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Some Characterizations of (r, s)-Fuzzy b-Open Sets with Applications in Double Fuzzy Topological Spaces

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Abstract. In this paper, we displayed and characterized a novel class of fuzzy open sets ($\mathcal{F}$-open sets) in double fuzzy topological spaces ($\mathcal{DFTS}s$) based on Šostak's sense, called (r, s)-fuzzy b-open sets ((r, s)-$\mathcal{F}$-b-open sets). This class is contained in the class of (r, s)-$\mathcal{F}$-β-open sets and contains all (r, s)-$\mathcal{F}$-α-open sets, (r, s)-$\mathcal{F}$-pre-open sets, and (r, s)-$\mathcal{F}$-semi-open sets. Next, we explored and studied the notion of $\mathcal{DF}$-b-continuity between $\mathcal{DFTS}s$ $(G, \Im, \Im^*)$ and $(Z, \digamma, \digamma^*)$. We also defined and discussed the notions of $\mathcal{DF}$-almost b-continuity and $\mathcal{DF}$-weakly b-continuity, which are weaker forms of $\mathcal{DF}$-b-continuity. Thereafter, we presented and investigated novel $\mathcal{DF}$-mappings via (r, s)-$\mathcal{F}$-b-open and (r, s)-$\mathcal{F}$-b-closed sets. Finally, we introduced some novel types of $\mathcal{DF}$-separation axioms, called (r, s)-$\mathcal{F}$-b-regular and (r, s)-$\mathcal{F}$-b-normal spaces, and studied some properties of them.

2020 Mathematics Subject Classifications: 54A05, 54A40, 54C05, 54C08, 54D15

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1. Introduction

The concept of a fuzzy set ($\mathcal{F}$-set) of a nonempty set G is a mapping $\mathcal{M} : G \to I$ (where $I = [0, 1]$). This concept was first defined in 1965 by Zadeh [1]. The concept of an $\mathcal{F}$-topology was presented in 1968 by the author of [2]. Several authors have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. According to Šostak [3], the notion of an $\mathcal{F}$-topology being a crisp subclass of the class of $\mathcal{F}$-sets and fuzziness in the notion of openness of an $\mathcal{F}$-set have not been considered, which seems to be a drawback in the process of fuzzification of a topological space. Thus, the author of [3] introduced a novel definition of an $\mathcal{F}$-topology as the concept of openness

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of $\mathcal{F}$-sets. It is an extension of an $\mathcal{F}$-topology introduced by Chang [2]. Also, many researchers (Ramadan [4], Chattopadhyay et. al. [5], El Gayyar et. al. [6], Höhle and Šostak [7], Ramadan et. al. [8], Kim et. al. [9], Abbas [10, 11], Kim and Abbas [12], Aygun and Abbas [13, 14], Li and Shi [15, 16], Shi and Li [17], Fang and Guo [18], El-Dardery et. al. [19], Kalaivani and Roopkumar [20], Solovyov [21], Minana and Šostak [22]) have redefined the same notion and studied $\mathcal{FTS}s$ being unaware of Šostak's work.

The notion of an intuitionistic $\mathcal{F}$-set was defined by Atanassov [23, 24], which is a generalization of an $\mathcal{F}$-set [1]. Coker [25, 26] presented the notion of an intuitionistic $\mathcal{F}$-topology based on Chang's sense [2]. After that, the notion of an intuitionistic $\mathcal{F}$-topology based on Šostak's sense [3] was introduced by the authors of [27, 28]. The name (intuitionistic) was replaced with the name (double) by Garcia and Rodabaugh [29]. In addition, the notions of (r, s)-$\mathcal{F}$-semi-open, (r, s)-$\mathcal{F}$-pre-open, and (r, s)-$\mathcal{F}$-α-open sets were introduced by the authors of [30, 31] based on Šostak's sense [3]. Also, lots of creative studies about the theories of an intuitionistic $\mathcal{F}$-set have been considered by several researchers; see [32–39].

The layout of this study is as follows.

• In Section 3, we present and investigate a novel class of $\mathcal{F}$-open sets in $\mathcal{DFTS}s$ based on Šostak's sense [3], called (r, s)-$\mathcal{F}$-b-open sets. Furthermore, we define and discuss the notions of $\mathcal{DF}$-b-closure operators and $\mathcal{DF}$-b-interior operators.

• In Section 4, we introduce and discuss the concept of $\mathcal{DF}$-b-continuity between $\mathcal{DFTS}s$ $(G, \Im, \Im^*)$ and $(Z, \digamma, \digamma^*)$. In addition, we display and characterize the concepts of $\mathcal{DF}$-weakly b-continuity and $\mathcal{DF}$-almost b-continuity, which are weaker forms of $\mathcal{DF}$-b-continuity.

