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Second-Order Differential Subordination with the Mittag-Leffler Operator

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Abstract. In this study, we employ the generalized Mittag-Leffler function and the Komatu integral operator to present a new linear operator in terms of the convolution and define related classes of admissible functions. Then, we derive several properties and characteristics of two-order differential subordinations and superordinations. Moreover, we establish sandwich-type results for a class of analytic functions on an open unit disc. Over and above, we derive various results involving univalent functions in some details.

2020 Mathematics Subject Classifications: 30C45, 30C80, 33E12, 26A33

Key Words and Phrases: p-valent function, Borel distribution, Inclusion relation, Integral operator, Convolution

1. Introduction

Let $\mathcal{A}$ be the class of complex-valued analytic functions of the subsequent form

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \tag{1}$$

which are defined on the open unit disc $\Delta = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$. Let S, ST and CV denote the familiar subclasses of $\mathcal{A}$ of univalent, starlike and convex functions on Δ, respectively ([1, 2]). In a very recent years, various researchers have studied a number of different subclasses of univalent functions in the context of geometric function theory (see for details [3–8]). For two analytic functions f and g belonging to $\mathcal{A}$, we say that the function f is subordinate to the function g (or g superordinate of function f), written as

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$f \prec g$ or $f(\zeta) \prec g(\zeta)$, if there exists a Schwartz function w where $w(0) = 0$, $|w(\zeta)| < 1$ and $f(\zeta) = g(w(\zeta))$. If g is univalent in Δ, then the subordination is equivalent to say $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ (see [9]). Further, the convolution (or Hadamard) product of two functions f and g, where f is given by (1) and

$$g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n,$$

is presented by Ruscheweyh [10] as $f(\zeta) * g(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n$. Let $p(\zeta)$ be an analytic function in Δ and suppose that $\psi(r_1, r_2, r_3, \zeta) : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ be a univalent function and $p(\zeta)$ satisfies the subsequent differential subordination

$$\psi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \prec \varphi(\zeta), \tag{2}$$

where $\varphi(\zeta) \in S$. Then, $p(\zeta)$ is said to be a solution of the differential subordination (2). An analytic function $\gamma(\zeta)$ is said to be a dominant to the solution (2), if $p(\zeta) \prec \gamma(\zeta)$ for all functions $p(\zeta)$ satisfies differential subordinate (2). A univalent function $\hat{\gamma}(\zeta)$ that satisfies $\hat{\gamma}(\zeta) \prec \gamma(\zeta)$ for all the subordinates $\gamma(\zeta)$ of (2) is called the best dominant of (2). The best dominant is unique to a rotation of Δ (see [11]).
Let $p(\zeta)$ be an analytic function in Δ and that $\phi(r_1, r_2, r_3, \zeta) : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ be a univalent function and $p(\zeta)$ satisfies the subsequent differential superordination

$$\varphi(\zeta) \prec \phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta), \tag{3}$$

where $\varphi(\zeta) \in S$. Then, $p(\zeta)$ is said to be a solution of differential superordination (3). An analytic function $\gamma(\zeta)$ is said to be dominant of the solution (3), if $\gamma(\zeta) \prec p(\zeta)$ for all functions $p(\zeta)$ satisfies differential superordinate (3).
A univalent function $\hat{\gamma}(\zeta)$ that satisfies $\gamma(\zeta) \prec \hat{\gamma}(\zeta)$ for all the superordinates $\gamma(\zeta)$ of (3) is called the best dominant of (3). The best dominant is unique to a rotation of Δ (see [11]).
Miller et al. [12] investigated sufficient conditions on the function p, γ and ξ for which if the $p(\zeta)$ satisfying (3) then $\gamma(\zeta) \prec p(\zeta)$.
Due to the result of Miller et al. [12], Bulboaca in [13] studied a subclass of first-order differential superordination whenever the superordination preserves operators. Moreover, Ali et al. [14] studied the sufficient condition for a function $f \in \mathcal{A}$ to satisfy

$$\gamma_1(\zeta) \prec \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \gamma_2(\zeta),$$

where $\gamma_1(\zeta)$ and $\gamma_2(\zeta)$ are analytic functions with $\gamma_1(0) = \gamma_2(0) = 1$.
A detailed investigation of subordination and superordination is given by many authors (see [15–17]).
Komatu [18] considered the linear integral operator for $\sigma \in \mathbb{C}$ as follows

$$F^{\sigma}(\zeta) = \frac{2^{\sigma}}{\gamma(\sigma)} \int_0^{\zeta} \left(\log \frac{t}{\zeta} \right)^{\sigma - 1} f(t) dt = \zeta + \sum_{n=2}^{\infty} \frac{a_n}{n^{\sigma}} \zeta^n.$$

The Pochhammer symbol, denoted by $(\mu)_m$, is defined by

$$(\mu)_m = \begin{cases} 0, & n = 0, \mu \neq 0, \\ \mu(\mu+1)...(\mu+n-1), & n \in \mathbb{N}. \end{cases} \tag{4}$$

Sharma and Jain [19] introduced the M-series as a function defined by means of the power series

$${}^{\alpha}_{p}M^{\beta}_{q}\left((a_j)_n,(b_j)_n,\zeta\right) = \sum_{n=0}^{\infty} \frac{(a_1)n...(a_p)_n}{(b_1)_n...(b_q)_n} \frac{\zeta^n}{\Gamma(\alpha n+\beta} \tag{5}$$

where $\alpha, \beta, \mu \in \mathbb{C}$, $Re(\alpha) > 0$ and $(a_j)_n, (b_j)_n$ are the Pochhammer symbols defined by (4). The series (5) is not defined if one of the parameter $a_j, b_s, j = 1, ..., p, s = 1, ..., q$ is a negative integer or zero.
The series (5) is convergent for all ζ if $p \leq q$ (see [19]).
The generalized Mittag-Leffler function is a M-series for $p = q = 1$, $a = \mu$ and $b = 1$. Thus, this function is defined by the power series [20]

$${}^{\alpha}_{1}M^{\beta}_{1}(\mu,1,\zeta) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!\Gamma(\alpha n+\beta)}\zeta^n, \qquad (\zeta \in \Delta). \tag{6}$$

where $\alpha, \beta, \mu \in \mathbb{C}$ and $Re(\alpha) > 0$.
It is clear that the series is convergent for all ζ.
A detailed investigation of analytic function by Mittag-Leffler is given by reserchers. (see [21, 22]).
The normalized form of ${}^{\alpha}_{1}M^{\beta}_{1}(\mu,1,\zeta)$ can be performed as follows

$${}_{\alpha}E^{\mu}_{\beta}(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)(\mu)_{n-1}}{(n-1)!\Gamma(\alpha(n-1)+\beta)}\zeta^n, \qquad (\zeta \in \Delta).$$

