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A Comprehensive Study of Bipolar Vague Soft Expert P-Open Sets in Bipolar Vague Soft Expert Topological Spaces with Applications to Cancer Diagnosis

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Abstract. We rigorously examine the concept of bipolar vague soft expert sets (BPVSESs) and their defining characteristics. Fundamental operations such as complement, union, and intersection are firmly established as foundational elements of the framework. Additionally, the notion of bipolar vague soft expert topology (BPVSET) is introduced, along with eight innovative definitions. Among these, the definition of the bipolar vague soft expert pre-open set, often abbreviated as the p-open set, is particularly significant for constructing diverse structures. This study also provides a strong and healthy articulation of the concepts of interior and closure, offering a detailed exploration of their interactions. Furthermore, it develops foundational topological concepts in bipolar vague soft expert topology by introducing and analyzing bases, sub-bases, and local bases. The notions of first and second countability in the bipolar vague soft expert topology context are formally defined, while separability is explored via countable dense sets. These results enhance the theoretical framework of bipolar vague soft expert topological space, supporting soft topological modeling under uncertainty and parameterization. A comprehensive investigation into these foundational concepts culminates in a series of compelling results concerning the basis of bipolar vague soft expert topological spaces. Finally, it introduces a decision-making framework based on bi-polar vague soft expert sets to support cancer diagnosis.
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1. Introduction

Soft set theory, introduced by Molodtsov [1], was developed as a mathematical framework to address uncertainty. Chen et al. [2] critiqued certain irrational and erroneous claims in [3], arguing that the characteristic reduction in rough set theory (RST) proposed in [3] was unnecessary for minimizing the parameters required to determine optimal objects. They also highlighted the fundamental differences between characteristic reduction in rough sets and parameterization reduction in soft sets. Maji et al. [3–5] extended the theory by combining soft sets with fuzzy sets to create fuzzy soft sets. Roy and Maji [6] applied this theory to various decision-making problems, thereby enhancing the practical utility and robustness of soft set theory. Alkhazaleh et al. [7] introduced the concept of soft multisets as a generalization of soft sets. They later proposed the fuzzy parameterized interval-valued fuzzy soft set (FPIVFSS) [8] , studying its operations, and further developed the idea of possibility fuzzy soft sets [9] . These sets were demonstrated to be applicable in medical diagnosis by utilizing a similarity measure between two possibility fuzzy soft sets. Building upon these advancements, Alkhazaleh and Salleh [10] integrated fuzzy soft sets with expert sets, resulting in the concept of soft expert sets. This innovation enabled users to access professional opinions even after performing operations on the sets, further broadening the scope of practical applications. O. Dalkilic and I. N. Cangul [11] aimed to analyze decision-making processes involving interactions between elements from two distinct universe sets. To achieve this, they first introduced the concepts of object interaction and inverse object interaction sets for these separate universes. They then extended these concepts specifically to binary soft sets, taking into account the presence of two distinct universe sets. By employing a parameter set, their method enabled the determination of interaction values between objects. Furthermore, they proposed two decision-making algorithms based on these concepts within the framework of binary soft sets. The significance and advantages of the proposed algorithms were demonstrated through a practical application. Demirtas et al. [12] proposed several strategies based on soft set theory to address scenarios involving various types of uncertainty in decision-making problems. To develop these algorithms, they introduced several novel concepts to the literature, including object code, personal object code, parameter importance weight, and new distance measures. Additionally, the authors presented application results and provided further illustrative examples. Dalkilic and Demirtac [13] focused on the parameterization tool of soft set theory, introducing factor sets to account for every possible influence on each parameter. This approach aims to yield more accurate results by determining the membership values of parameters in uncertain environments. In addition, several novel hybrid types of soft sets were proposed. One of the key advantages of these new hybrid mathematical tools is their ability to reduce the potential error margin for decision-makers. Furthermore, a decision-making algorithm was developed for the soft set type that offers the most comprehensive data under conditions of uncertainty. Finally, the proposed method was applied to solve an uncertainty problem.

1.1. Literature review

According to Bosc and Pivert [14], bi-polarity refers to the human mind's capacity to think and make decisions based on both positive and negative impacts?. Positive information outlines what is conceivable, acceptable, permissible, wanted, or thought to be acceptable. Negative statements, on the other hand, express what is impossibly feasible, rejected, or prohibited. Lee [15, 16] introduced the concept of (BFS). Majumder [17] proposed (BVF) sub semigroup, (BVF) bi-idea, bipolar valued fuzzy (1, 2) - ideal and (BVF) ideal. Bipolar fuzzy groups, often referred to as fuzzy d-ideals of groups under (T-S) norm, are applications of bipolar fuzzy sets in groups that Manemaran and Chellappa [18] investigated. The study of m-polar fuzzy sets by Chen et al. [19] demonstrates how many concepts have been defined using(BFSs). Alkhazaleh et al [20].s mapping example used in decision-making showed how the approximate function is defined from a set of fuzzy parameters. Soft expert sets were first proposed by Alkhazaleh and Salleh [21] , allowing users to access all expert sets' opinions at once. (BVSTS) now have the concepts of vague soft s-open set, vague soft s-interior, vague soft s-closer, and vague soft s-exterior added by Afzal et al. [22]. Saeed et al. [11] established a new idea of operators, such as the interior operator, exterior operator, and closure operator, in (BVSTSs). On the basis of these concepts, a few results in (BVSTS) are addressed. (BVSTS) are used to address some more results based on the appealing idea of a sequence's limit being the Last. There are four sections to this piece of writing. Dalkilic and Demirtas [23] expanded the existing methodology by incorporating bipolar fuzzy soft set theory, enabling the representation of two distinct types of medical knowledge within a unified framework. They also introduced a novel decision-making algorithm specifically designed for this enhanced model. The effectiveness of the proposed algorithm is demonstrated through practical applications in the medical field, highlighting its potential to enhance diagnostic accuracy and support decision-making in clinical practice. Dalkilic [24] introduced the first type semi-strong (α, β)-cuts, second type semi-strong (α, β)-cuts, strong (α, β)-cuts, inverse (α, β)-cuts, first type semi-weak inverse (α, β)-cuts, second type semi-weak inverse (α, β)-cuts, and weak inverse (α, β)-cuts of bipolar fuzzy soft sets, along with some of their properties. In addition, several distinguishing properties between (α, β)-cuts and inverse (α, β)-cuts were established. Furthermore, related theorems were formulated and proven. It was also demonstrated that both (α, β)-cuts and inverse (α, β)-cuts of bipolar fuzzy soft sets serve as effective tools in decision-making. Abdullah et al. [25] combined the concepts of bipolar fuzzy sets and soft sets to introduce the notion of a bipolar fuzzy soft set. They investigated its fundamental properties and explored basic operations such as extended union and intersection. Moreover, they demonstrated the applicability of bipolar fuzzy soft sets in solving decision-making problems and proposed a general algorithm for this purpose. Their work laid the groundwork for applying bipolar fuzzy soft sets to real-world scenarios. Mustafa et al. [26] advanced the field by focusing on bipolar fuzzy multicriteria decision-making methods. Their primary objective was to assist students in identifying the most suitable university by evaluating the factors influencing admission decisions. To address the complexities of such decisions, they integrated bipolar fuzzy sets with soft expert sets to create a robust multicriteria decision-making model. The study also involved developing structural hierarchical models of parameters and implementing a new algorithm to enhance the accuracy of decision-making. Their approach proved to be effective, particularly in the context of university selection, and demonstrated strong potential for broader application in the education sector. Hatamleh [27] explored the compactness and continuity of two-variable Uryson operators defined by integral equations in fuzzy functional analysis. The study also examined the convergence of operator sequences using a specific measure and the Carathéodory condition. Rajalakshmi et al. [28] introduced new types of neutrosophic structures in ordered Gamma-semigroups, such as subsemigroups, ideals, and bi-ideals. Their work extended existing definitions and explored the properties of these structures through level sets. Hatamleh et al. [29] introduced complex cubic intuitionistic fuzzy subbisemirings and studied their properties, including homomorphisms and level sets. The main results were illustrated with examples. Abubaker et al. [30] investigated the use of Lagrange polynomials to find numerical solutions for various neutrosophic boundary value problems.

1.2. Research Gap

Given that the bipolar fuzzy soft expert set framework predominantly emphasizes the membership function, it entirely neglects the non-membership function. In essence, it encapsulates only the "truth" aspect of information while disregarding the "false" or contradictory component of membership. This inherent limitation underscores the necessity for a more holistic and refined approach-one that can simultaneously capture both dimensions of uncertainty. Broadly speaking, the bipolar fuzzy soft expert set is a one-dimensional approach. To study non-membership effectively, an additional dimension is required. As a result, the new set, known as a vague set, emerges as a two-dimensional model capable of handling both membership and non-membership values simultaneously.

1.3. Motivation

The following studies have served as pivotal sources of inspiration and intellectual impetus for the present research. Al-Qudah and Hassan [31] extended the concepts of bipolar fuzzy sets and soft expert sets to develop the framework of bipolar fuzzy soft expert sets. They defined fundamental theoretical operations?namely complement, union, intersection, AND, and OR on bipolar fuzzy soft expert sets, supported by illustrative examples. The authors also examined several related properties and provided formal proofs. Furthermore, they established basic properties and relevant laws associated with this concept. An algorithm was constructed based on the proposed framework, which was subsequently applied to a decision-making problem to demonstrate its practicality. The results, illustrated through an example, confirmed the effectiveness of the proposed method in solving decision-making problems. M. Akram et al. [32] presented a new multi-criteria group decision-making (MCGDM) model that incorporates criteria evaluation by multiple experts. A novel hybrid framework, termed m-bipolar fuzzy soft expert set, was developed by integrating m-polar fuzzy sets with soft expert sets, thereby enabling the investigation of soft expert sets within an m-polar fuzzy environment. The characteristics of this hybrid model are explored through numerical examples. Additionally, its fundamental properties are examined, and operations such as subsethood, complement, intersection, union, as well as the OR and AND operators, are defined. The proposed model is applied to two well-known real-world problems: site selection for a dam and human trafficking analysis across different countries. The algorithm developed for the model demonstrates both efficiency and validity. A comparative analysis with existing mathematical methods is also provided to highlight its advantages. The study of these references leads us to conclude that the absence of a unified topological framework for handling hybrid uncertainty highlights the need to formalize Bipolar Vague Soft Expert Sets (BPVSETS), which integrate bipolarity, vagueness, soft sets, and expert opinions. Fundamental operations such as complement, union, and intersection have not been rigorously defined within this structure, and a corresponding bipolar vague soft expert topology (BPVSET) remains undeveloped. Key topological concepts including p-open sets, interior, closure, basis and local basis lack formal definitions and analysis in this context. Moreover, classical properties like countability and separability have yet to be explored under the BPVSET framework. This research addresses these gaps by constructing a comprehensive topological foundation for BPVSETS, enabling soft topological modeling under uncertainty and parameterization.

1.4. Novelty

This study introduces several groundbreaking concepts within the framework of bipolar vague soft expert sets (BPVSETS), marking a significant advancement in the field. Among its notable contributions is the development of a comprehensive bipolar vague soft expert topology, offering a novel perspective for analyzing relationships among uncertain and imprecise data. The introduction of eight innovative definitions, including the pivotal bipolar vague soft expert pre-open set, represents a paradigm shift in the understanding and application of vague soft expert set theory. This foundational definition not only deepens theoretical insights but also paves the way for practical applications across various domains, such as artificial intelligence, decision-making, and fuzzy logic. Furthermore, the study systematically explores the interplay between interior and closure concepts within BPVSETS, adding depth to the existing literature and enabling a more nuanced analysis of their interactions. This dual focus enhances the theoretical framework while simultaneously presenting practical implications for modeling and addressing complex problems under uncertainty. In essence, this research extends the boundaries of traditional set theory, providing novel tools and insights that promise to transform how experts approach and analyze vague information in multifaceted environments.

1.5. Organization of the Paper

This paper is organized into seven main sections 2: Introduces Bipolar Vague Soft Sets (BVSS) based on [10], [21], and [22], defining key operations (union, intersection, AND, OR), along with concepts like empty and absolute BVSS, supported by examples. 3: Presents eight new definitions, including the bipolar vague soft expert topology (BVSET). It emphasizes the role of p-open sets in constructing topological structures and explores interior and closure concepts in depth. 4: Develops topological foundations in BVSETS through bases, sub-bases, and local bases. Introduces countability and separability with key theorems for comparing and constructing BVSET topologies. 5: Applies BVSES to cancer diagnosis, handling uncertain, vague, and conflicting expert opinions based on medical parameters. 6: Compares the proposed work with the study in [31], highlighting theoretical advancements and innovations. Summary results are presented in Table 3. 7: Summarizes findings and suggests future directions for extending BVSES in decision-making and soft topological modeling under uncertainty.

2. Preliminaries

This section discusses references [10], [21], and [22], introducing bipolar vague soft sets (BVSS), which extend vague soft sets to handle both positive and negative information under uncertainty. It defines BVSS, formalizes key operations and their properties, and introduces concepts such as the empty and absolute BVSS, as well as union, intersection, AND, and OR operators, with examples illustrating their use and consistency.

Definition 1. *[22]*
Let X be universal set and E be a set of parameters. Let $P\langle X\rangle$ denotes power set of X then the vague soft set $\langle\langle \tilde{f}, E\rangle\rangle$ over X is a set given by $\tilde{f} : E \to P\langle X\rangle$ and in other words, $\langle\langle \tilde{f}, E\rangle\rangle = [(e, < x, g_{\tilde{f}\langle e\rangle}\langle x\rangle, h_{\tilde{f}\langle e\rangle}\langle x\rangle >: x\tilde{\in}X) : e\tilde{\in}X]$ where $g_{\tilde{f}\langle e\rangle}\langle x\rangle\tilde{\in}[0,1]$ and $h_{\tilde{f}\langle e\rangle}\langle x\rangle\tilde{\in}[0,1]$ with $0\tilde{\leq}g_{\tilde{f}\langle e\rangle}\langle x\rangle + h_{\tilde{f}\langle e\rangle}\langle x\rangle\tilde{\leq}2$. This means that each value is a typical value between 0 and 1.

Definition 2. *[22]*
Let X be universal set, E be a set of parameters. A bi-polar vague soft set $\langle BVSS\rangle$

$$\langle\langle \tilde{f}, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}\langle e\rangle}\langle x\rangle \end{pmatrix} \right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

where,

$$g^{\oplus}_{\tilde{f}\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}\langle e\rangle} \to [0,1], \quad g^{\ominus}_{\tilde{f}\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}\langle e\rangle} \to [-1,0].$$

Definition 3. *[22] Let $\langle\langle \tilde{f}, E\rangle\rangle$ be a bi-polar vague soft over X then complement of a bi-polarr vague rrsoft set $\langle\langle \tilde{f}, E\rangle\rangle$,is signified by $\langle\langle \tilde{f}, E\rangle\rangle^c$ and given as*

$$\langle\langle \tilde{f}, E\rangle\rangle^c = \left[\left(e, \left\langle x, \begin{pmatrix} h^{\oplus}_{\tilde{f}\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}\langle e\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}\langle e\rangle}\langle x\rangle \end{pmatrix} \right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Definition 4. *[22]*
The empty bi-polar vague soft set $\langle\langle \tilde{f}_{null}, E\rangle\rangle$ over ?? is defined by;

$$\langle\langle \tilde{f}_{null}, E\rangle\rangle = [(e, \langle x, (0, 0, -1, 0)\rangle : x\tilde{\in}X) : e\tilde{\in}E]$$

Absolute $\langle BVSS\rangle, \langle\langle X_{aboslute}, E\rangle\rangle$ *over X is defined by;*

$$\langle\langle X_{aboslute}, E\rangle\rangle = [(e, \langle x, (1,1,0,-1)\rangle : x\tilde{\in}X) : e\tilde{\in}E]$$

Definition 5. *[22]*

Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *be two bi-polar vague soft sets over X.* $\langle\langle \tilde{f}_1, E\rangle\rangle$ *is said to be bi-polar vague soft sub set of* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *if*

$$\begin{aligned} g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle &\precsim g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \\ h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle &\succsim h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle &\precsim g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle \succsim h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \quad \forall \quad \langle e, x\rangle\tilde{\in} \quad (EX\langle M\rangle)\end{aligned}$$

It is denoted by $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\subseteq}\langle\langle \tilde{f}_2, E\rangle\rangle$*.*$\langle\langle \tilde{f}_1, E\rangle\rangle$ *is said to be bi-polar vague soft equal to* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *if* $\langle\langle \tilde{f}_1, E\rangle\rangle$ *is bi-polar vague soft sub set of* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *is bi-polar vague soft sub set of* $\langle\langle \tilde{f}_1, E\rangle\rangle$ *and is signified by* $\langle\langle \tilde{f}_1, E\rangle\rangle = \langle\langle \tilde{f}_2, E\rangle\rangle$

Example 1. *[22]*

Let $X = \{x_1, x_2\}$ *and* $E = \{e_1, e_2\}$*, if* $\langle\langle \tilde{f}_1, E\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E\rangle\rangle$ *are two bi-polar vague soft sets as*

$$\langle\langle \tilde{f}_1, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (06\times10^{-1}, 05\times10^{-1}, -08\times10^{-1}, -04\times10^{-1})\rangle, \\ \langle x_2, (05\times10^{-1}, 04\times10^{-1}, -06\times10^{-1}, -03\times10^{-1})\rangle), \\ (e_2, \langle x_1, (05\times10^{-1}, 07\times10^{-1}, -06\times10^{-1}, -05\times10^{-1})\rangle, \\ \langle x_2, (03\times10^{-1}, 05\times10^{-1}, -04\times10^{-1}, -02\times10^{-1})\rangle), \end{pmatrix}$$

$$\langle\langle \tilde{f}_1, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (07\times10^{-1}, 08\times10^{-1}, -05\times10^{-1}, -06\times10^{-1})\rangle, \\ \langle x_2, (06\times10^{-1}, 06\times10^{-1}, -05\times10^{-1}, -07\times10^{-1})\rangle), \\ (e_2, \langle x_1, (06\times10^{-1}, 09\times10^{-1}, -04\times10^{-1}, -07\times10^{-1})\rangle, \\ \langle x_2, (04\times10^{-1}, 07\times10^{-1}, -03\times10^{-1}, -06\times10^{-1})\rangle), \end{pmatrix}$$

Then, $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\subseteq}\langle\langle \tilde{f}_2, E\rangle\rangle$

Definition 6. *[22] Let*

$$\langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right],$$

for i= 1,2 be two bi-polar vague soft sub sets over X. Then, their union is signified by $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\cup}\langle\langle \tilde{f}_2, E\rangle\rangle$ *and it is given as;*

$$\coprod_{i=1}^{2}\langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} max\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\}, min\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\} \\ max\{g^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\}, min\{h^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Definition 7. *[22] Let*

$$\langle\langle \tilde{f}_i, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_i\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_i\langle e\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right],$$

for i= 1,2 be two bi-polar vague soft sub sets over X then, their intersection is signified by $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\cap}\langle\langle \tilde{f}_2, E\rangle\rangle$ *and it is given as;*

$$\prod_{i=1}^{2}\langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} min\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\}, max\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\} \\ min\{g^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\}, max\{h^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Definition 8. *[22] Let*

$$\langle\langle \tilde{f}_i, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_i\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_i\langle e\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right],$$

for $i\tilde{\in}I$ be a family of bi-polar vague soft sub sets over X then

$$\coprod_{i\tilde{\in}I}\langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} sup\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\}, inf\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\} \\ sup\{g^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\}, inf\{h^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

$$\prod_{i\tilde{\in}I}\langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} inf\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\}, sup\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle\} \\ inf\{g^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\}, sup\{h^{\ominus}_{\tilde{B}_i\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Proposition 1. *[22] Let $\langle\langle \tilde{f}_{null}, E\rangle\rangle$ and $\langle\langle X_{aboslute}, E\rangle\rangle$ be empty bi-polar vague soft sub set and absolutes bi-polar vague soft sub set over X, respectively then,*

1. $\langle\langle \tilde{f}_{null}, E\rangle\rangle \quad \subseteq \quad \langle\langle X_{aboslute}, E\rangle\rangle$
2. $\langle\langle \tilde{f}_{null}, E\rangle\rangle \quad \tilde{\cup} \quad \langle\langle X_{aboslute}, E\rangle\rangle = \langle\langle X_{aboslute}, E\rangle\rangle$
3. $\langle\langle \tilde{f}_{null}, E\rangle\rangle \quad \tilde{\cap} \quad \langle\langle X_{aboslute}, E\rangle\rangle = \langle\langle \tilde{f}_{null}, E\rangle\rangle$

Proof. Straightforward.

