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On Jordan-Hölder Theorem under Intuitionistic Fuzzy Groups

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Abstract. The theory of intuitionistic fuzzy groups is an algebraic structure derivable from the utilization of groups in intuitionistic fuzzy sets. Many notions in group theory have been presented in intuitionistic fuzzy group theory. However, concepts like simple group, maximal normal subgroup, normal series, composition series, and the Jordan-Hölder Theorem are open problems in intuitionistic fuzzy group theory. Hence, this paper defines simple intuitionistic fuzzy groups, maximal normal intuitionistic fuzzy subgroups, normal series for intuitionistic fuzzy groups, and composition series for intuitionistic fuzzy groups with illustrations. In addition, the Jordan-Hölder Theorem in intuitionistic fuzzy group theory is verified. It is shown that every intuitionistic fuzzy group of a finite group possesses a composition series, and any two composition series for an intuitionistic fuzzy group of a finite group are equivalent.

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1. Introduction

The introduction of fuzzy set theory (FST) helped in the resolution of uncertainty that set theory could not handle by considering the membership degree of elements defined in a unit closed interval, $[0, 1]$. The presentation of FST by Zadeh [1] has transformed decision-making in real-life issues. Likewise, FST has prompted the study of fuzzy algebra in line with the classical algebraic structures. Rosenfeld [2] introduced fuzzy groups as the application of fuzzy sets to group theory and studied many group theoretic notions under

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fuzzy group setting. In addition, several group analogous concepts like fuzzy subgroups and Frattini fuzzy subgroups were studied [3, 4].

Because of the drawback of the FST (because it considers only the membership degree of elements), the idea of intuitionistic fuzzy sets (IFSs) was presented by Atanassov [5] and numerous properties of IFSs have been presented [6, 7]. Biswas [8, 9] applied IFSs to group theory by proposing intuitionistic fuzzy subgroups (IFSGs), Ahn [10] discussed the several forms of sublattice of lattice of IFSGs, and proved the connection of the sublattices of the lattice of IFGs. Certain properties of IFSGs were discussed in [11], Fathi and Salleh [12] presented intuitionistic fuzzy groups (IFGs) based on intuitionistic fuzzy space and discussed certain of their properties, and Yuan et al. [13] shared some additional insight on IFSGs. Bal et al. [14] presented a note of kernel subgroups on IFGs and discussed some properties of IFGs, Sharma [15] presented the direct product of IFSGs, and the homomorphism of IFGs was discussed in [16].

In addition, the α and β cuts of IFGs were discussed in [17] and the t-IFSGs was presented in [18]. Some important theorems of t-intuitionistic fuzzy isomorphism of t-IFSGs and the fuzzification of the prominent Lagrange's theorem were discussed in [19, 20]. Some algebraic descriptions of ε-IFSGs were discussed in [21], Shuaib et al. [22] discussed the idea of η-IFSG using η-IFSs and showed that each IFSG is an η-IFSG. In addition, the study defined η-intuitionistic fuzzy cosets and η-intuitionistic fuzzy normal subgroups (IFNSGs) of a given group. Gulzar et al. [23] presented some properties of t-IFSGs, Rasuli [24] discussed t- norn and s-norm of IFSGs, and Latif and Shuaib [25] applied t-IFSGs to discuss the famous Sylow theory.

Furthermore, the notion of intuitionistic anti-fuzzy subgroups was deliberated in [26] and Husban et al. [27] discussed complex intuitionistic fuzzy group (CIFG) by permitting the degrees of membership and non-membership to include complex numbers. Subsequently, Husban et al. [28] discussed normality in CIFGs with some properties, Al-Sharoa [29] presented $\alpha_{1,2}$ and $\beta_{1,2}$ cuts of CIFSGs with their algebraic properties, and Rasuli [30] discussed t- norn and s-norm for CIFSGs. Although many group theoretical properties have been discussed in IFGs, it is certain that simple intuitionistic fuzzy groups, maximal normal intuitionistic fuzzy subgroups, normal series for intuitionistic fuzzy groups, composition series for intuitionistic fuzzy groups, and the Jordan-Hölder Theorem under IFG have not been studied. The motivation for this work is to establish simple groups, maximal normal subgroups, normal series, composition series, and the Jordan-Hölder Theorem in the context of IFGs. The following are the contributions of the article:

(i) The idea of simple group and maximal normality are established in the context of IFGs.

(ii) Normal series and composition series for IFGs are presented, exemplied, and discussed with the aid of some theorems.

(iii) The famous Jordan-Hölder Theorem is investigated under IFG by using composition series for IFGs.