By making use of ${}_{\alpha}E^{\mu}_{\beta}$, we introduce the operator ${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} : \mathcal{A} \to \mathcal{A}$, defined in terms of the convolution as

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) = {}_{\alpha}E^{\mu}_{\beta}(\zeta) * F^{\sigma}(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)(\mu)_{n-1}}{n^{\sigma}(n-1)!\Gamma(\alpha(n-1)+\beta)}a_n\zeta^n, \qquad (\zeta \in \Delta),$$

where $\alpha, \beta, \mu, \sigma \in \mathbb{C}$ and $Re(\alpha) > 0$.
The operator ${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)$ indeed satisfies the following first-order differential recurrence relation

$$\zeta\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)\right)' = \mu{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta) - (\mu-1){}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta). \tag{7}$$

In this paper, we study a suitable class of admissible functions involving linear generalized Mittag-Leffler and Komatu integral operators. We also derive several sufficient conditions of two-order differential subordinations and superordinations of analytic univalent functions on an open unit disc Δ. Moreover, we obtain some Sandwich-type subordination of the subsequent form:

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta) \prec \gamma_2(\zeta),$$

where $\gamma_1(\zeta)$ and $\gamma_2(\zeta)$ are analytic functions with $\gamma_1(0) = 0$ and $\gamma_2(0) = 0$.

2. Preliminaries Lemma

The following lemmas are very useful in our investigation. We first recall some definitions.

Definition 1. *[12] Let $\tilde{\zeta} \in \Delta - E(f)$. The set of all functions $f(\zeta) \in S$ on $\Delta - E(f)$ such that $f'(\tilde{\zeta}) \neq 0$ is denoted by $\mathcal{H}$, where*

$$E(f) = \{\tilde{\zeta}, \tilde{\zeta} \in \partial\Delta : \lim_{\zeta \to \tilde{\zeta}} f(\zeta) = +\infty\}.$$

Definition 2. *[13] Let Ω be a subset of $\mathbb{C}$ and $\gamma \in \mathcal{H}$. The class $\Psi_n[\Omega, \gamma]$ of admissible functions, consists of the complex-valued functions $\psi : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$, which satisfy the following admissibility conditions:*

$$\psi(\theta_1, \theta_2, \theta_3; \zeta) \notin \Omega,$$

whenever

$$\theta_1 = \gamma(\tilde{\zeta}), \quad \theta_2 = m\tilde{\zeta}\gamma'(\tilde{\zeta}),$$

and

$$Re\left(\frac{\theta_3}{\theta_2} + 1\right) \geq mRe\left[\frac{\tilde{\zeta}\gamma''(\tilde{\zeta})}{\gamma'(\tilde{\zeta})} + 1\right],$$

where $\zeta \in \Delta$, $\tilde{\zeta} \in \partial\Delta - E(q)$ and $m \geq 1$.

Lemma 1. *[23] Let $\Omega \subseteq \mathbb{C}$ and $\phi \in \Psi[\Omega, \gamma]$. If $p \in \mathcal{H}$ satisfies the following condition*

$$\{\phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) : \zeta \in \Delta\} \in \Omega,$$

then we have the following differential subordination

$$p(\zeta) \prec \gamma(\zeta), \qquad (\zeta \in \Delta).$$

Definition 3. *[23] Let Ω be a subset of $\mathbb{C}$ and $\gamma \in \mathcal{H}$. The class $\Phi[\Omega, \gamma]$ of admissible complex-valued functions $\phi : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$, which satisfy the following admissibility conditions:*

$$\phi(\theta_1, \theta_2, \theta_3; \zeta) \in \Omega,$$

whenever

$$\theta_1 = \gamma(\tilde{\zeta}), \quad \theta_2 = \frac{\tilde{\zeta}\gamma'(\tilde{\zeta})}{m},$$

and

$$Re\left(\frac{\theta_3}{\theta_2} + 1\right) \geq \frac{1}{m} Re\left[\frac{\tilde{\zeta}\gamma''(\tilde{\zeta})}{\gamma'(\tilde{\zeta})} + 1\right],$$

where $\zeta \in \Delta$, $\tilde{\zeta} \in \partial\Delta - E(q)$ *and* $m \geq n$.

Lemma 2. *[23] Let* $\Omega \subseteq \mathbb{C}$ *and* $\phi \in \Phi[\Omega, \gamma]$. *If* $p \in \mathcal{H}$ *and* $\phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta)$ *is univalent in* Δ, *then*

$$\Omega \subset \{\phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) : \zeta \in \Delta\},$$

implies that the differential subordination is as follows

$$\gamma(\zeta) \prec p(\zeta), \qquad (\zeta \in \Delta).$$

Lemma 3. *[2] If* $f \in \mathcal{A}$ *is a univalent function such that* $g(\zeta) \prec f(\zeta)$. *Then* $|g(\zeta)| \leq |f(\zeta)|$, *for all* ζ *in the disc* $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$. *This radius is best possible.*

Lemma 4. *[2] If* $f \in \mathcal{A}$ *is a univalent function such that* $g(\zeta) \prec f(\zeta)$. *Then* $|g'(\zeta)| \leq |f'(\zeta)|$ *for all* ζ *in the disc* $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$. *This radius is best possible.*

3. Two-order subordination result

In this section, we derive a foundation result in the theory of second-order differential subordination. Furthermore, we will take several applications on the boundary of Δ.

Definition 4. *Let* Ω *be a subset of* $\mathbb{C}$, $\gamma \in \mathcal{H} \cap \mathcal{A}$ *and* $\mu \in \mathbb{C}, (\mu \neq 0, 1)$. *We define the class* $\Psi'(\Omega, \gamma)$ *of admissible complex valued functions* $\psi' : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ *such that the following admissibility conditions hold:*

$$\psi'(\tau_1, \tau_2, \tau_3; \zeta) \notin \Omega,$$

whenever

$$\tau_1 = \gamma(\tilde{\zeta}), \qquad \tau_2 = \frac{m\tilde{\zeta}\gamma'(\tilde{\zeta}) + (\mu - 1)\gamma(\tilde{\zeta})}{\mu},$$

and

$$Re\left(\frac{\mu^2\tau_3 - (\mu - 1)\tau_2}{\mu\tau_2 + (\mu - 1)\tau_1} - \mu + 1\right) \geq mRe\left(\frac{\tilde{\zeta}\gamma''(\tilde{\zeta})}{\gamma'(\tilde{\zeta})} + 1\right),$$

where $\zeta \in \Delta$, $\tilde{\zeta} \in \partial\Delta - E(\gamma)$ *and* $m \geq 1$.