Definition 9. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ be two bi-polar vague soft sub sets over X then, $\langle\langle \tilde{f}_1, E\rangle\rangle \setminus \langle\langle \tilde{f}_2, E\rangle\rangle = \langle\langle \tilde{f}_3, E\rangle\rangle$ and is signified by $\langle\langle \tilde{f}_3, E\rangle\rangle = \langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\cap}\langle\langle \tilde{f}_2, E\rangle\rangle^c$ as follows:*

$$\langle\langle \tilde{f}_3, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right],$$

Where

$$h^{\oplus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle = [min\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, g^{\ominus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle = min\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}],$$

$$h^{\oplus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle = [max\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, h^{\ominus}_{\tilde{f}_3\langle e\rangle}\langle x\rangle = max\{h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}].$$

Definition 10. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ be two bi-polar vague soft sub sets over X then, AND operation is given by $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\bigwedge}\langle\langle \tilde{f}_2, E\rangle\rangle = \langle\langle \tilde{f}_3, E\times E\rangle\rangle$ and is signified by*

$$\langle\langle \tilde{B}_3, E\times E\rangle\rangle = \left[\left((e_1,e_2), \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : (e_1,e_2)\tilde{\in}E\times E\right],$$

Where

$$g^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = [min\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, g^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = min\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}],$$

$$g^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = [max\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, h^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = max\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}].$$

Definition 11. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ be two bi-polar vague soft sub sets over X then, OR operation is given by $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigvee} \langle\langle \tilde{f}_2, E\rangle\rangle = \langle\langle \tilde{f}_3, E\times E\rangle\rangle$ and is signified by*

$$\langle\langle \tilde{f}_3, E\times E\rangle\rangle = \left[\left((e_1,e_2), \left\langle x, \begin{pmatrix} g^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle \\ g^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle \end{pmatrix} \right\rangle : x \tilde{\in} X\right) : (e_1,e_2)\tilde{\in} E\times E\right],$$

Where

$$g^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = [max\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, g^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = max\{g^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}].$$

$$h^{\oplus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = [min\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, g^{\ominus}_{\tilde{f}_3\langle e_1,e_2\rangle}\langle x\rangle = min\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}],$$

Example 2. *[22]*

Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2\}$, if $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ are two bi-polar vague soft sets such that

$$\langle\langle \tilde{f}_1, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -07\times 10^{-1})\rangle, \\ \langle x_2, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -08\times 10^{-1})\rangle\rangle, \\ (e_2, \langle x_1, (04\times 10^{-1}, 04\times 10^{-1}, -04\times 10^{-1}, -03\times 10^{-1})\rangle, \\ \langle x_2, (05\times 10^{-1}, 08\times 10^{-1}, -09\times 10^{-1}, -07\times 10^{-1})\rangle\rangle, \end{pmatrix}$$

$$\langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (04\times 10^{-1}, 06\times 10^{-1}, -03\times 10^{-1}, -09\times 10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 06\times 10^{-1}, -02\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ (e_2, \langle x_1, (03\times 10^{-1}, 03\times 10^{-1}, -06\times 10^{-1}, -08\times 10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 05\times 10^{-1}, -01\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \end{pmatrix}$$

Then,

$$\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (04\times 10^{-1}, 06\times 10^{-1}, -03\times 10^{-1}, -09\times rr10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 06\times 10^{-1}, -02\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ (e_2, \langle x_1, (04\times 10^{-1}, 04\times 10^{-1}, -04\times 10^{-1}, -08\times 10^{-1})\rangle, \\ \langle x_2, (05\times 10^{-1}, 08\times 10^{-1}, -01\times 10^{-1}, -07\times 10^{-1})\rangle\rangle, \end{pmatrix}$$

$$\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -07\times rr10^{-1})\rangle, \\ \langle x_2, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ (e_2, \langle x_1, (03\times 10^{-1}, 03\times 10^{-1}, -06\times 10^{-1}, -03\times 10^{-1})\rangle, \\ \langle x_2, (05\times 10^{-1}, 05\times 10^{-1}, -09\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \end{pmatrix}$$

$$\langle\langle \tilde{f}_1, E\rangle\rangle \setminus \langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} (e_1, \langle x_1, (03\times 10^{-1}, 04\times 10^{-1}, -07\times 10^{-1}, -05\times rr10^{-1})\rangle, \\ \langle x_2, (02\times 10^{-1}, 04\times 10^{-1}, -08\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ (e_2, \langle x_1, (04\times 10^{-1}, 04\times 10^{-1}, -04\times 10^{-1}, -03\times 10^{-1})\rangle, \\ \langle x_2, (03\times 10^{-1}, 05\times 10^{-1}, -09\times 10^{-1}, -06\times 10^{-1})\rangle\rangle, \end{pmatrix} \quad (1)$$

$$\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigwedge} \langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} ((e_1,e_2), \langle x_1, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -07\times rr10^{-1})\rangle, \\ \langle x_2, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ ((e_1,e_2), \langle x_1, (03\times 10^{-1}, 03\times 10^{-1}, -06\times 10^{-1}, -07\times 10^{-1})\rangle, \\ \langle x_2, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ ((e_1,e_2), \langle x_1, (04\times 10^{-1}, 04\times 10^{-1}, -04\times 10^{-1}, -03\times rr10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 06\times 10^{-1}, -02\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \\ ((e_1,e_2), \langle x_1, (03\times 10^{-1}, 03\times 10^{-1}, -06\times 10^{-1}, -03\times rr10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 05\times 10^{-1}, -09\times 10^{-1}, -03\times 10^{-1})\rangle\rangle, \end{pmatrix}$$

$$\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigvee} \langle\langle \tilde{f}_2, E\rangle\rangle = \begin{pmatrix} ((e_1,e_2), \langle x_1, (04\times 10^{-1}, 06\times 10^{-1}, -03\times 10^{-1}, -09\times 10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 06\times 10^{-1}, -02\times 10^{-1}, -08\times 10^{-1})\rangle), \\ ((e_1,e_2), \langle x_1, (03\times 10^{-1}, 05\times 10^{-1}, -05\times 10^{-1}, -08\times 10^{-1})\rangle, \\ \langle x_2, (04\times 10^{-1}, 05\times 10^{-1}, -01\times 10^{-1}, -08\times 10^{-1})\rangle), \\ ((e_1,e_2), \langle x_1, (04\times 10^{-1}, 06\times 10^{-1}, -03\times 10^{-1}, -09\times 10^{-1})\rangle, \\ \langle x_2, (05\times 10^{-1}, 08\times 10^{-1}, -02\times 10^{-1}, -07\times 10^{-1})\rangle), \\ ((e_1,e_2), \langle x_1, (04\times 10^{-1}, 04\times 10^{-1}, -04\times 10^{-1}, -08\times rr10^{-1})\rangle, \\ \langle x_2, (05\times 10^{-1}, 08\times 10^{-1}, -01\times 10^{-1}, -07\times 10^{-1})\rangle), \end{pmatrix}$$

Proposition 2. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle, \langle\langle \tilde{f}_2, E\rangle\rangle$ and $\langle\langle \tilde{f}_3, E\rangle\rangle$ be three bi-polar vague soft sub sets sets over X then,*

1. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} [\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_3, E\rangle\rangle] = [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_2, E\rangle\rangle] \tilde{\cup} \langle\langle \tilde{f}_3, E\rangle\rangle$,
   $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} [\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_3, E\rangle\rangle] = [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, E\rangle\rangle] \tilde{\cap} \langle\langle \tilde{f}_3, E\rangle\rangle$;
2. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} [\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_3, E\rangle\rangle] = [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_2, E\rangle\rangle] \tilde{\cup} [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_3, E\rangle\rangle]$,
   $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} [\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_3, E\rangle\rangle] = [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, E\rangle\rangle] \tilde{\cup} [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_3, E\rangle\rangle]$;
3. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_{null}, E\rangle\rangle = \langle\langle \tilde{f}_1, E\rangle\rangle$,
   $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_{null}, E\rangle\rangle = \langle\langle \tilde{f}_{null}, E\rangle\rangle$;
4. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle X_{absolute}, E\rangle\rangle = \langle\langle X_{absolute}, E\rangle\rangle$,
   $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle X_{absolute}, E\rangle\rangle = \langle\langle \tilde{f}_1, E\rangle\rangle$;
5. $\langle\langle \tilde{f}_{null}, E\rangle\rangle \setminus \langle\langle X_{absolute}, E\rangle\rangle = \langle\langle \tilde{f}_{null}, E\rangle\rangle$,
   $\langle\langle X_{absolute}, E\rangle\rangle \setminus \langle\langle \tilde{f}_{null}, E\rangle\rangle = \langle\langle X_{absolute}, E\rangle\rangle$;

Proof. Straightforward.

Proposition 3. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ be two bi-polar vague soft sub sets sets over X then,*

1. $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c \tilde{\cap} [\langle\langle \tilde{f}_2, E\rangle\rangle]^c$,

2. $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c \tilde{\cup} [\langle\langle \tilde{f}_2, E\rangle\rangle]^c$,

Proof. (i) For all $e \tilde{\in} E, x \tilde{\in} X$

$$\coprod_{i=1}^{2} \langle\langle \tilde{f}_1, E\rangle\rangle = \left[\left(e, \left\langle x, \begin{pmatrix} max\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, min\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \\ max\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, min\{h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \end{pmatrix} \right\rangle : x \tilde{\in} X \right) : e \tilde{\in} E \right].$$

$$\left[\coprod_{i=1}^{2} \langle\langle \tilde{f}_1, E\rangle\rangle\right]^c = \left[\left(e, \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \end{pmatrix} \right\rangle : x \tilde{\in} X \right) : e \tilde{\in} E \right].$$

Now

$$\langle\langle \tilde{f}_1, E\rangle\rangle^c = \left[\left(e, \left\langle x, \begin{pmatrix} h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \end{pmatrix} \right\rangle : x \tilde{\in} X \right) : e \tilde{\in} E \right],$$

$$\langle\langle \tilde{f}_2, E\rangle\rangle^c = \left[\left(e, \left\langle x, \begin{pmatrix} h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Then,

$$\prod_{i=1}^{2}\langle\langle \tilde{f}_i, E\rangle\rangle^c = \left[\left(e, \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

$$= \left[\left(e, \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle : x\tilde{\in}X\right) : e\tilde{\in}E\right].$$

Thus, $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\cup}\langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c\tilde{\cap}[\langle\langle \tilde{f}_2, E\rangle\rangle]^c$
(ii) Obvious.

Proposition 4. *[22] Let $\langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle$ be two bi-polar vague soft sub sets sets over X then,*

1. $[\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\bigvee}\langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c\tilde{\bigwedge}[\langle\langle \tilde{f}_2, E\rangle\rangle]^c$,

2. $[\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\bigwedge}\langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c\tilde{\bigvee}[\langle\langle \tilde{f}_2, E\rangle\rangle]^c$,

Proof. (i) For all $(e_1, e_2)\tilde{\in}E\times E, x\tilde{\in}X$

$$\bigvee_{i=1}^{2}\langle\langle \tilde{f}_i, E\rangle\rangle = \left[(e_1, e_2), \left\langle x, \begin{pmatrix} max\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, min\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \\ max\{g^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, min\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle\right],$$

$$\left[\bigvee_{i=1}^{2}\langle\langle \tilde{f}_i, E\rangle\rangle\right]^c = \left[(e_1, e_2), \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle\right].$$

Now

$$\langle\langle \tilde{f}_1, E\rangle\rangle^c = \left[e_1, \left\langle x, \begin{pmatrix} h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle \end{pmatrix}\right\rangle : e\tilde{\in}E\right],$$

$$\langle\langle \tilde{f}_2, E\rangle\rangle^c = \left[e_2, \left\langle x, \begin{pmatrix} h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle \\ h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle \end{pmatrix}\right\rangle : e\tilde{\in}E\right].$$

Then,

$$\bigwedge_{i=1}^{2}\langle\langle \tilde{f}_i, E\rangle\rangle^c = \left[(e_1, e_2), \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle\right].$$

$$= \left[(e_1, e_2), \left\langle x, \begin{pmatrix} min\{h^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\oplus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\oplus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \\ min\{h^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, h^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\}, max\{g^{\ominus}_{\tilde{f}_1\langle e_1\rangle}\langle x\rangle, g^{\ominus}_{\tilde{f}_2\langle e_2\rangle}\langle x\rangle\} \end{pmatrix}\right\rangle\right].r$$

Thus, $\langle\langle \tilde{f}_1, E\rangle\rangle\tilde{\bigvee}\langle\langle \tilde{f}_2, E\rangle\rangle]^c = [\langle\langle \tilde{f}_1, E\rangle\rangle]^c\tilde{\bigwedge}[\langle\langle \tilde{f}_2, E\rangle\rangle]^c$
(ii) Obvious.

Definition 12. *[21] A pair* $\langle\langle \tilde{f}, E\rangle\rangle$ *is called a soft expert set (SES) over* X *, where* $\tilde{f}$ *is a mapping given by* $\tilde{f} : E \to P(X)$ *where* $P(X)$ *is power set of* X*.*

Definition 13. *[21] For two soft expert sets* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *over* X *,* $\langle\langle \tilde{f}, E\rangle\rangle$ *is called a soft expert subset of* (G, B) *if:*

1. $E_1 \subseteq E_2$,

2. *for all* $\varepsilon \tilde{\in} E_2, \quad \tilde{f}_1(\varepsilon) \subseteq \tilde{f}_2(\varepsilon)$ *This relationship is denoted by* $\langle\langle \tilde{f}_1, E_1\rangle\rangle \subseteq \langle\langle \tilde{f}_2, E_2\rangle\rangle$ *In this case* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *is called a (SE) super set of* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$

Definition 14. *[21]*
Two soft expert sets $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *over* X *are said to be equal if* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *is a (SESs) of* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *is a (SESs)* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *.*

Definition 15. *[21]*
Let E *be a set of parameters and* X *a set of experts. The NOT set of* $\check{Z} = E \times X \times O$ *denoted by* $\check{I}\check{Z}$ *is defined by* $\check{I}\check{Z} = \{\check{I}e_i, x_j, o_k\} \quad \forall i, j, k,$ *where* $\check{I}e_i$ *is not* e_i *.*

Definition 16. *[21]*
The complement of a soft expert set $\langle\langle \tilde{f}, E\rangle\rangle$ *is denoted by* $\langle\langle \tilde{f}, E\rangle\rangle^c$ *and is defined by* $\langle\langle \tilde{f}, E\rangle\rangle^c = \langle\langle \tilde{f}^c, \check{I}E\rangle\rangle$ *where* $\tilde{f}^c : \check{I}E \to P(X)$ *is a mapping given by* $\tilde{f}^c(\alpha) = X - \tilde{f}(\check{I}\alpha), \forall \alpha \tilde{\in} \check{I}E$

Definition 17. *[21] An absolute soft expert set* $\langle\langle \tilde{f}, E\rangle\rangle_1$ *over* X *is a (SESS) of* $\langle\langle \tilde{f}, E\rangle\rangle$ *defined as follows:*

$$\langle\langle \tilde{f}, E\rangle\rangle_1 = \{\tilde{f}_1(\alpha) : \alpha \tilde{\in} E \times X \times \{1\}$$

Definition 18. *[21] A null soft expert set* $\langle\langle \tilde{f}, E\rangle\rangle_0$ *over* X *is a soft expert sub set of* $\langle\langle \tilde{f}, E\rangle\rangle$ *defined as follows:*

$$\langle\langle \tilde{f}, E\rangle\rangle_0 = \{\tilde{f}_0(\alpha) : \alpha \tilde{\in} E \times X \times \{0\}$$

Definition 19. *[21] The union of two soft expert sub sets* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *over* X *denoted by* $\langle\langle \tilde{f}_1, E_1\rangle\rangle \tilde{\cup} \langle\langle \tilde{f}_2, E_2\rangle\rangle$ *is (SES)* $(H, \check{C})$ *where* $\check{C} = E_1 \tilde{\cup} E_2, \quad \forall \varepsilon \tilde{\in} \check{C}$

$$H(\varepsilon) = \begin{cases} \tilde{f}_1(\varepsilon), & if \quad \varepsilon \tilde{\in} E_1 - E_2, \\ \tilde{f}_2(\varepsilon), & if \quad \varepsilon \tilde{\in} E_2 - E_1, \\ \tilde{f}_1(\varepsilon) \tilde{\cup} \tilde{f}_2(\varepsilon), & if \quad \varepsilon \tilde{\in} E_1 \tilde{\cap} E_2, \end{cases}$$

