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k-Geodetic Hop Domination Defect in a Graph

Jesica M. Anoche[1,2,*], Sergio R. Canoy, Jr.[1,2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Center of Mathematical and Theoretical Physical Sciences-PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let $G = (V(G), E(G))$ be a simple undirected graph. A set $S \subseteq V(G)$ is a geodetic hop dominating set in G if for every $v \in V(G) \setminus S$, there exist vertices $x, y, z \in S$ such that $d_G(x, v) = 2$ and v lies in a y-z geodesic, that is, $v \in I_G(y, z)$. The minimum cardinality of a geodetic hop dominating set of G, denoted by $\gamma_{hg}(G)$, is called the geodetic hop domination number of G. The minimality of $\gamma_{hg}(G)$ implies that if $S \subseteq V(G)$ such that $|S| < \gamma_{hg}(G)$, then there is at least one vertex of G that is not geodetically hop-dominated by S. The k-geodetic hop domination defect of G, denoted by $\zeta_k^{hg}(G)$, is the minimum number of vertices of G that is not geodetically hop-dominated by any subset of vertices of G with cardinality $\gamma_{hg}(G) - k$. A set $S \subseteq V(G)$ of cardinality $\gamma_{hg}(G) - k$ for which $|V(G) \setminus N_G^{hg}[S]| = \zeta_k^{hg}(G)$, where $N_G^{hg}[S] = N_G^2[S] \cap I_G[S]$, is called a ζ_k^{hg}-*set* of G. In this paper, we initiate the study of the concept of k-geodetic hop domination defect of a non-trivial graph G and investigate it for some known classes of graphs.

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Key Words and Phrases: Geodetic set, hop domination, geodetic hop domination, k-geodetic hop domination defect

1. Introduction

The domination number $\gamma(G)$ of a graph G refers to the smallest number of vertices required to dominate all the vertices of G. Hence, if a set S of vertices of G has cardinality strictly less than $\gamma(G)$, then definitely, there will be vertices of G that will not be dominated by any of the vertices in S. Recently, Das et al. [1] introduced and studied the notion of k-domination defect of a graph, where k is a positive integer strictly less than the domination number of the graph. The authors in this study established various bounds on the k-domination defect of a graph in terms of the maximum degree and domination number of the graph, and other parameters.

*Corresponding author.
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Email addresses: jesica.anoche@g.msuiit.edu.ph (J. Anoche), sergio.canoy@g.msuiit.edu.ph (S. Canoy, Jr.)

For hop domination, a concept which is studied by many researchers (see for example, [2], [3], [4], [5], [6], and [7]), Anoche et al. [8] introduced and studied the notion of k-hop domination defect in a graph. The study obtained some bounds on the k-hop domination defect of a graph in terms of its order and maximum hop degree. Moreover, the k-hop domination defects of some classes of graphs have been determined.

Some variants of domination and hop domination utilize the concept of geodetic set. The associated term geodetic number of a graph was introduced by Harary et al. [9]. Geodetic number and geodetic domination were considered in [10], [11], [12], [13], and [14]. Recently, Anoche et al. [15] introduced and studied the notion of k-geodetic domination defect of a graph. The authors in this study established some sharp bounds on the k-geodetic domination defect of a graph and computed its values for several well-known graphs. On the other hand, the notion of geodetic hop domination was introduced and investigated in [16], [17], [18] and [19]. Note that if G is a graph and S is a set of vertices of G with cardinality strictly less than the geodetic hop domination number $\gamma_{hg}(G)$ of G, then there is at least one vertex outside S that is not geodetically hop-dominated, that is, has no hop neighbor in S or is not in any shortest path joining any two vertices in S. In this paper, we introduce the notion k-geodetic hop domination defect and study it for some classes of graphs.

For a motivation of the study, consider an establishment with a large number of employees which needs to make an annual evaluation of their workers. The manager chooses some workers to form a team of assessors to evaluate the performance of their co-workers. To be cost effective or to minimize costs, the manager ensures that this team will consist of the smallest number of members that can do the task. Moreover, to avoid bias in the assessment, an inspector should be non-biased, that is, neither be close friends nor enemies with any of the workers he or she is assigned to assess. This situation can be modeled by constructing a graph where each vertex represents a worker and an edge between two workers represents possible bias, that is, if the two workers are either close friends or enemies. Here, every worker who is not in the team will be evaluated by a non-biased inspector who is at a distance two from him/her. Moreover, for the purpose of visibility and monitoring that the policy is strictly followed, it is imposed that every worker must be in a shortest path connecting two members of the team. However, due to possible budgetary constraints, the required minimum number of evaluators may not always be attained or sometimes, it may happen that during the course of the evaluation process an evaluator may be absent and, subsequently, unable to perform his or her task. Consequently, some workers may not be evaluated accordingly. Finding the number of unevaluated workers with respect to a given team of evaluators not reaching the required minimum number of membership may be of help to the management. Situations such as this led us to introduce the concept of geodetic hop domination defect in a graph.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a u-v *geodesic*. The distance between two subsets A and B of $V(G)$ is given by $d_G(A,B) = \min\{d_G(a,b) : a \in A \text{ and } b \in B\}$. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v \in A} N_G(v)$ is called the *open neighborhood of* A and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of* A. A vertex v of G is isolated if $|N_G(v)| = 0$. The set containing all the isolated vertices of G is denoted by $I(G)$. The *open hop neighborhood* of a point u is the set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$. The *closed hop neighborhood* of u is $N_G^2[u] = N_G^2(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G^2(A) = \bigcup_{v \in A} N_G^2(v)$ is called the *open hop neighborhood of* A and $N_G^2[A] = N_G^2(A) \cup A$ is called the *closed hop neighborhood of* A.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set S of G with $|S| = \gamma(G)$, is called a γ-set of G. The geodetic closure of a set $S \subseteq V(G)$, denoted by $I_G[S]$, is the union of the intervals $I_G[u,v]$, where $u, v \in S$. The set S is a geodetic set in G if $I_G[S] = V(G)$. The smallest cardinality among all geodetic sets in G, denoted by $g(G)$, is called the geodetic number of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set $S \subseteq V(G)$ is a geodetic dominating set in G if it is both a dominating and a geodetic set.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for each $x \in V(G) \setminus S$, there exists $z \in S$ such that $d_G(x,z) = 2$. The smallest cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A hop dominating set S of G with $|S| = \gamma_h(G)$ is called a γ_h-set of G.