The rest of the paper are outlined as follows: Section 2 presents IFSs, IFSGs, and their properties; Section 3 introduces the concepts of simple IFGs, maximal normal IFSGs, normal series for IFGs, composition series for IFGs, and the Jordan-Hölder Theorem under IFGs; and Section 4 concludes the paper.

2. Preliminaries

Throughout the paper, the symbols S and G represent a non-empty set and a group, respectively.

Definition 1 ([1])**.** A fuzzy subset ζ of S is presented as:

$$\zeta = \{\langle s, \zeta_m(s)\rangle \mid s \in S\}, \tag{1}$$

where $\zeta_m \colon X \to [0,1]$ is the membership degree of $s \in X$.

Definition 2 ([2])**.** A fuzzy subset ζ of G is a fuzzy subgroup of G if

(i) $\zeta_m(xy) \geq \min\Big\{\zeta_m(x), \zeta_m(y)\Big\}$ $\forall\ x, y \in G$,

(ii) $\zeta_m(x^{-1}) = \zeta_m(x)$ $\forall\ x \in G$.

In addition, $\zeta_m(e) = \zeta_m(xx^{-1}) \geq \min\Big\{\zeta_m(x), \zeta_m(x)\Big\} = \zeta_m(x)$ $\forall\ x \in G$, where e is the unit element of G.

Definition 3 ([5])**.** An IFS λ of S is presented as:

$$\left.\begin{aligned} \lambda &= \{\langle s, \lambda_m(s), \lambda_n(s)\rangle \mid s \in S\} \\ &= \{\langle \frac{\lambda_m(s), \lambda_n(s)}{s}\rangle \mid s \in S\} \end{aligned}\right\}, \tag{2}$$

where $\lambda_m \colon X \to [0,1]$ and $\lambda_n \colon X \to [0,1]$ are the membership and non-membership grades of $s \in S$ and $0 \leq \lambda_m(s) + \lambda_n(s) \leq 1$.

Definition 4 ([6])**.** Let λ and γ be IFSs of S. Then,

(i) $\lambda = \gamma \iff \lambda_m(s) = \gamma_m(s)$ and $\lambda_n(s) = \gamma_n(s)$ $\forall s \in S$,

(ii) $\lambda \subseteq \gamma \iff \lambda_m(s) \leq \gamma_m(s)$ and $\lambda_n(s) \geq \gamma_n(s)$ $\forall s \in S$,

(iii) $\lambda \cap \gamma = \Big\{\langle s, \min\{\lambda_m(s), \gamma_m(s)\}, \max\{\lambda_n(s), \gamma_n(s)\}\rangle \mid s \in S\Big\}$,

(iv) $\lambda \cup \gamma = \Big\{\langle s, \max\{\lambda_m(s), \gamma_m(s)\}, \min\{\lambda_n(s), \gamma_n(s)\}\rangle \mid s \in S\Big\}$.

Definition 5 ([8])**.** An IFS λ of G is an IFG/IFSG of G if

(i) $\lambda_m(xy) \geq \min\Big\{\lambda_m(x), \lambda_m(y)\Big\}$ and $\lambda_n(xy) \leq \max\Big\{\lambda_n(x), \lambda_n(y)\Big\}$ $\forall\ x, y \in G$,

(ii) $\lambda_m(x^{-1}) = \lambda_m(x)$ and $\lambda_n(x^{-1}) = \lambda_n(x)$ $\forall\ x \in G$.

In addition,

$$\left.\begin{aligned} \zeta_m(e) = \lambda_m(xx^{-1}) \geq \min\Big\{\lambda_m(x), \lambda_m(x)\Big\} = \lambda_m(x), \\ \lambda_n(e) = \lambda_n(xx^{-1}) \leq \max\Big\{\lambda_n(x), \lambda_n(x)\Big\} = \lambda_n(x) \end{aligned}\right\}, \tag{3}$$

$\forall\ x \in G$, where e is the unit element of G.

The order of λ is defined by

$$|\lambda| = \sum_{i=1}^{k} \lambda_m(x_i) + \sum_{i=1}^{k} \lambda_n(x_i)\ \forall x_i \in G. \tag{4}$$

$|\lambda|$ is defined by the finiteness of G or otherwise.

Definition 6 ([8])**.** Let λ and γ be IFGs of G. Then, λ is an IFSG of γ if $\lambda \subseteq \gamma$. Again, λ is a proper IFSG of γ if $\lambda \subseteq \gamma$ and $\beta \neq \gamma$.

Definition 7 ([17])**.** Let λ be an IFG of G. Then, the support of λ defined by:

$$\lambda_* = \{x \in G \mid \lambda_m(x) > 0 \text{ and } \lambda_n(x) < 0\}, \tag{5}$$

is a subgroup of G.