Definition 20. *[21] The intersection of two soft expert sub sets* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *over* X *denoted by* $\langle\langle \tilde{f}_1, E_1\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, E_2\rangle\rangle$ *is (SES)* $(H, \check{C})$ *where* $\check{C} = E_1 \tilde{\cup} E_2, \quad \forall \varepsilon \tilde{\in} \check{C}$

$$H(\varepsilon) = \begin{cases} \tilde{f}_1(\varepsilon), & if \quad \varepsilon \tilde{\in} E_1 - E_2, \\ \tilde{f}_2(\varepsilon), & if \quad \varepsilon \tilde{\in} E_2 - E_1, \\ \tilde{f}_1(\varepsilon) \tilde{\cap} \tilde{f}_2(\varepsilon), & if \quad \varepsilon \tilde{\in} E_1 \tilde{\cap} E_2, \end{cases}$$

Definition 21. *[21] If* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *are two soft expert sub set over* X *then* $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ *AND* $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ *denoted by* $\langle\langle \tilde{f}_1, E_1\rangle\rangle \bigwedge \langle\langle \tilde{f}_2, E_2\rangle\rangle$ *, is defined by*

$$\langle\langle \tilde{f}_1, E_1\rangle\rangle \bigwedge \langle\langle \tilde{f}_2, E_2\rangle\rangle = (H, E_1 \times E_2)$$

where $H(\alpha, \beta) = \tilde{f}_1(\alpha) \tilde{\cap} \tilde{f}_2(\beta) \quad \forall (\alpha, \beta) \tilde{\in} E_1 \times E_2$

Definition 22. *[21] If $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ are two soft expert sub set over X then $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ OR $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ denoted by $\langle\langle \tilde{f}_1, E_1\rangle\rangle \bigvee \langle\langle \tilde{f}_2, E_2\rangle\rangle$, is defined by*

$$\langle\langle \tilde{f}_1, E_1\rangle\rangle \bigvee \langle\langle \tilde{f}_2, E_2\rangle\rangle = (O, E_1 \times E_2)$$

where $O(\alpha, \beta) = \tilde{f}_1(\alpha)\tilde{\cup}\tilde{f}_2(\beta) \quad \forall(\alpha, \beta)\tilde{\in}E_1 \times E_2$

Definition 23. *[10] A pair $\langle\langle \tilde{f}, E\rangle\rangle$ is called a fuzzy soft expert sub sets over X where $\tilde{f}$ is a mapping given by $\tilde{f} : E \to I^X$ where I^X denotes the set of all fuzzy soft expert sub set of X.*

Definition 24. *[10]*

For two fuzzy soft expert sub set that is $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ over X , $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ called fuzzy soft expert sub set of $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ if:

1. $E_2 \subseteq E_1$,

2. $\forall\ \varepsilon\tilde{\in}E_1, \quad \tilde{f}_1(\varepsilon)$ *is fuzzy soft expert sub set of* $\tilde{f}_2(\varepsilon)$.
   This relationship is denoted by $\langle\langle \tilde{f}_1, E_1\rangle\rangle \subseteq \langle\langle \tilde{f}_2, E_2\rangle\rangle$ In this case $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ is called a (FSE) super-set of $\langle\langle \tilde{f}_1, E_1\rangle\rangle$

Definition 25. *[10]*

Two fuzzy soft expert set that is $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ over X are said to be equal, if $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ is a fuzzy soft expert sub set of $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ is a fuzzy soft expert sub set of $\langle\langle \tilde{f}_1, E_1\rangle\rangle$.

Definition 26. *[10] An (AFSES) $\langle\langle \tilde{f}, E\rangle\rangle_1$ over X is a fuzzy soft expert sub set of $\langle\langle \tilde{f}, E\rangle\rangle$ defined as follows:*

$$\langle\langle \tilde{f}, E\rangle\rangle_1 = \{\tilde{f}_1(\alpha) : \alpha\tilde{\in}E \times X \times \{1\}$$

Definition 27. *[10] An absolute fuzzy soft expert set $\langle\langle \tilde{f}, E\rangle\rangle_0$ over X is a fuzzy soft expert sub set of $\langle\langle \tilde{f}, E\rangle\rangle$ defined as follows:*

$$\langle\langle \tilde{f}, E\rangle\rangle_0 = \{\tilde{f}_0(\alpha) : \alpha\tilde{\in}E \times X \times \{0\}$$

Definition 28. *[10]*

The complement of a fuzzy soft expert set $\langle\langle \tilde{f}, E\rangle\rangle$ is denoted by $\langle\langle \tilde{f}, E\rangle\rangle^c$ and is defined by $\langle\langle \tilde{f}, E\rangle\rangle^c = \langle\langle \tilde{f}^c, \dagger E\rangle\rangle$ where $\tilde{f}^c : E \to P(X)$ is a mapping given by $\tilde{f}^c(\alpha) = c(\tilde{f}(\alpha)), \forall\alpha\tilde{\in}E$ Where c is a fuzzy complement.

Definition 29. *[10] The union of two fuzzy soft expert sets $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ over X denoted by $\langle\langle \tilde{f}_1, E_1\rangle\rangle\tilde{\cup}\langle\langle \tilde{f}_2, E_2\rangle\rangle$ is the fuzzy soft expert set $(H, \check{C})$ where $\check{C} = E_1\tilde{\cup}E_2, \quad \forall\varepsilon\tilde{\in}\check{C}$*

$$H(\varepsilon) = \begin{cases} \tilde{f}_1(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_1 - E_2, \\ \tilde{f}_2(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_2 - E_1, \\ s\tilde{f}_1(\varepsilon), \tilde{f}_2(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_1\tilde{\cap}E_2, \end{cases}$$

where s is an s-norm.

Definition 30. *[10] The intersection of two fuzzy soft expert sets $\langle\langle \tilde{f}_1, E_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, E_2\rangle\rangle$ over X denoted by $\langle\langle \tilde{f}_1, E_1\rangle\rangle\tilde{\cap}\langle\langle \tilde{f}_2, E_2\rangle\rangle$ is fuzzy soft expert set $(H, \check{C})$ where $\check{C} = E_1\tilde{\cap}E_2, \quad \forall\varepsilon\tilde{\in}\check{C}$*

$$H(\varepsilon) = \begin{cases} \tilde{f}_1(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_1 - E_2, \\ \tilde{f}_2(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_2 - E_1, \\ t\tilde{f}_1(\varepsilon), \tilde{f}_2(\varepsilon), & \text{if} \quad \varepsilon\tilde{\in}E_1\tilde{\cap}E_2, \end{cases}$$

where t is a t-norm.

Definition 31. *[10] If* $\langle\langle\tilde{f}_1, E_1\rangle\rangle$ *and* $\langle\langle\tilde{f}_2, E_2\rangle\rangle$ *are two fuzzy soft expert sets over* X *then* $\langle\langle\tilde{f}_1, E_1\rangle\rangle$ *AND* $\langle\langle\tilde{f}_2, E_2\rangle\rangle$ *denoted by* $\langle\langle\tilde{f}_1, E_1\rangle\rangle \bigwedge \langle\langle\tilde{f}_2, E_2\rangle\rangle$ *, is defined by*

$$\langle\langle\tilde{f}_1, E_1\rangle\rangle \bigwedge \langle\langle\tilde{f}_2, E_2\rangle\rangle = (H, E_1 \times E_2)$$

s.t. $H(\alpha, \beta) = t(\tilde{f}_1(\varepsilon), \tilde{f}_2(\varepsilon)) \quad \forall(\alpha, \beta)\tilde{\in}E_1 \times E_2$*, where* t *is a t-norm.*

Definition 32. *[10] If* $\langle\langle\tilde{f}_1, E_1\rangle\rangle$ *and* (G, B) *are two fuzzy soft expert sets over* X *then* $\langle\langle\tilde{f}_1, E_1\rangle\rangle$ *OR* $\langle\langle\tilde{f}_2, E_2\rangle\rangle$ *denoted by* $\langle\langle\tilde{f}_1, E_1\rangle\rangle \bigvee \langle\langle\tilde{f}_2, E_2\rangle\rangle$ *, is defined by*

$$\langle\langle\tilde{f}_1, E_1\rangle\rangle \bigvee \langle\langle\tilde{f}_2, E_2\rangle\rangle = (H, E_1 \times E_2)$$

s.t. $H(\alpha, \beta) = s(\tilde{f}_1(\varepsilon), \tilde{f}_2(\varepsilon)) \quad \forall(\alpha, \beta)\tilde{\in}E_1 \times E_2$ *, where* s *is an s-norm.*

3. Exhibition of Bipolar Vague Soft Expert Structure

This section introduces eight new definitions alongside several original contributions, including the bipolar vague soft expert topology (BPVST). Among these, the concept of the bipolar vague soft expert pre-open set, abbreviated as "p-open set," is highlighted as a pivotal and versatile tool for constructing diverse structures. The notions of interior and closure are explored in detail, and outcomes derived from these concepts are thoroughly addressed. Furthermore, additional insights are provided regarding the interaction between interior and closure, adding depth to the analysis. Let X be universal set, E a set of parameter, X a set of expert (agents), and $o = \{1 = agree, 0 = disagree\}$ a set of opinion. let $\check{Z} = E \times X \times O$ and $\bar{E} \subseteq \check{Z}$

Definition 33. *A pair* $\langle\langle H, \bar{E}\rangle\rangle$ *is called a bipolar vague soft expert set over* X*, where* H *is mapping given by* $H : \bar{E} \to P(X)$ *Where* $P(X)$ *denotes a power set of bipolar vague soft expert set of* X *and* $\langle\langle H, \bar{E}\rangle\rangle = \begin{pmatrix} \langle u, T^{+}_{H(e)}(u), \tilde{f}^{+}_{H(e)}, T^{-}_{H(e)}(u), \tilde{f}^{-}_{H(e)}(u)\rangle \\ \forall \quad e\tilde{\in}E, u\tilde{\in}X \end{pmatrix}$.
Where $T^{+}_{H(e)}(u), \tilde{f}^{+}_{H(e)} : X \to [0, 1]$ *and* $T^{-}_{H(e)}(u), \tilde{f}^{-}_{H(e)} : X \to [0, 1]$

Example 3. *Let* $E = \{cheap\ ,\ expensive\} = \{(e_1, e_2)\}.X = \{p, q, r\}$ *be a set of experts such that*

$$H(e_1, p, 1) = \begin{pmatrix} \langle u_1, 03 \times 10^{-1}, 07 \times 10^{-1}, \\ -02 \times 10^{-1}, -04 \times 10^{-1}\rangle, \\ \langle u_3, 05 \times 10^{-1}, 03 \times 10^{-1}, \\ -03 \times 10^{-1}, -01 \times 10^{-1}\rangle, \end{pmatrix} .H(e_1, q, 1) = \begin{pmatrix} \langle u_2, 08 \times 10^{-1}, 03 \times 10^{-1}, \\ -01 \times 10^{-1}, -05 \times 10^{-1}\rangle, \\ \langle u_3, 09 \times 10^{-1}, 07 \times 10^{-1}, \\ -04 \times 10^{-1}, -02 \times 10^{-1}\rangle, \end{pmatrix}.$$

$$H(e_1, r, 1) = \begin{pmatrix} \langle u_1, 04 \times 10^{-1}, 06 \times 10^{-1}, \\ -06 \times 10^{-1}, -04 \times 10^{-1}\rangle \end{pmatrix}.$$

$$H(e_2, p, 1) = \begin{pmatrix} \langle u_1, 04 \times 10^{-1}, 03 \times 10^{-1}, \\ -02 \times 10^{-1}, -01 \times 10^{-1}\rangle, \\ \langle u_2, 07 \times 10^{-1}, 03 \times 10^{-1}, \\ -03 \times 10^{-1}, -05 \times 10^{-1}\rangle, \end{pmatrix} .H(e_2, q, 1) = \begin{pmatrix} \langle u_3, 03 \times 10^{-1}, 02 \times 10^{-1}, \\ -05 \times 10^{-1}, -04 \times 10^{-1}\rangle, \end{pmatrix}.$$

$$H(e_2, r, 1) = \begin{pmatrix} \langle u_2, 03 \times 10^{-1}, 09 \times 10^{-1}, \\ -04 \times 10^{-1}, -01 \times 10^{-1}\rangle \end{pmatrix}.$$

$$H(e_1, p, 0) = \begin{pmatrix} \langle u_2, 05 \times 10^{-1}, 03 \times 10^{-1}, \\ -05 \times 10^{-1}, -03 \times 10^{-1}\rangle, \end{pmatrix} .H(e_1, q, 0) = \begin{pmatrix} \langle u_1, 06 \times 10^{-1}, 05 \times 10^{-1}, \\ -04 \times 10^{-1}, -06 \times 10^{-1}\rangle, \end{pmatrix}.$$

$$H(e_1,r,0)=\begin{pmatrix}\langle u_2, 07\times10^{-1}, 04\times10^{-1},\\ -03\times10^{-1}, -05\times10^{-1}\rangle\\ \langle u_3, 09\times10^{-1}, 07\times10^{-1},\\ -02\times10^{-1}, -05\times10^{-1}\rangle\end{pmatrix}.$$

$$H(e_2,p,0)=\begin{pmatrix}\langle u_3, 07\times10^{-1}, 06\times10^{-1},\\ -02\times10^{-1}, -04\times10^{-1}\rangle,\end{pmatrix}.H(e_2,q,0)=\begin{pmatrix}\langle u_1, 07\times10^{-1}, 06\times10^{-1},\\ -03\times10^{-1}, -04\times10^{-1}\rangle,\\ \langle u_2, 06\times10^{-1}, 05\times10^{-1},\\ -03\times10^{-1}, -04\times10^{-1}\rangle\end{pmatrix}.$$

$$H(e_2,r,0)=\begin{pmatrix}\langle u_1, 06\times10^{-1}, 05\times10^{-1},\\ -05\times10^{-1}, -02\times10^{-1}\rangle\\ \langle u_3, 07\times10^{-1}, 08\times10^{-1},\\ -06\times10^{-1}, -01\times10^{-1}\rangle\end{pmatrix}.$$

Definition 34. *Let $\langle\langle H,\bar{E}_1\rangle\rangle$ and $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ be two bipolar vague soft expert sub sets over X. $\langle\langle H,\bar{E}_1\rangle\rangle$ is said to be bipolar vague soft expert sub sets of $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$, if $\langle\langle H,\bar{E}_1\rangle\rangle\tilde{\subseteq}\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ if and only if $T^+_{H(e)}(u)\precsim T^+_{G(e)}(u), \tilde{f}^+_{H(e)}\succsim\tilde{f}^+_{G(e)}$ and*

$T^+_{H(e)}(u)\succsim T^+_{G(e)}(u), \tilde{f}^+_{H(e)}\precsim\tilde{f}^+_{G(e)}, \quad \forall e\tilde{\epsilon}\bar{E}_1, u\tilde{\epsilon}X.$ *$\langle\langle H,\bar{E}_1\rangle\rangle$ is said to be bipolar vague soft expert super set of $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ if $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ is a (BVSE S) of $\langle\langle H,\bar{E}_1\rangle\rangle$ denoted by $\langle\langle H,\bar{E}_1\rangle\rangle\tilde{\supseteq}\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$.*

Example 4. *Let $X=\{u_1,u_2,u_3\}$ be a master set, $E=\{u_1,u_2\}$ a set of parameters where $e_i(i=1,2)$ denotes the decision 'cheap' , 'expensive' respectively and Let $X=\{u_1,u_2,u_3\}$ be a set of experts. Suppose $\langle\langle H,\bar{E}_1\rangle\rangle$ and $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ be defined as follows:*

$$\langle\langle H,\bar{E}_1\rangle\rangle=\left\{\begin{array}{c}\left[\begin{array}{c}(e_1,p,1),\langle u_1, 03\times10^{-1}, 06\times10^{-1}, -02\times10^{-1}, -04\times10^{-1}\rangle, \langle u_2, 05\times10^{-1}, 03\times10^{-1},\\ -04\times10^{-1}, -05\times10^{-1}\rangle,\end{array}\right]\\ \left[(e_1,p,0),\langle u_2, 02\times10^{-1}, 07\times10^{-1}, -05\times10^{-1}, -03\times10^{-1}\rangle,\right]\\ \left[\begin{array}{c}(e_1,q,1),\langle u_1, 06\times10^{-1}, 05\times10^{-1}, -06\times10^{-1}, -05\times10^{-1}\rangle, \langle u_2, 06\times10^{-1}, 03\times10^{-1},\\ -05\times10^{-1}, -03\times10^{-1}\rangle,\end{array}\right]\\ \left[(e_1,r,0),\langle u_1, 02\times10^{-1}, 03\times10^{-1}, -04\times10^{-1}, -05\times10^{-1}\rangle,\right]\\ \left[\begin{array}{c}(e_2,r,1),\langle u_2, 03\times10^{-1}, 09\times10^{-1}, -03\times10^{-1}, -04\times10^{-1}\rangle, \langle u_3, 07\times10^{-1}, 08\times10^{-1},\\ -05\times10^{-1}, -06\times10^{-1}\rangle,\end{array}\right]\end{array}\right\}$$

$$\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle=\left\{\begin{array}{c}\left[\begin{array}{c}(e_1,p,1),\langle u_1, 03\times10^{-1}, 07\times10^{-1}, -02\times10^{-1}, -06\times10^{-1}\rangle, \langle u_2, 05\times10^{-1}, 03\times10^{-1},\\ -01\times10^{-1}, -07\times10^{-1}\rangle,\end{array}\right]\\ \left[(e_2,p,0),\langle u_2, 02\times10^{-1}, 07\times10^{-1}, -02\times10^{-1}, -05\times10^{-1}\rangle,\right]\\ \left[\begin{array}{c}(e_1,q,1),\langle u_1, 06\times10^{-1}, 05\times10^{-1}, -01\times10^{-1}, -08\times10^{-1}\rangle, \langle u_2, 06\times10^{-1}, 03\times10^{-1},\\ -03\times10^{-1}, -04\times10^{-1}\rangle,\end{array}\right]\end{array}\right\}$$

Therefore $\langle\langle H,\bar{E}_1\rangle\rangle\tilde{\supseteq}\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$.