• In Section 5, we explore and characterize some novel $\mathcal{DF}$-mappings using (r, s)-$\mathcal{F}$-b-open and (r, s)-$\mathcal{F}$-b-closed sets. We also introduce novel types of $\mathcal{DF}$-separation axioms, called (r, s)-$\mathcal{F}$-b-regular and (r, s)-$\mathcal{F}$-b-normal spaces, and discuss some properties of them.

• In Section 6, we close this paper with conclusions and proposed future papers.

2. Preliminaries

In this study, nonempty sets will be denoted by G, Z, Q, etc. On G, I^G is the class of all $\mathcal{F}$-sets. For any $\mathcal{F}$-set $\mathcal{M} \in I^G$, $\mathcal{M}^c(g) = 1 - \mathcal{M}(g)$, for each $g \in G$. Also, for $\theta \in I$, $\underline{\theta}(g) = \theta$, for each $g \in G$.

An $\mathcal{F}$-point g_θ on G is an $\mathcal{F}$-set, and is defined as follows: $g_\theta(v) = \theta$ if $v = g$, and $g_\theta(v) = 0$ for any $v \in G - \{g\}$. Moreover, we say that g_θ belongs to $\mathcal{M} \in I^G$ ($g_\theta \in \mathcal{M}$), if

$\theta \leq \mathcal{M}(g)$. On G, $P_\theta(G)$ is the class of all $\mathcal{F}$-points.

On G, an $\mathcal{F}$-set $\mathcal{M} \in I^G$ is a quasi-coincident with $\mathcal{N} \in I^G$ ($\mathcal{M}\, q\, \mathcal{N}$), if there is $g \in G$, with $\mathcal{M}(g) + \mathcal{N}(g) > 1$. Otherwise, $\mathcal{M}$ is not a quasi-coincident with $\mathcal{N}$ ($\mathcal{M}\, \overline{q}\, \mathcal{N}$).

Lemma 1. *[40]* Let $\mathcal{M}, \mathcal{N} \in I^G$. Thus,

(i) $\mathcal{M}\, q\, \mathcal{N}$ iff there is $g_\theta \in \mathcal{M}$ such that $g_\theta\, q\, \mathcal{N}$,

(ii) $\mathcal{M} \wedge \mathcal{N} \neq \underline{0}$ if $\mathcal{M}\, q\, \mathcal{N}$,

(iii) $\mathcal{M}\, \overline{q}\, \mathcal{N}$ iff $\mathcal{M} \leq \mathcal{N}^c$,

(iv) $\mathcal{M} \leq \mathcal{N}$ iff $g_\theta \in \mathcal{M}$ implies $g_\theta \in \mathcal{N}$ iff $g_\theta\, q\, \mathcal{M}$ implies $g_\theta\, q\, \mathcal{N}$,

(v) $g_\theta\, \overline{q}\, \bigvee_{i\in\Gamma} \mathcal{M}_i$ iff there is $i_\circ \in \Gamma$ such that $g_\theta\, \overline{q}\, \mathcal{M}_{i_\circ}$.

Definition 1. *[27, 35, 38]* A double fuzzy topology ($\mathcal{DFT}$) on G is a pair $(\Im, \Im^*)$ of the mappings $\Im, \Im^* : I^G \longrightarrow I$, which satisfy the following conditions:

(i) $\Im(\mathcal{M}) + \Im^*(\mathcal{M}) \leq 1$, $\forall\ \mathcal{M} \in I^G$.

(ii) $\Im(\mathcal{M} \wedge \mathcal{N}) \geq \Im(\mathcal{M}) \wedge \Im(\mathcal{N})$ and $\Im^*(\mathcal{M} \wedge \mathcal{N}) \leq \Im^*(\mathcal{M}) \vee \Im^*(\mathcal{N})$, $\forall\ \mathcal{M}, \mathcal{N} \in I^G$.

(iii) $\Im(\bigvee_{i\in\Gamma} \mathcal{M}_i) \geq \bigwedge_{i\in\Gamma} \Im(\mathcal{M}_i)$ and $\Im^*(\bigvee_{i\in\Gamma} \mathcal{M}_i) \leq \bigvee_{i\in\Gamma} \Im^*(\mathcal{M}_i)$, $\forall\ \{\mathcal{M}_i\}_{i\in\Gamma} \subset I^G$.

Thus, $(G, \Im, \Im^*)$ is said to be an $\mathcal{DFTS}$ based on Šostak's sense [3].