A set $S \subseteq V(G)$ is geodetic hop dominating if it is both a geodetic and a hop dominating set. The geodetic hop domination number $\gamma_{hg}(G)$ of G is the minimum cardinality among all geodetic hop dominating sets in G. Any geodetic hop dominating set of G with cardinality $\gamma_{hg}(G)$ is called a γ_{hg}-set.

Let G be a non-trivial graph of order n and let $1 \leq k < \gamma_{hg}(G)$. Let $S \subseteq V(G)$ with cardinality $|S| = \gamma_{hg}(G) - k$ and let $N_G^{hg}[S] = N_G^2[S] \cap I_G[S]$, the set of geodetically hop-dominated set of vertices of G. The set $V(G) \setminus N_G^{hg}[S]$ is called the *k-geodetic hop domination defect set* of S and the *k-geodetic hop domination defect* of S is $\zeta_k^{hg}(S) = |V(G) \setminus N_G^{hg}[S]| = n - |N_G^{hg}[S]|$. The minimum cardinality of a *k-geodetic hop domination defect set* in G, denoted by $\zeta_k^{hg}(G)$, is called the *k-geodetic hop domination defect* of G, i.e.,

$$\zeta_k^{hg}(G) = \min\{\zeta_k^{hg}(S) : S \subseteq V(G) \text{ with } |S| = \gamma_{hg}(G) - k\}.$$

A set $S \subseteq V(G)$ of cardinality $\gamma_{hg}(G) - k$ for which $|V(G) \setminus N_G^{hg}[S]| = \zeta_k^{hg}(G)$ is called a *ζ_k^{hg}-set* of G. Thus, $\left\langle N_G^{hg}[S] \right\rangle$ is an induced subgraph of G with $n - \zeta_k^{hg}(G)$ vertices and geodetic hop domination number $\gamma_{hg} - k$.

Consider the graph G in Figure 1. Then $D = \{a, b, x, y\}$ is a γ_{hg}-set of G, i.e.,

$\gamma_{hg}(G) = 4$. If $k = 1$, then $S_1 = \{a, b, x\}$ is a ζ_1^{hg}-set of G. Since $N_G^{hg}[S_1] = \{a, b, c, d, u, v, x\}$, it follows that

$$\zeta_1^{hg}(G) = \zeta_1^{hg}(S) = |V(G)| - |N_G^{hg}[S_1]| = 8 - 7 = 1.$$

Clearly, the set $\{a, b, d\}$ is not a ζ_1^{hg}-set of G because $N_G^{hg}[\{a, b, d\}] = \{a, b, c, d, u, v\}$, i.e., $\zeta_1^{hg}(\{a, b, d\}) = 8 - 6 = 2$. If $k = 2$, then $S_2 = \{a, v\}$ is a ζ_2^{hg}-set of G and $N_G^{hg}[S_2] = \{a, c, d, u, v\}$. Hence, $\zeta_2^{hg}(G) = 8 - 5 = 3$. Observe that the sets $\{a, x\}$ and $\{b, x\}$ are not ζ_2^{hg}-sets of G. Finally, if $k = 3$, then any 1-element subset S_3 of $V(G)$ is a ζ_3^{hg}-set of G. Since $N_G^{hg}[S_3] = S_3$, it follows that $\zeta_3^{hg}(G) = |V(G)| - |N_G^{hg}[S_3]| = 8 - 1 = 7$.

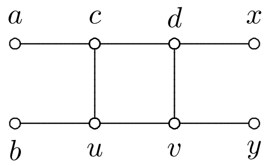

Figure 1: Graph G with $\gamma_{hg}(G) = 4$

3. Results

Theorem 1 ([19]). *Let n be positive integer. Then each of the following holds.*

(i) *For a path P_n on n vertices, we have*

$$\gamma_{hg}(P_n) = \begin{cases} n & \text{if } n = 1, 2 \\ \frac{n+6}{3} & \text{if } n \equiv 0 (mod\ 3) \\ \frac{n+2}{3} & \text{if } n \equiv 1 (mod\ 3) \\ \frac{n+4}{3} & \text{if } n \equiv 2 (mod\ 3). \end{cases}$$

(ii) *For a cycle C_n on n vertices, we have*

$$\gamma_{hg}(C_n) = \begin{cases} 3 & \text{if } n = 3, 4, 5 \\ \frac{n}{3} & \text{if } n \equiv 0 (mod\ 3) \\ \frac{n+2}{3} & \text{if } n \equiv 1 (mod\ 3) \\ \frac{n+4}{3} & \text{if } n \equiv 2 (mod\ 3). \end{cases}$$

(iii) *For a complete graph K_n, we have $\gamma_{hg}(K_n) = n$.*

Corollary 1 ([18]). *Let G and H be any two graphs of orders m and n, respectively. Then*

(i) *$\gamma_{hg}(G + H) = m + n$ if G and H are complete;*

(ii) $\gamma_{hg}(K_1, n-1) = \gamma_{hg}(K_1 + K_{n-1}) = n$ *for* $n \geq 2$*;*

(iii) $\gamma_{hg}(F_n) = 1 + \rho_{2pnd}(P_n)$*;*

(iv) $\gamma_{hg}(W_n) = 1 + \rho_{2pnd}(C_n)$*; and*

(v) $\gamma_{hg}(K_{m,n}) = \begin{cases} 3 & \text{if } m = 2 \text{ or } n = 2 \\ 4 & \text{otherwise.} \end{cases}$

Remark 1. *Let* $G_1, G_2, \cdots, G_r$ *be the components of a graph* G*. Then each of the following holds:*

(i) $\gamma_{hg}(G) = \sum_{j=1}^{r} \gamma_{hg}(G_j)$.

(ii) If $A_j \subseteq V(G_j)$ *for each* $j \in [r] = \{1, 2, \cdots, r\}$ *and* $A = \cup_{j=1}^{r} A_j$*, then* $N_G^{hg}[A] = \cup_{j=1}^{r} N_G^{hg}[A_j]$ *(a disjoint union).*