Definition 8 ([31])**.** Let λ and γ be IFGs of G. Then, the product $\lambda \circ \gamma$ is an IFS defined as:

$$(\lambda \circ \gamma)(g) = \begin{cases} \bigvee_{x=yz} \min\Big\{\lambda_m(y), \gamma_m(z)\Big\}, \bigwedge_{x=yz} \max\Big\{\lambda_n(y), \gamma_n(z)\Big\}, \\ \text{if } \exists\, x, y \in G \text{ such that } x = yz \\ 0, \qquad \text{otherwise.} \end{cases} \tag{6}$$

Definition 9 ([31])**.** An IFG λ of G is commutative if $\lambda_m(xy) = \lambda_m(yx)$ and $\lambda_n(xy) = \lambda_n(yx)$ $\forall x, y \in G$. Succinctly, an IFG β is commutative if G is a commutative group.

Definition 10 ([31])**.** Let λ and γ be IFGs of G such that $\lambda \subseteq \gamma$. Then, λ is an intuitionistic fuzzy normal subgroup (IFNSG) of γ denoted as $\lambda \triangleleft \gamma$ if $\lambda_m(xy) = \lambda_m(yx)$ and $\lambda_n(xy) = \lambda_n(yx) \iff \lambda_m(y) = \lambda_m(x^{-1}yx)$ and $\lambda_n(y) = \lambda_n(x^{-1}yx)$ $\forall\ x, y \in G$.

Definition 11 ([32])**.** If λ is an IFSG of G. Then, $y\lambda$ for $y \in G$ defined by $(y\lambda)_m(x) = \lambda_m(y^{-1}x)$ and $(y\lambda)_n(x) = \lambda_n(y^{-1}x)$ $\forall\ x \in G$ is called the left intuitionistic fuzzy coset (IFCS) of G. Similarly, λy for $y \in G$ defined by $(\lambda y)_m(x) = \lambda_m(xy^{-1})$ and $(\lambda y)_n(x) = \lambda_n(xy^{-1})$ $\forall\ x \in G$ is called the right IFCS of G.

Definition 12 ([33])**.** Suppose λ and γ are IFGs of G and $\lambda \triangleleft \gamma$. Then, the collection of the left/right IFCSs of λ such that $x\lambda \circ y\lambda = xy\lambda$ $\forall\ x, y \in G$ is called an intuitionistic fuzzy factor group (IFFG) of γ by λ, represented by γ/λ.

3. Main Results

The Jordan-Hölder therorem has been discused in group [34] and the decomposition of a crossed square [35]. The idea of the Jordan-Hölder therorem has not been investigated under IFG. Before we introduce normal series for IFGs, composition series for IFGs, and Jordan-Hölder therorem under IFGs, the notions of maximal IFNSG of a IFG and simple IFGs are defined. Using Definition 10, we define a maximal IFNSG of an IFG as follows:

Definition 13. Let λ and γ be IFGs of G with $\lambda \lhd \gamma$. Then

(i) λ is a maximal non-trivial IFNSG if it is the largest proper non-trivial IFNSG of γ.

(ii) γ is simple if it has no proper non-trivial IFNSG.

Remark 1. An IFSG λ of an IFG γ in G is non-trivial if $\lambda_* = \gamma_* = G$ and $\lambda \neq \gamma$. Otherwise, it is trivial.

Example 1. For a given cyclic group $G = \{1, a, a^2, a^3\}$ where $a^4 = 1$, $a^{-1} = a^3$, $(a^2)^{-1} = a^2$, and $(a^3)^{-1} = a$, we define an IFG λ in G as:

$$\lambda = \{\langle\frac{0.9, 0.05}{1}\rangle, \langle\frac{0.5, 0.3}{a}\rangle, \langle\frac{0.6, 0.2}{a^2}\rangle, \langle\frac{0.5, 0.3}{a^3}\rangle\}.$$

Certainly, λ is commutative IFG. Then, the following are the IFSGs of λ:

$$\begin{aligned}
\lambda_1 &= \{\langle\frac{0.4, 0.55}{1}\rangle, \langle\frac{0.0, 0.8}{a}\rangle, \langle\frac{0.1, 0.5}{a^2}\rangle, \langle\frac{0.0, 0.8}{a^3}\rangle\},\\
\lambda_2 &= \{\langle\frac{0.5, 0.45}{1}\rangle, \langle\frac{0.1, 0.7}{a}\rangle, \langle\frac{0.2, 0.6}{a^2}\rangle, \langle\frac{0.1, 0.7}{a^3}\rangle\},\\
\lambda_3 &= \{\langle\frac{0.6, 0.35}{1}\rangle, \langle\frac{0.2, 0.6}{a}\rangle, \langle\frac{0.3, 0.5}{a^2}\rangle, \langle\frac{0.2, 0.6}{a^3}\rangle\},\\
\lambda_4 &= \{\langle\frac{0.7, 0.25}{1}\rangle, \langle\frac{0.3, 0.5}{a}\rangle, \langle\frac{0.4, 0.4}{a^2}\rangle, \langle\frac{0.3, 0.5}{a^3}\rangle\},\\
\lambda_5 &= \{\langle\frac{0.8, 0.15}{1}\rangle, \langle\frac{0.4, 0.4}{a}\rangle, \langle\frac{0.5, 0.3}{a^2}\rangle, \langle\frac{0.4, 0.4}{a^3}\rangle\},\\
\lambda_6 &= \{\langle\frac{0.9, 0.05}{1}\rangle, \langle\frac{0.5, 0.3}{a}\rangle, \langle\frac{0.6, 0.2}{a^2}\rangle, \langle\frac{0.5, 0.3}{a^3}\rangle\}.
\end{aligned}$$

Here, λ_6 is a trivial IFSG of λ because $\lambda_6 = \lambda$, and λ_i for $i = 1, 2, 3, 4, 5$ are proper non-trivial IFNSG of λ, and so λ is not a simple IFG. In addition, λ_5 is the maximal non-trivial IFNSG of λ.

Example 2. In a symmetry group, S_k for $k = 3$, $A_3 = \{\rho_0, \rho_1, \rho_2\} \subseteq S_3$ is a simple group ($\rho_1^{-1} = \rho_2$ and $\rho_2^{-1} = \rho_1$). Then, an IFG of A_3 is:

$$\gamma = \{\langle\frac{0.6, 0.3}{\rho_0}\rangle, \langle\frac{0.4, 0.3}{\rho_1}\rangle, \langle\frac{0.4, 0.3}{\rho_2}\rangle\}.$$

Since $\rho_1.\rho_2 = \rho_2.\rho_1 = \rho_0$, then γ is commutative. Then, the IFSGs of γ are:

$$\begin{aligned}
\gamma_1 &= \{\langle\frac{0.2, 0.7}{\rho_0}\rangle, \langle\frac{0.0, 0.7}{\rho_1}\rangle, \langle\frac{0.0, 0.7}{\rho_2}\rangle\}, \\
\gamma_2 &= \{\langle\frac{0.3, 0.6}{\rho_0}\rangle, \langle\frac{0.1, 0.6}{\rho_1}\rangle, \langle\frac{0.1, 0.6}{\rho_2}\rangle\}, \\
\gamma_3 &= \{\langle\frac{0.4, 0.5}{\rho_0}\rangle, \langle\frac{0.2, 0.5}{\rho_1}\rangle, \langle\frac{0.2, 0.5}{\rho_2}\rangle\}, \\
\gamma_4 &= \{\langle\frac{0.5, 0.4}{\rho_0}\rangle, \langle\frac{0.3, 0.4}{\rho_1}\rangle, \langle\frac{0.3, 0.4}{\rho_2}\rangle\}, \\
\gamma_5 &= \{\langle\frac{0.6, 0.3}{\rho_0}\rangle, \langle\frac{0.4, 0.3}{\rho_1}\rangle, \langle\frac{0.4, 0.3}{\rho_2}\rangle\}.
\end{aligned}$$

Here, γ_5 is a trivial IFSG of γ because $\gamma_5 = \gamma$, and γ_i for $i = 1, 2, 3, 4$ are proper non-trivial IFNSG of γ, and so γ is not simple. In addition, γ_4 is the maximal non-trivial IFNSG of γ.

Example 3. Suppose we have an IFG $\delta = \{\langle\frac{0.5,0.3}{\rho_0}\rangle, \langle\frac{0.0,1}{\rho_1}\rangle, \langle\frac{0.0,1}{\rho_2}\rangle\}$ defined in $A_3 = \{\rho_0, \rho_1, \rho_2\} \subseteq S_3$. It is observed that δ has no proper non-trivial IFSG. Thus, δ is a simple IFG.

Theorem 1. *Every IFG which has no non-trivial IFSGs is simple.*

Proof. The proof is simple from Example 3.