Definition 2. *[28, 30, 35]* In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$ (where $I_\circ = (0, 1]$ and $I_1 = [0, 1)$), we define $\mathcal{DF}$-operators $C_{\Im^*}$ and $I_{\Im^*} : I^G \times I_\circ \times I_1 \to I^G$ as follows:

$$C_{\Im^*}(\mathcal{M}, r, s) = \bigwedge \{\mathcal{N} \in I^G : \mathcal{M} \leq \mathcal{N},\ \Im(\mathcal{N}^c) \geq r, \Im^*(\mathcal{N}^c) \leq s\}.$$

$$I_{\Im^*}(\mathcal{M}, r, s) = \bigvee \{\mathcal{N} \in I^G : \mathcal{N} \leq \mathcal{M},\ \Im(\mathcal{N}) \geq r, \Im^*(\mathcal{N}) \leq s\}.$$

Definition 3. *[30, 31, 35]* Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-set $\mathcal{M} \in I^G$ is said to be (r,s)-$\mathcal{F}$-regularly-open (resp. (r,s)-$\mathcal{F}$-pre-open, (r,s)-$\mathcal{F}$-semi-open, (r,s)-$\mathcal{F}$-β-open, and (r,s)-$\mathcal{F}$-α-open) if $\mathcal{M} = I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)$ (resp. $\mathcal{M} \leq I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)$, $\mathcal{M} \leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s)$, $\mathcal{M} \leq C_{\Im^*}(I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s),r,s)$, and $\mathcal{M} \leq I_{\Im^*}(C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s),r,s)$).

Definition 4. *[28, 35, 38]* An $\mathcal{F}$-mapping $\mathbb{P}: (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is said to be

(i) $\mathcal{DF}$-continuous if $\Im(\mathbb{P}^{-1}(\mathcal{N})) \geq \digamma(\mathcal{N})$ and $\Im^*(\mathbb{P}^{-1}(\mathcal{N})) \leq \digamma^*(\mathcal{N})$, $\forall\, \mathcal{N} \in I^Z$;

(ii) $\mathcal{DF}$-open if $\digamma(\mathbb{P}(\mathcal{M})) \geq \Im(\mathcal{M})$ and $\digamma^*(\mathbb{P}(\mathcal{M})) \leq \Im^*(\mathcal{M})$, $\forall\, \mathcal{M} \in I^G$;

(iii) $\mathcal{DF}$-closed if $\digamma((\mathbb{P}(\mathcal{M}))^c) \geq \Im(\mathcal{M}^c)$ and $\digamma^*((\mathbb{P}(\mathcal{M}))^c) \leq \Im^*(\mathcal{M}^c)$, $\forall\, \mathcal{M} \in I^G$.

Definition 5. *[30, 31, 35]* Let $(G, \Im, \Im^*)$ and $(Z, \digamma, \digamma^*)$ be $\mathcal{DFTS}s$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-mapping $\mathbb{P}: I^G \longrightarrow I^Z$ is said to be $\mathcal{DF}$-α-continuous (resp. $\mathcal{DF}$-pre-continuous, $\mathcal{DF}$-semi-continuous, and $\mathcal{DF}$-β-continuous) if $\mathbb{P}^{-1}(\mathcal{N})$ is an (r,s)-$\mathcal{F}$-α-open set (resp. (r,s)-$\mathcal{F}$-pre-open set, (r,s)-$\mathcal{F}$-semi-open set, and (r,s)-$\mathcal{F}$-β-open set), for every $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$.

Some basic notations and results that we need in the sequel are found in [27, 28, 30, 31, 35, 36].

3. On (r,s)-fuzzy b-open and b-closed sets

Here, we present and study a new class of $\mathcal{F}$-open sets, called (r,s)-$\mathcal{F}$-b-open sets in $\mathcal{DFTS}$ $(G, \Im, \Im^*)$ based on Šostak's sense [3]. Also, we explore and investigate the concepts of $\mathcal{DF}$-b-interior operators and $\mathcal{DF}$-b-closure operators.

Definition 6. Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-set $\mathcal{M} \in I^G$ is said to be an (r,s)-$\mathcal{F}$-b-open set if $\mathcal{M} \leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s) \vee I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)$.

Definition 7. Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-set $\mathcal{M} \in I^G$ is said to be an (r,s)-$\mathcal{F}$-b-closed set if $\mathcal{M} \geq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s) \wedge I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)$.

Remark 1. The complement of (r,s)-$\mathcal{F}$-b-open sets (resp. (r,s)-$\mathcal{F}$-b-closed sets) are (r,s)-$\mathcal{F}$-b-closed sets (resp. (r,s)-$\mathcal{F}$-b-open sets).

Proposition 1. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$, then

(i) every (r,s)-$\mathcal{F}$-pre-open set is (r,s)-$\mathcal{F}$-b-open;

(ii) every (r,s)-$\mathcal{F}$-b-open set is (r,s)-$\mathcal{F}$-β-open;

(iii) every (r,s)-$\mathcal{F}$-semi-open set is (r,s)-$\mathcal{F}$-b-open.

Proof.
(i) If $\mathcal{M}$ is an (r,s)-$\mathcal{F}$-pre-open set, then

$$\begin{aligned}\mathcal{M} &\leq I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)\\ &\leq I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s) \vee I_{\Im^*}(\mathcal{M},r,s)\\ &\leq I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s) \vee C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s).\end{aligned}$$

Thus, $\mathcal{M}$ is (r,s)-$\mathcal{F}$-b-open.