Theorem 2. *Let* $G_1, G_2, \cdots, G_r$ *be the components of graph* G *and let* $\zeta_1^{hg}(G_i)$ *be the 1-geodetic hop domination defect of* G_i *for each* $i \in [r] = \{1, 2, \cdots, r\}$*. Then*

$$\zeta_1^{hg}(G) = \min\{\zeta_1^{hg}(G_i) : i \in [r]\}.$$

Proof. Let $\gamma_{hg}(G_i)$ and $\gamma_{hg}(G)$ be the geodetic hop domination numbers of G_i and G, respectively. By Remark 1(i), $\gamma_{hg}(G) = \sum_{i=1}^{r} \gamma_{hg}(G_i)$. For each $i \in [r]$, let D_i be a ζ_1^{hg}-set of G_i. Then $|D_i| = \gamma_{hg}(G_i) - 1$ and $\zeta_1^{hg}(G_i) = |V(G_i) - N_G^{hg}[D_i]|$. Let $j \in [r]$ be such that $\zeta_1^{hg}(G_j) = \min\{\zeta_1^{hg}(G_i) : i \in [r]\}$. Let S_i be a γ_{hg}-set in G_i for each $i \in [r]$ and let $S = (\cup_{i \in [r] \setminus \{j\}} S_i) \cup D_j$. Then

$$|S| = \sum_{i \in [r] \setminus \{j\}} |S_i| + |D_j| = \gamma_{hg}(G) - 1$$

and, by Remark 1(ii),

$$\begin{aligned} |N_G^{hg}[S]| &= |N_{G_j}^{hg}[D_j]| + \sum_{i \in [r] \setminus \{j\}} |N_{G_i}^{hg}[S_i]| \\ &= |V(G_j)| - \zeta_1^{hg}(G_j) + \sum_{i \in [r] \setminus \{j\}} |V(G_i)| \\ &= \sum_{i=1}^{r} |V(G_i)| - \zeta_1^{hg}(G_j). \end{aligned}$$

Thus, in G, $\zeta_1^{hg}(S) = |V(G)| - |N_G^{hg}[S]| = \zeta_1^{hg}(G_j)$. We claim that $\zeta_1^{hg}(S)$ is the minimum among all subsets of $V(G)$ with cardinality $\gamma_{hg}(G) - 1$. To this end, suppose there exists $Q \subseteq V(G)$ such that $|Q| = \gamma_{hg}(G) - 1$ and $\zeta_1^{hg}(Q) < \zeta_1^{hg}(S)$. Let $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_r$ where $Q_i \subseteq V(G_i)$ for each $i \in [r]$. Since $|Q| = \gamma_{hg}(G) - 1$, at least one Q_t is not a geodetic hop dominating set of G_t by Remark 1(i). Thus, $|Q_t| = \gamma_{hg}(G_t) - 1$ and $\zeta_1^{hg}(Q_t) \geq \zeta_1^{hg}(G_t) \geq \zeta_1^{hg}(G_j)$. Hence,

$$\begin{aligned}\zeta_1^{hg}(Q) = |V(G)| - |N_G^{hg}[Q]| &= \sum_{i=1}^{r}(|V(G_i)| - |N_{G_i}^{hg}[Q_i]|) \\ &\geq |V(G_t)| - |N_{G_t}^{hg}[Q_t]| \\ &= \zeta_1^{hg}(Q_t) \\ &\geq \zeta_1^{hg}(G_j) \\ &= \zeta_1^{hg}(S),\end{aligned}$$

contrary to the assumption that $\zeta_1^{hg}(Q) < \zeta_1^{hg}(S)$. Therefore, $\zeta_1^{hg}(G) = \zeta_1^{hg}(S) = \zeta_1^{hg}(G_j)$. □

Theorem 3. *Let G be a graph with $I(G) \neq \varnothing$ and suppose $|I(G)| = r$. Then $\zeta_j^{hg}(G) = j$ for every $j \in [r] = \{1, 2, \cdots, r\}$ and $\zeta_k^{hg}(G) = r + \zeta_{k-r}^{hg}(G')$ for every $k \in \{r+1, \cdots, \gamma_{hg}(G) - 1\}$, where $G' = \langle V(G) \setminus I(G) \rangle$.*

Proof. Let $I(G) = \{v_1, v_2, \cdots, v_r\}$ and let S be a γ_{hg}-set in G. Then $I(G) \subseteq S$. Let $j \in [r]$. Then $D = S \setminus \{v_1, v_2, \cdots, v_j\}$ is a ζ_j^{hg}-set of G and $|N_G^{hg}[D]| = |N_G^{hg}[S]| - |N_G^{hg}[\{v_1, v_2, \cdots, v_j\}]| = |V(G)| - j$. Hence, $\zeta_j^{hg}(G) = |V(G)| - (|V(G)| - j) = j$.

Next, let $k \in \{r+1, \cdots, \gamma_{hg}(G) - 1\}$. Then $S_0 = S \setminus I(G)$ is γ_{hg}-set in $G' = \langle V(G) \setminus I(G) \rangle$. Hence, $\gamma_{hg}(G') = \gamma_{hg}(G) - r$. Since $k \leq \gamma_{hg}(G) - 1$, $k - r \leq \gamma_{hg}(G) - (r+1) < \gamma_{hg}(G) - r$. Let S' be a ζ_{k-r}^{hg}-set of G'. Then $|S'| = (\gamma_{hg}(G) - r) - (k - r) = \gamma_{hg}(G) - k$ and $\zeta_{k-r}^{hg}(G') = |V(G')| - |N_{G'}^{hg}[S']| = (|V(G)| - r) - |N_G^{hg}[S']|$. This implies that $|V(G)| - |N_G^{hg}[S']| = r + \zeta_{k-r}^{hg}(G')$. Therefore, since S' is also a ζ_k^{hg}-set of G, $\zeta_k^{hg}(G) = r + \zeta_{k-r}^{hg}(G')$. □

Theorem 4. *Let G be a non-trivial graph of order n and let k be a positive integer with $k \leq \gamma_{hg}(G) - 1$. Then $\zeta_k^{hg}(G) \leq n - \gamma_{hg}(G) + k$.*

Proof. Let k be a positive integer with $k \leq \gamma_{hg}(G) - 1$ and let S be a ζ_k^{hg}-set of G. Then $|S| = \gamma_{hg}(G) - k$ and $\zeta_k^{hg}(G) = n - |N_G^{hg}[S]|$. Since $S \subseteq N_G^{hg}[S]$, it follows that

$$\zeta_k^{hg}(G) = n - |N_G^{hg}[S]| \leq n - |S| = n - (\gamma_{hg}(G) - k) = n - \gamma_{hg}(G) + k.$$

This proves the assertion. □

Remark 2. *The bound in Theorem 4 is sharp. Strict inequality is also possible.*

To see this, consider the graph $G = C_5$ in Figure 2. By Theorem 1(ii), $\gamma_{hg}(G) = 3$. Let $k = 1$. Then any set $S_1 \subseteq V(G)$ with $|S_1| = 2$ is a ζ_1^{hg}-set of G and $N_G^{hg}[S_1] = S_1$.