Definition 35. *Let $\langle\langle H,\bar{E}_1\rangle\rangle$ and $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ be two bipolar vague soft expert sub sets . $\langle\langle H,\bar{E}_1\rangle\rangle$ is said to be bipolar vague soft expert equal $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ and we write $\langle\langle H,\bar{E}_1\rangle\rangle=\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ if and only if $T^+_{H(e)}(u)=T^+_{G(e)}(u), \tilde{f}^+_{H(e)}=\tilde{f}^+_{G(e)}$ and*

$T^-_{H(e)}(u)=T^-_{G(e)}(u), \tilde{f}^-_{H(e)}=\tilde{f}^-_{G(e)}, \quad \forall e\tilde{\epsilon}\bar{E}_1, u\tilde{\epsilon}X.$

Definition 36. *Let $E=\{e_1,e_2,...,e_n\}$ be a set of parameters. The NOT set of E is denoted by $\neg E=\{\neg e_1,\neg e_2,...,\neg e_n\}$ where $\neg e_i=note_i, \forall i=1,2,...,n$.*

Example 5. *Consider Example ref 3.2 Here* $\neg E = \{notcheap, notexpensive\}$

Definition 37. *The complement of a bipolar vague soft expert set* $\langle\langle H, \bar{E}_1\rangle\rangle$ *denoted by* $\langle\langle H, \bar{E}_1\rangle\rangle^c$ *and is defined as* $\langle\langle H, \bar{E}_1\rangle\rangle^c = \langle\langle H^c, \neg\bar{E}_1\rangle\rangle$ *where* $H^c = \neg\bar{E}_1 \to P(X)$ *is mapping given by* $H^c(u) = T^+_{H^c(u)} = \tilde{f}^+_{H(u)}, \tilde{f}^+_{H^c(u)} = T^+_{H^c(u)}$ *and* $T^-_{H^c(u)} = \tilde{f}^-_{H(e)}, \tilde{f}^-_{H^c(u)} = T^-_{H(u)}$.

Example 6. *Consider the 4 Example. Then* $\langle\langle H, \bar{\bar{Z}}\rangle\rangle^c$ *is*

$$\langle\langle H, \bar{\bar{Z}}\rangle\rangle^c = \left\{\begin{array}{c}
\left[(\neg e_1, p, 1), \langle u_2, 03\times10^{-1}, 05\times10^{-1}, -03\times10^{-1}, -05\times10^{-1}\rangle,\right],\\
\left[(\neg e_1, q, 1), \langle u_1, 05\times10^{-1}, 06\times10^{-1}, -04\times10^{-1}, -03\times10^{-1}\rangle,\right],\\
\left[\begin{array}{c}(\neg e_1, r, 1), \langle u_2, 04\times10^{-1}, 07\times10^{-1}, -03\times10^{-1}, -02\times10^{-1}\rangle, \langle u_3, 0\times10^{-1}, 09\times10^{-1},\\ -01\times10^{-1}, -03\times10^{-1}\rangle,\end{array}\right],\\
\left[(\neg e_2, p, 1), \langle u_3, 06\times10^{-1}, 07\times10^{-1}, -04\times10^{-1}, -02\times10^{-1}\rangle,\right],\\
\left[\begin{array}{c}(\neg e_2, q, 1), \langle u_1, 06\times10^{-1}, 07\times10^{-1}, -05\times10^{-1}, -03\times10^{-1}\rangle, \langle u_2, 05\times10^{-1}, 06\times10^{-1},\\ -03\times10^{-1}, -06\times10^{-1}\rangle,\end{array}\right],\\
\left[\begin{array}{c}(\neg e_2, r, 1), \langle u_1, 05\times10^{-1}, 06\times10^{-1}, -06\times10^{-1}, -04\times10^{-1}\rangle, \langle u_3, 08\times10^{-1}, 07\times10^{-1},\\ -03\times10^{-1}, -01\times10^{-1}\rangle,\end{array}\right],\\
\left[\begin{array}{c}(\neg e_1, p, 0), \langle u_1, 07\times10^{-1}, 03\times10^{-1}, -04\times10^{-1}, -03\times10^{-1}\rangle, \langle u_3, 03\times10^{-1}, 05\times10^{-1},\\ -06\times10^{-1}, -05\times10^{-1}\rangle,\end{array}\right],\\
\left[\begin{array}{c}(\neg e_1, q, 0), \langle u_2, 03\times10^{-1}, 08\times10^{-1}, -03\times10^{-1}, -07\times10^{-1}\rangle, \langle u_3, 09\times10^{-1}, 07\times10^{-1},\\ -07\times10^{-1}, -05\times10^{-1}\rangle,\end{array}\right],\\
\left[(\neg e_1, r, 0), \langle u_1, 06\times10^{-1}, 04\times10^{-1}, -04\times10^{-1}, -05\times10^{-1}\rangle,\right],\\
\left[\begin{array}{c}(\neg e_2, p, 0), \langle u_1, 03\times10^{-1}, 04\times10^{-1}, -03\times10^{-1}, -04\times10^{-1}\rangle, \langle u_2, 03\times10^{-1}, 07\times10^{-1},\\ -06\times10^{-1}, -01\times10^{-1}\rangle,\end{array}\right],\\
\left[(\neg e_2, q, 0), \langle u_3, 02\times10^{-1}, 03\times10^{-1}, -07\times10^{-1}, -03\times10^{-1}\rangle,\right],\\
\left[(\neg e_2, r, 0), \langle u_2, 09\times10^{-1}, 03\times10^{-1}, -08\times10^{-1}, -05\times10^{-1}\rangle,\right]
\end{array}\right\}$$

Definition 38. *The set* $\langle\langle H, \bar{E}_1\rangle\rangle$ *is termed to be bi-polar vague soft expert null if* $T^+_{H(e)}(u) = T^+_{G(e)}(u) = 0, \quad \tilde{f}^+_{H(e)}(u) = \tilde{f}^+_{G(e)}(u) = 0$ *and* $T^-_{H(e)}(u) = T^-_{G(e)}(u) = 0, \tilde{f}^-_{H(e)}(u) = \tilde{f}^-_{G(e)}(u) = 0$, $\forall \quad e\tilde{\epsilon}\bar{E}_1, u\tilde{\epsilon}X$.

Example 7. *Let* $X = \{u_1, u_2, u_3\}, E = \{quality\} = \{e_1\}$ *and Let* $X = \{p, q\}$ *be a set of experts*

$$\bar{\breve{E}}_1 = (NBNSES) = \left\{\begin{array}{c}
\left[(e_1, p, 1), \langle u_1, 0, 0, 0, 0\rangle, \langle u_2, 0, 0, 0, 0\rangle,\right]\\
\left[(e_1, q, 1), \langle u_1, 0, 0, 0, 0\rangle, \langle u_2, 0, 0, 0, 0\rangle,\right]\\
\left[(e_1, p, 0), \langle u_3, 0, 0, 0, 0\rangle,\right]\\
\left[(e_1, q, 1), \langle u_3, 0, 0, 0, 0\rangle,\right]
\end{array}\right\}$$

Definition 39. *A bi-polar vague soft expert set* $\langle\langle H, \bar{E}_1\rangle\rangle_1$ *soft expert subset of* $\langle\langle H, \bar{E}_1\rangle\rangle$ *is defined as follows:*

$\langle\langle H, \bar{E}_1\rangle\rangle_1 = \{H_1(u) : u\tilde{\epsilon}E \times X \times \{1\}$.

Example 8. *Consider the 4 Example 3.2. Then bi-polar vague soft expert set* $\langle\langle H, \bar{E}_1\rangle\rangle_1$ *over* X *is*

$$\langle\langle H, \bar{E}_1\rangle\rangle_1 = \left\{\begin{array}{c}
\left[\begin{array}{c}(\neg e_1, p, 1), \langle u_1, 03\times10^{-1}, 07\times10^{-1}, -02\times10^{-1}, -04\times10^{-1}\rangle, \langle u_3, 05\times10^{-1}, 03\times10^{-1},\\ -03\times10^{-1}, -01\times10^{-1}\rangle,\end{array}\right],\\
\left[\begin{array}{c}(\neg e_1, q, 1), \langle u_2, 08\times10^{-1}, 03\times10^{-1}, -01\times10^{-1}, -05\times10^{-1}\rangle, \langle u_3, 09\times10^{-1}, 07\times10^{-1},\\ -04\times10^{-1}, -02\times10^{-1}\rangle,\end{array}\right],\\
\left[(\neg e_1, r, 1), \langle u_1, 04\times10^{-1}, 06\times10^{-1}, -06\times10^{-1}, -04\times10^{-1}\rangle\right],\\
\left[\begin{array}{c}(\neg e_2, p, 1), \langle u_1, 04\times10^{-1}, 03\times10^{-1}, -02\times10^{-1}, -01\times10^{-1}\rangle, \langle u_2, 07\times10^{-1}, 03\times10^{-1},\\ -03\times10^{-1}, -05\times10^{-1}\rangle,\end{array}\right],\\
\left[(\neg e_2, q, 1), \langle u_3, 03\times10^{-1}, 02\times10^{-1}, -05\times10^{-1}, -04\times10^{-1}\rangle,\right],\\
\left[(\neg e_2, r, 1), \langle u_2, 03\times10^{-1}, 09\times10^{-1}, -04\times10^{-1}, -01\times10^{-1}\rangle\right],
\end{array}\right\}$$

Definition 40. *A bipolar vague soft expert set* $\langle\langle H,\bar{E}_1\rangle\rangle_0$ *over* X *is a bi-polar vague soft expert set of* $\langle\langle H,\bar{E}_1\rangle\rangle$ *defined as fallows;* $\langle\langle H,\bar{E}_1\rangle\rangle_0=\{H_0(u):u\tilde{\epsilon}E\times X\times\{0\}$

Example 9. *Consider the 4 Example 3.2. Then the bipolar vague soft expert set* $\langle\langle H,\bar{E}_1\rangle\rangle_0$ *over* X *is*

$$\langle\langle H,\bar{E}_1\rangle\rangle_0=\left\{\begin{array}{c}\left[(\neg e_1,p,0),\langle u_2,05\times10^{-1},03\times10^{-1},-05\times10^{-1},-03\times10^{-1}\rangle,\right],\\ \left[(\neg e_1,q,0),\langle u_1,06\times10^{-1},05\times10^{-1},-04\times10^{-1},-06\times10^{-1}\rangle,\right],\\ \left[\begin{array}{c}(\neg e_1,r,0),\langle u_2,07\times10^{-1},04\times10^{-1},-03\times10^{-1},-05\times10^{-1}\rangle\langle u_3,09\times10^{-1},07\times10^{-1},\\ -02\times10^{-1},-05\times10^{-1}\rangle,\end{array}\right],\\ \left[(\neg e_2,p,0),\langle u_3,07\times10^{-1},06\times10^{-1},-02\times10^{-1},-04\times10^{-1}\rangle,\right],\\ \left[\begin{array}{c}(\neg e_2,q,0),\langle u_1,07\times10^{-1},06\times10^{-1},-03\times10^{-1},-04\times10^{-1}\rangle,\langle u_2,06\times10^{-1},05\times10^{-1},\\ -03\times10^{-1},-04\times10^{-1}\rangle\end{array}\right],\\ \left[\begin{array}{c}(\neg e_2,r,0),\langle u_1,06\times10^{-1},05\times10^{-1},-05\times10^{-1},-02\times10^{-1}\rangle\langle u_3,07\times10^{-1},08\times10^{-1},\\ -06\times10^{-1},-01\times10^{-1}\rangle\end{array}\right],\end{array}\right\}$$

Definition 41. *The union of two bipolar vague soft expert sets. Let*

$$\langle\langle H,\bar{E}_1\rangle\rangle=\left\{\begin{array}{c}\langle u,T^+_{H(e)}(u),\tilde{f}^+_{H(e)}(u),T^-_{H(e)}(u),\tilde{f}^-_{H(e)}(u)\rangle\\ :\forall e\tilde{\epsilon}X,u\tilde{\epsilon}X\end{array}\right\}$$

and

$$\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle=\left\{\begin{array}{c}\langle u,T^+_{G(e)}(u),\tilde{f}^+_{G(e)}(u),T^-_{G(e)}(u),\tilde{f}^-_{G(e)}(u)\rangle\\ :\forall e\tilde{\epsilon}B,u\tilde{\epsilon}X\end{array}\right\}$$

be two bipolar vague soft expert sets then their union is defined as:

$$\langle\langle H,\bar{E}_1\rangle\rangle\tilde{\cup}\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle(u)=\begin{pmatrix}max(T^+_{H(e)}(u),T^+_{G(e)}(u)),min(\tilde{f}^+_{H(e)}(u),\tilde{f}^+_{G(e)}(u))\\ min(T^-_{H(e)}(u),T^-_{G(e)}(u)),max(\tilde{f}^-_{H(e)}(u),\tilde{f}^-_{G(e)}(u))\end{pmatrix}$$

Example 10. *Let* $\langle\langle H,\bar{E}_1\rangle\rangle$ *and* $\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle$ *be two bipolar vague soft expert sets*

$$\langle\langle H,\bar{E}_1\rangle\rangle=\left\{\begin{array}{c}\left[\begin{array}{c}(e_1,p,1),\langle u_1,02\times10^{-1},08\times10^{-1},-04\times10^{-1},-05\times10^{-1}\rangle,\langle u_3,02\times10^{-1},05\times10^{-1},\\ -02\times10^{-1},-04\times10^{-1}\rangle,\end{array}\right]\\ \left[\begin{array}{c}(e_1,q,1),\langle u_1,05\times10^{-1},06\times10^{-1},-02\times10^{-1},-03\times10^{-1}\rangle,\langle u_2,08\times10^{-1},03\times10^{-1},\\ -02\times10^{-1},-01\times10^{-1}\rangle,\end{array}\right]\end{array}\right\}$$

$$\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle=\left\{\left[\begin{array}{c}(e_1,p,1),\langle u_1,01\times10^{-1},02\times10^{-1},-03\times10^{-1},-04\times10^{-1}\rangle,\langle u_2,04\times10^{-1},08\times10^{-1},\\ -01\times10^{-1},-05\times10^{-1}\rangle,\end{array}\right]\right\}$$

Therefor $\langle\langle H,\bar{E}_1\rangle\rangle\tilde{\cup}\langle\langle\tilde{f}_2,\bar{E}_2\rangle\rangle=\langle\langle R,\bar{C}\rangle\rangle$

$$\langle\langle R,\bar{C}\rangle\rangle=\left\{\begin{array}{c}\left[\begin{array}{c}(e_1,p,1),\langle u_1,02\times10^{-1},02\times10^{-1},-04\times10^{-1},-04\times10^{-1}\rangle,\\ \langle u_2,04\times10^{-1},08\times10^{-1},-01\times10^{-1},-05\times10^{-1}\rangle,\langle u_3,02\times10^{-1},05\times10^{-1},\\ -02\times10^{-1},-04\times10^{-1}\rangle,\end{array}\right]\\ \left[\begin{array}{c}(e_1,q,1),\langle u_1,05\times10^{-1},06\times10^{-1},-02\times10^{-1},-03\times10^{-1}\rangle,\langle u_2,04\times10^{-1},08\times10^{-1},\\ -01\times10^{-1},-05\times10^{-1}\rangle,\end{array}\right]\end{array}\right\}$$

Definition 42. *The intersection of two bipolar vague soft expert sets*

$$\langle\langle H,\bar{E}_1\rangle\rangle=\left\{\begin{array}{c}\langle u,T^+_{H(e)}(u),\tilde{f}^+_{H(e)}(u),T^-_{H(e)}(u),\tilde{f}^-_{H(e)}(u)\rangle\\ :\forall e\tilde{\epsilon}X,u\tilde{\epsilon}X\end{array}\right\}$$

and

$$\langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle = \left\{ \begin{matrix} \langle u, T^{+}_{G(e)}(u), \tilde{f}^{+}_{G(e)}(u), T^{-}_{G(e)}(u), \tilde{f}^{-}_{G(e)}(u)\rangle \\ : \forall e\tilde{\epsilon}B, u\tilde{\epsilon}X \end{matrix} \right\}$$

be two bipolar vague soft expert sets then their intersection is defined as:

$$\langle\langle H, \bar{E}_1\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle(u) = \begin{pmatrix} min(T^{+}_{H(e)}(u), T^{+}_{G(e)}(u)), max(\tilde{f}^{+}_{H(e)}(u), \tilde{f}^{+}_{G(e)}(u)) \\ max(T^{-}_{H(e)}(u), T^{-}_{G(e)}(u)), min(\tilde{f}^{-}_{H(e)}(u), \tilde{f}^{-}_{G(e)}(u)) \end{pmatrix}$$

Example 11. *Let $\langle\langle H, \bar{E}_1\rangle\rangle$ and $\langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle$ be two bipolar vague soft expert sets*

$$\langle\langle H, \bar{E}_1\rangle\rangle = \left\{ \begin{matrix} \left[\begin{matrix} (e_1, p, 1), \langle u_1, 02\times 10^{-1}, 08\times 10^{-1}, -04\times 10^{-1}, -05\times 10^{-1}\rangle, \langle u_3, 02\times 10^{-1}, 05\times 10^{-1}, \\ -02\times 10^{-1}, -04\times 10^{-1}\rangle, \end{matrix} \right] \\ \left[\begin{matrix} (e_1, q, 1), \langle u_1, 05\times 10^{-1}, 06\times 10^{-1}, -02\times 10^{-1}, -03\times 10^{-1}\rangle, \langle u_2, 08\times 10^{-1}, 03\times 10^{-1}, \\ -02\times 10^{-1}, -01\times 10^{-1}\rangle, \end{matrix} \right] \end{matrix} \right\}$$

$$\langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle = \left\{ \left[\begin{matrix} (e_1, p, 1), \langle u_1, 01\times 10^{-1}, 02\times 10^{-1}, -03\times 10^{-1}, -04\times 10^{-1}\rangle, \langle u_2, 04\times 10^{-1}, 08\times 10^{-1}, \\ -01\times 10^{-1}, -05\times 10^{-1}\rangle, \end{matrix} \right] \right\}$$

Therefor $\langle\langle H, \bar{E}_1\rangle\rangle \tilde{\cap} \langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle = \langle\langle R, \bar{C}\rangle\rangle$

$$\langle\langle R, \bar{C}\rangle\rangle = \left\{ \left[(e_1, p, 1), \langle u_1, 01\times 10^{-1}, 08\times 10^{-1}, -03\times 10^{-1}, -05\times 10^{-1}\rangle \right] \right\}$$

Definition 43. *Let bi-polar vague soft expert set $\langle\langle X, E\rangle\rangle$ be family of all bipolar vague soft expert sets over X and $J^{BVSES} \tilde{\subseteq} BVSE$ set $\langle\langle X_{absolute}, E\rangle\rangle$, then J^{BVSES} is said to be a bipolar vague soft expert topology $\langle\langle BVSET\rangle\rangle$ on X . if*

1. $\langle\langle \tilde{f}_{null}, E\rangle\rangle$ *and* $\langle\langle X_{absolute}, E\rangle\rangle \ \tilde{\epsilon}\ J^{BVSES}$.