(ii) If $\mathcal{M}$ is an (r,s)-$\mathcal{F}$-b-open set, then

$$\begin{aligned}\mathcal{M} &\leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s) \vee I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)\\ &\leq C_{\Im^*}(I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s),r,s) \vee I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s)\\ &\leq C_{\Im^*}(I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s),r,s).\end{aligned}$$

Thus, $\mathcal{M}$ is (r,s)-$\mathcal{F}$-β-open.

(iii) If $\mathcal{M}$ is an (r,s)-$\mathcal{F}$-semi-open set, then

$$\begin{aligned}\mathcal{M} &\leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s)\\ &\leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s) \vee I_{\Im^*}(\mathcal{M},r,s)\\ &\leq C_{\Im^*}(I_{\Im^*}(\mathcal{M},r,s),r,s) \vee I_{\Im^*}(C_{\Im^*}(\mathcal{M},r,s),r,s).\end{aligned}$$

Thus, $\mathcal{M}$ is (r,s)-$\mathcal{F}$-b-open.

Remark 2. From the previous discussions and definitions, we have the following diagram.

		(r,s)-$\mathcal{F}$-**pre-open set**		
	$\nearrow$	$\downarrow$		
(r,s)-$\mathcal{F}$-α-**open set**	$\longrightarrow$	(r,s)-$\mathcal{F}$-**b-open set**	$\longrightarrow$	(r,s)-$\mathcal{F}$-β-**open set**
	$\searrow$	$\uparrow$		
		(r,s)-$\mathcal{F}$-**semi-open set**		

Remark 3. The converse of the above diagram fails as Examples 1, 2, and 3 will show.

Example 1. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.4}, \frac{g_2}{0.3}\}$, $\mathcal{N} = \{\frac{g_1}{0.2}, \frac{g_2}{0.6}\}$, $\mathcal{U} = \{\frac{g_1}{0.5}, \frac{g_2}{0.7}\}$. Define $\Im, \Im^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N} \wedge \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N} \vee \mathcal{M}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N} \wedge \mathcal{M}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N} \vee \mathcal{M}, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{U}$ is an $(\frac{1}{4}, \frac{1}{2})$-$\mathcal{F}$-$b$-open set, but it is neither $(\frac{1}{4}, \frac{1}{2})$-$\mathcal{F}$-pre-open nor $(\frac{1}{4}, \frac{1}{2})$-$\mathcal{F}$-$\alpha$-open.

Example 2. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}\}$, $\mathcal{N} = \{\frac{g_1}{0.7}, \frac{g_2}{0.8}\}$, $\mathcal{U} = \{\frac{g_1}{0.5}, \frac{g_2}{0.4}\}$. Define $\Im, \Im^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{U}$ is an $(\frac{1}{3}, \frac{1}{2})$-$\mathcal{F}$-$b$-open set, but it is not $(\frac{1}{3}, \frac{1}{2})$-$\mathcal{F}$-semi-open.

Example 3. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.5}, \frac{g_2}{0.4}\}$, $\mathcal{U} = \{\frac{g_1}{0.4}, \frac{g_2}{0.5}\}$. Define $\Im, \Im^* : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{U}$ is an $(\frac{1}{3}, \frac{1}{2})$-$\mathcal{F}$-$\beta$-open set, but it is not $(\frac{1}{3}, \frac{1}{2})$-$\mathcal{F}$-$b$-open.

Corollary 1. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, $r \in I_\circ$, and $s \in I_1$, we have the following properties:

(i) the union of (r, s)-$\mathcal{F}$-b-open sets is (r, s)-$\mathcal{F}$-b-open;

(ii) the intersection of (r, s)-$\mathcal{F}$-b-closed sets is (r, s)-$\mathcal{F}$-b-closed.

Proof. This is easily proved by Definitions 6 and 7.

Corollary 2. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each (r, s)-$\mathcal{F}$-b-closed set $\mathcal{M} \in I^G$:

(i) If $\mathcal{M}$ is (r, s)-$\mathcal{F}$-regularly-open, then $\mathcal{M}$ is (r, s)-$\mathcal{F}$-pre-closed.

(ii) If $\mathcal{M}$ is (r, s)-$\mathcal{F}$-regularly-closed, then $\mathcal{M}$ is (r, s)-$\mathcal{F}$-semi-closed.

(iii) If $I_{\Im^*}(\mathcal{M}, r, s) = \underline{0}$, then $\mathcal{M}$ is (r, s)-$\mathcal{F}$-semi-closed.

(iv) If $C_{\Im^*}(\mathcal{M}, r, s) = \underline{0}$, then $\mathcal{M}$ is (r, s)-$\mathcal{F}$-pre-closed.

Proof. The proof follows by Definitions 3 and 7.