Thus,

$$\zeta_1^{hg}(G) = |V(G)| - \gamma_{hg}(G) + k = 5 - 3 + 1 = 3.$$

If $k = 2$, then any 1-element subset S_2 is a ζ_2^{hg}-set of G and $N_G^{hg}[S_2] = S_2$. Thus, $\zeta_2^{hg}(G) = 5 - 1 = 4$. Since $|V(G)| - \gamma_{hg}(G) + k = 5 - 3 + k = 4$, the equality $\zeta_k^{hg}(C_5) = n - \gamma_{hg}(C_5) + k$ holds.

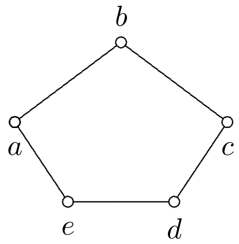

Figure 2: $G = C_5$ and $\gamma_{hg}(G) = 3$

For strict inequality, consider $H = P_{10}$ in Figure 3. By Theorem 1(i), $\gamma_{hg}(H) = 4$. Let $P_{10} = [v_1, v_2, \cdots, v_{10}]$ and let $k = 1$. It can easily be verified that $S_1 = \{v_1, v_4, v_7\}$ is a ζ_1^{hg}-set of H and $N_H^{hg}[S_1] = \{v_1, v_2, v_3, \cdots, v_7\}$. Hence,

$$\zeta_1^{hg}(H) = 10 - 7 = 3 < 7 = |V(H)| - \gamma_{hg}(H) + k.$$

If $k = 2$, then $S_2 = \{v_1, v_4\}$ is a ζ_2^{hg}-set of H and $N_H^{hg}[S_2] = \{v_1, v_2, v_3, v_4\}$. This implies that

$$\zeta_2^{hg}(H) = 6 < 8 = |V(H)| - \gamma_{hg}(H) + k.$$

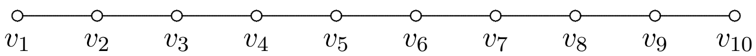

Figure 3: $H = P_{10}$ and $\gamma_{hg}(H) = 4$

Theorem 5. *Let G be a graph of order $n \geq 2$ and $k \leq \gamma_{hg}(G) - 1$, then $1 \leq \zeta_k^{hg}(G) \leq n - 1$.*

Proof. Let S be a ζ_k^{hg}-set of G. Then $|S| = \gamma_{hg}(G) - k$. Hence, $V(G) \setminus N_G^{hg}[S] \neq \varnothing$. It follows that $\zeta_k^{hg}(G) = |V(G)| - |N_G^{hg}[S]| \geq 1$. Also, since $|N_G^{hg}[S]| \geq 1$, $\zeta_k^{hg}(G) = |V(G)| - |N_G^{hg}[S]| \leq n - 1$. This proves the assertion. □

Theorem 6. *Let G be a non-trivial graph of order n. Then $\zeta_1^{hg}(G) = 1$ if and only if there exists $v \in V(G)$ such that $\gamma_{hg}(G - v) = \gamma_{hg}(G) - 1$.*

Proof. Suppose $\zeta_1^{hg}(G) = 1$ and let S be a ζ_1^{hg}-set of G. Then $|S| = \gamma_{hg}(G) - 1$ and $\zeta_k^{hg}(G) = |V(G) \setminus N_G^{hg}[S]| = 1$. Let $v \in V(G) \setminus N_G^{hg}[S]$. Then $N_G^{hg}[S] = V(G) \setminus \{v\}$. Therefore, $\gamma_{hg}(G - v) = \gamma_{hg}(\langle N_G^{hg}[S] \rangle) = \gamma_{hg}(G) - 1$.

Conversely, let $v \in V(G)$ such that $\gamma_{hg}(G - v) = \gamma_{hg}(G) - 1$. Then there exists $D \subseteq V(G)$ with $|D| = \gamma_{hg}(G) - 1$ and $N_G^{hg}[D] = V(G) \setminus \{v\}$. This implies that $\zeta_1^{hg}(G) = |V(G) \setminus N_G^{hg}[D]| = |\{v\}| = 1$. Therefore, $\zeta_1^{hg}(G) = 1$. □

Lemma 1. *Let G be a non-trivial graph of order n. If $k = \gamma_{hg}(G) - 1$, then $\zeta_k^{hg}(G) = n - 1$.*

Proof. Let $k = \gamma_{hg}(G) - 1$ and let S be a ζ_k^{hg}-set of G. Then $|S| = 1$, say, $S = \{x\}$ and $\zeta_k^{hg}(G) = \zeta_k^{hg}(S) = n - |N_G^{hg}[S]|$. Since $x \in N_G^2[S]$ and $I_G[S] = S$, it follows that $N_G^{hg}[S] = S$. It follows that $\zeta_k^{hg}(G) = n - |N_G^{hg}[S]| = n - 1$. □

Lemma 2. *Let G be a connected graph of order n. If S is a clique in G, then $N_G^{hg}[S] = S$.*

Proof. Let S is be clique in G. Then $N_G^2[S] = I_G[S] = S$. Thus, $N_G^{hg}[S] = N_G^2[S] \cap I_G[S] = S$. □

The next result shows that the bound given in Theorem 4 is sharp.

Theorem 7. *If K_n is a complete graph on n vertices, where $n \geq 2$, and $1 \leq k \leq n - 1$, then $\zeta_k^{hg}(K_n) = k$.*

Proof. Let k be a positive integer with $k \leq \gamma_{hg}(K_n) - 1$. Since $\gamma_{hg}(K_n) = n$, $k \leq n - 1$. Let S be a ζ_k^{hg}-set of K_n. Then S is a clique, $|S| = n - k$ and $\zeta_k^{hg}(G) = n - |N_G^{hg}[S]|$. Therefore, by Lemma 2, $\zeta_k^{hg}(G) = n - (n - k) = k$. □

Theorem 8. *For a path P_n with n vertices,*

$$\zeta_k^{hg}(P_n) = \begin{cases} 1 & \text{if } \ n = 2 \ \text{or} \ n = 3r \ \text{and} \ \ k = 1 \\ 3k - 4 & \text{if } \ n = 3r \ \text{and} \ \ k \geq 2 \\ 3k & \text{if } \ n = 3r + 1 \ \text{and} \ \ k \leq r \\ 3k - 2 & \text{if } \ n = 3r + 2 \ \text{and} \ \ k \leq r + 1. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \cdots, v_n]$ and let $r \geq 1$. Consider the following cases:

Case 1: $n = 3r$.
By Theorem 1 (i), $\gamma_{hg}(P_n) = \frac{3r+6}{3}$. Let $k = 1$ and let $S_1 = \{v_1, v_4, \cdots, v_{3r-2}\} \cup \{v_{3r-1}\}$. Then $N_G^{hg}[S_1] = V(P_n) \setminus \{v_{3r}\}$. Hence, $\zeta_k^{hg}(S_1) = n - |N_G^{hg}[S_1]| = 3r - (3r - 1) = 1$. Thus, $\zeta_k^{hg}(P_n) = 1$. Next, let $k \geq 2$. Then $\frac{3r+6}{3} - k = r - k + 2$. Choose an $(r - k + 2)$-element set $S_2 = \{v_1, v_4, \cdots, v_{3r-3k+4}\}$. Then S_2 is a ζ_k^{hg}-set of P_n and $N_G^{hg}[S_2] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+4}\}$. This implies that

$$\zeta_k^{hg}(P_n) = \zeta_k^{hg}(S_2) = n - |N_G^{hg}[S_2]| = 3r - (3r - 3k + 4) = 3k - 4.$$

Case 2: $n = 3r + 1$.
By Theorem 1 (i), $\gamma_{hg}(P_n) = \frac{3r+3}{3}$. Then $k \leq r$ and $\frac{3r+3}{3} - k = r - k + 1$. Choose

an $(r-k+1)$-element set $S_3 = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then S_3 is a ζ_k^{hg}-set of P_n and $N_G^{hg}[S_3] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}\}$. Thus,

$$\zeta_k^{hg}(P_n) = \zeta_k^{hg}(S_3) = n - |N_G^{hg}[S_3]| = (3r+1) - (3r-3k+1) = 3k.$$

Case 3: $n = 3r+2$.
By Theorem 1 (i), $\gamma_{hg}(P_n) = \frac{3r+6}{3} = r+2$. Here, $k \leq r+1$ and $\gamma_{hg}(P_n) - k = r-k+2$. Consider an $(r-k+2)$-element set $S_4 = \{v_1, v_4, \cdots, v_{3r-3k+4}\}$. The set S_4 is a ζ_k^{hg}-set of P_n and $N_G^{hg}[S_4] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+4}\}$. Therefore,

$$\zeta_k^{hg}(P_n) = \zeta_k^{hg}(S_4) = n - |N_G^{hg}[S_4]| = 3r+2-(3r-3k+4) = 3k-2. \quad \square$$

Theorem 9. *If C_n is the cycle with n vertices and $k \leq \gamma_{hg}(C_n) - 1$, then*

$$\zeta_k^{hg}(C_n) = \begin{cases} n+k-3 & \text{if } n = 3,4,5 \\ 3k+2 & \text{if } n = 3r,\ r \geq 2 \text{ and } k = \gamma_{hg}(C_n)-1 \text{ or} \\ & r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_{hg}(C_n)-2 \\ 3k & \text{if } n = 3r,\ r \geq 4 \text{ and},\ 1 \leq k \leq \gamma_{hg}(C_n)-3 \\ & \text{or } n = 3r,\ r \text{ is even},\ r \geq 4, \text{ and } k = \gamma_{hg}(C_n)-2 \\ & \text{or } n = 3r+1 \text{ and } k = \gamma_{hg}(C_n)-1 \\ & \text{or } n = 3r+1,\ r \text{ is even}, \text{ and } k = \gamma_{hg}(C_n)-2 \\ 1 & \text{if } n = 3r+2,\ r \geq 2, \text{ and } k = 1 \\ 2 & \text{if } n = 3r+1,\ r \geq 3 \text{ and } k = 1 \\ 3k-2 & \text{if } n = 3r+1,\ r \geq 4, \text{ and } 2 \leq k \leq \gamma_{hg}(C_n)-3 \\ & \text{or } n = 3r+1,\ r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_{hg}(C_n)-2 \\ & \text{or } n = 3r+2 \text{ and } k = \gamma_{hg}(C_n)-1 \\ & \text{or } n = 3r+2,\ r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_{hg}(C_n)-2 \\ & \text{or } n = 3r+2,\ r = 2, \text{ and } k = \gamma_{hg}(C_n)-2 \\ 3k-4 & \text{if } n = 3r+2, r \geq 3 \text{ and } 2 \leq k \leq \gamma_{hg}(C_n)-3 \\ & \text{or } n = 3r+2,\ r \text{ is even},\ r \geq 4, \text{ and } k = \gamma_{hg}(C_n)-2 \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \cdots, v_n, v_1]$ and let $k \leq \gamma_{hg}(C_n) - 1$. Consider the following cases:

Case 1: $n = 3$ or 4, or 5.

By Theorem 1(ii), $\gamma_{hg}(C_n) = 3$. Choose a $(3-k)$-element set $D_1 = \{v_1, v_{3-k}\}$. This set is a ζ_k^{hg}-set of C_n. Since D_1 is a clique, $N_G^{hg}[D_1] = D_1$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_1) = n - |N_G^{hg}[D_1]| = n - (3-k) = n + k - 3.$$

Case 2: $n = 3r$ where $r \geq 2$.

By Theorem 1(ii), $\gamma_{hg}(C_n) = \frac{3r}{3} = r$. Thus, $k \leq r-1$. Consider the following subcases:

Subcase 1: $k = \gamma_{hg}(C_n) - 1$.

By Lemma 1, $\zeta_k^{hg}(C_n) = n - 1 = 3r - 1 = 3k + 2$.

Subcase 2: r is odd and $k = \gamma_{hg}(C_n) - 2$.

Let $D_2 = \{v_1, v_4\}$. Then D_2 is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_2] = \{v_1, v_2, v_3, v_4\}$. Thus,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_2) = 3r - |N_G^{hg}[D_2]| = 3r - 4 = 3k + 2.$$

Subcase 3: $r \geq 4$ is even and $k = \gamma_{hg}(C_n) - 2$.

Let $D_3 = \{v_1, v_{\frac{3r+2}{2}}\}$. Then D_3 is a ζ_k^{hg}-set of C_n and

$$N_G^{hg}[D_3] = \{v_1, v_3, v_{\frac{3r-2}{2}}, v_{\frac{3r+2}{2}}, v_{\frac{3r+6}{2}}, v_{3r-1}\}.$$

Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_3) = 3r + 1 - |N_G^{hg}[D_3]| = 3r - 6 = 3k.$$

Subcase 4: $r \geq 4$ and $1 \leq k \leq \gamma_{hg}(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor$ and let $D_4 = \{v_1, v_4, \cdots, v_{3r-3k-2}\}$. Then D_4 is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_4] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-2}, v_{3r-3k}, v_{3r-1}\}$. Thus, $\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_4) = n - |N_G^{hg}[D_4]| = 3r - [(3r-3k-2)+2] = 3k$. Next, let $\lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor < k \leq \gamma_{hg}(C_n) - 3$. Choose an $(r-k)$-element set $D_5 = \{v_{\lceil \frac{3r+2}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k-5}\}$. Then D_5 is a ζ_k^{hg}-set of C_n and

$$N_G^{hg}[D_5] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-5}, v_{3r-3k-3}, v_{\lceil \frac{3r+2}{2} \rceil - 2}, v_{\lceil \frac{3r+2}{2} \rceil}, v_{\lceil \frac{3r+2}{2} \rceil + 2}, v_{3r-1}\}.$$

This implies that $\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_5) = n - |N_G^{hg}[D_5]| = 3r - [(3r-3k-5)+5] = 3k$.