Definition 14. Let G be a finite group and λ be an IFG of G. Then, λ possesses a normal series if there exist:

$$\left.\begin{aligned}
\lambda_{1_m}(x) \leq \lambda_{2_m}(x) \leq \cdots \leq \lambda_{k_m}(x) = \lambda_m(x) \\
\lambda_{1_n}(x) \geq \lambda_{2_n}(x) \geq \cdots \geq \lambda_{k_n}(x) = \lambda_n(x)
\end{aligned}\right\}, \tag{7}$$

$\forall x \in G$ such that $\lambda_{1_*} = \lambda_{2_*} = \cdots = \lambda_{k_*} = \lambda_* = G$ and $\lambda_i \lhd \lambda_{i+1}$ $\forall$ $1 \leq i \leq k$. Eq. (7) can be simply written as:

$$\lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_k = \lambda \tag{8}$$

such that $\lambda_{1_*} = \lambda_{2_*} = \cdots = \lambda_{k_*} = \lambda_* = G$ and $\lambda_i \lhd \lambda_{i+1}$ $\forall$ $1 \leq i \leq k$ or

$$\lambda_1 \lhd \lambda_2 \lhd \cdots \lhd \lambda_k = \lambda \tag{9}$$

for $\lambda_{1_*} = \lambda_{2_*} = \cdots = \lambda_{k_*} = \lambda_* = G$.

Example 4. Using the IFSGs λ in Example 1, the normal series is:

$$\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3 \subseteq \lambda_4 \subseteq \lambda_5 \subseteq \lambda_6 = \lambda,$$

where $\lambda_i \lhd \lambda_{i+1}$ $\forall$ $1 \leq i \leq k$.

Example 5. Using the IFSGs of γ in Example 2, the normal series is:

$$\gamma_1 \subseteq \gamma_2 \subseteq \gamma_3 \subseteq \gamma_4 \subseteq \gamma_5 = \gamma,$$

because $\gamma_i \triangleleft \gamma_{i+1} \ \forall\ 1 \leq i \leq k$.

Next, we define the concept of composition series for IFG.

Definition 15. Let G be a finite group and λ be an IFG of G. Then, λ possesses a composition series if there exist:

$$\left.\begin{aligned} \lambda_{1_m}(x) \leq \lambda_{2_m}(x) \leq \cdots \leq \lambda_{k_m}(x) = \lambda_m(x) \\ \lambda_{1_n}(x) \geq \lambda_{2_n}(x) \geq \cdots \geq \lambda_{k_n}(x) = \lambda_n(x) \end{aligned}\right\}, \tag{10}$$

$\forall x \in G$ such that $\lambda_{1_*} = \lambda_{2_*} = \cdots = \lambda_{k_*} = \lambda_* = G$ with the properties

(i) $\lambda_i \triangleleft \lambda_{i+1} \ \forall\ 1 \leq i \leq k$,

(ii) λ_{i+1}/λ_i is simple $\forall\ 1 \leq i \leq k$.

Eq. (10) can be simply written as:

$$\lambda_1 \triangleleft \lambda_2 \triangleleft \cdots \triangleleft \lambda_k = \lambda \tag{11}$$

for $\lambda_{1_*} = \lambda_{2_*} = \cdots = \lambda_{k_*} = \lambda_* = G$ and λ_{i+1}/λ_i is simple $\forall\ 1 \leq i \leq k$.

Using the IFSGs of λ in Example 1, the composition series for λ is:

$$\lambda_1 \triangleleft \lambda_2 \triangleleft \lambda_3 \triangleleft \lambda_4 \triangleleft \lambda_5 \triangleleft \lambda_6 = \lambda,$$

where λ_{i+1}/λ_i is simple $\forall\ 1 \leq i \leq k$.

Using the IFSGs of γ in Example 2, the composition series is:

$$\gamma_1 \triangleleft \gamma_2 \triangleleft \gamma_3 \triangleleft \gamma_4 \triangleleft \gamma_5 = \gamma,$$

because λ_{i+1}/λ_i is simple $\forall\ 1 \leq i \leq k$.

Theorem 2. *Every finite IFG of a finite group has a composition series.*

Proof. Let γ be an IFG of a finite group G. We present the proof using the principle of induction. Assume every IFG of order less than $|\gamma|$ has a composition series. If γ is simple, then it has a trivial composition series. Otherwise, if γ is not simple, then it has at least one non-trivial proper IFNSG and a maximal non-trivial IFNSG denoted as δ in γ, since γ is finite. Then $|\delta| < |\gamma|$.

By the principle of induction, $|\delta|$ has a composition series, which is:

$$\left.\begin{aligned} \delta_{1_m}(x) \leq \delta_{2_m}(x) \leq \cdots \leq \delta_{k_m}(x) = \delta_m(x) \\ \delta_{1_n}(x) \geq \delta_{2_n}(x) \geq \cdots \geq \delta_{k_n}(x) = \delta_n(x) \end{aligned}\right\}, \tag{12}$$

$\forall x \in G$ such that $\delta_{1_*} = \delta_{2_*} = \cdots = \delta_{k_*} = G$ with $\delta_i \triangleleft \delta_{i+1}$ $\forall$ $1 \le i \le k$ and δ_{i+1}/δ_i is simple $\forall$ $1 \le i \le k$.