Theorem 1. *Let* Ω *be a subset of* $\mathbb{C}$ *and* $\psi' \in \Psi'(\Omega, \gamma)$*. If* $f \in \mathcal{A}$ *satisfies*

$$\left\{\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right), \quad \zeta \in \Delta\right\} \subseteq \Omega,$$

then we have

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma(\zeta). \tag{8}$$

Proof. Assume that

$$p(\zeta) = {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta). \tag{9}$$

Then, by making (7) and (9), we obtain that

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta) = \frac{\zeta p'(\zeta) + (\mu - 1)p(\zeta)}{\mu}, \qquad (\zeta \in \Delta).$$

Moreover, a simple computation shows that

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta) = \frac{\zeta^2 p''(\zeta) + (2\mu - 1)\zeta p'(\zeta) + (\mu - 1)^2 p(\zeta)}{\mu^2}, \qquad (\zeta \in \Delta).$$

Now, we define

$$\tau_1 = \theta_1, \quad \theta_2 = \frac{\theta_2 + (\mu - 1)\theta_1}{\mu}, \quad \text{and} \quad \tau_3 = \frac{\theta_3 + (2\mu - 1)\theta_2 + (\mu - 1)^2\theta_1}{\mu^2}.$$

Further, we define the transformation h from $\mathbb{C}^3 \times \Delta$ to $\mathbb{C}$ as

$$h(\theta_1, \theta_2, \theta_3; \zeta) = \psi'(\tau_1, \tau_2, \tau_3; \zeta) = \psi'\left(\theta_1, \frac{\theta_2 + (\mu - 1)\theta_1}{\mu}, \frac{\theta_3 + (2\mu - 1)\theta_2 + (\mu - 1)^2\theta_1}{\mu^2}; \zeta\right). \tag{10}$$

From the equations (9) to (10), we have

$$h(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta), \zeta) = \psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta); \zeta\right). \tag{11}$$

Hence, the assertion (11) becomes

$$\psi'(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \notin \Omega.$$

We note that

$$\frac{\theta_3}{\theta_2} + 1 = \frac{\mu^2\tau_3 - (\mu - 1)\tau_2}{\mu\tau_2 - (\mu - 1)\tau_3} - \mu + 1.$$

Since the admissibility conditions for $\psi' \in \Psi'(\Omega, \gamma)$ are equivalent to $\psi \in \Psi(\Omega, \gamma)$ as given in Definition 2, then, by using Lemma 1 we have

$$p(\zeta) \prec \gamma(\zeta), \qquad (\zeta \in \Delta).$$

This shows that the desired differential subordination (8) is established.

The result can be extended to the case $\Omega = h(\Delta)$ in which the complex-valued function $h(\zeta)$ is a conformal mapping of Δ onto Ω. In this case, we write $\Psi'(\Omega, \gamma) = \Psi'(h, \gamma)$.

Theorem 2. *Let $\psi' \in \Psi'(h,\gamma)$. If $\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ is univalent in Δ and*

$$\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h(\zeta),$$

then, we have

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma(\zeta).$$

Proof. Following similar proof to that of Theorem [[11], Theorem 2.3c], we can proof Theorem 2. So, it is omitted.

We next consider the behaviour of γ on the boundary of Δ. The following result is an interesting consequence of Theorem 1.

Theorem 3. *Let $0 < \rho < 1$ and $h(\zeta), \gamma(\zeta) \in S$ satisfy the conditions $\gamma_\rho(\zeta) = \gamma(\rho\zeta)$ and $h_\rho(\zeta) = h(\rho\zeta)$. Let $\psi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ satisfy one of the subsequent conditions:*

(i) $\psi' \in \Psi'(h, \gamma_\rho)$,

(ii) *there exist $\rho_0 \in (0,1)$ such that $\psi' \in \Psi'(h_\rho, \gamma_\rho)$, for all $\rho \in (\rho_0, 1)$.*

If $\psi' \in \Psi'(h,\gamma)$, $\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ is analytic in Δ and

$$\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h(\zeta),$$

then we have

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma(\zeta).$$

Proof. By following the proof Theorem [[11], Theorem 2.3d], we can proof Theorem 3. So, it has been omitted.

Theorem 4. *Let $k \in \{2,3,4,...\}$, $0 < \rho < 1$, $h \in S$ and $\psi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), k\zeta^{k-1}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), k^2\zeta^{2(k-1)}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta); \zeta\right) = h(\zeta) \tag{12}$$

has a solution $\gamma(\zeta)$ with $\gamma(0) = 0$ and one of the subsequent conditions is satisfied:

(i) $\gamma \in \mathcal{H}$ *and* $\psi' \in \Psi'(h,\gamma)$,

(ii) $\gamma \in S$ *and* $\psi' \in \Psi'(h,\gamma_\rho)$, *or*

(iii) $\gamma \in S$ *and there exists* $\rho_0 \in (0,1)$ *such that*

$$\psi' \in \Psi'(h_\rho, \gamma_\rho) \quad \text{for all} \quad \rho \in (0,1).$$

If

$$p(\zeta) = {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta^k) \tag{13}$$

and $\psi'(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \in \mathcal{A}$ *such that*

$$\psi'(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \prec h(\zeta), \tag{14}$$

then $p(\zeta) \prec \gamma(\zeta)$ *and* γ *is the best dominant.*

Proof. Because of Theorems 2 and 3, we deduce that γ is dominant (14). From (7) and (13), we obtain that

$$k\zeta^{k-1} {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta^k) = \frac{\zeta p'(\zeta) + (\mu - 1)k\zeta^{k-1}p(\zeta)}{\mu}. \tag{15}$$

Moreover, a simple computation shows that

$$k^2\zeta^{2(k-1)} {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta^k) = \frac{\zeta^2 p''(\zeta) + \left(1 - k + 2(\mu - 1)k\zeta^{k+1}\right)\zeta p'(\zeta) + (\mu - 1)^2 k^2 \zeta^{2k-1} p(\zeta)}{\mu^2}. \tag{16}$$

Similar to the proof of the Theorem 1, we define the transformation $h : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ as follows

$$h\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right) = \psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta^k), k\zeta^{k-1} {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta^k), k^2\zeta^{2(k-1)} {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta^k); \zeta\right).$$

Therefore, from (12) we obtain that

$$h\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right) = \gamma(\zeta^k) \prec \gamma(\zeta).$$

Since $p(\Delta) = \gamma(\Delta)$, we conclude that γ is the best dominant. This completes the proof of Theorem 4.