2. *The union of any number of $\langle\langle BVSE\rangle\rangle$ sets in $J^{BVSES}\ \tilde{\epsilon}\ J^{BVSES}$*

3. *The intersection of finite number of $\langle\langle BVSE\rangle\rangle$ sets in $J^{BVSES}\ \tilde{\epsilon}\ J^{BVSES}$*

Then $\langle\langle X_{absolute}, J^{BVSES}, E\rangle\rangle$ is said to be a $J^{BVSESTS}$ over X.

Definition 44. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle$ be a bi-polar vague soft expert set set over X then $\langle\langle \tilde{f}_2, \bar{E}_2\rangle\rangle$ is said to be bi-polar vague soft expert set closed set iff its complemt is a $\langle\langle BVSE\rangle\rangle$ open set.*

Definition 45. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space and $\langle\langle \tilde{f}, \bar{E}\rangle\rangle$ be a $\langle\langle BVSE\rangle\rangle$ set over X then $\langle\langle \tilde{f}, \bar{E}\rangle\rangle$ is called bipolar vague soft expert*

1. *Semi-open if $\langle\langle \tilde{f}, \bar{E}\rangle\rangle \tilde{\subseteq} BVEScl(BVSEint\langle\langle \tilde{f}, \bar{E}\rangle\rangle)$ and BVSE semi-close if $\langle\langle \tilde{f}, \bar{E}\rangle\rangle \tilde{\supseteq} BVESinterior(BVSEcl\langle\langle \tilde{f}, \bar{E}\rangle\rangle)$*

2. *Pre-open if $\langle\langle \tilde{f}, \bar{E}\rangle\rangle \tilde{\subseteq} BVESinterior(BVSEcl\langle\langle \tilde{f}, \bar{E}\rangle\rangle)$ and BVSE pre close if $\langle\langle \tilde{f}, \bar{E}\rangle\rangle \tilde{\supseteq} BVEScl(BVSEinterior\langle\langle \tilde{f}, \bar{E}\rangle\rangle)$.*

3. *$\alpha-$open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVESinterior(BVSEcl(BVSEinterior)\langle\langle\tilde{f},\bar{E}\rangle\rangle)$ *and BVSE* $\alpha_c lose$ *if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVEScl(BVSEint(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle))$.

4. *$\beta-$ open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVEScl(BVESinterior(BVSEcl)\langle\langle\tilde{f},\bar{E}\rangle\rangle)$ *and BVSE $\beta-$ open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVESinterior(BVSEcl(BVESinterior\langle\langle\tilde{f},\bar{E}\rangle\rangle))$.

5. *b−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVEScl(BVESinterior\langle\langle\tilde{f},\bar{E}\rangle\rangle)\,\tilde{\cup}\, BVESinterior(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle)$ *and BVSE b−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVESinterior(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle)\,\tilde{\cap}\, BVEScl(BVESinterior\langle\langle\tilde{f},\bar{E}\rangle\rangle)$.

6. *∗b−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVEScl(BVESinterior\langle\langle\tilde{f},\bar{E}\rangle\rangle)\,\tilde{\cap}\, BVESinterior(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle)$ *and BVSE ∗b−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVESinterior(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle)\,\tilde{\cup}\, BVEScl(BVESinterior\langle\langle\tilde{f},\bar{E}\rangle\rangle)$.

7. *b∗∗−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVESint(BVSEcl(BVSEint\langle\langle\tilde{f},\bar{E}\rangle\rangle))\,\tilde{\cup}\, BVEScl(BVSEint(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle))$ *and BVSE $b**-$ open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVEScl(BVSEint(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle))\,\tilde{\cap}$ $BVESint(BVSEcl(BVSEint\langle\langle\tilde{f},\bar{E}\rangle\rangle))$.

8. *∗∗b−open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\subseteq} BVESint(BVSEcl(BVSEint\langle\langle\tilde{f},\bar{E}\rangle\rangle))\,\tilde{\cap}\, BVEScl(BVSEint(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle))$ *and BVSE $**b-$ open if* $\langle\langle\tilde{f},\bar{E}\rangle\rangle \tilde{\supseteq} BVEScl(BVSEint(BVSEcl\langle\langle\tilde{f},\bar{E}\rangle\rangle))\,\tilde{\cup}$ $BVESint(BVSEcl(BVSEint\langle\langle\tilde{f},\bar{E}\rangle\rangle))$.

Proposition 5. *Let* $\langle\langle X, J^{BVSES}, E\rangle\rangle$ *be a bi-polar vague soft expert set topological space over* X *. Then*

1. $\langle\langle\tilde{f}_{null},\bar{E}\rangle\rangle$, $\langle\langle X_{absolute},E\rangle\rangle$ *are* $\langle\langle BVSE\rangle\rangle$ *p- closed sets over* X.

2. *The intersection of any number of* $\langle\langle BVSE\rangle\rangle$ *p- closed sets* $\langle\langle \check{C}S\rangle\rangle$ *is a* $\langle\langle BVS\rangle\rangle$ *s-closed sets* $\langle\langle \check{C}S\rangle\rangle$ *over* X.

3. *The union of finite number of* $\langle\langle BVSE\rangle\rangle$ *p- closed sets* $\langle\langle \check{C}S\rangle\rangle$ *is a* $\langle\langle BVSE\rangle\rangle$ *p-closed sets* $\langle\langle \check{C}S\rangle\rangle$ *over* X.

Proof. Given $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a $\langle\langle BVSETS\rangle\rangle$ over X.

1. Since $\langle\langle\tilde{f}_{null},\bar{E}\rangle\rangle$ and $\langle\langle X_{absolute},E\rangle\rangle$ are $\langle\langle BVSE\rangle\rangle$ open sets and hence p-open sets because these sets belong to J^{BVSES} . We see that the complement of $\langle\langle\tilde{f}_{null},\bar{E}\rangle\rangle$ is $\langle\langle X_{absolute},E\rangle\rangle$ which is open and hence it is s-open. This implies that $\langle\langle\tilde{f}_{null},\bar{E}\rangle\rangle$ is p-closed and hence it is p-closed. Similarly, the complement of $\langle\langle X_{absolute},E\rangle\rangle$ is $\langle\langle\tilde{f}_{null},\bar{E}\rangle\rangle$ which is open and hence p-open. This implies that $\langle\langle X_{absolute},E\rangle\rangle$ is closed and hence it is s-closed over X

2. Suppose $\{(\tilde{f},E)_i : i\tilde{\epsilon}I\}$ be collection of s-closed subsets of X then $(\tilde{f},E)_i$ is p-closed for all $i\tilde{\epsilon}I$ this implies that $(\tilde{f},E)_i^c$ is s-open for all $i\tilde{\epsilon}I$. Since the union of any number of p-open sets is p-open, so $\bigcup_{i\in I}$ $(\tilde{f},E)_i^c$ is p-open this implies thatc is p-open this implies that $\bigcap_{i\in I}(\tilde{f},E)_i$ is p-closed.

3. Let $(\tilde{f},E)_1$, $(\tilde{f},E)_2$, $(\tilde{f},E)_3$, $(\tilde{f},E)_4$, ..., $(\tilde{f},E)_n$, are any finite number of p-closed sets, then $(\tilde{f},E)_i$ is p-closed for all $i = 1,2,3,...,n$ this implies $(\tilde{f},E)_i^c$ is s-open for all $i = 1,2,3,...,n$ but the intersection of any finite number of s-open sets is p-open, so $\bigcap_{i\in I}^{n}$ $(\tilde{f},E)_i^c$ is p-open. This implies that $(\bigcup_{i\in I}(\tilde{f},E)_i)^c$ is p-open this implies that $\bigcup_{i\in I}(\tilde{f},E)_i$ is p-closed.

Definition 46. *The family of all $\langle\langle BVSE\rangle\rangle$ sets over X is denoted by $BVSES\langle\langle X_{absolute}, E\rangle\rangle$.*

1. *If $J^{BVSES} = \{\langle\langle \tilde{f}_{null}, E\rangle\rangle, \langle\langle X_{absolute}, E\rangle\rangle\}$, , then J^{BVSES} is said to be $\langle\langle BVSE\rangle\rangle$ indiscrete topology (X, J^{BVSES}, E) is said to be a $\langle\langle BVSE\rangle\rangle$ indiscrete topological space over X.*
2. *If $J^{BVSES} = BVSS\langle\langle M_{absolute}, E\rangle\rangle$ then J^{BVSES} is said to be $\langle\langle BVSE\rangle\rangle$ discrete topology. (X, J^{BVSES}, E) is said to be a $\langle\langle BVSE\rangle\rangle$ discrete topological space over X.*

Proposition 6. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ and $\langle\langle X, J_2^{BVSES}, E\rangle\rangle$ be two bi-polar vague soft expert set topological space over X . Then $\langle\langle X, J_1^{BVSES}, \tilde{\bigcap} J_2^{BVSES}, E\rangle\rangle$ is $\langle\langle BVSETS\rangle\rangle$ over X.*

Proof.

1. since $\langle\langle \tilde{f}_{null}, E\rangle\rangle$, $\langle\langle X_{absolute}, E\rangle\rangle$ $\tilde{\epsilon} J^{BVSES}$ and $\langle\langle \tilde{f}_{null}, E\rangle\rangle$, $\langle\langle X_{absolute}, E\rangle\rangle$ $\tilde{\epsilon} J_2^{BVSES}$ then $\langle\langle \tilde{f}_{null}, E\rangle\rangle$, $\langle\langle X_{absolute}, E\rangle\rangle$ $\tilde{\epsilon} J_1^{BVSES} \tilde{\bigcap} J_2^{BVSES}$
2. Let $\{\langle\langle \tilde{f}, E\rangle\rangle : i \in I\}$ be a family of $\langle\langle BVSES\rangle\rangle$ sets in $J_1^{BVSES} \tilde{\bigcap} J_2^{BVSES}$. Then $(\tilde{f}_i, E)$ $\tilde{\epsilon}$ J_1^{BVSES} . $(\tilde{f}_i, E)$ $\tilde{\epsilon}$ J_2^{BVSES} $\forall i \tilde{\epsilon} I$, so $\coprod_{i\in I} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ J_1^{BVSES} and $\coprod_{i\in I} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ J_2^{BVSES} Thus $\coprod_{i\in I} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ $J_1^{BVSES} \tilde{\bigcap} J_2^{BVSES}$.
3. Let $\{\langle\langle \tilde{f}, E\rangle\rangle : i = \overline{1, n}\}$ be a family of finite number of $\langle\langle BVSES\rangle\rangle$ sets in $J_1^{BVSES} \tilde{\bigcap} J_2^{BVSES}$. Then $(\tilde{f}_i, E)$ $\tilde{\epsilon}$ J_1^{BVSES} . $(\tilde{f}_i, E)$ $\tilde{\epsilon}$ J_2^{BVSES} $\forall i = \overline{1, n}$, so $\prod_{i=1}^{n} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ J_1^{BVSES} and $\prod_{i=1}^{n} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ J_2^{BVSES} Thus $\prod_{i=1}^{n} \langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ $J_1^{BVSES} \tilde{\bigcap} J_2^{BVSES}$.

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Definition 47. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ is a bi-polar vague soft expert set topological space over X , $\langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ BVSES $\langle\langle X, E\rangle\rangle$ be a $\langle\langle BVSES\rangle\rangle$ set. Then, $\langle\langle BVSES\rangle\rangle$ interior of $\langle\langle \tilde{f}, E\rangle\rangle$, , denoted as $\langle\langle \tilde{f}, E\rangle\rangle^o$, is defined as $\langle\langle BVSES\rangle\rangle$ union of all $\langle\langle BVSES\rangle\rangle$ p-open subsets of $\langle\langle \tilde{f}, E\rangle\rangle$. Clearly, $\langle\langle \tilde{f}, E\rangle\rangle^o$ is biggest $\langle\langle BVSES\rangle\rangle$ p-open set contained by $\langle\langle \tilde{f}, E\rangle\rangle$*

Theorem 1. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ is a bi-polar vague soft expert set topological space over X , $\langle\langle \tilde{f}, E\rangle\rangle$ $\tilde{\epsilon}$ BVSES $\langle\langle X, E\rangle\rangle$. $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVSES\rangle\rangle$ p-open set iff $\langle\langle \tilde{f}, E\rangle\rangle = \langle\langle \tilde{f}, E\rangle\rangle^o$*

Proof. Let $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVSES\rangle\rangle$ p-open set then the biggest $\langle\langle BVSES\rangle\rangle$ p-open set that is contained by $\langle\langle \tilde{f}, E\rangle\rangle$ is equal to $\langle\langle \tilde{f}, E\rangle\rangle$. Hence , $\langle\langle \tilde{f}, E\rangle\rangle = \langle\langle \tilde{f}, E\rangle\rangle^o$
Conversely, $\langle\langle \tilde{f}, E\rangle\rangle^o$ is a $\langle\langle BVSES\rangle\rangle$ p-open set, if $\langle\langle \tilde{f}, E\rangle\rangle = \langle\langle \tilde{f}, E\rangle\rangle^o$, , then $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVSES\rangle\rangle$ set.

Theorem 2. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ is a bi-polar vague soft expert set topological space over X. $\langle\langle \tilde{f}_1, E\rangle\rangle$, $\langle\langle \tilde{f}_2, E\rangle\rangle$ $\tilde{\epsilon}$ BVSES $\langle\langle X, E\rangle\rangle$. Then,*

1. $[\langle\langle \tilde{f}_1, E\rangle\rangle^o]^o = \langle\langle \tilde{f}_1, E\rangle\rangle^o$,
2. $\langle\langle \tilde{f}_{null}, E\rangle\rangle^o = \langle\langle \tilde{f}_{null}, E\rangle\rangle$ *and* $\langle\langle X_{absolute}, E\rangle\rangle^o = \langle\langle X_{absolute}, E\rangle\rangle$,
3. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \langle\langle \tilde{f}_2, E\rangle\rangle \Longrightarrow \langle\langle \tilde{f}_1, E\rangle\rangle^o \tilde{\subseteq} \langle\langle \tilde{f}_2, E\rangle\rangle^o$,
4. $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]^o = \langle\langle \tilde{f}_1, E\rangle\rangle^o \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle^o$,
5. $\langle\langle \tilde{f}_1, E\rangle\rangle^o \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle^o \tilde{\subseteq} [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]^o$.

Proof.

1. Let $\langle\langle \tilde{f}_1, E\rangle\rangle^o = \langle\langle \tilde{f}_2, E\rangle\rangle$. Then $\langle\langle \tilde{f}_2, E\rangle\rangle \,\tilde{\epsilon}\, \tau^{BN}$ iff $\langle\langle \tilde{f}_2, E\rangle\rangle = \langle\langle \tilde{f}_2, E\rangle\rangle^o$. So, $[\langle\langle \tilde{f}_1, E\rangle\rangle^o]^o = \langle\langle \tilde{f}_1, E\rangle\rangle^o$,

2. Since $\langle\langle \tilde{f}_{null}, E\rangle\rangle$ and $\langle\langle X_{absolute}, E\rangle\rangle$ are always $\langle\langle BVSES\rangle\rangle$ p-open this implies $\langle\langle \tilde{f}_{null}, E\rangle\rangle^o = \langle\langle \tilde{f}_{null}, E\rangle\rangle$ and $\langle\langle X_{absolute}, E\rangle\rangle^o = \langle\langle X_{absolute}, E\rangle\rangle$ because BVSES $\langle\langle \tilde{f}, E\rangle\rangle$ of $\langle\langle BVSES\rangle\rangle$ p-open if $\langle\langle \tilde{f}_1, E\rangle\rangle^o = \langle\langle \tilde{f}_2, E\rangle\rangle$.

3. Let $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle$, $\langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle$. Since $\langle\langle \tilde{f}_2, E\rangle\rangle^o$ is the biggest $\langle\langle BVSES\rangle\rangle$ p-open set covered in $\langle\langle \tilde{f}_2, E\rangle\rangle$ so, $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o$.

4. Since $\langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle$, then $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle^o$, $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o$, and so $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o$. On the other hand, since $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_2, E\rangle\rangle$, then $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle$ besides $[\langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle]^o \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle$ and it is the biggest $\langle\langle BVSES\rangle\rangle$ p-open set. Therefore, $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, [\langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle]^o$ Thus, $[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]^o = \langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o$

5. Since $\langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcup}\, \langle\langle \tilde{f}_2, E\rangle\rangle$ and $\langle\langle \tilde{f}_2, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}_1, E\rangle\rangle \,\tilde{\bigcup}\, \langle\langle \tilde{f}_2, E\rangle\rangle$ then $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\subseteq}\, [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]^o$ and $\langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]^o$. Therefore, $\langle\langle \tilde{f}_1, E\rangle\rangle^o \,\tilde{\bigcap}\, \langle\langle \tilde{f}_2, E\rangle\rangle^o \,\tilde{\subseteq}\, [\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]^o$.