But $\delta \triangleleft \gamma$ because δ is maximal in γ, then we have

$$\left.\begin{array}{r}\delta_{1_m}(x) \le \delta_{2_m}(x) \le \cdots \le \delta_{k_m}(x) = \delta_m(x) \le \gamma_m(x)\\ \delta_{1_n}(x) \ge \delta_{2_n}(x) \ge \cdots \ge \delta_{k_n}(x) = \delta_n(x) \ge \gamma_n(x)\end{array}\right\} \tag{13}$$

$\forall x \in G$, which proves that γ has a composition series.

Theorem 3 (The Jordan-Hölder Theorem)**.** *Every IFG of a finite group has at least two equivalent composition series.*

Proof. Let γ be an IFG of a finite group G. Assume γ has two composition series:

$$\left.\begin{array}{r}\gamma_{1_m}(x) \le \gamma_{2_m}(x) \le \cdots \le \gamma_{k_m}(x) = \gamma_m(x)\\ \gamma_{1_n}(x) \ge \gamma_{2_n}(x) \ge \cdots \ge \gamma_{k_n}(x) = \gamma_n(x)\end{array}\right\}, \tag{14}$$

$$\left.\begin{array}{r}\delta_{1_m}(x) \le \delta_{2_m}(x) \le \cdots \le \delta_{l_m}(x) = \delta_m(x)\\ \delta_{1_n}(x) \ge \delta_{2_n}(x) \ge \cdots \ge \delta_{l_n}(x) = \delta_n(x)\end{array}\right\}, \tag{15}$$

such that $\gamma_i \triangleleft \gamma_{i+1}$ with γ_{i+1}/γ_i simple $\forall$ $1 \le i \le k$ and $\delta_i \triangleleft \delta_{i+1}$ with δ_{i+1}/δ_i simple $\forall$ $1 \le i \le l$. In each series, $\gamma_{1_*} = \gamma_{2_*} = \cdots = \gamma_{k_*} = G$ and $\delta_{1_*} = \delta_{2_*} = \cdots = \delta_{l_*} = G$.

Now, we show that $k = l$ and $(\gamma_2/\gamma_1, \gamma_3/\gamma_2, \cdots, \gamma_k/\gamma_{k-1})$ is identical to $(\delta_2/\delta_1, \delta_3/\delta_2, \cdots, \delta_l/\delta_{l-1})$. We establish the proof by induction on $|\gamma|$. The proof is trivial if $|\gamma| = 1$. Assume $\gamma_{(k-1)} = \delta_{(l-1)}$, then the result is straightforward by induction. On the contrary, assume $\gamma_{(k-1)} \ne \delta_{(l-1)}$. Set $A = \gamma_{(k-1)}$, $B = \delta_{(l-1)}$, and $C = A \cap B$, where C is a maximal IFNSG of $\gamma_{(k-1)}$ and $\delta_{(l-1)}$.

But, C has a composition series,

$$\left.\begin{array}{r}C_{1_m}(x) \le C_{2_m}(x) \le \cdots \le C_{t_m}(x) = C_m(x)\\ C_{1_n}(x) \ge C_{2_n}(x) \ge \cdots \ge C_{t_n}(x) = C_n(x)\end{array}\right\}, \tag{16}$$

$\forall x \in G$. Then,

$$\left.\begin{array}{r}\gamma_{1_m}(x) \le \gamma_{2_m}(x) \le \cdots \le \gamma_{(k-1)_m}(x) = A_m(x)\\ \gamma_{1_n}(x) \ge \gamma_{2_n}(x) \ge \cdots \ge \gamma_{(k-1)_n}(x) = A_n(x)\end{array}\right\} \tag{17}$$

$\forall x \in G$, and

$$\left.\begin{array}{r}C_{1_m}(x) \le C_{2_m}(x) \le \cdots \le C_{t_m}(x) = C_m(x) \le A_m(x)\\ C_{1_n}(x) \ge C_{2_n}(x) \ge \cdots \ge C_{t_n}(x) = C_n(x) \le A_n(x)\end{array}\right\} \tag{18}$$

$\forall x \in G$ are the composition series for A. By induction, we have $k-1 = t+1 \Rightarrow k-2 = t$, and

$$(\gamma_2/\gamma_1, \gamma_3/\gamma_2, \cdots, \gamma_{(k-1)}/\gamma_{(k-2)}) \sim (C_2/C_1, C_3/C_2, \cdots, C_t/C_{t-1}, A/C). \tag{19}$$

