1}}\Gamma(\beta+1)\Gamma(\gamma+1),\ \mathrm{Re}(\kappa)>0\text{ and }\mathrm{Re}(\beta)>-1.$$

(iii)

$$S_\tau H_\upsilon(e^{\beta\tau+\gamma\upsilon}) = \frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} e^{\beta\tau+\gamma\upsilon} d\tau d\upsilon$$

$$= \left(\frac{1}{\kappa}\int_0^\infty e^{\beta\tau-\frac{\tau}{\kappa}} d\tau\right)\left(\int_0^\infty e^{\gamma\upsilon-\frac{\lambda\upsilon}{\mu}} d\upsilon\right) = \frac{1}{1-\kappa\beta} \times \frac{\mu}{\lambda-\gamma\mu}$$

$$= \frac{\mu}{(1-\kappa\beta)(\lambda-\gamma\mu)}, \ \mathrm{Re}(\frac{1}{\kappa}) > \mathrm{Re}(\beta).$$

3.2. Existence condition for the DSHT

Definition 3. *A function $q(\tau,\upsilon)$ is said to be of exponential orders β and γ on $0 \le \tau < \infty$ and $0 \le \upsilon < \infty$. If there exist $B, X, Y > 0$ such that $|q(\tau,\upsilon)| \le Be^{\beta\tau+\gamma\upsilon}$, for all $\tau > X$, $\upsilon > Y$.*

Theorem 1. *Let $q(\tau,\upsilon)$ be a continuous function on the region $[0,\infty)\times[0,\infty)$ of exponential orders β and γ. Then $Q(\kappa,\lambda,\mu)$ exists for κ, λ and μ whenever $\mathrm{Re}(\frac{1}{\kappa}) > \beta$ and $\mathrm{Re}\left(\frac{\lambda}{\mu}\right) > \gamma$.*

Proof.

$$\begin{aligned}
|Q(\kappa,\lambda,\mu)| &= \left|\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)\ d\tau d\upsilon\right| \le \frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} |q(\tau,\upsilon)|\, d\tau d\upsilon \\
&\le \frac{B}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} e^{\beta\tau+\gamma\upsilon} d\tau d\upsilon = \frac{B}{\kappa}\int_0^\infty e^{-(\frac{1}{\kappa}-\beta)\tau} d\tau \int_0^\infty e^{-(\frac{\lambda}{\mu}-\gamma)\upsilon} d\upsilon \\
&= \frac{B}{\kappa(\frac{1}{\kappa}-\beta)(\frac{\lambda}{\mu}-\gamma)} = \frac{B\mu}{(1-\kappa\beta)(\lambda-\frac{\gamma}{\mu})}
\end{aligned}$$

where $\mathrm{Re}(\frac{1}{\kappa}) > \beta$ and $\mathrm{Re}\left(\frac{\lambda}{\mu}\right) > \gamma$.

3.3. Linearity

The transform $S_\tau H_\upsilon(q(\tau,\upsilon))$ exhibits linearity. For any nonzero constants β and γ, this property is expressed as:

$S_\tau H_\upsilon(\beta q_1(\tau,\upsilon)+\gamma q_2(\tau,\upsilon))$

$$\begin{aligned}
&= \frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}\left(\beta q_1(\tau,\upsilon)+\gamma q_2(\tau,\upsilon)\right)\, d\tau d\upsilon,\\
&= \beta\times\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}q_1(\tau,\upsilon)\, d\tau d\upsilon+\gamma\times\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}q_2(\tau,\upsilon)\, d\tau d\upsilon\\
&= \beta S_\tau H_\upsilon(q_1(\tau,\upsilon))+\gamma S_\tau H_\upsilon(q_2(\tau,\upsilon)).
\end{aligned}$$

4. Properties of the DSHT

In this section, we explore the fundamental properties of the DSHT

4.1. Derivatives properties

Let $Q(\kappa,\lambda,\mu)=S_\tau H_\upsilon(q(\tau,\upsilon))$. Then

(i)

$$S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\tau}\right)=\frac{Q(\kappa,\lambda,\mu)}{\kappa}-\frac{H(q(0,\upsilon))}{\kappa} \tag{12}$$

(ii)

$$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau^2}\right)=\frac{Q(\kappa,\lambda,\mu)}{\kappa^2}-\frac{H(q(0,\upsilon))}{\kappa^2}-\frac{H(q_\tau(0,\upsilon))}{\kappa} \tag{13}$$

(iii)

$$S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\upsilon}\right)=\frac{\lambda}{\mu}Q(\kappa,\lambda,\mu)-S(q(\tau,0)) \tag{14}$$

(iv)

$$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2}\right)=\frac{\lambda^2}{\mu^2}Q(\kappa,\lambda,\mu)-\frac{\lambda}{\mu}S(q(\tau,0))-S(q_\upsilon(\tau,0)) \tag{15}$$

(v)

$$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon}\right)=\frac{\lambda}{\kappa\mu}Q(\kappa,\lambda,\mu)-\frac{1}{\kappa}S(q(\tau,0))-\frac{\lambda}{\kappa\mu}H(q(0,\upsilon))+\frac{1}{\kappa}q(0,0) \tag{16}$$