Case 3: $n = 3r + 1$.

By Theorem 1(ii), $\gamma_{hg}(C_n) = \frac{3r+1+2}{3} = r + 1$. Thus, $k \leq r$. Consider the following subcases:

Subcase 1: $k = \gamma_{hg}(C_n) - 1$.

By Lemma 1, $\zeta_k^{hg}(C_n) = n - 1 = 3r + 1 - 1 = 3r = 3k$.

Subcase 2: r is even and $k = \gamma_{hg}(C_n) - 2$.

Let $D_6 = \{v_1, v_4\}$. Then D_6 is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_6] = \{v_1, v_2, v_3, v_4\}$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_6) = 3r + 1 - |N_G^{hg}[D_6]| = 3r + 1 - 4 = 3k.$$

Subcase 3: $r \geq 3$ and $k = 1$.

Let $D_7 = \{v_1, v_4, v_7, \cdots, v_{3r-2}\}$ be an r-element set. Then $N_G^{hg}[D_7] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-2}, v_{3r}\}$. Thus, D_7 is a ζ_1^{hg}-set of C_n. Hence, $\zeta_1^{hg}(C_n) = \zeta_1^{hg}(D_7) = n - |N_G^{hg}[D_7]| = 3r + 1 - (3r - 1) = 2$.

Subcase 4: $r \geq 4$ and $2 \leq k \leq \gamma_{hg}(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor$ and $D_8 = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then D_8 is a ζ_k^{hg}-set of C_n and

$$N_G^{hg}[D_8] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}, v_{3r-3k+3}, v_{3r}\}.$$

This implies that $\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_8) = n - |N_G^{hg}[D_8]| = 3r+1-[(3r-3k+1)+2] = 3k-2$. Hence, $\zeta_k^{hg}(C_n) = 3k - 2$. Next, let $\lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor < k \leq \gamma_{hg}(C_n) - 3$. Choose an $(r - k + 1)$-element set $D_9 = \{v_{\lceil \frac{3r+3}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k-2}\}$. Then D_9 is a ζ_k^{hg}-set of C_n and

$$N_G^{hg}[D_9] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-2}, v_{3r-3k}, v_{\lceil \frac{3r+3}{2} \rceil - 2}, v_{\lceil \frac{3r+3}{2} \rceil}, v_{\lceil \frac{3r+3}{2} \rceil + 2}, v_{3r}\}.$$

This implies that $\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_9) = n - |N_G^{hg}[D_9]| = 3r+1-[(3r-3k-2)+5] = 3k-2$. Hence, $\zeta_k^{hg}(C_n) = 3k - 2$.

Subcase 5: $r \geq 3$ is odd and $k = \gamma_{hg}(C_n) - 2$.

Let $D_{10} = \{v_1, v_{\frac{3r+3}{2}}\}$. Then D_{10} is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_{10}] = \{v_1, v_3, v_{\frac{3r-1}{2}}, v_{\frac{3r+3}{2}}, v_{\frac{3r+7}{2}}, v_{3r}\}$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{10}) = 3r + 1 - |N_G^{hg}[D_{10}]| = 3r + 1 - 6 = 3k - 2.$$

Case 4: $n = 3r + 2$ where $r \geq 2$.

By Theorem 1(ii), $\gamma_{hg}(C_n) = \frac{3r+6}{3} = r + 2$. Thus, $k \leq r + 1$. Consider the following subcases:

Subcase 1: $k = 1$.

Let $D_{11} = \{v_1, v_4, \cdots, v_{3r-2}, v_{3r+1}\}$ be an $(r+1)$-element set. Then $N_G^{hg}[D_{11}] = V(C_n) \setminus \{v_{3r+2}\}$. Thus, D_{11} is a ζ_1^{hg}-set of C_n. Hence, $\zeta_1^{hg}(C_n) = \zeta_1^{hg}(D_{11}) = n - |N_G^{hg}[D_{11}]| = 3r + 2 - (3r + 1) = 1$.

Subcase 2: $k = \gamma_{hg}(C_n) - 1$.

By Lemma 1, $\zeta_k^{hg}(C_n) = n - 1 = 3r + 2 - 1 = 3r + 1 = 3k - 2$.

Subcase 3: r is odd and $k = \gamma_{hg}(C_n) - 2$.

Let $D_{12} = \{v_1, v_4\}$. Then D_{12} is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_{12}] = \{v_1, v_2, v_3, v_4\}$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{12}) = 3r + 2 - |N_G^{hg}[D_{12}]| = 3r + 2 - 4 = 3k - 2.$$

Subcase 4: r is even and $k = \gamma_{hg}(C_n) - 2$.

Let $r = 2$. Then $D_{13} = \{v_1, v_4\}$ is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_{13}] = \{v_1, v_2, v_3, v_4\}$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{13}) = 3r + 2 - |N_G^{hg}[D_{13}]| = 3r + 2 - 4 = 3k - 2.$$

Suppose $r \geq 4$. Then $D_{14} = \{v_1, v_{\frac{3r+4}{2}}\}$ is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_{14}] = \{v_1, v_3, v_{\frac{3r+3}{2}-2}, v_{\frac{3r+3}{2}}, v_{\frac{3r+3}{2}+2}, v_{3r+1}\}$. Hence,

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{14}) = 3r + 1 - |N_G^{hg}[D_{14}]| = 3r + 2 - 6 = 3k - 4.$$