In this particular case, we define the function $\gamma_2 : \Delta \longrightarrow \mathbb{C}$ as follows

$$\gamma_2(\zeta) = z - \beta_1 z^2, \qquad |\beta_1| < 1. \tag{17}$$

Now, we introduce and investigate the class $\Psi'(\Delta, \gamma_2)$ consisting of all admissible functions.

Definition 5. *Let* $\gamma_2(\zeta)$ *be given by* (22) *and* $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. *Then, we define the class of admissible functions* $\Psi'(\Delta, \gamma_2)$ *to be the set of all functions* $\psi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ *satisfying the following admissibility conditions:*

$$\Psi'(\tau_1', \tau_2', \tau_3', \zeta) \notin \Delta$$

where

$$\tau_1' = z - \beta_1 z^2, \qquad \tau_2' = \frac{me^{i\theta}(1 - 2\beta_1 e^{i\theta}) + (\mu - 1)(e^{i\theta} - 2\beta_1 e^{2i\theta})}{\mu},$$

$$\tau_3' = \frac{L + (2\mu - 1)me^{i\theta}(1 - 2\beta_1 e^{i\theta}) + (\mu - 1)^2(e^{i\theta} - \beta_1 e^{i\theta}}{\mu^2},$$

such that

$$Re\left\{\frac{Le^{-i\theta}}{1 - 2\beta_1 e^{i\theta}}\right\} \geq m^2 \frac{2\beta(1 - \cos\theta)}{1 + 2\beta(1 - 2\cos\theta)},$$

where $\zeta \in \Delta$, $\theta \in \mathbb{R}$, $|\beta_1| < 1$ *and* $m \geq 1$.

Theorem 5. *Let* $\gamma_2(\zeta)$ *be given by* (22), $\psi' \in \Psi'(\Delta, \gamma_2)$ *and* $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left\{\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right), \quad \zeta \in \Delta\right\} \in \Delta,$$

then, we have

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec z - \beta_1 z^2, \qquad (|\beta_1| < 1).$$

Proof. Similar to the proof of Theorem 1, we can proof Theorem 5

Theorem 6. *Let* $h(\zeta)$ *be a conformal mapping from* Δ *onto* Δ, $\gamma_2(\zeta)$ *be given by* (22), $\psi' \in \Psi'(\Delta, \gamma_2)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ *and* $\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ *is univalent in* Δ. *If*

$$\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h(\zeta),$$

then, we have

$$\left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)\right| \leq 1 + |\beta_1|, \tag{18}$$

for all ζ *in the disc* $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$ *and* $|\beta_1| < 1$. *This radius is best possible.*

Proof. In view of Theorem 2, we obtain that

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma_2(\zeta).$$

Now, by applying Lemma 3, we get

$$\left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)\right| \leq |\gamma_2(\zeta)|,$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$.
From the Maximum-Modulus Principle, we have

$$|\gamma_2(\zeta)| \leq 1 + |\beta_1|.$$

This establishes inequality (34). By using Lemma 3 we conclude that this radius is best possible. Hence, the proof of Theorem 6 is completed.

Plots of the suggested function $\gamma_2(\zeta) = \zeta - \beta_1\zeta^2$ in the unit disc Δ are illustrated in figure 1(a). The parameter was $\beta_1 = \frac{1}{2}$.
By putting $\psi'(\tau_1', \tau_2', \tau_3', \zeta) = \tau_2'$ in Theorem (6) we obtain the following corollary.

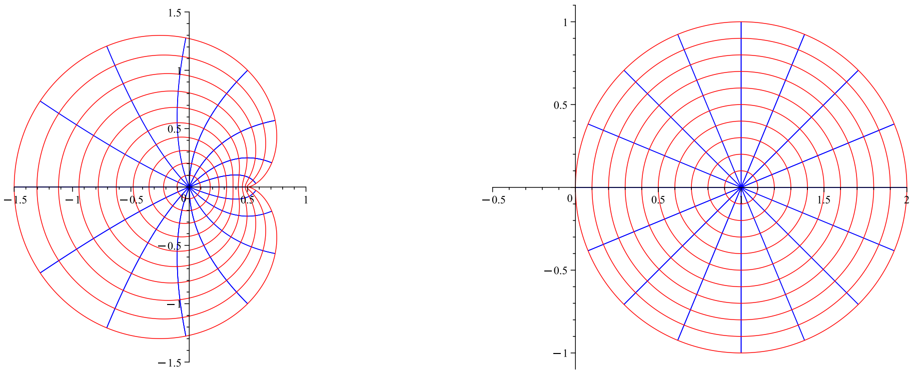

Figure 1: (a)$\gamma_2(\zeta) = \zeta - \beta\zeta^2$ for $\beta = \frac{1}{2}$, (b)$\gamma_1'(\zeta) = 1 - 2\beta\zeta$ for $\beta = \frac{1}{2}$

Corollary 1. *Let $h(\zeta)$ be a conformal mapping of Δ onto Δ, $\gamma_2(\zeta)$ be given by (22), $\psi' \in \Psi'(\Delta, \gamma_2)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\psi'\left({}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+2} f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$ {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta) \prec h(\zeta), $$

then we have

$$ \left| {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta) \right| \leq 1 + |\beta_1|, $$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$ and $|\beta_1| < 1$. This radius is best possible.

Theorem 7. *Let $h(\zeta)$ be a conformal mapping of Δ onto Δ, $\gamma_2(\zeta)$ be given by (22), $\psi' \in \Psi'(\Delta, \gamma_2)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\psi'\left({}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+2} f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$ \psi'\left({}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+2} f(\zeta), \zeta\right) \prec h(\zeta), $$

then we have

$$ \left| \frac{1}{\zeta} \left[\mu {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta) - (\mu - 1) {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta) \right] \right| \leq 1 + 2|\beta_1|, \tag{19} $$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$ and $|\beta_1 < 1$. This radius is best possible.