Proof. (1) $S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\tau}\right)=\frac{1}{\kappa}\int_0^\infty\int_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}\frac{\partial q(\tau,\upsilon)}{\partial\tau}\, d\tau d\upsilon=\frac{1}{\kappa}\int_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\int_0^\infty e^{-\frac{\tau}{\kappa}}\frac{\partial q(\tau,\upsilon)}{\partial\tau}d\tau d\upsilon.$
By integrating by parts, we get
$S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\tau}\right)=\frac{1}{\kappa}\int_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\left(-q(0,\upsilon)+\frac{1}{\kappa}\int_0^\infty e^{-\frac{\tau}{\kappa}}q(\tau,\upsilon)\, d\tau\right)d\upsilon$

$= -\frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q(0,\upsilon)d\upsilon + \frac{1}{\kappa} \times \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)\ d\tau d\upsilon$

$= \frac{Q(\kappa,\lambda,\mu)}{\kappa} - \frac{H(q(0,\upsilon))}{\kappa}.$

(2) $S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau^2}\right) = \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\tau^2}\ d\tau d\upsilon = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\tau^2} d\tau d\upsilon.$

By integrating by parts, we get

$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau^2}\right) = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\left(-q_\tau(0,\upsilon) - \frac{1}{\kappa}q(0,\upsilon) + \frac{1}{\kappa^2}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} q(\tau,\upsilon)d\tau\right)d\upsilon$

$= -\frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q_\tau(0,\upsilon)d\upsilon - \frac{1}{\kappa^2}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q(0,\upsilon)d\upsilon + \frac{1}{\kappa^2} \times \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)d\tau d\upsilon$

$= \frac{Q(\kappa,\lambda,\mu)}{\kappa^2} - \frac{H(q(0,\upsilon))}{\kappa^2} - \frac{H(q_\tau(0,\upsilon))}{\kappa}.$

(3) $S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\upsilon}\right) = \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} \frac{\partial q(\tau,\upsilon)}{\partial\upsilon}\ d\tau d\upsilon = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} \frac{\partial q(\tau,\upsilon)}{\partial\upsilon} d\upsilon d\tau.$

By integrating by parts, we get

$S_\tau H_\upsilon\left(\frac{\partial q(\tau,\upsilon)}{\partial\upsilon}\right) = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}}\left(-q(\tau,0) + \frac{\lambda}{\mu}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)d\upsilon\right)d\tau$

$= -\frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} q(\tau,0)d\tau + \frac{\lambda}{\mu} \times \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)\ d\upsilon d\tau$

$= \frac{\lambda}{\mu}Q(\kappa,\lambda,\mu) - S(q(\tau,0)).$

(4) $S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2}\right) = \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2}\ d\tau d\upsilon = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2} d\upsilon d\tau.$

By integrating by parts, we get

$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2}\right) = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}}\left(-q_\upsilon(\tau,0) - \frac{\lambda}{\mu}q(\tau,0) + \frac{\lambda^2}{\mu^2}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)d\upsilon\right)d\tau$

$= -\frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} q_\upsilon(\tau,0)d\tau - \frac{\lambda}{\mu} \times \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} q(\tau,0)d\tau + \frac{\lambda^2}{\mu^2} \times \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau,\upsilon)d\upsilon d\tau$

So, $S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\upsilon^2}\right) = \frac{\lambda^2}{\mu^2}Q(\kappa,\lambda,\mu) - \frac{\lambda}{\mu}S(q(\tau,0)) - S(q_\upsilon(\tau,0)).$

(5) $S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon}\right) = \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon} d\tau d\upsilon = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} \frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon} d\tau d\upsilon$

By integrating by parts, we get

$$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon}\right) = \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}}\left(-q_\upsilon(0,\upsilon) + \frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\tau}{\kappa}} q_\upsilon(\tau,\upsilon)\, d\tau\right) d\upsilon$$

$$= -\frac{1}{\kappa}\int\limits_0^\infty e^{-\frac{\lambda\upsilon}{\mu}} q_\upsilon(0,\upsilon)d\upsilon + \frac{1}{\kappa}\times\frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q_\upsilon(\tau,\upsilon)d\tau d\upsilon$$

$$= -\frac{1}{\kappa}H(q_\upsilon(0,\upsilon)) + \frac{1}{\kappa}S_\tau H_\upsilon\left(q_\upsilon(\tau,\upsilon)\right)$$

Using Equations 9 and 14, we get

$$S_\tau H_\upsilon\left(\frac{\partial^2 q(\tau,\upsilon)}{\partial\tau\partial\upsilon}\right) = \frac{\lambda}{\kappa\mu}Q(\kappa,\lambda,\mu) - \frac{1}{\kappa}S(q(\tau,0)) - \frac{\lambda}{\kappa\mu}H(q(0,\upsilon)) + \frac{1}{\kappa}q(0,0).$$

4.2. Convolution Theorem of the DSHT

The Heaviside unit step function $M(\tau,\upsilon)$ is defined as

$$M(\tau-\beta,\upsilon-\gamma) = \begin{cases} 1, & \tau>\beta \text{ and } \upsilon>\gamma \\ 0, & \text{otherwise} \end{cases}$$

Then we have the following lemma

Lemma 1. $S_\tau H_\upsilon(q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)) = e^{-\frac{\beta}{\kappa}-\frac{\lambda\gamma}{\mu}} S_\tau H_\upsilon(q(\tau,\upsilon)$