Subcase 5: $r \geq 3$ and $2 \leq k \leq \gamma_{hg}(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor$ and let $D_{15} = \{v_1, v_4, \cdots, v_{3r-3k+4}\}$. Then D_{15} is a ζ_k^{hg}-set of C_n and $N_G^{hg}[D_{15}] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+4}, v_{3r-3k+6}, v_{3r+1}\}$. Thus, $\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{15}) = n - |N_G^{hg}[D_{15}]| = 3r + 2 - [(3r - 3k + 4) + 2] = 3k - 4$. Next, let $\lfloor \frac{\gamma_{hg}(C_n)-2}{2} \rfloor < k \leq \gamma_{hg}(C_n) - 3$. Choose an $(r - k + 2)$-element set $D_{16} = \{v_{\lceil \frac{3r+5}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then D_{16} is a ζ_k^{hg}-set of C_n and

$$N_G^{hg}[D_{16}] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}, v_{3r-3k+3}, v_{\lceil \frac{3r+5}{2} \rceil - 2}, v_{\lceil \frac{3r+5}{2} \rceil}, v_{\lceil \frac{3r+5}{2} \rceil + 2}, v_{3r+1}\}.$$

This implies that

$$\zeta_k^{hg}(C_n) = \zeta_k^{hg}(D_{16}) = n - |N_G^{hg}[D_{16}]| = 3r + 2 - [(3r - 3k + 1) + 5] = 3k - 4.$$

This proves the assertion. □

Theorem 10. *If $G = K_{m,n}$ is a complete bipartite graph with $1 \leq m \leq n$ and k is a postive integer with $k \leq \gamma_{hg}(K_m, n) - 1$, then*

$$\zeta_k^{hg}(G) = \begin{cases} k & \text{if } \ m = 1 \ \text{ and } \ k \leq n \\ n + k - 1 & \text{if } \ m = 2 \ \text{ and } \ k \leq 2 \\ m - 2 & \text{if } \ m \geq 3 \ \text{ and } \ k = 1 \\ m + n + k - 4 & \text{if } \ m \geq 3 \ \text{ and } \ k = 2, 3. \end{cases}$$

Proof. Let $k \leq \gamma_{hg}(G) - 1$. Consider the following cases:

Case 1: $m = 1$.

If $n = 1$, then $k = 1$ and $\zeta_1^{hg}(G) = 1$. Suppose $n \geq 2$. By Corollary 1(ii), $\gamma_{hg}(G) = n + 1$. Let $k \leq n$ and let $S \subseteq V(K_{1,n})$ with $|S| = n - k + 1$. Let v_0 be the central vertex of $K_{1,n}$ and $V(K_{1,n}) = \{v_0, w_1, w_2, \cdots, w_n\}$. Suppose $w_j \in V(G) \setminus S$ for some $1 \leq j \leq n$. Since $w_j \notin I_G[S]$, $w_j \notin N_G^{hg}[S]$. Suppose $v_0 \notin S$. Since $v_0 \notin N_G^2[S]$, $v_0 \notin N_G^{hg}[S]$. Therefore, $N_G^{hg}[S] \cap (V(G) \setminus S) = \varnothing$, i.e., $N_G^{hg}[S] = S$. This implies that

$$\begin{aligned} \zeta_k^{hg}(G) = \zeta_k^{hg}(S) &= |V(G)| - |N_G^{hg}[S]| \\ &= (n + 1) - (n - k + 1) \\ &= k. \end{aligned}$$

Case 2: $m = 2$.

By Corollary 1(v), $\gamma_{hg}(G) = 3$. Hence, $k \leq 2$ and any set $S \subseteq V(G)$ with $|S| = 3 - k$ is a ζ_k^{hg}-set of G. Clearly, $N_G^{hg}[S] = S$. Hence, $\zeta_k^{hg}(G) = (n + 2) - (3 - k) = n + k - 1$.

Case 3: $m \geq 3$.

By Corollary 1(v), $\gamma_{hg}(G) = 4$. This implies that $k \leq 3$. If $k = 3$, then $\zeta_3^{hg}(G) = m + n - 1 = m + n + k - 4$ by Lemma 1. If $k = 2$, then any set $S \subseteq V(G)$ with $|S| = 2$ is a ζ_2^{hg}-set of G and $N_G^{hg}[S] = S$. Hence, $\zeta_2^{hg}(G) = m + n - |S| = m + n - 2 = m + n + k - 4$. Suppose $k = 1$. Let A and B be the partite sets of G with $|A| = m$ and $|B| = n$, respectively, and let S be a set of vertices of G with $|S| = 3$. Consider the following subcases:

Subcase 1: $S \subset A$ or $S \subset B$.

Then $N_G^{hg}[S] = S$ and $\zeta_1^{hg}(S) = m + n - 3$.

Subcase 2: $|S \cap A| = 1$ and $|S \cap B| = 2$.

Then $N_G^{hg}[S] = A \cup (S \cap B)$. Thus, $N_G^{hg}[S] = m + 2$. Hence, $\zeta_1^{hg}(S) = m + n - (m + 2) =$

$n-2$.

Subcase 3: $|S \cap A| = 2$ and $|S \cap B| = 1$.

Then $N_G^{hg}[S] = (S \cap A) \cup B$. Thus, $N_G^{hg}[S] = 2 + n$. Hence, $\zeta_1^{hg}(S) = m + n - (n+2) = m-2$.

Therefore, S is a ζ_1^{hg}-set of G if $|S \cap A| = 2$ and $|S \cap B| = 1$ because $m-2 \leq n-2 \leq m+n-3$. Accordingly, $\zeta_1^{hg}(G) = m-2$. □

Theorem 11. *If $G = K_{m_1,m_2,\cdots,m_r}$ is a complete multipartite graph with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$, where $r \geq 3$, then*

$$\gamma_{hg}(G) = \begin{cases} r+1 & \text{if } m_1 = 2 \\ r+2 & \text{if } m_1 \geq 3. \end{cases}$$

Proof. Let $Q_1, Q_2, \cdots, Q_r$ be the partite sets of G with $|Q_j| = m_j$ for each $j \in [r]$. Let S be a geodetic hop dominating set of G. Suppose there exists $j \in [r] = \{1, 2, \cdots, r\}$ such that $S \cap Q_j = \varnothing$. Then $Q_j \subseteq N_G(S)$. This implies that $Q_j \cap N_G^2(S) = \varnothing$, contrary to the assumption that S is a hop dominating set. Therefore, $S \cap Q_j \neq \varnothing$ for each $j \in [r]$. Suppose $|S \cap Q_j| = 1$ for each $j \in [r]$. Then $[Q_1 \setminus (S \cap Q_1)] \cap I_G[S] = \varnothing$, contrary to the assumption that S is a geodetic set. Thus, there exists $t \in [r]$ such that $|S \cap Q_t| \geq 2$. This implies that $\gamma_{hg}(G) \geq r+1$. If $m_1 = 2$, then choose S_1 such that $|S_1 \cap Q_1| = 2$ and $|S_1 \cap Q_j| = 1$ for each $j \in [r] \setminus \{1\}$. Then S_1 is a γ_{hg}-set of G. Hence, $\gamma_{hg}(G) = r+1$. Suppose $m_1 \geq 3$. Let S_2 be a γ_{hg}-set of G. Suppose there exists exactly a single set Q_t with $|S_2 \cap Q_t| = 2$. Then $[Q_1 \setminus (S_2 \cap Q_1)] \cap I_G[S_2] = \varnothing$, a contradiction. This would imply that $\gamma_{hg}(G) = |S_2| \geq r+2$. Consider a set D with the property that $|D \cap Q_1| = |D \cap Q_2| = 2$ and $|D \cap Q_j| = 1$ for each $j \in [r] \setminus \{1, 2\}$. Then D is a geodetic hop dominating set of G and $|D| = r+2$. Since S_2 is a γ_{hg}-set of G, it follows that $\gamma_{hg}(G) = |S_2| = |D| = r+2$. □