Proof. In view of Theorem 2, we obtain that

$$ {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta) \prec \gamma_2(\zeta). $$

Now, by applying Lemma 4, we get

$$ \left| \left({}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta) \right)' \right| \leq |\gamma_2'(\zeta)|, \tag{20} $$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3-\sqrt{8})$.
From the Maximum-Modulus Principle, we have

$$|\gamma_2'(\zeta)| \leq 1 + 2|\beta_1|. \tag{21}$$

From the inequalities (7), (21) and (20), we establish ineqality (35). By using Lemma 4 we conclude that this radius is best possible. Hence, the proof of Theorem 7 is completed.

Plots of the suggested function $\gamma_2'(\zeta) = 1 - 2\beta_1\zeta$ in the unit disc Δ are illustrated in figure 1(b). The parameter was $\beta_1 = \frac{1}{2}$.
Similarly, putting $\psi'(\tau_1', \tau_2', \tau_3', \zeta) = \tau_2'$ in Theorem (6) leads to the following corollary.

Corollary 2. *Let $h(\zeta)$ be a conformal mapping of Δ onto Δ, $\gamma_2(\zeta)$ be given by (22), $\psi' \in \Psi'(\Delta, \gamma_2)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\psi'\left({}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta), {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+2} f(\zeta), \zeta\right)$ is univalent in Δ. If*

$${}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta) \prec h(\zeta),$$

then we have

$$\left|\frac{1}{\zeta}\left[\mu {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu+1} f(\zeta) - (\mu-1) {}_{\alpha}^{\sigma}\mathcal{Q}_{\beta}^{\mu} f(\zeta)\right]\right| \leq 1 + 2|\beta_1|,$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3-\sqrt{8})$ and $|\beta_1 < 1$. This radius is best possible.

[rgb]1.00,0.00,0.00For another example, we define the function $\gamma_3 : \Delta \longrightarrow \Delta$ as follow

$$\gamma_3(\zeta) = \frac{-2\zeta}{1+\zeta}. \tag{22}$$

Now, we introduce and investigate the class of $\Psi'(\Delta, \gamma_3)$ consisting of admissible functions.

Definition 6. *Let $\gamma_3(\zeta)$ that is given by (22) and $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. Then, we define the class of admissible functions $\Psi'(\Delta, \gamma_3)$ to be the set of all function $\psi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ satisfying the following admissibility conditions:*

$$\Psi'(\tau_1^1, \tau_2^1, \tau_3^1, \zeta) \notin \Delta$$

where

$$\tau_1^1 = \gamma_3(\zeta), \qquad \tau_2^1 = \zeta\gamma_3'(\zeta)\left(m - (\mu-1)(1+\zeta)\right),$$
$$\tau_3^1 = \zeta\gamma_3''(\zeta)\left[\zeta - \frac{(2\mu-1)(1+\zeta)}{2} - \frac{(\mu-1)^2(1+\zeta)^2}{2}\right].$$

such that

$$Re\left(\frac{\mu^2\tau_3^1 - (\mu-1)\tau_2^1}{\mu\tau_2^1 + (\mu-1)\tau_1^1} - \mu + 1\right) \geq 0$$

where $\zeta \in \Delta$, $\zeta \in \partial\Delta - E(q)$ and $m \geq 1$.

Theorem 8. *Let $\gamma_3(\zeta)$ be given by (22), $\psi' \in \Psi'(\Delta, \gamma_3)$ and $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(v), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(v), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(v), v\right), \quad v \in \Delta\right\} \subseteq \Delta.$$

Then, we have

$${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \frac{-2\zeta}{1+\zeta}.$$

Proof. Similar to the proof of Theorem 1, we can proof of Theorem 8

Plots of the suggested function $\gamma_3(\zeta) = \frac{-2\zeta}{1+\zeta}$ in the unit disc Δ are illustrated in figure 2.

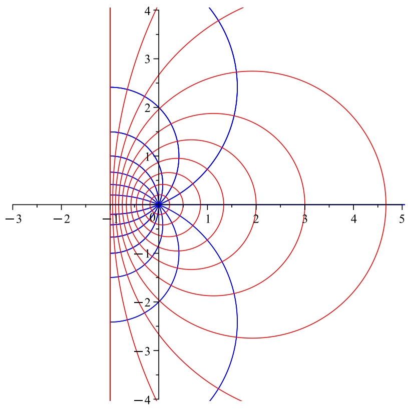

Figure 2: $\gamma_3(\zeta) = \frac{-2\zeta}{1+\zeta}$

4. Two-order superordination result

In this section, we investigate the following new class of admissible functions which yield a result of two-order differential superordination for the operator $\mathcal{Q}^{\mu}_{\beta}f(\zeta)$.

Definition 7. *Let Ω be a subset of $\mathbb{C}$, $\gamma \in \mathcal{H} \cap \mathcal{A}$ and $\mu \in \mathbb{C}, (\mu \neq 0, 1)$. We define the set $\Phi'(\Omega, \gamma)$ of admissible complex valued functions $\phi' : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ such that the subsequent admissibility conditions hold:*

$$\phi'(\tau_1, \tau_2, \tau_3; \zeta) \in \Omega,$$

whenever

$$\tau_1 = \gamma(\tilde{\zeta}), \qquad \tau_2 = \frac{\tilde{\zeta}\gamma'(\tilde{\zeta})/m_1 + \mu\gamma(\tilde{\zeta})}{\mu - 1},$$

and

$$Re\left(\frac{\mu^2\tau_3-(\mu-1)\tau_2}{\mu\tau_2+(\mu-1)\tau_1}-\mu+1\right)\geq\frac{1}{m_1}Re\left(\frac{\tilde{\zeta}\gamma''(\tilde{\zeta})}{\gamma'(\tilde{\zeta})}+1\right),$$

where $\zeta\in\Delta$, $\tilde{\zeta}\in\partial\Delta-E(\gamma)$ *and* $m_1\geq 1$.