Proof. We have

$$S_\tau H_\upsilon(q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma))$$
$$= \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)d\tau d\upsilon$$
$$= \frac{1}{\kappa}\int\limits_\beta^\infty\int\limits_\gamma^\infty e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}} q(\tau-\beta,\upsilon-\gamma)d\tau d\upsilon. \tag{17}$$

Now, by making the substitution $s=\tau-\beta$ and $r=\upsilon-\gamma$, equation (17) becomes:

$$\begin{aligned} S_\tau H_\upsilon(q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)) &= \frac{1}{\kappa}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{(s+\beta)}{\kappa}-\frac{\lambda(r+\gamma)}{\mu}} q(s,r)dsdr \\ &= e^{-\frac{\beta}{\kappa}-\frac{\lambda\gamma}{\mu}} S_\tau H_\upsilon(q(\tau,\upsilon)). \end{aligned}$$

Definition 4. *Let $q(\tau,\upsilon)$ and $p(\tau,\upsilon)$ be continuous functions. We define the convolution in the DSHT as*

$$(q**p)(\tau,\upsilon)=\int_0^{\tau}\int_0^{\upsilon}q(\tau-\beta,\upsilon-\gamma)p(\beta,\gamma))d\beta d\gamma.$$

The following theorem provides the computation of the DSHT for the convolution of two functions

Theorem 2. *Let $Q(\kappa,\lambda,\mu)=S_\tau H_\upsilon(q(\tau,\upsilon))$ and $P(\kappa,\lambda,\mu)=S_\tau H_\upsilon(p(\tau,\upsilon))$. Then*

$$S_\tau H_\upsilon((q**p)(\tau,\upsilon))=\kappa Q(\kappa,\lambda,\mu)P(\kappa,\lambda,\mu).$$

Proof.

$S_\tau H_\upsilon((q**p)(\tau,\upsilon))$

$$\begin{aligned}
&= \frac{1}{\kappa}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}(q**p)(\tau,\upsilon)d\tau d\upsilon\\
&= \frac{1}{\kappa}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}\left(\int_0^{\tau}\int_0^{\upsilon}q(\tau-\beta,\upsilon-\gamma)p(\beta,\gamma))d\beta d\gamma\right)d\tau d\upsilon. \qquad (18)
\end{aligned}$$

By incorporating the Heaviside unit step function, equation (18) can be rewritten as:

$S_\tau H_\upsilon((q**p)(\tau,\upsilon))$

$$\begin{aligned}
&=\frac{1}{\kappa}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}\left(\int_0^{\infty}\int_0^{\infty}q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)p(\beta,\gamma))d\beta d\gamma\right)d\tau d\upsilon\\
&=\int_0^{\infty}\int_0^{\infty}p(\beta,\gamma)\left(\frac{1}{\kappa}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\tau}{\kappa}-\frac{\lambda\upsilon}{\mu}}q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)d\tau d\upsilon\right)d\beta d\gamma
\end{aligned}$$

So by Lemma 1, we have

$$\begin{aligned}
S_\tau H_\upsilon((q**p)(\tau,\upsilon)) &= Q(\kappa,\lambda,\mu)\int_0^{\infty}\int_0^{\infty}p(\beta,\gamma)e^{-\frac{\beta}{\kappa}-\frac{\lambda\gamma}{\mu}}d\beta d\gamma\\
&= \kappa Q(\kappa,\lambda,\mu)P(\kappa,\lambda,\mu).
\end{aligned}$$