Definition 48. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X . $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\epsilon}\, BVSES\, \langle\langle X, E\rangle\rangle$ be a $\langle\langle BVSES\rangle\rangle$ set then, $\langle\langle BVSES\rangle\rangle$ p-closure of $\langle\langle \tilde{f}, E\rangle\rangle$, , denoted $\overline{\langle\langle \tilde{f}, E\rangle\rangle}$, , is defined as $\langle\langle BVSES\rangle\rangle$ soft intersection of all $\langle\langle BVSES\rangle\rangle$ p-closed supersets of $\langle\langle \tilde{f}, E\rangle\rangle$.*
Clearly, $\overline{\langle\langle \tilde{f}, E\rangle\rangle}$ is the smallest $\langle\langle BVSES\rangle\rangle$ p-closed set covering by $\langle\langle \tilde{f}, E\rangle\rangle$

Definition 49. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\epsilon}\, BVSES\, \langle\langle X, E\rangle\rangle$ be a $\langle\langle BVSES\rangle\rangle$ set then the boundary of $\langle\langle \tilde{f}, E\rangle\rangle$ is dented by $Fr\langle\langle \tilde{f}, E\rangle\rangle$ is defined as a $\langle\langle BVSES\rangle\rangle$ point $\tau^m_{(\rho_1,\rho_2)}$ is called boundary of $\langle\langle \tilde{f}, E\rangle\rangle$ if every $\langle\langle BVSES\rangle\rangle$ p- open set containing $\tau^m_{(\rho_1,\rho_2)}$ contains at least one point of $\langle\langle \tilde{f}, E\rangle\rangle$ and least one $\langle\langle BVSES\rangle\rangle$ point of $\langle\langle \tilde{f}, E\rangle\rangle^c$.*

Definition 50. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\epsilon}\, BVSES\, \langle\langle X, E\rangle\rangle$ be a $\langle\langle BVSES\rangle\rangle$ set then $\langle\langle BVSES\rangle\rangle$ exterior of $\langle\langle \tilde{f}, E\rangle\rangle$ s dented by $Ext\langle\langle \tilde{f}, E\rangle\rangle$ is defined as a $\langle\langle BVSES\rangle\rangle$ point $\tau^m_{(\rho_1,\rho_2)}$ is called exterior of $\langle\langle \tilde{f}, E\rangle\rangle$ if $\tau^m_{(\rho_1,\rho_2)}$ if $\langle\langle BVSES\rangle\rangle$ point $\tau^m_{(\rho_1,\rho_2)}$ is $\langle\langle BVSES\rangle\rangle$ interior of $\langle\langle \tilde{f}, E\rangle\rangle^c$ that is there exists $\langle\langle BVSES\rangle\rangle$ p- open set $\langle\langle \tilde{g}, E\rangle\rangle$ such that $\tau^m_{(\rho_1,\rho_2)} \,\tilde{\epsilon}\, \langle\langle \tilde{g}, E\rangle\rangle \,\tilde{\subseteq}\, \langle\langle \tilde{f}, E\rangle\rangle^c$.*

Theorem 3. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\epsilon}\, BVSES\, \langle\langle X, E\rangle\rangle$. $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVSES\rangle\rangle$ p- closed set iff $\langle\langle \tilde{f}, E\rangle\rangle = \overline{\langle\langle \tilde{f}, E\rangle\rangle}$.*

Proof.
If $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVSES\rangle\rangle$ p- closed set then this means that $\langle\langle \tilde{f}, E\rangle\rangle$ contains all of its limit points that $\langle\langle \tilde{f}, E\rangle\rangle^{/} \,\tilde{\subseteq}\, \langle\langle \tilde{f}, E\rangle\rangle$ this implies that $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\bigcup}\, \langle\langle \tilde{f}, E\rangle\rangle^{/} = \langle\langle \tilde{f}, E\rangle\rangle$ this implies that $\overline{\langle\langle \tilde{f}, E\rangle\rangle} = \langle\langle \tilde{f}, E\rangle\rangle$ and conversely, let $\overline{\langle\langle \tilde{f}, E\rangle\rangle} = \langle\langle \tilde{f}, E\rangle\rangle$ this implies that $\langle\langle \tilde{f}, E\rangle\rangle \,\tilde{\bigcup}\, \langle\langle \tilde{f}, E\rangle\rangle^{/} = \langle\langle \tilde{f}, E\rangle\rangle$ this implies that $\langle\langle \tilde{f}, E\rangle\rangle^{/} \,\tilde{\subseteq}\, \langle\langle \tilde{f}, E\rangle\rangle$ this implies that $\langle\langle \tilde{f}, E\rangle\rangle$ is p-closed.

Theorem 4. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle \tilde{f}_1, E\rangle\rangle$, $\langle\langle \tilde{f}_2, E\rangle\rangle \,\tilde{\epsilon}\, BVSES\, \langle\langle X, E\rangle\rangle$. Then,*

1. $\overline{[\overline{\langle\langle \tilde{f}_1, E\rangle\rangle}]} = \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$,

2. $\overline{\langle\langle \tilde{f}_{null}, E\rangle\rangle} = \langle\langle \tilde{f}_{null}, E\rangle\rangle$ *and* $\overline{\langle\langle X_{absolute}, E\rangle\rangle} = \langle\langle X_{absolute}, E\rangle\rangle$,

3. $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \langle\langle \tilde{f}_2, E\rangle\rangle \Longrightarrow \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\subseteq} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$,

4. $\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]} \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcap} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$

5. $\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]} = \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcup} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$

Proof.

1. Let $\overline{\langle\langle \tilde{f}_1, E\rangle\rangle} = \langle\langle \tilde{f}_2, E\rangle\rangle$, then $\langle\langle \tilde{f}_2, E\rangle\rangle$ is a $\langle\langle BVSETS\rangle\rangle$ p-closed set. Hence $\langle\langle \tilde{f}_2, E\rangle\rangle = \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$. So, $\overline{[\overline{\langle\langle \tilde{f}_1, E\rangle\rangle}]} = \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$.

2. By theorem 3 Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a $\langle\langle BVSETS\rangle\rangle$ over X, $\langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon}$ BVSES $\langle\langle X, E\rangle\rangle$. $\langle\langle \tilde{f}, E\rangle\rangle$ is a $\langle\langle BVES\rangle\rangle$ p-closed set iff $\langle\langle \tilde{f}_1, E\rangle\rangle = \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$. Since $\langle\langle \tilde{f}_{null}, E\rangle\rangle$ and $\langle\langle X_{absolute}, E\rangle\rangle$ are p-closed sets. So using this results we have $\overline{\langle\langle \tilde{f}_{null}, E\rangle\rangle} = \langle\langle \tilde{f}_{null}, E\rangle\rangle$ and $\overline{\langle\langle X_{absolute}, E\rangle\rangle} = \langle\langle X_{absolute}, E\rangle\rangle$.

3. It is known that $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$, $\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$. Since $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$. Since $\overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$ is the smallest $\langle\langle BVSETS\rangle\rangle$ p closed set covering then so, $\overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\subseteq} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$.

4. Since $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle$, $\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle$, then $\overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\subseteq} \overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]}$, $\overline{\langle\langle \tilde{f}_2, E\rangle\rangle} \tilde{\subseteq} \overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]}$ and so, $\overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcup} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle} \tilde{\subseteq} \overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]}$.
   Contrary-wise,
   since $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle}$, $\langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$ then $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcup} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$. Besides ,$\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]}$ is the smallest $\langle\langle BVSETS\rangle\rangle$ p-closed set covering $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle$.
   Therefore, ,$\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcup} \langle\langle \tilde{f}_2, E\rangle\rangle]} \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcup} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$.
   Since $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcap} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$ and $\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]}$ is then smallest $\langle\langle BVSETS\rangle\rangle$ p-closed set covering $\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle$, then $\overline{[\langle\langle \tilde{f}_1, E\rangle\rangle \tilde{\bigcap} \langle\langle \tilde{f}_2, E\rangle\rangle]} \tilde{\subseteq} \overline{\langle\langle \tilde{f}_1, E\rangle\rangle} \tilde{\bigcap} \overline{\langle\langle \tilde{f}_2, E\rangle\rangle}$

Theorem 5. *Let $\langle\langle M, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over M, $\langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon}$ BVSETS $\langle\langle M, E\rangle\rangle$.*

1. $[\overline{\langle\langle \tilde{f}, E\rangle\rangle}]^c = [\langle\langle \tilde{f}, E\rangle\rangle^c]^o$,

2. $[\langle\langle \tilde{f}, E\rangle\rangle^o]^c = \overline{[\langle\langle \tilde{f}, E\rangle\rangle^c]}$.

Proof.

1. $\overline{\langle\langle\tilde{f},E\rangle\rangle} = \prod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\epsilon}(J^{BVSES})^c : \langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\supseteq}\langle\langle\tilde{f},E\rangle\rangle\}$,
$\Longrightarrow [\overline{\langle\langle\tilde{f},E\rangle\rangle}]^c = [\prod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\epsilon}(J^{BVSES})^c : \langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\supseteq}\langle\langle\tilde{f},E\rangle\rangle \quad \forall i\in I\}]^c$
$= \coprod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\epsilon}(J^{BVSES}) : \langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\subseteq}\langle\langle\tilde{f},E\rangle\rangle^c\} = [\langle\langle\tilde{f},E\rangle\rangle^c]^o$.

2. $\langle\langle\tilde{f},E\rangle\rangle^o = \coprod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\epsilon}(J^{BVSES}) : \langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\subseteq}\langle\langle\tilde{f},E\rangle\rangle \coprod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\epsilon}(J^{BVSES}) : \langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\subseteq}\langle\langle\tilde{f},E\rangle\rangle^c$
$\Longrightarrow [\langle\langle\tilde{f},E\rangle\rangle^o]^c = [\coprod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\epsilon}(J^{BVSES}) : \langle\langle\tilde{f}_i,E\rangle\rangle\tilde{\subseteq}\langle\langle\tilde{f},E\rangle\rangle\}]^c$
$= \prod_{i\in I}\{\langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\epsilon}(J^{BVSES})^c : \langle\langle\tilde{f}_i,E\rangle\rangle^c\tilde{\supseteq}\langle\langle\tilde{f},E\rangle\rangle^c\} = \overline{[\langle\langle\tilde{f},E\rangle\rangle^c]}$.

Theorem 6. *Let $\langle\langle X, J^{BVSES}, E\rangle\rangle$ be a bi-polar vague soft expert set topological space over X, $\langle\langle\tilde{f}_1,E\rangle\rangle$, $\langle\langle\tilde{f}_2,E\rangle\rangle$ $\tilde{\epsilon}$ BVSES $\langle\langle X,E\rangle\rangle$ then*

1. $Ext(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle) = Ext\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Ext\langle\langle\tilde{f}_2,E\rangle\rangle$.

2. $Ext(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)\tilde{\supseteq}Ext\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Ext\langle\langle\tilde{f}_2,E\rangle\rangle$.

3. $Fr(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle g,l\rangle\rangle)\tilde{\subseteq}Fr\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Fr\langle\langle\tilde{f}_2,E\rangle\rangle$.

4. $Fr(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)\tilde{\subseteq}Fr\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Fr\langle\langle\tilde{f}_2,E\rangle\rangle$.

Proof.

1. Since , $Ext(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle) = ((\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle)^c)^o = (\langle\langle\tilde{f}_1,E\rangle\rangle^c\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle^c)^o =$ $(\langle\langle\tilde{f}_1,E\rangle\rangle^c)^o\tilde{\bigcap}(\langle\langle\tilde{f}_2,E\rangle\rangle^c)^9 = Ext\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}Ext\langle\langle g,l\rangle\rangle$

2. $Ext(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle) = ((\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)^c)^o = (\langle\langle\tilde{f}_1,E\rangle\rangle^c\tilde{\bigcup}$ $\langle\langle\tilde{f}_2,E\rangle\rangle^c)^o\tilde{\supseteq}(\langle\langle\tilde{f}_1,E\rangle\rangle^c)^o\tilde{\bigcup}(\langle\langle\tilde{f}_2,E\rangle\rangle^c)^o = Ext\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Ext\langle\langle\tilde{f}_2,E\rangle\rangle$, that is $Ext(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)\tilde{\supseteq}Ext\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Ext\langle\langle\tilde{f}_2,E\rangle\rangle$.

3. $Fr(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle) = \overline{\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle}\tilde{\bigcap}\overline{(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle)^c} =$
$(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle})\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle^c} \tilde{\subseteq} (\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle)\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c}$
$= \{\overline{\langle\langle\tilde{f},l\rangle\rangle}\tilde{\bigcup}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c}\}\tilde{\bigcap}\overline{\langle\langle g,l\rangle\rangle^c} =$
$\{(\overline{\langle\langle\tilde{f}_1,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c})\tilde{\bigcup}(\overline{\langle\langle\tilde{f}_2,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c})\} \tilde{\bigcup} \{\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c}\}$
$= \{Fr\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c}\} \tilde{\bigcup} \{Fr\langle\langle\tilde{f}_2,E\rangle\rangle\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c}\} \tilde{\subseteq} Fr\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Fr\langle\langle\tilde{f}_2,E\rangle\rangle$.

4. $Fr(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle) = \overline{(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)}\tilde{\bigcap}\overline{(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\langle\langle\tilde{f}_2,E\rangle\rangle)^c} \tilde{\subseteq} (\overline{\langle\langle\tilde{f}_1,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle})$
$\tilde{\bigcap}\ \overline{(\langle\langle\tilde{f}_1,E\rangle\rangle^c\tilde{\bigcup}\langle\langle\tilde{f}_2,E\rangle\rangle^c)} = \{(\overline{\langle\langle\tilde{f}_1,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle})\tilde{\bigcap}\overline{\langle\langle\tilde{f}_1,E\rangle\rangle^c}\} \tilde{\bigcup} \{(\overline{\langle\langle\tilde{f}_1,E\rangle\rangle}\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle})\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle^c}\}$
$= \{Fr(\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcap}\overline{\langle\langle\tilde{f}_2,E\rangle\rangle})\}\tilde{\bigcup}\{\overline{\langle\langle\tilde{f}_1,E\rangle\rangle}\tilde{\bigcap}Fr\langle\langle g,l\rangle\rangle\} \tilde{\subseteq} Fr\langle\langle\tilde{f}_1,E\rangle\rangle\tilde{\bigcup}Fr\langle\langle\tilde{f}_2,E\rangle\rangle$.

4. Characterization of few more results in terms of Basis Concerning P-Open Sets

This work develops foundational topological concepts in BVSETS, introducing and analyzing bases, sub-bases, and local bases. It defines first and second BVSE countability, explores separability via countable dense sets, and characterizes BVSE limit points using local bases. Key theorems establish criteria for comparing BVSET topologies and constructing subspace topologies, including p-closure operations. These results enhance the theoretical framework of BVSETS, supporting soft topological modeling under uncertainty and parameterization.

Definition 51. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X *and* B^{BVSES} *be a sub-family of* τ^{BVSES}. B^{BVSES} *is said to be a BVSE base or p-open base or basis for the BVSET* τ^{BVSES} *if given any non-empty BVS*$\langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon} \tau^{BVSES}$ *this implies that there exists* $\mathbf{B_1} \tilde{\subseteq} B^{BVSES}$, *such that* $\langle\langle \tilde{f}, E\rangle\rangle = \bigcup\{B : B \tilde{\epsilon} \mathbf{B_1}\}$. *In other words* B^{BVSES} *is said to be base for BVSET if* $x^e_{(\alpha)} \tilde{\epsilon} \langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon} \tau^{BVSES}$ *implies that there exists* $B \tilde{\epsilon} \mathbf{B}$ *such that* $x^e_{(\alpha)}$ $\tilde{\epsilon} B \tilde{\epsilon} \langle\langle \tilde{f}, E\rangle\rangle$.

Definition 52. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X *and* S^{BVSES} *be a sub-family of* τ^{BVSES} . S^{BVSES} *is said to be a BVSE sub-base or P-open sub-base or basis for the BVSET* τ^{BVSES} *on* X *if finite intersections of the members of* S^{BVSES} *form a base for the BVSET* τ^{BVSES} *on* X . *That is, the union of the members of* S^{BVSES} *give all the members of* τ^{BVSES}. *The elements of* S^{BVSES} *are referred to as sub-basic BPVSE P-open sets. If given any non-empty BVSE* $\langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon} \tau^{BVSES}$ *this implies that there exists* $\mathbf{B_1}$ $\tilde{\epsilon} B^{BVSES}$ *such that* $\langle\langle \tilde{f}, E\rangle\rangle = \bigcup\{B : B \tilde{\epsilon} \mathbf{B_1}\}$ *In other words* B^{BVSES} *is said to be base for BVSET if* $x^e_{(\alpha)} \tilde{\epsilon} \langle\langle \tilde{f}, E\rangle\rangle \tilde{\epsilon} \tau^{BVSES}$ *implies that there exists* $B \tilde{\epsilon} \mathbf{B}$ *such that* $x^e_{(\alpha)} \tilde{\epsilon} B \tilde{\epsilon} \langle\langle \tilde{f}, E\rangle\rangle$.

Definition 53. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X *A family* $\mathbf{B}_\alpha$ *of BVSE p-open subsets of* X X *is said to be BVSE local base at* $x^e_{(\alpha)} \tilde{\epsilon} X$ *for the BVSETS on* X *if*

1. *Any* $B \tilde{\epsilon} B_{x^e_{(\alpha)}} \Longrightarrow x^e_{(\alpha)} \tilde{\epsilon} B$.

2. *Any* $\langle\langle \tilde{f}, E\rangle\rangle \tau^{BVSES}$ *with* $y^e_{(\alpha)} \tilde{\epsilon} \langle\langle \tilde{f}, E\rangle\rangle \Longrightarrow \exists B \tilde{\epsilon} B_{x^e_{(\alpha)}}$ *such that* $y^e_{(\alpha)} \tilde{\epsilon} B \tilde{\subseteq} \langle\langle \tilde{f}, E\rangle\rangle$.

Definition 54. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X. *The space* X *is satisfy the first axioms of BVSE soft countability if* X *has a BVSE countable local base at each* $x^e_{(\alpha)} \tilde{\epsilon} X$. *The BVSE space* X, *in this case, is called first BVSE countable space.*

Definition 55. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X. *The space* X *is satisfy the second axioms of BVSE countability if there exists a BVSE countable base for* τ^{BVSES} *on* X. *The BVSE space* X, *in this case, is called second BVSE countable space. A second BVSE countable space is also called BVSE completely separable space.*

Definition 56. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X. *A property* P^{NSHP} *of* X *is said to be hereditary if the property is possessed by every subspace of* X *e.g., BVSE first countable, BVSE second countable are hereditary properties whereas BVSE p-closed sets, BVSE p- open sets, are not hereditary properties.*

Definition 57. *Let* (X, τ^{BVSES}, E) *be a bi-polar vague soft expert set topological space over* X. *This space is said to be if and only if* X *contains a BVSE countable dense BVSE subset, if and only if there exists BVSE countable subset* $(\tilde{k}, E)$ *of ?? such that* $\overline{(\tilde{k}, E)} = X$

Theorem 7. *Let (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over X and B^{BVSES} be a BVSE basis for τ^{BVSES}. Then, τ^{BVSES} equals to the collection of all BVSE unions of elements of B^{BVSES}.*

Proof.
This is easily seen from the definition of BVSE basis.