Likewise, we get

$$\left.\begin{aligned} \delta_{1_m}(x) \leq \delta_{2_m}(x) \leq \cdots \leq \delta_{(l-1)_m}(x) = B_m(x) \\ \delta_{1_n}(x) \geq \delta_{2_n}(x) \geq \cdots \geq \delta_{(l-1)_n}(x) = B_n(x) \end{aligned}\right\} \tag{20}$$

$\forall x \in G$ and

$$\left.\begin{aligned} C_{1_m}(x) \leq C_{2_m}(x) \leq \cdots \leq C_{t_m}(x) = C_m(x) \leq B_m(x) \\ C_{1_n}(x) \geq C_{2_n}(x) \geq \cdots \geq C_{t_n}(x) = C_n(x) \leq B_n(x) \end{aligned}\right\} \tag{21}$$

$\forall x \in G$, which are the composition series for B. Thus, $l-1 = t+1 \Rightarrow l-2 = t$, and

$$(\delta_2/\delta_1, \delta_3/\delta_2, \cdots, \delta_{(l-1)}/\delta_{(l-2)}) \sim (C_2/C_1, C_3/C_2, \cdots, C_t/C_{(t-1)}, B/C). \tag{22}$$

From $k-1 = t+1 = l-1$, we have $k = l$. By attaching γ/A to both sides of (19), we have

$$(\gamma_2/\gamma_1, \cdots, \gamma_{(k-1)}/\gamma_{(l-2)}, \gamma/\gamma_{(k-1)}) \sim (C_2/C_1, \cdots, C_t/C_{(t-1)}, A/C, \gamma/A). \tag{23}$$

Likewise, attaching γ/B to both sides of (22), we have

$$(\delta_2/\delta_1, \cdots, \delta_{(l-1)}/\delta_{(l-2)}, \delta/\delta_{(l-1)}) \sim (C_2/C_1, \cdots, C_t/C_{(t-1)}, B/C, \gamma/B). \tag{24}$$

The right hand side of (23) and (24) are equal except $(A/C, \gamma/A)$ and $(B/C, \gamma/B)$. Hence, $(A/C, \gamma/A) \sim (B/C, \gamma/B)$ and so

$$(\gamma_2/\gamma_1, \cdots, \gamma_k/\gamma_{(k-1)}) \sim (\delta_2/\delta_1, \cdots, \delta_l/\delta_{(l-1)}).$$

4. Conclusion

In this paper, the notions of simple IFGs, maximal normal IFSGs, normal series for IFGs, and composition series for IFGs were defined and described with examples and particular results. In addition, it was verified that every IFG of a finite group has a composition series. Finally, the Jordan-Hölder Theorem was discussed in IFS context, and it was proved that any two composition series of an IFG of a finite group are equivalent. This explorations have further enrich the study of fuzzy algebra and pave the way for the study of nilpotency under IFGs. In addition, this work can be extended to other variants of FST to enhance algebra under uncertain environments.

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References

[1] L A Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[2] A Rosenfeld. Fuzzy groups. *Journal of Mathematical Analysis and Applications*, 35(3):512–517, 1971.

[3] J M Anthony and H Sherwood. A characterization of fuzzy subgroups. *Fuzzy Sets and Systems*, 7:297–305, 1982.

[4] P A Ejegwa and J A Otuwe. Frattini fuzzy subgroups of fuzzy groups. *Journal of Universal Mathematics*, 2(2):175–182, 2019.

[5] K T Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[6] K T Atanassov. New operations defined over the intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 61:137–142, 1994.

[7] P A Ejegwa, S O Akowe, P M Otene, and J M Ikyule. An overview on intuitionistic fuzzy sets. *International Journal of Scientific and Technological Research*, 3(3):142–145, 2014.

[8] R Biswas. Intuitionistic fuzzy subgroups. *Mathematical Forum*, 10:37–46, 1989.

[9] R Biswas. Intuitionistic fuzzy subgroups. *Notes on Intuitionistic Fuzzy Sets*, 2:53–60, 1997.

[10] T C Ahn, K W Jang, S B Roh, and K Hur. A note on intuitionistic fuzzy subgroups. *Proceedings of KFIS Autumn Conference 2005*, 15(2):496–499, 2005.

[11] T C Ahn, K Hur, K W Jang, and S B Roh. Intuitionistic fuzzy subgroups. *Honam Mathematical Journal*, 28(1):31–44, 2006.