In Table 1, we have the DSHT of some basic functions

Table 1: Table of the DSHT

$q(\tau,\upsilon)$	$S_\tau H_\upsilon(q(\tau,\upsilon))$
$w(\tau)z(\upsilon)$	$S(w(\tau))H(z(\upsilon))$
1	$\frac{\mu}{\lambda},\ \mathrm{Re}(\kappa)>0$
$\tau^\beta\upsilon^\gamma$	$\frac{\kappa^\beta\mu^{\gamma+1}}{\lambda^{\gamma+1}}\Gamma(\beta+1)\Gamma(\gamma+1),\ \mathrm{Re}(\kappa)>0$ and $\mathrm{Re}(\beta)>-1$
$e^{\beta\tau+\gamma\upsilon}$	$\frac{\mu}{(1-\kappa\beta)(\lambda-\gamma\mu)},\ \mathrm{Re}(\frac{1}{\kappa})>\mathrm{Re}(\beta)$
$e^{i(\beta\tau+\gamma\upsilon)}$	$\frac{i\mu}{(i+\kappa\beta)(\lambda-i\gamma\mu)},\ \mathrm{Im}(\beta)+\mathrm{Re}(\frac{1}{\kappa})>0$
$\sin(\beta\tau+\gamma\upsilon)$	$\frac{\mu(\kappa\lambda\beta+\mu\gamma)}{(1+\kappa^2\beta^2)(\lambda^2+\gamma^2\mu^2)},\ \lvert\mathrm{Im}(\beta)\rvert<\mathrm{Re}(\frac{1}{\kappa})$
$\cos(\beta\tau+\gamma\upsilon)$	$\frac{\mu(\lambda-\kappa\mu\beta\gamma)}{(1+\kappa^2\beta^2)(\lambda^2+\gamma^2\mu^2)},\ \lvert\mathrm{Im}(\beta)\rvert<\mathrm{Re}(\frac{1}{\kappa})$
$\sinh(\beta\tau+\gamma\upsilon)$	$\frac{\mu(\kappa\lambda\beta+\mu\gamma)}{(\kappa^2\beta^2-1)(\lambda^2-\gamma^2\mu^2)},\ \mathrm{Re}(\frac{1}{\kappa})>\mathrm{Re}(\beta)$ and $\mathrm{Re}(\frac{1}{\kappa}+\beta)>0$
$\cos(\beta\tau+\gamma\upsilon)$	$\frac{\mu(\lambda+\kappa\mu\beta\gamma)}{(\kappa^2\beta^2-1)(\lambda^2-\gamma^2\mu^2)},\ \mathrm{Re}(\frac{1}{\kappa})>\mathrm{Re}(\beta)$ and $\mathrm{Re}(\frac{1}{\kappa}+\beta)>0$
$J_0(c\sqrt{\tau\upsilon})$	$\frac{4\mu}{4\lambda+c^2\kappa\mu},\ \mathrm{Re}\left(\frac{1}{\kappa}+\frac{c^2\mu}{4\lambda}\right)>0$
$q(\tau-\beta,\upsilon-\gamma)M(\tau-\beta,\upsilon-\gamma)$	$e^{-\frac{\beta}{\kappa}-\frac{\lambda\gamma}{\mu}}S_\tau H_\upsilon(q(\tau,\upsilon))$
$(q**p)(\tau,\upsilon)$	$\kappa S_\tau H_\upsilon(q(\tau,\upsilon))S_\tau H_\upsilon(p(\tau,\upsilon))$

5. Applications

In this section, we use the DHST for solving PDEs

Consider the PDE of the form

$$B_1q_{\tau\tau}+B_2q_{\tau\upsilon}+B_3q_{\upsilon\upsilon}+B_4q_\tau+B_5q_\upsilon+B_6q(\tau,\upsilon)=k(\tau,\upsilon) \tag{19}$$

With ICs

$q(\tau,0)=r_1(\tau),\ q_\upsilon(\tau,0)=r_2(\tau)$

and BCs

$q(0,\upsilon)=t_1(\upsilon),\ q_\tau(0,\upsilon)=t_2(\upsilon)$

and assuming $q(0,0)=\Psi$

Given that $q(\tau,\upsilon)$ is the unknown function, $k(\tau,\upsilon)$ is the source term, and $B_1,B_2,...,B_6$ and Ψ are constants, we aim to apply the DHST to Equation (19).

To do this, we begin by applying the single Sumudu transform to the ICs and the single Shehu transform to the BCs

$S\left(r_1\left(\tau\right)\right)=R_1(\tau)$, $S\left(r_2\left(\tau\right)\right)=R_2(\tau)$, $H\left(t_1\left(\upsilon\right)\right)=T_1(\upsilon)$ and $H\left(t_2\left(\upsilon\right)\right)=T_2(\upsilon)$

By applying the DHST to Equation (19), we have

$$\begin{aligned} &B_1S_\tau H_\upsilon\left(q_{\tau\tau}\right)+B_2S_\tau H_\upsilon\left(q_{\tau\upsilon}\right)+B_3S_\tau H_\upsilon\left(q_{\upsilon\upsilon}\right)+B_4S_\tau H_\upsilon\left(q_\tau\right)\\ &+B_5S_\tau H_\upsilon\left(q_\upsilon\right)+B_6S_\tau H_\upsilon\left(q\left(\tau,\upsilon\right)\right)=S_\tau H_\upsilon\left(k\left(\tau,\upsilon\right)\right) \end{aligned} \tag{20}$$

By the properties of the derivatives in Equations $(12)-(16)$, we get

$$\begin{aligned} &B_1\left(\frac{1}{\kappa^2}Q(\kappa,\lambda,\mu)-\frac{1}{\kappa^2}T_1(\upsilon)-\frac{1}{\kappa}T_2(\upsilon)\right)\\ &+B_2\left(\frac{\lambda}{\kappa\mu}Q(\kappa,\lambda,\mu)-\frac{1}{\kappa}R_1(\tau)-\frac{\lambda}{\mu}T_1(\upsilon)+\Psi\right)\\ &+B_3\left(\frac{\lambda^2}{\mu^2}Q(\kappa,\lambda,\mu)-\frac{\lambda}{\mu}R_1(\tau)-R_2(\tau)\right)+B_4\left(\frac{1}{\kappa}Q(\kappa,\lambda,\mu)-\frac{1}{\kappa}T_1(\upsilon)\right)\\ &+B_5\left(\frac{\lambda}{\mu}Q(\kappa,\lambda,\mu)Q(\kappa,\lambda,\mu)-R_1(\tau)\right)+B_6Q(\kappa,\lambda,\mu)=K(\kappa,\lambda,\mu) \end{aligned} \tag{21}$$