Theorem 8. *Let (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over X. A sub-collection B^{BVSES} of τ^{BVSES} is a base for τ^{BVSES} If and only if for each BVSE p-open set $(\tilde{f}, E)$ and each BVSE point $x^e_{(\alpha)}$ in $(\tilde{f}, E)$, there exists a basis elements $B_{x^e_{(\alpha)}}$ such that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$.*

Proof.
Given that (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over X and B^{BVSES} is a collection of BVSE p-open sets. Let B^{BVSES} is base for a BVSET τ^{BVSES} , then by definition every BVSE p-open set $(\tilde{f}, E)$ is the union of some members of B^{BVSES} i.e. $(\tilde{f}, E) = \bigcup_{i\in I} B_i$ where $B_i \tilde{\epsilon} B^{BVSES}$ for all $i \in I$. Let $x^e_{(\alpha)}$ be an arbitrary BVSE point of $(\tilde{f}, E)$, we are to prove that there exists a BVSE basis element $B_{x^e_{(\alpha)}}$ containing $x^e_{(\alpha)}$ such that $B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$. Since $x^e_{(\alpha)} \;\tilde{\epsilon}\; (\tilde{f}, E)$ but $(\tilde{f}, E) = \bigcup_{i\in I} B_i$ implies that $x^e_{(\alpha)} \;\tilde{\epsilon}\; \bigcup_{i\in I} B_i$ implies that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_i$ for some $i \in I$. Let $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_i$ for $i = x^e_{(\alpha)}$, then $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}}$ and $B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; \bigcup_{i\in I} B_i$ as $B_i \;\tilde{\subseteq}\; \bigcup_{i\in I} B_i$ for all i implies that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; \bigcup_{i\in I} B_i$ implies that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$ where $B_{x^e_{(\alpha)}} \;\tilde{\epsilon}\; B^{BVSES}$. Conversely, suppose for each BVSE point $x^e_{(\alpha)}$ of a BVSE p-open set $(\tilde{f}, E)$, there exists BVSE set $x^e_{(\alpha)} \;\tilde{\epsilon}\; B^{BVSES}$ such that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$. We are to prove that B^{BVSES} is a BVSE basis for BVSET τ^{BVSES} and for this we will prove that every BVSE P-open set $(\tilde{f}, E)$ can be written as a union of some members of B^{BVSES}. Since $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}}$ $\tilde{\subseteq}\; (\tilde{f}, E)$ implies that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_{x^e_{(\alpha)}}$ and $B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$ or $\{x^e_{(\alpha)}\} \;\tilde{\subseteq}\; B_{x^e_{(\alpha)}}$ and $B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$ implies that $\bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)} \{x^e_{(\alpha)}\} \;\tilde{\subseteq}\; \bigcup_{i\in I} B_i$ and $\bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)} B_{x^e_{(\alpha)}} \;\tilde{\subseteq}\; \bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)} (\tilde{f}, E)$ implies that $(\tilde{f}, E) \;\tilde{\subseteq}\; \bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)} B_{x^e_{(\alpha)}}$ and $\bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)} B_{x^e_{(\alpha,\beta,\gamma)}} \;\tilde{\subseteq}\; (\tilde{f}, E)$ therefore $(\tilde{f}, E) = \bigcup_{x^e_{(\alpha)} \in (\tilde{f}, E)}$ $B_{x^e_{(\alpha)}}$. Since each $B_{x^e_{(\alpha)}} \;\tilde{\epsilon}\; B^{BVSES}$, so $(\tilde{f}, E)$ is the union of some members of B^{BVSES}. But $(\tilde{f}, E)$ is arbitrary BVSE p-open set, so every BVSE p-open set is the union of some members of B^{BVSES}. Therefore, B^{BVSES} is a BVSE basis for the BVSE τ^{BVSES}.

Theorem 9. *Let (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over X. A sub-collection B^{BVSES} of τ^{BVSES} is a base for τ^{BVSES} If and only if ,*

1. *Every BVSE point of X is in some $B \;\tilde{\epsilon}\; B^{BVSES}$.*

2. *For $B_1, B_2 \;\tilde{\epsilon}\; B^{BVSES}$ and $x^e_{(\alpha)} \;\tilde{\epsilon}\; B_1 \bigcap B_2$, there is a ?? $\tilde{\epsilon}\; B^{BVSES}$ such that $x^e_{(\alpha)} \;\tilde{\epsilon}\; B \;\tilde{\subseteq}\; B_1 \bigcap B_2$.*

Proof. Let B^{BVSES} is a base for τ^{BVSES}.

1. Let $x^e_{(\alpha)}$ be an arbitrary BVSE point of X.
Since of X is BVSE p-open set, so there exists $B_{x^e_{(\alpha)}} \;\tilde{\epsilon}\; B^{BVSES}$ such that $x^e_{(\alpha)}$ $x^e_{(\alpha)}$ $B_{x^e_{(\alpha)}}$ $\tilde{\subseteq}$ X implies that $\bigcup_{x^e_{(\alpha)\in X}} \{x^e_{(\alpha)}\} \;\tilde{\subseteq}\; \bigcup_{x^e_{(\alpha)\in X}} B_{x^e_{(\alpha)}} \;\tilde{\subseteq}$ X implies that $\bigcup_{x^e_{(\alpha)\in X}} B_{x^e_{(\alpha)}} \;\tilde{\subseteq}$ X this shows that each BVSE point of X is some B $\tilde{\epsilon}\; B^{BVSES}$

2. For $B_1, B_2 \tilde{\epsilon} B^{BVSES}$ and $x^e_{(\alpha)} \tilde{\epsilon} B_1 \bigcap B_2$ since B_1 and B_2 are BVSE p-open set, so $B_1 \bigcap B_2$ is also a BVSE P-open set, and therefore there exists B $\tilde{\epsilon} B^{BVSES}$ such that $x^e_{(\alpha)}$ $\tilde{\epsilon}$ B $\tilde{\subseteq} B_1 \bigcap B_2$. Conversely let (1) and (2) are true then we prove that A sub-collection B^{BVSES} of τ^{BVSES} is a base for τ^{BVSES} . For this we prove that sub-collection B^{BVSES} of τ^{BVSES} satisfy the three conditions of BVSET. For this we proceed as fallows the BVSE null set $0_{(\pi,h)}$ being the union of BVSE null collection of BVSE subsets in B^{BVSES} is in τ^{BVSES}. Since X is BVSE P-open, so for any $x^e_{(\alpha)} \tilde{\epsilon}$ X the condition (1), gives a $B_{x^e_{(\alpha)}} \tilde{\epsilon}$ B^{BVSES} such that $x^e_{(\alpha)} \tilde{\epsilon} B_{x^e_{(\alpha)}} \tilde{\subseteq}$ X then $\bigcup_{x^e_{(\alpha)\in X}} \tilde{\subseteq} \bigcup_{x^e_{(\alpha)\in X}} B_{x^e_{(\alpha)}} \tilde{\subseteq}$ X implies that X $\tilde{\subseteq}$ $\bigcup_{x^e_{(\alpha)\in X}} B_{x^e_{(\alpha)}} \tilde{\subseteq}$ X implies that X $= \bigcup_{x^e_{(\alpha)\in X}} B_{x^e_{(\alpha)}}$ this shows that X, being the union of members of B^{BVSES}, is in τ^{BVSES}. Next we proceed for the second condition as fallows. The union of any number of members of τ^{BVSES}, being the union of members of B^{BVSES} is in τ^{BVSES}. Next we proceed for the third condition as fallows. Let $(\tilde{f}, E)_1$, $(\tilde{f}, E)_2 \tilde{\epsilon}$ τ^{BVSES}, then by definition of τ^{BVSES} we have $(\tilde{f}, E)_1 = \bigcup B_\alpha$, $(\tilde{f}, E)_2 = \bigcup B_\beta$ for some α, β ranging over sub-collection of B^{BVSES}. Therefore,

$$(\tilde{f}, E)_1 \tilde{\cap} (\tilde{f}, E)_2 = \tilde{\cup} B_\alpha \tilde{\cap} B_\beta$$

$$\Longrightarrow (\tilde{f}, E)_1 \tilde{\cap} (\tilde{f}, E)_2 = \tilde{\cap} B_\alpha \tilde{\cup} B_\beta, ...(i)$$

By the (2), for any $x^e_{(\alpha)} \tilde{\epsilon} (\tilde{f}, E)_1 \tilde{\cap} (\tilde{f}, E)_2$, there is a $B_{x^e_{(\alpha)}} \tilde{\epsilon} B^{BVSES}$ such that $x^e_{(\alpha)} \tilde{\epsilon}$ $B_{x^e_{(\alpha)}} \tilde{\subseteq} B_\alpha \tilde{\cup} B_\beta$.

implies that,

$\bigcup_{x^e_{(\alpha)\in B_\alpha \cap B_\beta}} \{x^e_{(\alpha,\beta,\gamma)}\} \tilde{\subseteq} \bigcup_{x^e_{(\alpha)\in B_\alpha \cap B_\beta}} B_{x^e_{(\alpha)}} \tilde{\subseteq} B_\alpha \tilde{\cap} B_\beta$

$B_\alpha \tilde{\cap} B_\beta \tilde{\subseteq} \bigcup_{x^e_{(\alpha)\in B_\alpha \cap B_\beta}} B_{x^e_{(\alpha,\beta,\gamma)}} \tilde{\subseteq} B_\alpha \tilde{\cap} B_\beta \Longrightarrow B_\alpha \tilde{\cap} B_\beta = \bigcup_{x^e_{(\alpha)\in B_\alpha \cap B_\beta}} B_{x^e_{(\alpha)}}$.

Putting this value in (i) , we have $(\tilde{f}, E)_1 \tilde{\cap} (\tilde{f}, E)_2 = \tilde{\cup} B_\alpha \tilde{\cap} B_\beta = \tilde{\cup}(\bigcup_{x^e_{(\alpha)\in B_\alpha \cap B_\beta}} B_{x^e_{(\alpha)}})$

This shows that $(\tilde{f}, E)_1 \tilde{\cap} (\tilde{f}, E)_2$ is the union of members of B^{BVSES} , so in in τ^{BVSES} . In the same way, we prove that the intersection of any finite number of members of τ^{BVSES} is in τ^{BVSES} . Since all the conditions of the is BVSET are satisfied, so τ^{BVSES} is BVSET on X. Consequently, B^{BVSES} is a BVSE p-base for τ^{BVSES} .

Theorem 10. *Let (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over ??. A BVSE point $x^e_{(\alpha)}$ in BVSETS is a BVSE limit point of $(\tilde{F}, E) \tilde{\subseteq} X$ if and only if every member of any BVSE p-local base $B_{x^e_{(\alpha)}}$ at $x^e_{(\alpha)}$ contains a point of $(\tilde{F}, E)$ different from $x^e_{(\alpha)}$.*

Proof. Let (X, τ^{BVSES}, E) be a bi-polar vague soft expert set topological space over ?? and $(\tilde{F}, E) \tilde{\subseteq}$ X . Let $x^e_{(\alpha)} \tilde{\epsilon}$ X be a BVSE limit point of $(\tilde{F}, E)$. Let $B_{x^e_{(\alpha)}}$ be a local base at $x^e_{(\alpha)}$ for BVSET on X.

To prove that $(B - x^e_{(\alpha)}) \tilde{\cap} (\tilde{F}, E) \neq 0_{(X,E)}$ $\forall$ B $\tilde{\epsilon} B_{x^e_{(\alpha)}}$. By hypothesis, $x^e_{(\alpha)}$ BVSE limit point of $(\tilde{F}, E)$ and so $((\tilde{F}, E) - x^e_{(\alpha)}) \tilde{\cap} (\tilde{F}, E) \neq 0_{(X,E)}$ $\forall$ BVSE p-open sets $(\tilde{f}, E)$ that is $(\tilde{f}, E) \tilde{\epsilon} \tau^{BVSES}$. By definition of BVSE p-local base, $(\tilde{f}, E) \tilde{\epsilon} B_{x^e_{(\alpha)}}$ implies that $(\tilde{f}, E) \tilde{\epsilon} \tau^{BVSES}$ with $x^e_{(\alpha)} \tilde{\epsilon}$ $(\tilde{f}, E)$ then the foregoing statement takes the form $((\tilde{F}, E) - x^e_{(\alpha)}) \tilde{\cap} (\tilde{F}, E) \neq 0_{(X,E)}$ $\forall$ $(\tilde{f}, E) \tilde{\epsilon}$ $B_{x^e_{(\alpha)}}$ that is $(B - x^e_{(\alpha)}) \tilde{\cap} (\tilde{F}, E) \neq 0_{(X,E)}$ $\forall$ B $\tilde{\epsilon} B_{x^e_{(\alpha)}}$. Conversely, suppose $B_{x^e_{(\alpha)}}$ be BVSE p-local base at $x^e_{(\alpha)} \tilde{\epsilon}$ X for some BVSET τ^{BVSES} on X. Also suppose that $(B - x^e_{(\alpha)}) \tilde{\cap} (\tilde{F}, E) \neq 0_{(X,E)}$ $\forall$ B $\tilde{\epsilon} B_{x^e_{(\alpha)}}$. Where $((\tilde{F}, E) \tilde{\epsilon}$ X. Let $(\tilde{f}, E) \tilde{\epsilon} \tau^{BVSES}$ be an arbitrary such that $x^e_{(\alpha)} \tilde{\epsilon} (\tilde{f}, E)$

then by definition of BVSE P-local base there exists B $\tilde{\epsilon}$ $B_{x^e_{(\alpha)}}$ such that $x^e_{(\alpha)}$ $\tilde{\epsilon}$ B $\tilde{\epsilon}$ $((\tilde{F},E)$. consequently, $((\tilde{f},E)-x^e_{(\alpha)})\tilde{\cap}(\tilde{F},E) \tilde{\supseteq} (B-x^e_{(\alpha)})\tilde{\cap}(\tilde{F},E) \neq 0_{(X,E)}$. This implies that $((\tilde{f},E)-x^e_{(\alpha)})\tilde{\cap}(\tilde{F},E) \neq 0_{(X,E)}$, meaning that there by $x^e_{(\alpha)}$ is a BVSE limit point of $(\tilde{F},E)$

Theorem 11. *Let τ_1^{BVSES} and τ_2^{BVSES} be two bi-polar vague soft expert set topological spaces over X generated by BVSE p-bases B_1^{BVSES} and B_2^{BVSES} , respectively. Then $\tau_1^{BVSES} \tilde{\subseteq} \tau_2^{BVSES}$ iff for each $x^e_{(\alpha)}$ $\tilde{\epsilon}$ BVSES (X,E) and for each $(\tilde{B}_1,E) \tilde{\subseteq} B_1^{BVSES}$ containing $x^e_{(\alpha)}$ there exists $(\tilde{B}_2,E) \tilde{\subseteq} B_2^{BVSES}$such that $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_2,E) \tilde{\subseteq} (\tilde{B}_1,E)$.*

Proof. Let $\tau_1^{BVSES} \tilde{\subseteq} \tau_2^{BVSES}$ and $x^e_{(\alpha)} \tilde{\subseteq}$ BVSES (X,E) , $(\tilde{B}_1,E)$ $\tilde{\epsilon}$ B_1^{BVSES} such that $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_1,E)$. Since B_1^{BVSES} is a BVSE p-basis for BVSET τ_1^{BVSES} over X, then $(\tilde{B}_1,E) \tilde{\subseteq} \tau_1^{BVSES} \Longrightarrow x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_1,E)$ $\tilde{\epsilon}$ $B_2^{BVSES} \tilde{\subseteq} \tau_1^{BVSES}$ i.e, $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_1,E)$ $\tilde{\epsilon}$ τ_2^{BVSES}. Since B_2^{BVSES} is a BVSE P-basis for τ_2^{BVSES} , so for $(\tilde{B}_2,E)$ $\tilde{\epsilon}$ B_2^{BVSES} we have $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_2,E) \tilde{\subseteq} (\tilde{B}_1,E)$.
Conversely, assume that the hypothesis holds. Let $(\tilde{F},E)$ $\tilde{\epsilon}$ τ_1^{BVSES} . Since B_1^{BVSES} is a BVSE p-basis for BVSET τ_1^{BVSES} , then for $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{F},E)$ there exist $(\tilde{B}_1,E)$ $\tilde{\epsilon}$ B_1^{BVSES} such that $x^e_{(\alpha)}$ $\tilde{\epsilon}$ $(\tilde{B}_1,E) \tilde{\subseteq} (\tilde{F},E)$. No by hypothesis, there exist $(\tilde{B}_2,E)$ $\tilde{\epsilon}$ B_2^{BVSES} such that $(\tilde{B}_2,E) \tilde{\subseteq} (\tilde{B}_1,E) \Longrightarrow (\tilde{B}_2,E) \tilde{\subseteq} (\tilde{B}_1,E) \tilde{\subseteq} (\tilde{F},E) \Longrightarrow (\tilde{B}_2,E) \tilde{\subseteq} (\tilde{F},E)$ $\Longrightarrow (\tilde{F},E)$ $\tilde{\epsilon}$ τ_2^{BVSES} .This show that $\tau_1^{BVSES} \tilde{\subseteq} \tau_2^{BVSES}$.

Theorem 12. *Let (X,τ^{BVSES},E) be a bi-polar vague soft expert set topological spaces over of $(\tilde{F},E)$, $(\tilde{K},E)$ $\tilde{\epsilon}$ BVSES (X,E) .*

1. *If B^{BVSES} is a BVSE P-base for τ^{BVSES} , then $B^{BVSES}_{(\tilde{F},E)} = \{(\tilde{B},E)\tilde{\cap}(\tilde{F},E) : (\tilde{B},E)\tilde{\epsilon}B^{BVSES}\}$ is a BVSE p-base for the BVSSET $\tau^{BVSES}_{(\tilde{F},E)}$,*

2. *If $(\tilde{F},E)$ is a BVSE in $\tau^{BVSES}_{(\tilde{F},E)}$ and $(\tilde{F},E)$ is a BVSE p-closed set in $\tau^{BVSES}_{(\tilde{F},E)}$, then $(\tilde{F},E)$ is a BVSE P-closed in $\tau^{BVSES}_{(\tilde{F},E)}$*

3. *Let $(\tilde{F},E) \tilde{\subseteq} (\tilde{F},E)$. If $\overline{(\tilde{f},E)}$ is BVSE p-closure (X,τ^{BVSES},E), then $\overline{(\tilde{f},E)}$ $\tilde{\cap}$ $(\tilde{F},E)$ is a BVSE p-closure in $(X_{(\tilde{F},E)},\tau^{BVSES}_{(\tilde{F},E)},E)$.*

Proof.