Corollary 3. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each (r, s)-$\mathcal{F}$-b-open set $\mathcal{N} \in I^G$:

(i) If $\mathcal{N}$ is (r, s)-$\mathcal{F}$-regularly-open, then $\mathcal{N}$ is (r, s)-$\mathcal{F}$-semi-open.

(ii) If $\mathcal{N}$ is (r, s)-$\mathcal{F}$-regularly-closed, then $\mathcal{N}$ is (r, s)-$\mathcal{F}$-pre-open.

(iii) If $I_{\Im^*}(\mathcal{N}, r, s) = \underline{0}$, then $\mathcal{N}$ is (r, s)-$\mathcal{F}$-pre-open.

(iv) If $C_{\Im^*}(\mathcal{N}, r, s) = \underline{0}$, then $\mathcal{N}$ is (r, s)-$\mathcal{F}$-semi-open.

Proof. The proof follows by Definitions 3 and 6.

Definition 8. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$, we define an $\mathcal{DF}$-b-closure operator $bC_{\Im^*} : I^G \times I_\circ \times I_1 \longrightarrow I^G$ as follows: $bC_{\Im^*}(\mathcal{M}, r, s) = \bigwedge \{\mathcal{N} \in I^G : \mathcal{M} \leq \mathcal{N},\ \mathcal{N}$ is (r,s)-$\mathcal{F}$-b-closed$\}$.

Proposition 2. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-set $\mathcal{M}$ is (r, s)-$\mathcal{F}$-b-closed iff $bC_{\Im^*}(\mathcal{M}, r, s) = \mathcal{M}$.

Proof. This is easily proved from Definition 8.

Theorem 1. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M}, \mathcal{N} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{DF}$-operator $bC_{\Im^*} : I^G \times I_\circ \times I_1 \longrightarrow I^G$ satisfies the following properties.

(i) $bC_{\Im^*}(\underline{0}, r, s) = \underline{0}$.

(ii) $\mathcal{M} \leq bC_{\Im^*}(\mathcal{M}, r, s) \leq C_{\Im^*}(\mathcal{M}, r, s)$.

(iii) $bC_{\Im^*}(\mathcal{M}, r, s) \leq bC_{\Im^*}(\mathcal{N}, r, s)$ if $\mathcal{M} \leq \mathcal{N}$.

(iv) $bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s) = bC_{\Im^*}(\mathcal{M}, r, s)$.

(v) $bC_{\Im^*}(\mathcal{M} \vee \mathcal{N}, r, s) \geq bC_{\Im^*}(\mathcal{M}, r, s) \vee bC_{\Im^*}(\mathcal{N}, r, s)$.

(vi) $bC_{\Im^*}(C_{\Im^*}(\mathcal{M}, r, s), r, s) = C_{\Im^*}(\mathcal{M}, r, s)$.

Proof. (i), (ii), and (iii) are easily proved by Definition 8.

(iv) From (ii) and (iii), $bC_{\Im^*}(\mathcal{M}, r, s) \leq bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)$. Now, we show $bC_{\Im^*}(\mathcal{M}, r, s) \geq bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)$. If $bC_{\Im^*}(\mathcal{M}, r, s)$ does not contain $bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)$, there is $g \in G$ and $\theta \in (0, 1)$ with $bC_{\Im^*}(\mathcal{M}, r, s)(g) < \theta < bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)(g)$. $(\mathcal{G})$

Since $bC_{\Im^*}(\mathcal{M}, r, s)(g) < \theta$, by Definition 8, there is $\mathcal{U} \in I^G$ as an (r, s)-$\mathcal{F}$-b-closed set and $\mathcal{M} \leq \mathcal{U}$ with $bC_{\Im^*}(\mathcal{M}, r, s)(g) \leq \mathcal{U}(g) < \theta$. Since $\mathcal{M} \leq \mathcal{U}$, then $bC_{\Im^*}(\mathcal{M}, r, s) \leq \mathcal{U}$. Again, by the definition of $bC_{\Im^*}$, then $bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s) \leq \mathcal{U}$.

Hence, $bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)(g) \leq \mathcal{U}(g) < \theta$, which is a contradiction for $(\mathcal{G})$. Thus, $bC_{\Im^*}(\mathcal{M}, r, s) \geq bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s)$. Therefore, $bC_{\Im^*}(bC_{\Im^*}(\mathcal{M}, r, s), r, s) = bC_{\Im^*}(\mathcal{M}, r, s)$.

(v) Since $\mathcal{M} \leq \mathcal{M} \vee \mathcal{N}$ and $\mathcal{N} \leq \mathcal{M} \vee \mathcal{N}$, hence by (iii), $bC_{\Im^*}(\mathcal{M}, r, s) \leq bC_{\Im^*}(\mathcal{M} \vee \mathcal{N}, r, s)$ and $bC_{\Im^*}(\mathcal{N}, r, s) \leq bC_{\Im^*}(\mathcal{M} \vee \mathcal{N}, r, s)$. Thus, $bC_{\Im^*}(\mathcal{M} \vee \mathcal{N}, r, s) \geq bC_{\Im^*}(\mathcal{M}, r, s) \vee bC_{\Im^*}(\mathcal{N}, r, s)$.