Simplify Equation 21 as following

$$Q(\kappa,\lambda,\mu)=\frac{\left(B_1\frac{1}{\kappa^2}+B_2\frac{\lambda}{\mu}+B_4\frac{1}{\kappa}\right)T_1+B_1\frac{1}{\kappa}T_2+\left(B_2\frac{1}{\kappa}+B_3\frac{\lambda}{\mu}+B_5\right)R_1+B_3R_2-B_2\Psi+K}{B_1\frac{1}{\kappa^2}+B_2\frac{\lambda}{\kappa\mu}+B_3\frac{\lambda^2}{\mu^2}+B_4\frac{1}{\kappa}+B_5\frac{\lambda}{\mu}+B_6} \tag{22}$$

Example 1. *Consider the heat equation*

$$q_{\tau\tau}=2q_\upsilon-3q(\tau,\upsilon)+3,\ \textit{where}\ \tau,\upsilon\geq 0$$

With IC
$q(\tau,0)=1-2\sin\tau$
and BCs
$q\left(0,\upsilon\right)=1,\ q_\tau\left(0,\upsilon\right)=-2e^\upsilon$

Solution 1. *By applying the single Sumudu transform to the IC and the single Shehu transform to the BCs, we get*
$R_1=1-\frac{2\kappa}{1+\kappa^2}$, $T_1=\frac{\mu}{\lambda}$, $T_2=\frac{-2\mu}{\lambda-\mu}$
and $K(\kappa,\lambda,\mu)=S_\tau H_\upsilon\left(3\right)=\frac{3\mu}{\lambda}$

Substitute in Equation (22) $B_1 = 1$, $B_5 = -2$, $B_6 = 3$, $B_2 = B_3 = B_4 = 0$ *and the values of* R_1, T_1, T_2 *and* K, *we get*

$$\begin{aligned} Q(\kappa,\lambda,\mu) &= \frac{\frac{\mu}{\kappa^2\lambda} - \frac{2\mu}{\kappa(\lambda-\mu)} - 2 + \frac{4\kappa}{1+\kappa^2} + \frac{3\mu}{\lambda}}{\frac{1}{\kappa^2} - \frac{2\lambda}{\mu} + 3} \\ &= \frac{\frac{3\kappa^2\mu-2\kappa^2\lambda+\mu}{\kappa^2\lambda} - \frac{2\left(3\kappa^2\mu-2\kappa^2\lambda+\mu\right)}{\kappa(1+\kappa^2)(\lambda-\mu)}}{\frac{3\kappa^2\mu-2\kappa^2\lambda+\mu}{\kappa^2\mu}} \\ &= \frac{\mu}{\lambda} - \frac{2\kappa\mu}{(1+\kappa^2)(\lambda-\mu)} \end{aligned}$$

So,

$$q(\tau,\upsilon) = S_\tau^{-1} H_\upsilon^{-1}\left(\frac{\mu}{\lambda} - \frac{2\kappa\mu}{(1+\kappa^2)(\lambda-\mu)}\right) = 1 - 2e^{\upsilon}\sin\tau$$

The graph of the exact solution is

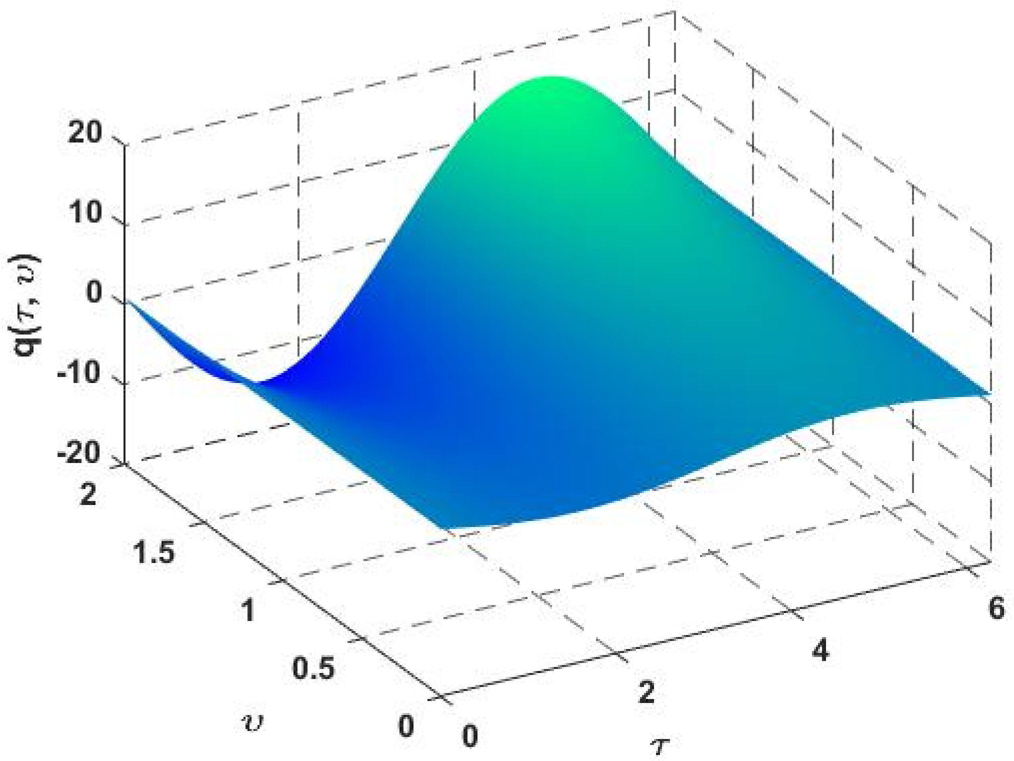

Figure 1: The solution $q(\tau,\upsilon)$ of Example 1

Example 2. *Consider the Advection-Diffusion equation*

$$q_{\upsilon} = q_{\tau\tau} - 2q_{\tau}, \text{ where } \tau, \upsilon \geq 0$$