Theorem 9. *Let* Ω *be a subset of* $\mathbb{C}$ *and* $\phi'\in\Phi'(\Omega,\gamma)$. *If* $f\in\mathcal{A}$ *satisfies*

$$\Omega\subseteq\left\{\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta),\zeta\right),\quad\zeta\in\Delta\right\},$$

then we have

$$\gamma(\zeta)\prec{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta).\tag{23}$$

Proof. Assume that

$$p(\zeta)={}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta).\tag{24}$$

Similar to the proof of Theorem 1, we can obtain the transformation $h_1:\mathbb{C}^3\times\Delta\to\mathbb{C}$ as follows

$$h_1(p(\zeta),\zeta p'(\zeta),\zeta^2p''(\zeta);\zeta)=\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta),\zeta\right).\tag{25}$$

From equations (23) and (25), we obtain

$$\Omega\subseteq\{h_1(p(\zeta),\zeta p'(\zeta),\zeta^2p''(\zeta);\zeta)\}.$$

Because of (25), we conclude that the admissibility condition $\phi'\in\Phi'(\Omega,\gamma)$ and the admissibility condition for ϕ in Definition 3 is equivalent.
Thus, by using Lemma 2, we conclude that

$$\gamma(\zeta)\prec p(\zeta).\tag{26}$$

Hence, the differential subordination (26) is equivalent to (23). This completes the proof of Theorem 9.

The result can be extended to the case $\Omega=h(\Delta)$ in which the complex-valued function $h(z)$ is a conformal mapping of Δ onto Ω. In this function, we write $\Phi'(\Omega,\gamma)=\Phi'(h,\gamma)$.

Theorem 10. *Let* $\phi'\in\Phi'(h,\gamma)$. *If* $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta),\zeta\right)$ *is univalent in* Δ *and*

$$h(\zeta)\prec\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta),{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta),\zeta\right),$$

then we have

$$\gamma(\zeta)\prec{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta).$$

Proof. Similar to the proof of Theorem 2, we can prove Theorem 10.

Theorem 11. *Let $0 < \rho < 1$ and $h(\zeta), \gamma(\zeta) \in S$ satisfy the conditions $\gamma_\rho(\zeta) = \gamma(\rho\zeta)$ and $h_\rho(\zeta) = h(\rho\zeta)$. Let $\phi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$ satisfy one of the following conditions:*

(i) $\phi' \in \Phi'(h, \gamma_\rho)$,

(ii) there exist $\rho_0 \in (0,1)$ such that $\phi' \in \Phi'(h_\rho, \gamma_\rho)$, for all $\rho \in (\rho_0, 1)$.

If $\phi' \in \Phi'(h, \gamma)$, $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right)$ is analytic in Δ and

$$h(\zeta) \prec \phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right).$$

Then, we have

$$\gamma(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta).$$

Proof. Similar to the proof of Theorem 3, we can prove Theorem 11.

Theorem 12. *Let $k \in \{2, 3, 4, ...\}$, $0 < \rho < 1$, $h \in S$ and $\phi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), k\zeta^{k-1}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), k^2\zeta^{2(k-1)}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta); \zeta\right) = h(\zeta), \tag{27}$$

has a solution $\gamma(\zeta)$ with $\gamma(0) = 0$ and one of the following conditions is satisfied:

(i) $\gamma \in \mathcal{H}$ and $\phi' \in \Phi'(h, \gamma)$,

(ii) $\gamma \in S$ and $\phi' \in \Phi'(h, \gamma_\rho)$, or

(iii) $\gamma \in S$ and there exists $\rho_0 \in (0,1)$ such that

$$\phi' \in \Phi'(h_\rho, \gamma_\rho) \quad \text{for all} \quad \rho \in (0,1).$$

If

$$p(\zeta) = {}^{\tau}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta^k),$$

and $\phi'(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \in \mathcal{A}$ such that

$$h(\zeta) \prec \phi'(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta), \tag{28}$$

then $\gamma(\zeta) \prec p(\zeta)$ and γ is the best dominant.

Proof. In view of the Theorems 10 and 11, we deduce that $\gamma(\zeta)$ is a dominant (28). By following similar proof to the proof of Theorem (9) and using the assertions (15) and (16), we define the transformation $h_1 : \mathbb{C} \times \Delta \to \mathcal{C}$ as follows

$$h_1\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right) = \phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta^k), k\zeta^{k-1}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta^k), k^2\zeta^{2(k-1)}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta^k); \zeta\right).$$

Therefore from (27), we obtain that

$$\gamma(\zeta) \prec \gamma(\zeta^k) = h_1\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right).$$

Since $p(\Delta) = \gamma(\Delta)$, we conclude that $\gamma(\zeta)$ is the best dominant. This completes the proof of Theorem 12.

In this particular case, we define the function $\gamma_1 : \Delta \longrightarrow \mathbb{C}$ as follows

$$\gamma_1(\zeta) = \zeta e^{\lambda\zeta}, \qquad 0 < \lambda \leq 1. \tag{29}$$

In what follows, we introduce the class $\Phi'(\Delta, \gamma_1)$ of admissible functions.

Definition 8. *Let $\gamma_1(\zeta)$ be given by (29) and $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. We define the class $\Phi'(\Delta, \gamma_1)$ of admissible functions $\phi' : \mathbb{C}^3 \times \Delta \longrightarrow \mathbb{C}$, which satisfy the following admissibility conditions:*

$$\Phi'(\tau_1'', \tau_2''\tau_3'', \zeta) \in \Omega,$$

whenever

$$\tau_1'' = \bar{\zeta} e^{\lambda\bar{\zeta}}, \qquad \tau_2'' = \bar{\zeta} e^{\lambda\bar{\zeta}} \frac{1 + \lambda\bar{\zeta} + m\mu}{m(\mu - 1)},$$

$$\tau_3'' = \frac{L + \bar{\zeta} e^{\lambda\bar{\zeta}}[\mu(2\mu - 1)(1 + \lambda\bar{\zeta} + m\mu) - m(\mu - 1)^2]}{m\mu^2(\mu - 1)},$$

such that

$$Re\left\{\frac{L}{\bar{\zeta} e^{\lambda\bar{\zeta}}(1 + \lambda\bar{\zeta} + m\mu)}\right\} \geq \frac{\mu}{m^2(\mu - 1)} Re\left\{\frac{\lambda}{1 + \lambda\bar{\zeta}} + \lambda + 1\right\},$$

where $\zeta \in \Delta$, $\tilde{\zeta} \in \partial\Delta - E(q)$ and $m \geq 1$.

Theorem 13. *Let $\gamma_1(\zeta)$ be given by (29), $\phi' \in \Phi'(\Delta, \gamma_2)$ and $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$. If $f \in \mathcal{A}$ satisfies*

$$\Delta \subseteq \left\{\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right), \quad \zeta \in \Delta\right\},$$

then we have

$$\zeta e^{\lambda\zeta} \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta).$$

Proof. Similar proof to the proof of Theorem 9, we can proof Theorem 13.