(vi) From Proposition 2 and the fact that $C_{\Im^*}(\mathcal{M}, r, s)$ is an (r, s)-$\mathcal{F}$-b-closed set, then $bC_{\Im^*}(C_{\Im^*}(\mathcal{M}, r, s), r, s) = C_{\Im^*}(\mathcal{M}, r, s)$.

Definition 9. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$, we define an $\mathcal{DF}$-b-interior operator $bI_{\Im^*} : I^G \times I_\circ \times I_1 \longrightarrow I^G$ as follows: $bI_{\Im^*}(\mathcal{M}, r, s) = \bigvee \{\mathcal{N} \in I^G : \mathcal{N} \leq \mathcal{M},\ \mathcal{N}$ is (r, s)-$\mathcal{F}$-b-open$\}$.

Proposition 3. Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$. Then

(i) $bC_{\Im^*}(\mathcal{M}^c, r, s) = (bI_{\Im^*}(\mathcal{M}, r, s))^c$;

(ii) $bI_{\Im^*}(\mathcal{M}^c, r, s) = (bC_{\Im^*}(\mathcal{M}, r, s))^c$.

Proof. (i) For each $\mathcal{M} \in I^G$, we have $bC_{\Im^*}(\mathcal{M}^c, r, s) = \bigwedge\{\mathcal{N} \in I^G : \mathcal{M}^c \leq \mathcal{N},\ \mathcal{N}$ is (r, s)-$\mathcal{F}$-b-closed$\} = [\bigvee\{\mathcal{N}^c \in I^G : \mathcal{N}^c \leq \mathcal{M},\ \mathcal{N}^c$ is (r, s)-$\mathcal{F}$-b-open$\}]^c = (bI_{\Im^*}(\mathcal{M}, r, s))^c$.

(ii) This is similar to that of (i).

Proposition 4. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{F}$-set $\mathcal{M}$ is (r, s)-$\mathcal{F}$-b-open iff $bI_{\Im^*}(\mathcal{M}, r, s) = \mathcal{M}$.

Proof. This is easily proved from Definition 9.

Theorem 2. In an $\mathcal{DFTS}$ $(G, \Im, \Im^*)$, for each $\mathcal{M}, \mathcal{N} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{DF}$-operator $bI_{\Im^*} : I^G \times I_\circ \times I_1 \longrightarrow I^G$ satisfies the following properties.

(i) $bI_{\Im^*}(\underline{1}, r, s) = \underline{1}$.

(ii) $I_{\Im^*}(\mathcal{M}, r, s) \leq bI_{\Im^*}(\mathcal{M}, r, s) \leq \mathcal{M}$.

(iii) $bI_{\Im^*}(\mathcal{M}, r, s) \leq bI_{\Im^*}(\mathcal{N}, r, s)$ if $\mathcal{M} \leq \mathcal{N}$.

(iv) $bI_{\Im^*}(bI_{\Im^*}(\mathcal{M}, r, s), r, s) = bI_{\Im^*}(\mathcal{M}, r, s)$.

(v) $bI_{\Im^*}(\mathcal{M}, r, s) \wedge bI_{\Im^*}(\mathcal{N}, r, s) \geq bI_{\Im^*}(\mathcal{M} \wedge \mathcal{N}, r, s)$.

Proof. The proof is similar to that of Theorem 1.

4. On double fuzzy b-continuity and b-irresoluteness

Here, we display and discuss the concept of $\mathcal{DF}$-b-continuity between $\mathcal{DFTS}s$ based on Šostak's sense [3]. Moreover, we present and study the notions of $\mathcal{DF}$-almost b-continuity and $\mathcal{DF}$-weakly b-continuity.

Definition 10. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is called $\mathcal{DF}$-b-continuous if $\mathbb{P}^{-1}(\mathcal{N})$ is an (r, s)-$\mathcal{F}$-b-open set, for each $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$.

Remark 4. From the previous definitions, we have the following diagram.

$$\begin{array}{ccccc} & & \mathcal{DF}\textbf{-pre-continuity} & & \\ & \nearrow & \downarrow & & \\ \mathcal{DF}\text{-}\alpha\textbf{-continuity} & \longrightarrow & \mathcal{DF}\textbf{-b-continuity} & \longrightarrow & \mathcal{DF}\text{-}\beta\textbf{-continuity} \\ & \searrow & \uparrow & & \\ & & \mathcal{DF}\textbf{-semi-continuity} & & \end{array}$$

Remark 5. The converse of the above diagram fails as Examples 4, 5, and 6 will show.