With IC
$q(\tau, 0) = e^{2\tau} - \tau$
and BCs
$q(0, \upsilon) = 2\upsilon + 1,\ q_{\tau}(0, \upsilon) = 1$

Solution 2. *By applying the single Sumudu transform to the IC and the single Shehu transform to the BCs, we get*
$R_1 = \frac{1}{1-2\kappa} - \kappa,\ T_1 = \frac{2\mu^2}{\lambda^2} + \frac{\mu}{\lambda},\ T_2 = \frac{\mu}{\lambda}$
Substitute in Equation (22) $B_1 = 1,\ B_4 = -2,\ B_5 = -1,\ B_2 = B_3 = B_6 = 0$ *and the values of* R_1, T_1 *and* T_2, *we get*

$$Q(\kappa, \lambda, \mu) = \frac{\frac{-4\kappa\mu^2 + \lambda\mu - 2\kappa\lambda\mu + 2\mu^2}{\kappa^2\lambda^2} + \frac{\mu}{\kappa\lambda} - \frac{1}{1-2\kappa} + \kappa}{\frac{1}{\kappa^2} - \frac{2}{\kappa} - \frac{\lambda}{\mu}}$$

By simplify,

$$Q(\kappa, \lambda, \mu) = \frac{2\mu^2}{\lambda^2} + \frac{\mu}{(1-2\kappa)\lambda} - \frac{\kappa\mu}{\lambda}$$

$$q(\tau, \upsilon) = S_{\tau}^{-1} H_{\upsilon}^{-1} \left(\frac{2\mu^2}{\lambda^2} + \frac{\mu}{(1-2\kappa)\lambda} - \frac{\kappa\mu}{\lambda} \right) = \upsilon + e^{2\tau} - \tau$$

The graph of the exact solution is

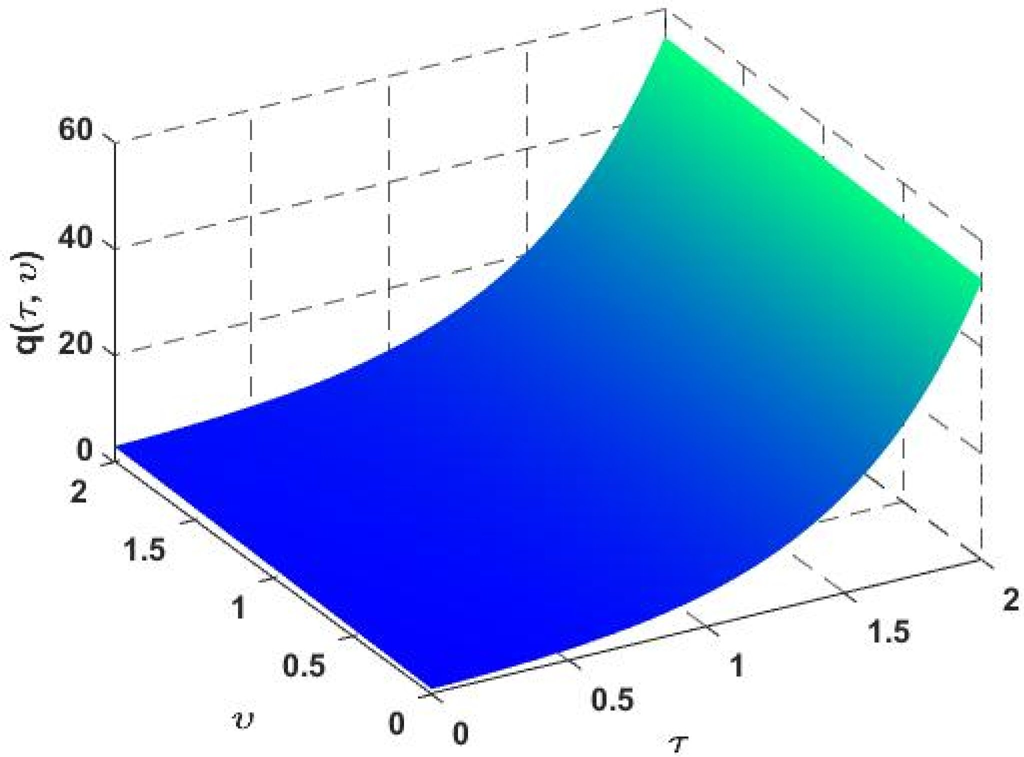

Figure 2: The solution $q(\tau, \upsilon)$ of Example 2

Example 3. *Consider the Klein-Gordon equation*

$$q_{\tau\tau} - q_{\upsilon\upsilon} - 4q(\tau, \upsilon) = -2\sinh\tau\cos\upsilon$$