1. Since B^{BVSES} is a BVSE p-base for τ^{BVSES} so for arbitrary $(\tilde{U},E)$ $\tilde{\epsilon}$ τ^{BVSES} , we have $(\tilde{U},E) = \bigcup_{(\tilde{B},E)\epsilon B^{BVSES}}(\tilde{B},E)$ .In case,
$$(\tilde{U},E)\tilde{\cap}(\tilde{F},E) = \left(\bigcup_{(\tilde{B},E)\epsilon B^{BVSES}}(\tilde{B},E)\right)\tilde{\cap}(\tilde{F},E) = \left(\bigcup_{(\tilde{B},E)\epsilon B^{BVSES}}((\tilde{B},E)\tilde{\cap}(\tilde{F},E))\right.$$
for $(\tilde{U},E)\tilde{\cap}(\tilde{F},E)$ $\tilde{\epsilon}$ $\tau^{BVSES}_{(\tilde{F},E)}$. Since arbitrary member $\tau^{BVSES}_{(\tilde{F},E)}$ can be expressed as the union of members of $B^{BVSES}_{(\tilde{F},E)}$

2. We first show that if $(\tilde{f},E)$ is a BVSE p-closed set in $\tau^{BVSES}_{(\tilde{F},E)}$ then there exist a closed set $(\tilde{V},E) \tilde{\subseteq} (\tilde{K},E)$ i.e, $(\tilde{V},E) \notin \tau^{BVSES}$ such that $(\tilde{f},E) = (\tilde{F},E)$ $\tilde{\cap}$ $(\tilde{F},E)$.
Let $(\tilde{f},E)$ be a p-closed in $\tau^{BVSES}_{(\tilde{F},E)}$. Then $(\tilde{g},E)^c$ is a BVSE p-open set in $\tau^{BVSES}_{(\tilde{F},E)}$ i.e, $(\tilde{f},E)^c$, $(\tilde{f},E)^c = (\tilde{U},E)$ $\tilde{\cap}$ $(\tilde{F},E)$ for $(\tilde{U},E)$ $\tilde{\epsilon}\tau^{BVSES}$
$\Longrightarrow ((\tilde{f},E)^c)^c = (\tilde{F},E)$ $\tilde{\cap}$ $((\tilde{U},E)\tilde{\cap}(\tilde{F},E))^c = (\tilde{U},E)^c$ $\tilde{\cap}$ $(\tilde{F},E)$. Here $(\tilde{U},E)^c \notin \tau^{BVSES}$

i.e, $(\tilde{U},E)^c$ is a p-closed in τ^{BVSES}. So here acts as $(\tilde{V},E)\tilde{\subseteq}(\tilde{K},E)$. Conversely, suppose that $(\tilde{f},E)=(\tilde{V},E)\tilde{\cap}(\tilde{F},E)$ where $(\tilde{F},E)\tilde{\subseteq}(\tilde{K},E)$ and $(\tilde{V},E)$ is P-closed in $\tau^{BVSES}_{(\tilde{F},E)}$. Clearly, $(\tilde{V},E)^c\,\tilde{\epsilon}\,\tau^{BVSES}$ so that $(\tilde{V},E)^c\,\tilde{\cap}\,(\tilde{F},E)\,\tilde{\epsilon}\,\tau^{BVSES}_{(\tilde{F},E)}$ Now, $(\tilde{V},E)^c\,\tilde{\cap}\,(\tilde{F},E)=((\tilde{K},E)\backslash(\tilde{V},E))\,\tilde{\cap}\,(\tilde{F},E)=((\tilde{K},E)\tilde{\cap}(\tilde{V},E))\backslash((\tilde{V},E)\tilde{\cap}(\tilde{F},E))=(\tilde{F},E)\backslash(\tilde{f},E)$. This implies $\overline{(\tilde{F},E)}\setminus(\tilde{f},E)$ is a BVSE in $(\tilde{F},E)$ i,e., $(\tilde{f},E)$ is a BVSE p-closed set in $\tau^{BVSES}_{(\tilde{F},E)}$. $\overline{(\tilde{f},E)}=\tilde{\cap}\,\{(\tilde{f}_i,E):(\tilde{f}_i,E) isBVSEp-closedand (\tilde{f}_i,E)\tilde{\supseteq}(\tilde{f},E)\}$ is the BVSETS P-closure of $(\tilde{f},E)$ and so $\overline{(\tilde{f},E)}$ is a BVSETS p-closed set. Now, $\overline{(\tilde{f},E)}\,\tilde{\cap}\,(\tilde{F},E)=\tilde{\cap}\{(\tilde{f}_i,E):(\tilde{f}_i,E),(\tilde{f}_i,E)\tilde{\supseteq}(\tilde{f},E)\}\,\tilde{\cap}\,(\tilde{F},E)=\tilde{\cap}\,((\tilde{f}_i,E)\tilde{\cap}\tilde{F},E))$. Since each $(\tilde{f}_i,E)$ is p-closed then each $(\tilde{f}_i,E)\tilde{\cap}\tilde{F},E)$ is p-closed in $\tau^{BVSES}_{(\tilde{F},E)}$. Now $(\tilde{f},E)\tilde{\subseteq}(\tilde{f}_i,E)$ and $(\tilde{f},E)\tilde{\subseteq}(\tilde{F},E)$. So $((\tilde{f},E)\tilde{\cap}\tilde{F},E))\tilde{\subseteq}((\tilde{f}_i,E)\tilde{\cap}\tilde{F},E))$
$\Longrightarrow(\tilde{f},E)\tilde{\subseteq}(\tilde{f}_i,E)\tilde{\cap}\tilde{F},E)$ Therefore, $\overline{(\tilde{f},E)}\,\tilde{\cap}\,(\tilde{F},E)=\tilde{\cap}\,\{((\tilde{f}_i,E)\tilde{\cap}\tilde{F},E)):(\tilde{f}_i,E)\tilde{\cap}\tilde{F},E) isp-closedand((\tilde{f}_i,E)\tilde{\cap}\tilde{F},E))\tilde{\supseteq}((\tilde{f}_i,E)\}$. Thus $\overline{(\tilde{f},E)}\,\tilde{\cap}\,(\tilde{F},E)$ is a BVSE p- closure of $(\tilde{f},E)$ in $\tau^{BVSES}_{(\tilde{F},E)}$

5. Decision-Making Problem Using Bipolar Vague Soft Expert Sets in Cancer Diagnosis

In the medical field, diagnosing cancer often involves analyzing uncertain, vague, and conflicting expert opinions based on various medical parameters (e.g., symptoms, test results, imaging). Bipolar Vague Soft Expert Sets (BVSES) provide a powerful tool to represent such complex information involving multiple experts and both positive (supportive) and negative (opposing) evaluations.
Let $X=\{u_1,u_2,u_3\}$ is a set of patients. $E=\{e_1,e_2\}$ is a set of parameters.

1. e_1 : Tumor Marker Level
2. e_2 : Imaging Results (e.g., CT Scan)
3. Experts: $\{p,q,r\}$
4. Decision values: 1 (Yes, indication of cancer), 0 (No, less likely to have cancer).
5. Experts: p,q,r
   Decision values: 1 (Yes, indication of cancer), 0 (No, less likely to have cancer). The bipolar vague soft expert set ??H,??? assigns a tuple to each patient under each parameter and expert.
6. $(\tilde{T}^+,\tilde{f}^+,\tilde{T}^-,\tilde{f}^-)$ = (truth, indeterminacy, counter-truth, counter-indeterminacy).

Sample Evaluations: Let?s extract only the evaluations for decision-making purposes. We'll simplify a subset: For decision 1(Yes, cancer suspected):

Expert	**Parameter**	**Patient**	$(\tilde{T}^+,\tilde{f}^+,\tilde{T}^-,\tilde{f}^-)$
p	e_1	u_1	$(0.3,0.7,-0.2,-0.4)$
p	e_2	u_1	$(0.4,0.3,-0.2,-0.1)$
q	e_1	u_2	$(0.8,0.3,-0.1,-0.5)$
q	e_2	u_3	$(0.3,0.2,-0.5,-0.4)$
r	e_1	u_1	$(0.4,0.6,-0.6,-0.4)$

Table 1: Decision-Making Step: Score Function

We define a score function for each tuple $(\tilde{T}^{+}, \tilde{f}^{+}, \tilde{T}^{-}, \tilde{f}^{-})$ as: Score We define a score function for each tuple $(\tilde{T}^{+} - \tilde{f}^{+} - \tilde{T}^{-} - \tilde{f}^{-})$. A higher score suggests stronger positive evidence with lower uncertainty and opposition. Score Calculation:
Patient u_1

1. From p, $e_1 : S_1 = 0.3 - 0.2 - 0.7 - 0.4 = -1.0$
2. From p, $e_2 : S_2 = 0.4 - 0.2 - 0.3 - 0.1 = -0.2$
3. From r, $e_1 : S_3 = 0.4 - 0.6 - 0.6 - 0.4 = -1.2$

Total score for $u_1 : S_{(}u_1) = -1.0 + (-0.2) + (-1.2) = -2.4$

Patient u_2

1. From q, $e_1 : S = 0.8 - 0.1 - 0.3 - 0.5 = -0.1$

Patient u_3

1. From q, $e_2 : S = 0.8 - 0.1 - 0.3 - 0.5 = -0.1$
   $S = 0.3 - 0.5 - 0.2 - 0.4 = -0.8$

Total score for $u_3 : S_{(}u_3) = -0.8$

Final decision scores of patients is given in Table 1.

Patient	**Total score** $S_{(}u_i)$	**Diagnosis suggestion**
u_2	e_1	Most likely to have cancer
u_3	e_2	Moderate likelihood
u_1	e_1	Least likely to have cancer

Conclusion : : Patient u_2 has the least negative score, meaning the evaluations show less opposition and lower uncertainty about the presence of cancer, hence u_2 is more likely to have cancer based on the bipolar vague soft expert decision model. **Interpretation** :This examp demonstrates how Bipolar Vague Soft Expert Sets can model real-world uncertainty in cancer diagnosis by incorporating multiple expert opinions with varying confidence levels, and allowing decision-makers to derive conclusions using aggregated scoring mechanisms.

6. Comparative Analysis

This section presents a comprehensive and critical comparison between the published study referenced in [31] and the proposed research, highlighting the advancements, innovations, and theoretical depth introduced in the current work. This given in Table 3

Aspect	**Published Work [31]**	**Proposed Work**
Core Concept	Bipolar Fuzzy Soft Expert Sets (BFSES)	Bipolar Vague Soft Expert Sets (BPVSES)
Theoretical Focus	Defined fundamental operations: complement, union, intersection, AND, OR	Defined operations: complement, union, intersection, and introduced new topological structures
Innovations	Algorithm development and application to decision-making	TIntroduction of BPVSET, 8 new definitions, and the novel concept of p-open sets
Topological Extension	Not addressed.	Extensive development of bipolar vague soft expert topology (BPVSET)
Interior and Closure	Not explored	Defined and analyzed interior, closure, and their interactions
Advanced Concepts	Basic properties and laws of operations.	Bases, sub-bases, local bases; first and second countability; separability via dense sets
Main Theme	MCDM using bipolar fuzzy soft sets.	Topological modeling in vague bipolar soft expert sets
Application Area	University selection.	Cancer diagnosis
Theoretical Depth	Moderate	High.
Innovation	Algorithm + hierarchy model	TNew definitions + topological framework.
Real-World Impact	Education guidance	Medical diagnostics under uncertainty

7. Conclusion and Future Work

In this study, we have rigorously developed the theoretical foundation of bipolar vague soft expert sets (BPVSESs) by introducing and formalizing their essential operations, such as complement, union, and intersection. Building upon these core elements, we advanced the framework further by proposing the concept of bipolar vague soft expert topology (BPVSET), supported by eight novel and meaningful definitions. Notably, the introduction of the p-open set emerged as a powerful and versatile construct for shaping complex topological structures under uncertainty. We presented a thorough articulation of interior and closure operations, examining their properties and mutual interactions, which underpin the topological behavior of BPVSETS. Furthermore, we extended classical topological notions?such as bases, sub-bases, local bases, countability, and separability?into the context of bipolar vague soft expert topology. These contributions significantly enrich the theoretical framework, making it robust for applications involving parameterized and uncertain environments. The study culminates in a practical decision-making framework, with an application in cancer diagnosis, illustrating the capability of BPVSESs to model and resolve real-world problems involving vague, bipolar, and expert-driven information. The work not only strengthens the mathematical foundations of soft expert set theory but also opens new directions for research in soft topology, uncertainty modeling, and intelligent decision-making systems. Future work will focus on developing new structures within bipolar vague soft expert bi-topological spaces, with an emphasis on the role of soft points and the integration of bipolar vague soft expert semi-open sets. Results concerning the basis of these structures will be outlined. Furthermore, we will plan to apply the Encrypted K-Mean Clustering method, a novel approach for K-mean clustering on encrypted data, which ensures the confidentiality of sensitive information. We will also apply the Elbow method, a strategy to determine the optimal value of ?? (the number of clusters) in clustering analysis. This method enhances consistency in cluster design and helps identify natural groupings within datasets, enabling a deeper understanding of data diversity. In our work, we will try to implement an encrypted version of the Elbow method on our dataset.

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References

[1] D. Molodtsov. Soft set theory—first results. *Computers & Mathematics with Applications*, 37(4-5):19–31, 1999.

[2] D. Chen, E. C. C. Tsang, S. S. Yeung, and X. Wang. The parameterization reduction of soft sets and its application. *Computers & Mathematics with Applications*, 49(5-6):757–763, 2005.

[3] P. K. Maji, A. R. Roy, and R. Biswas. An application of soft sets in a decision making problem. *Computers & Mathematics with Applications*, 44(8-9):1077–1083, 2002.

[4] P. K. Maji, A. R. Roy, and R. Biswas. Soft set theory. *Computers & Mathematics with Applications*, 45(4-5):555–562, 2003.

[5] P. K. Maji, A. R. Roy, and R. Biswas. Fuzzy soft sets. *Journal of Fuzzy Mathematics*, 9(3):589–602, 2001.

[6] R. Roy and P. K. Maji. A fuzzy soft set theoretic approach to decision making problems. *Journal of Computational and Applied Mathematics*, 203(2):412–418, 2007.

[7] S. Alkhazaleh, A. R. Salleh, and N. Hassan. Soft multisets theory. *Applied Mathematical Sciences*, 5(72):3561–3573, 2011.

[8] S. Alkhazaleh, A. R. Salleh, and N. Hassan. Fuzzy parameterized interval-valued fuzzy soft set. *Applied Mathematical Sciences*, 5(67):3335–3346, 2011.

[9] S. Alkhazaleh, A. R. Salleh, and N. Hassan. Possibility fuzzy soft set. *Advances in Decision Sciences*, 2011:479756, 2011.

[10] S. Alkhazaleh and A. R. Salleh. Fuzzy soft expert set and its application. *Applied Mathematical Sciences*, 8(28):1349–1368, 2014.

[11] Orhan Dalkılç and İsmail Naci Cangül. Determining interactions between objects from different universes: (inverse) object interaction set for binary soft sets. *Soft Computing*, 28(17-18):9981–9989, 2024.

[12] Naime Demirtaş, Orhan Dalkılç, Muhammad Riaz, and Ashraf Al-Quran. Mathematical analysis of parameters belonging to the universe in the soft set theory with new distance measures. *Journal of Intelligent & Fuzzy Systems*, 46(2):3975–3985, 2024.

[13] Orhan Dalkılıç and Naime Demirtaş. Novel hybrid soft set theories focusing on decision-makers by considering the factors affecting the parameters. *Journal of Experimental & Theoretical Artificial Intelligence*, pages 1–13, 2024.

[14] P. Bosc and O. Pivert. On a fuzzy bipolar rational algebra. *Information Sciences*, 219:1–16, 2013.

[15] K. M. Lee. Bipolar-valued fuzzy sets and their operations. *Proceedings of the International Conference on Intelligent Technologies*, pages 307–312, 2000.

[16] K. J. Lee. Bipolar fuzzy sub-algebras and bipolar fuzzy ideals of BCK/BCI-algebras. *Bulletin of the Malaysian Mathematical Sciences Society*, 32(3):361–373, 2009.

[17] S. K. Majumder. Bipolar valued fuzzy sets in Γ-semigroups. *Mathematica Aeterna*, 2(3):203–213, 2012.

[18] S. V. Manemaran and B. Chellappa. Structures on bipolar fuzzy groups and bipolar fuzzy D-ideals under (T, S) norms. *International Journal of Computer Applications*, 9(12):7–10, 2010.

[19] J. Chen, S. Li, S. Ma, and X. Wang. m-polar fuzzy sets: an extension of bipolar fuzzy sets. *The Scientific World Journal*, 2014:416530, 2014.

[20] S. Alkhazaleh, A. R. Salleh, and N. Hassan. Fuzzy parameterized interval-valued fuzzy soft set. *Applied Mathematical Sciences*, 5(67):3335–3346, 2011.

[21] S. Alkhazaleh and A. R. Salleh. Soft expert sets. *Advances in Decision Sciences*, 2011:757868, 2011.

[22] M. M. Saeed, S. A. Ghour, A. Mehmood, M. M. A. Shomrani, C. Park, and J. R. Lee. Bipolar vague soft topological structures in terms of operators and convergence of sequences. *Journal of Intelligent & Fuzzy Systems*, 44(1):1099–1116, 2023.

[23] Orhan Dalkılıç and Naime Demirtaş. Bipolar fuzzy soft set theory applied to medical diagnosis. *Turkish Journal of Mathematics and Computer Science*, 16(2):314–324, 2024.

[24] Orhan Dalkılıç. (α, β)-cuts and inverse (α, β)-cuts in bipolar fuzzy soft sets. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 70(2):582–599, 2021.

[25] Saleem Abdullah, Muhammad Aslam, and Kifayat Ullah. Bipolar fuzzy soft sets and its applications in decision making problem. *Journal of Intelligent & Fuzzy Systems*, 27(2):729–742, 2014.

[26] Saima Mustafa, Neelofar Safdar, Murrium Bibi, A. F. Sayed, Muhammad Ghaffar Khan, and Zabidin Salleh. A study of bipolar fuzzy soft sets and its application in decision-making problems. *Mathematical Problems in Engineering*, 2021:5742288, 2021.

[27] Raed Hatamleh. On the compactness and continuity of Uryson's operator in Orlicz space. *International Journal of Neutrosophic Science*, 24(3):233–239, 2024.

[28] A. Rajalakshmi, Nasreen Kausar, Brikena Vrioni, K. Lenin Muthu Kumaran, Nezir Aydin, and Murugan Palanikumar. Characterization of various (b, l) neutrosophic ideals of an ordered Γ-semigroups. *International Journal of Neutrosophic Science*, 25(2):269–281, 2025.

[29] Raed Hatamleh, Abdallah Al-Husban, N. Sundarakannan, and M. S. Malchijah Raj. Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism. *Communications on Applied Nonlinear Analysis*, 32(3):418–435, 2025.

[30] Ahmad A. Abubaker, Raed Hatamleh, Khaled Matarneh, and Abdallah Al-Husban. On the numerical solutions for some neutrosophic singular boundary value problems by using (LPM) polynomials. *International Journal of Neutrosophic Science*, 25(2):197–205, 2024.

[31] Yousef Al-Qudah and Nasruddin Hassan. Bipolar fuzzy soft expert set and its application in decision making. *International Journal of Applied Decision Sciences*, 10(2):175–191, 2017.

[32] Muhammad Akram, Ghous Ali, Muhammad Arif Butt, and Jose Carlos R. Alcantud. Novel MCGDM analysis under m-polar fuzzy soft expert sets. *Neural Computing and Applications*, 33(18):12051–12071, 2021.