Example 4. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.4}, \frac{g_2}{0.3}\}$, $\mathcal{N} = \{\frac{g_1}{0.2}, \frac{g_2}{0.6}\}$, $\mathcal{U} = \{\frac{g_1}{0.5}, \frac{g_2}{0.7}\}$. Define $\Im, \Im^*, \digamma, \digamma^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N} \wedge \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N} \vee \mathcal{M}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N} \wedge \mathcal{M}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N} \vee \mathcal{M}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\digamma(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 0, & \text{otherwise}, \end{cases} \qquad \digamma^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, \digamma, \digamma^*)$ is $\mathcal{DF}$-b-continuous, but it is neither $\mathcal{DF}$-pre-continuous nor $\mathcal{DF}$-α-continuous.

Example 5. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}\}$, $\mathcal{N} = \{\frac{g_1}{0.7}, \frac{g_2}{0.8}\}$, $\mathcal{U} = \{\frac{g_1}{0.5}, \frac{g_2}{0.4}\}$. Define $\Im, \Im^*, \digamma, \digamma^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\digamma(\mathcal{V})=\begin{cases}1, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if} \quad \mathcal{V}=\mathcal{U},\\ 0, & \text{otherwise,}\end{cases} \qquad \digamma^{*}(\mathcal{V})=\begin{cases}0, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \mathcal{V}=\mathcal{U},\\ 1, & \text{otherwise.}\end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P}:(G,\Im,\Im^{*})\longrightarrow(G,\digamma,\digamma^{*})$ is $\mathcal{DF}$-b-continuous, but it is not $\mathcal{DF}$-semi-continuous.

Example 6. Let $G=\{g_1,g_2\}$ and define $\mathcal{M},\mathcal{U}\in I^G$ as follows: $\mathcal{M}=\{\frac{g_1}{0.5},\frac{g_2}{0.4}\}$, $\mathcal{U}=\{\frac{g_1}{0.4},\frac{g_2}{0.5}\}$. Define $\Im,\Im^{*},\digamma,\digamma^{*}:I^G\longrightarrow I$ as follows:

$$\Im(\mathcal{V})=\begin{cases}1, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \mathcal{V}=\mathcal{M},\\ 0, & \text{otherwise,}\end{cases} \qquad \Im^{*}(\mathcal{V})=\begin{cases}0, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \mathcal{V}=\mathcal{M},\\ 1, & \text{otherwise,}\end{cases}$$

$$\digamma(\mathcal{V})=\begin{cases}1, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if} \quad \mathcal{V}=\mathcal{U},\\ 0, & \text{otherwise,}\end{cases} \qquad \digamma^{*}(\mathcal{V})=\begin{cases}0, & \text{if} \quad \mathcal{V}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \mathcal{V}=\mathcal{U},\\ 1, & \text{otherwise.}\end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P}:(G,\Im,\Im^{*})\longrightarrow(G,\digamma,\digamma^{*})$ is $\mathcal{DF}$-β-continuous, but it is not $\mathcal{DF}$-b-continuous.

Theorem 3. An $\mathcal{F}$-mapping $\mathbb{P}:(G,\Im,\Im^{*})\longrightarrow(Z,\digamma,\digamma^{*})$ is $\mathcal{DF}$-b-continuous iff for any $g_\theta\in P_\theta(G)$ and any $\mathcal{N}\in I^Z$ with $\digamma(\mathcal{N})\geq r$ and $\digamma^{*}(\mathcal{N})\leq s$ containing $\mathbb{P}(g_\theta)$, there is $\mathcal{M}\in I^G$ that is (r,s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M})\leq\mathcal{N}$.

Proof. ($\Rightarrow$) Let $g_\theta\in P_\theta(G)$ and $\mathcal{N}\in I^Z$ with $\digamma(\mathcal{N})\geq r$ and $\digamma^{*}(\mathcal{N})\leq s$ containing $\mathbb{P}(g_\theta)$, and then $\mathbb{P}^{-1}(\mathcal{N})\leq bI_{\Im^{*}}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$. Since $g_\theta\in\mathbb{P}^{-1}(\mathcal{N})$, then we obtain $g_\theta\in bI_{\Im^{*}}(\mathbb{P}^{-1}(\mathcal{N}),r,s)=\mathcal{M}$ (say). Hence, $\mathcal{M}\in I^G$ is (r,s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M})\leq\mathcal{N}$.

($\Leftarrow$) Let $g_\theta\in P_\theta(G)$ and $\mathcal{N}\in I^Z$ with $\digamma(\mathcal{N})\geq r$ and $\digamma^{*}(\mathcal{N})\leq s$ containing $\mathbb{P}(g_\theta)$. According to the assumption there is $\mathcal{M}\in I^G$ that is (r,s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M})\leq\mathcal{N}$. Hence, $g_\theta\in\mathcal{M}\leq\mathbb{P}^{-1}(\mathcal{N})$ and $g_\theta\in bI_{\Im^{*}}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$. Thus, $\mathbb{P}^{-1}(\mathcal{N})\leq bI_{\Im^{*}}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$, so $\mathbb{P}^{-1}(\mathcal{N})$ is an (r,s)-$\mathcal{F}$-b-open set. Then, $\mathbb{P}$ is $\mathcal{DF}$-b-continuous.