With ICs
$q(\tau, 0) = \sinh\tau,\ q_{\upsilon}(\tau, 0) = 0$
and BCs
$q\,(0, \upsilon) = 0,\ q_{\tau}\,(0, \upsilon) = \cos\upsilon$

Solution 3. *By applying the single Sumudu transform to the ICs and the single Shehu transform to the BCs, we get*
$R_1 = \frac{\kappa}{1-\kappa^2},\ R_2 = 0,\ T_1 = 0,\ T_2 = \frac{\lambda\mu}{\lambda^2+\mu^2}$
and $K(\kappa, \lambda, \mu) = S_{\tau}H_{\upsilon}\,(-2\sinh\tau\cos\upsilon) = \frac{-2\kappa\lambda\mu}{(1-\kappa^2)(\lambda^2+\mu^2)}$
Substitute in Equation (22) $B_1 = 1,\ B_3 = -1,\ B_6 = -4,\ B_2 = B_4 = B_5 = 0$ *and the values of* R_1, R_2, T_1, T_2 *and* K, *we get*

$$Q(\kappa, \lambda, \mu) \quad = \quad \frac{\frac{\lambda\mu}{\kappa(\lambda^2+\mu^2)} - \frac{\kappa\lambda}{(1-\kappa^2)\mu} - \frac{2\kappa\lambda\mu}{(1-\kappa^2)(\lambda^2+\mu^2)}}{\frac{1}{\kappa^2} - \frac{\lambda^2}{\mu^2} - 4}$$

$$= \frac{\frac{\lambda\left(\mu^2-\kappa^2\lambda^2-4\kappa^2\mu^2\right)}{\kappa\mu(1-\kappa^2)(\lambda^2+\mu^2)}}{\frac{\mu^2-\kappa^2\lambda^2-4\kappa^2\mu^2}{\kappa^2\mu^2}}$$

$$= \frac{\kappa\lambda\mu}{(1-\kappa^2)\left(\lambda^2+\mu^2\right)}$$

So,

$$q(\tau,\upsilon) = S_\tau^{-1}H_\upsilon^{-1}\left(\frac{\kappa\lambda\mu}{(1-\kappa^2)\left(\lambda^2+\mu^2\right)}\right) = \sinh\tau\cos\upsilon$$

The graph of the exact solution is

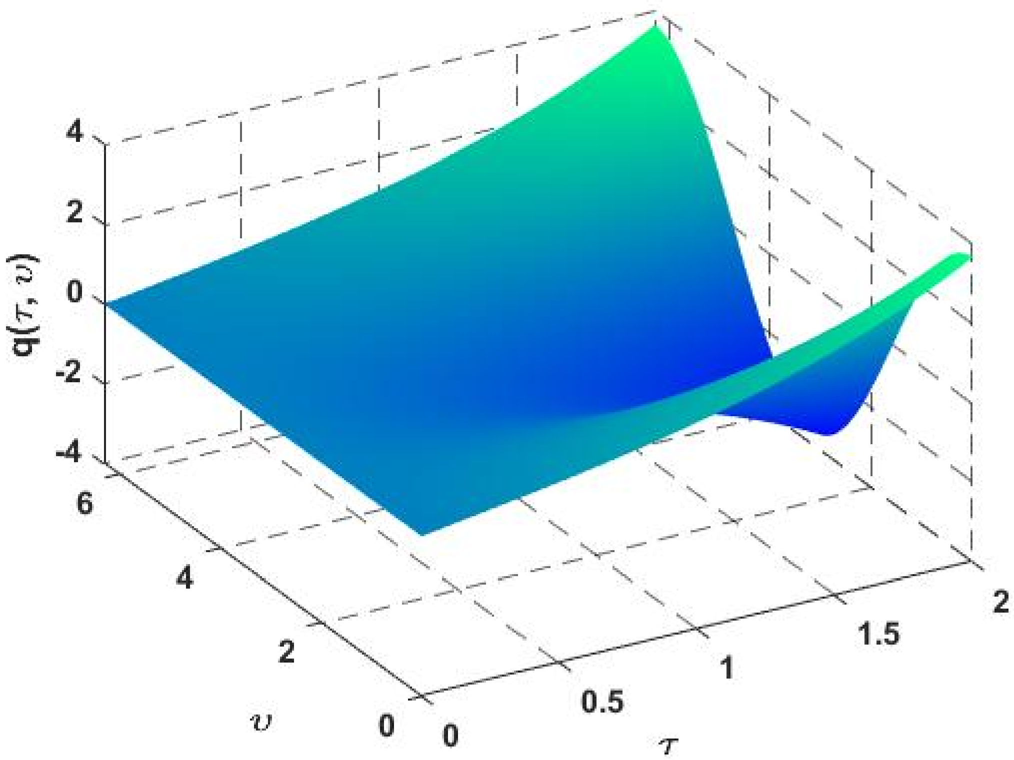

Figure 3: The solution $q(\tau,\upsilon)$ of Example 3

Example 4. *Consider the telegraph equation*

$$q_{\tau\tau} = q_{\upsilon\upsilon} - 2q_\upsilon - q(\tau,\upsilon), \text{ where } \tau,\upsilon \geq 0$$

With ICs
$q(\tau,0) = \sin\tau,\ q_\upsilon(\tau,0) = 2\sin\tau$
and BCs
$q(0,\upsilon) = 0,\ q_\tau(0,\upsilon) = e^{2\upsilon}$