Theorem 14. *Let $h(\zeta)$ be a conformal mapping of Δ onto $\mathbb{C}$, $\gamma_1(\zeta)$ be given by (29), $\phi' \in \Phi'(\Delta, \gamma_2)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right)$ is univalent in Δ . If*

$$h(\zeta) \prec \phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right),$$

then we have

$$\frac{1}{e^{\lambda}} \leq \left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta)\right|, \tag{30}$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$ and $0 < \lambda \leq 1$. This radius is best possible.

Proof. In view of Theorem 10, we obtain that

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta).$$

Now, by applying Lemma 3, we get

$$|\gamma_1(\zeta)| \leq \left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta)\right|,$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$.
From the Maximum-Modulus Principle, we have

$$\frac{1}{e^{\lambda}} \leq |\gamma_1(\zeta)|.$$

This establish inequality (30). By using Lemma 3 we conclude that this radius is best possible. Hence, the proof of Theorem 14 is completed.

Plots of the suggested function $\gamma_1(\zeta) = \zeta e^{\lambda\zeta}$ in the unit disc Δ are illustrated in figure 2(c). The parameter was $\lambda = \frac{1}{4}$.
Putting $\psi'(\tau_1', \tau_2', \tau_3', \zeta) = \tau_2'$ in Theorem (14) yields the following corollary.

Corollary 3. *Let $h(\zeta)$ be a conformal mapping of Δ onto $\mathbb{C}$, $\gamma_1(\zeta)$ be given by (29), $\phi' \in \Phi'(\Delta, \gamma_1)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right)$ is univalent in Δ . If*

$$h(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta),$$

then we have

$$\frac{1}{e^{\lambda}} \leq \left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta)\right|,$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$ and $0 < \lambda \leq 1$. This radius is best possible.

Theorem 15. *Let $h(\zeta)$ be a conformal mapping of Δ onto $\mathbb{C}$, $\gamma_1(\zeta)$ be given by (29), $\phi' \in \Phi'(\Delta, \gamma_1)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$h(\zeta) \prec \phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right),$$

then we have

$$\frac{1-\lambda}{e^{\lambda}} \leq \left| \frac{1}{\zeta} \left[\mu {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta) - (\mu - 1) {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta) \right] \right|, \tag{31}$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$ and $|\beta_1 < 1$. This radius is best possible.

Proof. In view of Theorem 10, we obtain that

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta).$$

Now, by applying Lemma 4, we get

$$|\gamma_1'(\zeta)| \leq \left| \left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta) \right)' \right|, \tag{32}$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$.
From the Maximum-Modulus Principle, we have

$$\frac{1-\lambda}{e^{\lambda}} \leq |\gamma_1'(\zeta)|. \tag{33}$$

From the inequalities (7), (33) and (32), we establish inequality (31). By using Lemma 4 we conclude that this radius is best possible. Hence, the proof of Theorem 15 is completed.

Plots of the suggested function $\gamma_1'(\zeta) = e^{\lambda\zeta} + \lambda\zeta e^{\lambda\zeta}$ in the unit disc Δ are illustrated in figure 2(d). The parameter was $\lambda = \frac{1}{4}$.
Similarly, putting $\phi'(\tau_1', \tau_2', \tau_3', \zeta) = \tau_2'$ in the Theorem (15) leads to the following corollary.

Corollary 4. *Let $h(\zeta)$ be a conformal mapping of Δ onto $\mathbb{C}$, $\gamma_1(\zeta)$ be given by (22), $\phi' \in \Phi'(\Delta, \gamma_1)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\phi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta} f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$h(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta),$$

then we have

$$\frac{1-\lambda}{e^{\lambda}} \leq \left| \frac{1}{\zeta} \left[\mu {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta} f(\zeta) - (\mu - 1) {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta} f(\zeta) \right] \right|,$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$ and $0 < \lambda < 1$. This radius is best possible.

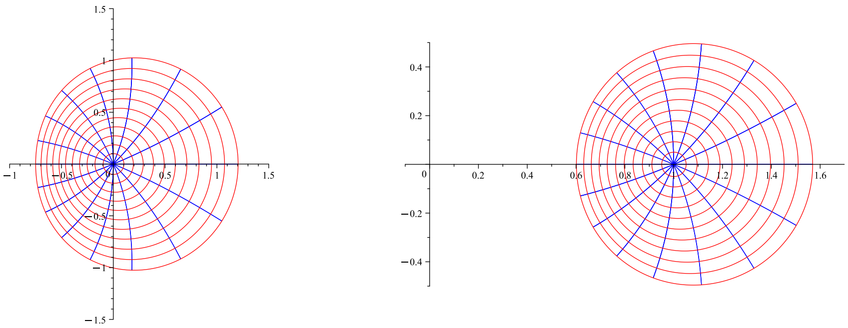

Figure 3: (c)$\gamma_1(\zeta) = \zeta e^{\lambda\zeta}$ for $\lambda = \frac{1}{4}$, (d)$\gamma_1'(\zeta) = e^{\lambda\zeta} + \frac{\zeta}{4}e^{\lambda\zeta}$ for $\lambda = \frac{1}{4}$

5. Result on Sandwich Theorems

In this section, we employ the results obtained in sections 3 and 4 and derive the sandwich-type theorem.

Theorem 16. *Let* Ω *be a subset of* $\mathbb{C}$ *and* $\varphi \in \Phi'(\Omega, \gamma_1) \bigcap \Psi'(\Omega, \gamma_2)$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left\{\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right), \quad \zeta \in \Delta\right\} = \Omega,$$

then we have

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma_2(\zeta).$$

Proof. We can combine the Theorems 1 and 9 and obtain the Theorem 16.

Theorem 17. *Let* h_1 *and* h_2 *be two conformal mapping of* Δ *onto* Ω, γ_1 *and* γ_2 *be two analytic functions in* Δ *with* $\gamma_1(0) = \gamma_2(0) = 0$ *and* $\varphi \in \Phi'(h, \gamma_1) \bigcap \Psi'(h, \gamma_2)$. *If* $f \in \mathcal{A}$, ${}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \in A \bigcap \mathcal{H}$ *and*

$$\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$$

is univalent in Δ, *then*

$$\gamma_1(\zeta) \prec \varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec \gamma_2(\zeta),$$

implies the following subordination

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma_2(\zeta).$$

Proof. We can combine Theorems 2 and 10 and obtain Theorem 17.