Theorem 12. *For a complete multipartite graph $G = K_{m_1,m_2,\cdots,m_r}$ where $r \geq 3$ and $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$, we have*

$$\zeta_k^{hg}(G) = \begin{cases} n-1 & \text{if } k = \gamma_{hg}(G) - 1 \\ n-2 & \text{if } k = \gamma_{hg}(G) - 2 \\ \sum_{j=2}^{k+1} m_j & \text{if } m_1 = 2 \text{ and } 1 \leq k \leq r-2 \\ \sum_{j=1}^{k} m_j - 2 & \text{if } m_1 \geq 3 \text{ and } 1 \leq k \leq r-2, \end{cases}$$

where $n = \sum_{j=1}^{r} m_j$.

Proof. Let $Q_1, Q_2, \cdots, Q_r$ be the partite sets of G with $|Q_j| = m_j$ for each $j \in [r]$. Consider the following cases:

Case 1: $k = \gamma_{hg}(G) - 1$.

By Lemma 1, $\zeta_k^{hg}(G) = n - 1 = \sum_{j=1}^{r} m_j - 1$.

Case 2: $k = \gamma_{hg}(G) - 2$.

Then any 2-element subset S of $V(G)$ is a ζ_k^{hg}-set of G and $N_G^{hg}[S] = S$. Hence, $\zeta_k^{hg}(G) = \zeta_k^{hg}(S) = n - |N_G^{hg}[S]| = n - 2 = \sum_{j=1}^{r} m_j - 2$.

Case 3: $m_1 = 2$ and $1 \leq k \leq r - 2$.

By Theorem 11, $\gamma_{hg}(G) = r + 1$. Consider the set $D = \{q_1^1, q_1^2, q_{k+2}^1, q_{k+3}^1, \cdots, q_r^1\}$ where $q_j^t \in Q_j$ for each $j \in \{1, k+2, k+3, \cdots, r\}$ and $t \in \{1, 2\}$. Then $N_G^{hg}[D] = \cup_{j=k+2}^{r} Q_j \cup \{q_1^1, q_1^2\}$. It follows that

$$\begin{aligned}
\zeta_k^{hg}(G) &\leq \zeta_k^{hg}(D) \\
&= |V(G)| - |N_G^{hg}[D]| \\
&= \sum_{j=1}^{r} m_j - [\sum_{j=k+2}^{r} m_j + 2] \\
&= [2 + \sum_{j=2}^{r} m_j] - [\sum_{j=k+2}^{r} m_j + 2] \\
&= \sum_{j=2}^{k+1} m_j.
\end{aligned}$$

Next, let S be a ζ_k^{hg}-set of G. Then $|S| = r - k + 1$ and $\zeta_k^{hg}(G) = \zeta_k^{hg}(S)$. Suppose $S \subseteq Q_j$ for some $j \in [r]$. Then $N_G^{hg}[S] = S$ and $\zeta_k^{hg}(S) = n - r + k - 1$. Suppose $|S \cap Q_i| = 1$ for each $i \in R = \{t \in [r] : S \cap Q_t \neq \varnothing\}$. Then $|R| = |S|$, $N_G^{hg}[S] = S$ and $\zeta_k^{hg}(S) = n - r + k - 1$. For both cases, we have

$$\zeta_k^{hg}(G) = \zeta_k^{hg}(S) = n - r + k - 1 = [\sum_{j=2}^{k+1} m_j + 2 + m_{k+2} + \cdots + m_r] - (r - k + 1) \geq \sum_{j=2}^{k+1} m_j.$$

Since only two vertices, say x and y, from a partite set are needed for the elements of the other partite sets to be in the interval $I_G(x, y)$, it can be shown that S always yields a k-geodetic hop domination defect greater than or equal to $\sum_{j=2}^{k+1} m_j$. Therefore, $\zeta_k^{hg}(G) = \zeta_k^{hg}(S) = \sum_{j=2}^{k+1} m_j$.

Case 4: $m_1 \geq 3$ and $1 \leq k \leq r - 2$.

By Theorem 11, $\gamma_{hg}(G) = r + 2$. Consider the set $D' = \{q_1^1, q_1^2, q_{k+1}^1, q_{k+2}^1, \cdots, q_r^1\}$ where $q_j^t \in Q_j$ for each $j \in \{1, k+1, k+2, \cdots, r\}$ and $t \in \{1, 2\}$. Then $N_G^{hg}[D'] = \cup_{j=k+1}^r Q_j \cup \{q_1^1, q_1^2\}$. It follows that

$$\begin{aligned} \zeta_k^{hg}(G) &\leq \zeta_k^{hg}(D') \\ &= |V(G)| - |N_G^{hg}[D']| \\ &= \sum_{j=1}^{r} m_j - [\sum_{j=k+1}^{r} m_j + 2] \\ &= \sum_{j=1}^{k} m_j - 2. \end{aligned}$$

By following the arguments of the preceding case, it can be shown that if S' is a ζ_k^{hg}-set of G, then $\zeta_k^{hg}(G) = \zeta_k^{hg}(S') \geq \sum_{j=1}^k m_j - 2$. Therefore, $\zeta_k^{hg}(G) = \sum_{j=1}^k m_j - 2$. □

4. Conclusion

In this paper, we introduced a new graph invariant called the k-geodetic hop domination defect. Some bounds of the parameter were obtained. Also, we computed the k-geodetic hop domination defects of several well-known graphs. It is recommended that further investigation of this newly defined parameter be done especially on graphs trees and graphs resulting from some graph operations. Moreover, complexity of the k-geodetic hop domination defect may be a worthwhile aspect to study.

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