[12] M Fathi and A R Salleh. Intuitionistic fuzzy groups. *Asian Journal of Algebra*, 2:1–10, 2009.

[13] X H Yuan, H X Li, and E S Lee. On the definition of the intuitionistic fuzzy subgroups. *Computers and Mathematics with Applications*, 59(9):3117–3129, 2010.

[14] M Bal, K D Ahmad, A A Hajjari, and R Ali. A short note on the kernel subgroup of intuitionistic fuzzy groups. *Journal of Neutrosophic and Fuzzy Systems*, 2(1):14–20, 2022.

[15] P K Sharma. On the direct product of intuitionistic fuzzy subgroups. *International Mathematical Forum*, 7(211):523–530, 2012.

[16] P K Sharma. Homomorphism of intuitionistic fuzzy groups. *International Mathematical Forum*, 6(64):3169–3178, 2011.

[17] P K Sharma. (α, β)-Cut of intuitionistic fuzzy groups. *International Mathematical Forum*, 6(53):2605–2614, 2011.

[18] P K Sharma. t-Intuitionistic fuzzy subgroups. *International Journal of Fuzzy Mathematics and Systems*, 3:233–243, 2012.

[19] L Latif, U Shuaib, H Alolaiyan, and A Razaq. On fundamental theorems of t-intuitionistic fuzzy isomorphism of t-intuitionistic fuzzy subgroups. *IEEE Access*, 6:74547–74556, 2018.

[20] H Alolaiyan, U Shuaib, L Latif, and A Razaq. t-Intuitionistic fuzzification of lagrange's theorem of t-intuitionistic fuzzy subgroup. *IEEE Access*, 7:158419–158426, 2019.

[21] U Shuaib, M Amin, S Dilbar, and F Tahir. On algebraic attributes of $\sum$-intuitionistic fuzzy subgroups. *International Journal of Mathematics and Computer Science*, 15(1):1–17, 2019.

[22] U Shuaib, H Alolaiyan, A Razaq, S Dilbar, and F Tahir. On some algebraic aspects of η-intuitionistic fuzzy subgroups. *Journal of Taibah University for Science*, 14(1):463–469, 2020.

[23] M Gulzar, D Alghazzawi, M H Mateen, and N Kausar. A certain class of t-intuitionistic fuzzy subgroups. *IEEE Access*, 14:163260–163268, 2020.

[24] R Rasuli. Intuitionistic fuzzy subgroups with respect to norms (t, s). *Engineering and Applied Science Letters*, 3:40–53, 2020.

[25] L Latif and U Shuaib. Application of t-intuitionistic fuzzy subgroup to sylow theory. *Heliyon*, 9:e19822, 2023.

[26] D Y Li, C Y Zhang, and S Q Ma. *The intuitionistic anti-fuzzy subgroup in group G, In: Cao BY, Zhang CY, Li TF (eds) Fuzzy Information and Engineering, Advances in Soft Computing, vol 54.* Springer, Berlin, Heidelberg, 2009.

[27] R A Husban and A R Salleh A G B Ahmad. Complex intuitionistic fuzzy group. *Global Journal of Pure and Applied Mathematics*, 12:4929–4949, 2016.

[28] R A Husban and A R Salleh A G B Ahmad. Complex intuitionistic fuzzy normal subgroup. *International Journal of Pure and Applied Mathematics*, 115:455–466, 2017.

[29] D Al-Sharoa. $(\alpha_{1,2}, \beta_{1,2})$-complex intuitionistic fuzzy subgroups and its algebraic structure. *AIMS Mathematics*, 8:8082–8116, 2023.

[30] R Rasuli. Intuitionistic fuzzy complex subgroups with respect to norms (t,s). *Journal of Fuzzy Extension and Applications*, 4:92–114, 2023.

[31] K Hur and S Y Jang. The lattice of intuitionistic fuzzy congruences. *International Mathematical Forum*, 1:211–236, 2006.

[32] P K Sharma. Intuitionistic fuzzy groups. *International Journal of Data Ware housing and Mining*, 1:86–94, 2011.

[33] C Xu. *New structures of intuitionistic fuzzy groups, In: Huang DS, Wunsch DC, Levine DS, Jo KH (eds) Advanced Intelligent Computing Theories and Applications. With Aspects of Contemporary Intelligent Computing Techniques, Communications in Computer and Information Science, vol 15.* Springer, Berlin, Heidelberg, 2008.

[34] C E Watts. A Jordan-Hölder theorem. *Pacific Journal of Mathematics*, 14(2):731–734, 1964.

[35] U Gürdal. A Jordan-Hölder theorem for crossed squares. *Kuwait Journal of Science*, 50(2):83–90, 2023.