Theorem 18. *Let* $0 < \rho < 1$, h_1, h_2 *be two conformal mapping of* Δ *onto* Ω *satisfying the conditions* $h_{1\rho}(\zeta) = h_1(\rho\zeta)$ *and* $h_{2\rho}(\zeta) = h_2(\rho\zeta)$. *Let* γ_1 *and* γ_2 *be two analytic functions in* Δ *with* $\gamma_1(0) = \gamma_2(0) = 0$ *satisfying the conditions,* $\gamma_{1\rho}(\zeta) = \gamma_1(\rho\zeta)$ *and one of the following conditions:*

(i) $\varphi \in \Phi'(h, \gamma_\rho) \bigcap \Psi'(h, \gamma_\rho)$, *or*

(ii) there exist $\rho_0 \in (0,1)$ *such that* $\varphi \in \Phi'(h_\rho, \gamma_\rho) \bigcap \Psi'(h_\rho, \gamma_\rho)$, *for all* $\rho \in (\rho_0, 1)$.

If $\varphi \in \Phi'(h,\gamma) \bigcap \Psi'(h,\gamma)$, $\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ *is univalent in* Δ *and*

$$h_1(\zeta) \prec \varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h_2(\zeta),$$

then, we have

$$\gamma_1(\zeta) \prec {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta) \prec \gamma_2(\zeta).$$

Proof. We can combine Theorems 3 and 11, and then obtain Theorem 18.

Theorem 19. *Let* $k \in \{2,3,4,...\}$, $0 < \rho < 1$ *and* h_1, h_2 *be two conformal mapping of* Δ *onto* Ω *satisfying the conditions* $h_{1\rho}(\zeta) = h_1(\rho\zeta)$ *and* $h_{2\rho}(\zeta) = h_2(\rho\zeta)$. *Let* γ_1 *and* γ_2 *be two analytic functions in* Δ *with* $\gamma_1(0) = \gamma_2(0) = 0$ *satisfying the conditions,* $\gamma_{1\rho}(\zeta) = \gamma_1(\rho\zeta)$. *Suppose that the differential equation*

$$\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), k\zeta^{k-1}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), k^2\zeta^{2(k-1)}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta); \zeta\right) = h_1(\zeta),$$

has a solution $\gamma_1(\zeta)$ *and*

$$\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), k\zeta^{k-1}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), k^2\zeta^{2(k-1)}{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta); \zeta\right) = h_2(\zeta),$$

has a solution $\gamma_2(\zeta)$ *and one of the following conditions is satisfied:*

(i) $\gamma_1, \gamma_2 \in \mathcal{H}$ *and* $\varphi \in \Psi'(h_1, \gamma_1) \bigcap \Phi'(h_2, \gamma_2)$,

(ii) $\gamma_1, \gamma_2 \in S$ *and* $\varphi \in \Psi'(h_1, \gamma_{1\rho}) \bigcap \Phi'(h_2, \gamma_{2\rho})$, *or*

(ii) $\gamma_1, \gamma_2 \in S$ *and there exists* $\rho_0 \in (0,1)$ *such that*

$$\varphi \in \Psi'(h_{1\rho}, \gamma_{1\rho)} \bigcap \Phi'(h_{2\rho}, \gamma_{2\rho)} \quad \textit{for all} \quad \rho \in (0,1).$$

If

$$p(\zeta) = {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta^k),$$

and $\varphi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \in \mathcal{A}$ *such that*

$$h_1(\zeta) \prec \varphi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \prec h_2(\zeta),$$

then $\gamma_1(\zeta) \prec p(\zeta) \prec \gamma_2(\zeta)$ *and* γ_1, γ_2 *are the best dominant.*

Proof. We can combine Theorems 4 and 12 and obtain Theorem 19.

Theorem 20. *Let $h_1(\zeta)$ and $h_2(\zeta)$ be two conformal mapping of Δ onto $\mathbb{C}$, $\gamma_2(\zeta)$ that is given by* (22)*, $\gamma_1(\zeta)$ that is given by* (29)*, $\varphi \in \Psi'(\Delta, \gamma_2) \bigcap \Phi'(\Delta, \gamma_1)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$h_1(\zeta) \prec \psi'\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h_2(\zeta),$$

then we have

$$\frac{1}{e^{\lambda}} \leq \left|{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)\right| \leq 1 + |\beta_1|, \tag{34}$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{5})$, $0 < \lambda < 1$ and $|\beta_1| < 1$. This radius is best possible.

Proof. We can combine Theorems 6 and 14 and obtain Theorem 20.

Theorem 21. *Let $h_1(\zeta)$ and $h_2(\zeta)$ be two conformal mapping of Δ onto $\mathbb{C}$, $\gamma_2(\zeta)$ given by* (22)*, $\gamma_1(\zeta)$ is given by* (29)*, $\varphi \in \Psi'(\Delta, \gamma_2) \bigcap \Phi'(\Delta, \gamma_1)$, $\mu \in \mathbb{C}$, $(\mu \neq 0, 1)$ and $\varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right)$ is univalent in Δ. If*

$$h_1(\zeta) \prec \varphi\left({}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta), {}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+2}_{\beta}f(\zeta), \zeta\right) \prec h_2(\zeta),$$

then we have

$$\frac{1-\lambda}{e^{\lambda}} \leq \left|\frac{1}{\zeta}\left[\mu{}^{\sigma}_{\alpha}\mathcal{Q}^{\mu+1}_{\beta}f(\zeta) - (\mu - 1){}^{\sigma}_{\alpha}\mathcal{Q}^{\mu}_{\beta}f(\zeta)\right]\right| \leq 1 + 2|\beta_1|, \tag{35}$$

for all ζ in the disc $|\zeta| \leq \frac{1}{2}(3 - \sqrt{8})$, $0 < \lambda < 1$ and $|\beta_1 < 1$. This radius is best possible.

Proof. We can combine Theorems 7 and 15 and establish Theorem 21.

6. Conclusion

In this paper, new linear operator of analytic univalent functions was defined by the linear Mittag-Leffler and the Komatu integral operators. Several suitable classes of admissible functions are defined. Some reliable results for the two-order differential subordinations and superordinations are presented. Further, various Sandwich-type theorems are investigated for a class of analytic functions involving the new linear operator.

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