Solution 4. *By applying the single Sumudu transform to the ICs and the single Shehu transform to the BCs, we get*
$R_1 = \frac{\kappa}{1+\kappa^2}$, $R_2 = \frac{2\kappa}{1+\kappa^2}$, $T_1 = 0$, $T_2 = \frac{\mu}{\lambda-2\mu}$
Substitute in Equation (22) $B_1 = 1$, $B_3 = -1$, $B_5 = 2$, $B_6 = 1$, $B_2 = B_4 = 0$ *and the values of* R_1, R_2, T_1 *and* T_2, *we get*

$$\begin{aligned} Q(\kappa,\lambda,\mu) &= \frac{\frac{\mu}{\kappa(\lambda-2\mu)} - \frac{\kappa\lambda}{(1+\kappa^2)\mu} + \frac{2\kappa}{1+\kappa^2} - \frac{2\kappa}{1+\kappa^2}}{\frac{1}{\kappa^2} - \frac{\lambda^2}{\mu^2} + \frac{2\lambda}{\mu} + 1} \\ &= \frac{\frac{\kappa^2\mu^2-\kappa^2\lambda^2+2\kappa^2\lambda\mu+\mu^2}{\kappa(1+\kappa^2)(\lambda-2\mu)\mu}}{\frac{\kappa^2\mu^2-\kappa^2\lambda^2+2\kappa^2\lambda\mu+\mu^2}{\kappa^2\mu^2}} \\ &= \frac{\kappa\mu}{(1+\kappa^2)(\lambda-2\mu)} \end{aligned}$$

So,

$$q(\tau,\upsilon) = S_\tau^{-1} H_\upsilon^{-1}\left(\frac{\kappa\mu}{(1+\kappa^2)(\lambda-2\mu)}\right) = \sin\tau e^{2\upsilon}$$

The graph of the exact solution is

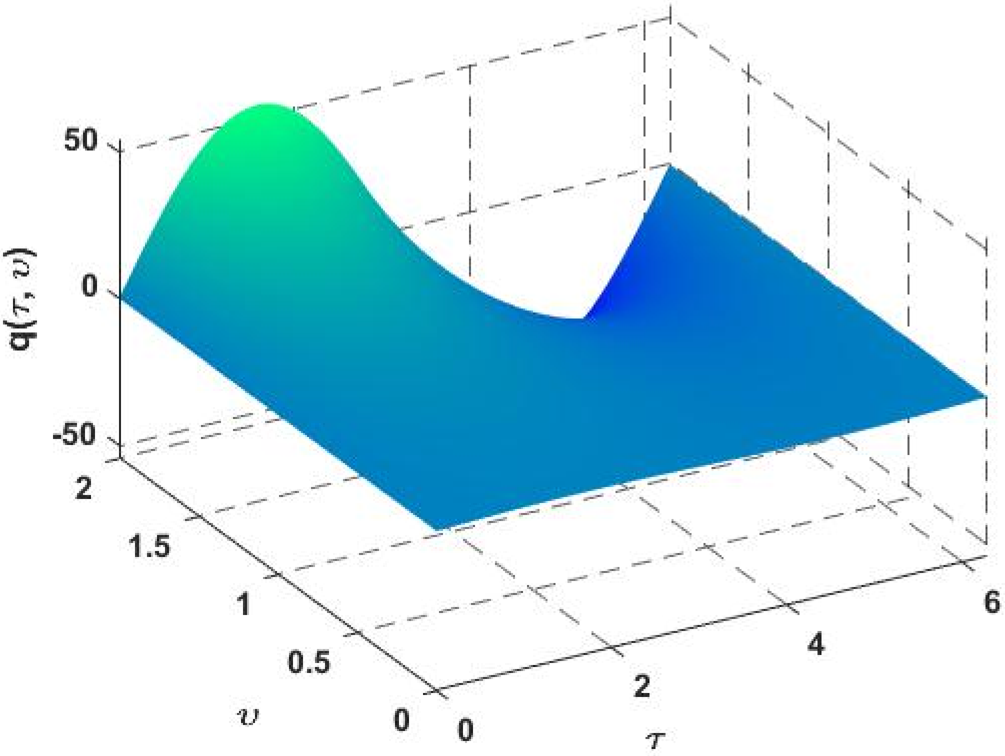

Figure 4: The solution $q(\tau,\upsilon)$ of Example 4

6. Conclusion

In this research, we introduce a novel approach termed the DSHT (Double Sumudu-Shehu Transform), offering a fresh perspective in the field of mathematical analysis. We explore the fundamental properties of this innovative double transform and demonstrate its application in solving partial differential equations and integral equations. Through carefully selected examples, we illustrate the effectiveness of the DSHT in obtaining exact solutions, highlighting its potential as a powerful tool in analytical problem-solving. Looking ahead, we anticipate further developments in DSHT, particularly in its application to conformable PDEs with variable coefficients, paving the way for new discoveries and advancements in this area. Further results and applications in this domain, including extensions to conformable PDEs, can be found in references [12–14].

Author contribution statement

All authors listed have significantly contributed to the development and the writing of this article.

Data availability statement

No data was used for the research described in the article.

Conflict of interest

The authors declare that they have no conflict of interest.

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