Theorem 4. Let $\mathbb{P}:(G,\Im,\Im^{*})\longrightarrow(Z,\digamma,\digamma^{*})$ be an $\mathcal{F}$-mapping, $r\in I_\circ$, and $s\in I_1$. Then the following statements are equivalent for every $\mathcal{M}\in I^G$ and $\mathcal{N}\in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-continuous.

(ii) $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-closed, for every $\mathcal{N} \in I^Z$ with $F(\mathcal{N}^c) \geq r$ and $F^*(\mathcal{N}^c) \leq s$.

(iii) $\mathbb{P}(bC_{\Im^*}(\mathcal{M},r,s)) \leq C_{F^*}(\mathbb{P}(\mathcal{M}),r,s)$.

(iv) $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(C_{F^*}(\mathcal{N},r,s))$.

(v) $\mathbb{P}^{-1}(I_{F^*}(\mathcal{N},r,s)) \leq bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$.

Proof. (i) $\Leftrightarrow$ (ii) The proof follows by $\mathbb{P}^{-1}(\mathcal{N}^c) = (\mathbb{P}^{-1}(\mathcal{N}))^c$ and Definition 10.

(ii) $\Rightarrow$ (iii) Let $\mathcal{M} \in I^G$. By (ii), we have $\mathbb{P}^{-1}(C_{F^*}(\mathbb{P}(\mathcal{M}),r,s))$ is (r,s)-$\mathcal{F}$-b-closed. Thus,

$$bC_{\Im^*}(\mathcal{M},r,s) \leq bC_{\Im^*}(\mathbb{P}^{-1}(\mathbb{P}(\mathcal{M})),r,s) \leq bC_{\Im^*}(\mathbb{P}^{-1}(C_{F^*}(\mathbb{P}(\mathcal{M}),r,s)),r,s) = \mathbb{P}^{-1}(C_{F^*}(\mathbb{P}(\mathcal{M}),r,s)).$$

Therefore, $\mathbb{P}(bC_{\Im^*}(\mathcal{M},r,s)) \leq C_{F^*}(\mathbb{P}(\mathcal{M}),r,s)$.

(iii) $\Rightarrow$ (iv) Let $\mathcal{N} \in I^Z$. By (iii), $\mathbb{P}(bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s)) \leq C_{F^*}(\mathbb{P}(\mathbb{P}^{-1}(\mathcal{N})),r,s) \leq C_{F^*}(\mathcal{N},r,s)$. Thus, $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(\mathbb{P}(bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s))) \leq \mathbb{P}^{-1}(C_{F^*}(\mathcal{N},r,s))$.

(iv) $\Leftrightarrow$ (v) The proof follows by $\mathbb{P}^{-1}(\mathcal{N}^c) = (\mathbb{P}^{-1}(\mathcal{N}))^c$ and Proposition 3.

(v) $\Rightarrow$ (i) Let $\mathcal{N} \in I^Z$ with $F(\mathcal{N}) \geq r$ and $F^*(\mathcal{N}) \leq s$. By (v), we obtain $\mathbb{P}^{-1}(\mathcal{N}) = \mathbb{P}^{-1}(I_{F^*}(\mathcal{N},r,s)) \leq bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(\mathcal{N})$. Then, $bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) = \mathbb{P}^{-1}(\mathcal{N})$. Thus, $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-open, so $\mathbb{P}$ is $\mathcal{DF}$-b-continuous.

Definition 11. An $\mathcal{F}$-mapping $\mathbb{P} : (G,\Im,\Im^*) \longrightarrow (Z,F,F^*)$ is called $\mathcal{DF}$-b-irresolute if $\mathbb{P}^{-1}(\mathcal{N})$ is an (r,s)-$\mathcal{F}$-b-open set, for every (r,s)-$\mathcal{F}$-b-open set $\mathcal{N} \in I^Z$.

Lemma 2. Every $\mathcal{DF}$-b-irresolute mapping is $\mathcal{DF}$-b-continuous.

Proof. The proof follows by Definitions 10 and 11.

Remark 6. The converse of Lemma 2 fails as Example 7 will show.

Example 7. Let $G = \{g_1, g_2\}$ and define $\mathcal{M},\mathcal{N} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.5}, \frac{g_2}{0.5}\}$, $\mathcal{N} = \{\frac{g_1}{0.5}, \frac{g_2}{0.4}\}$. Define $\Im, \Im^*, F, F^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise,} \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\digamma(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 0, & \text{otherwise}, \end{cases} \qquad \digamma^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, \digamma, \digamma^*)$ is $\mathcal{DF}$-b-continuous, but it is not $\mathcal{DF}$-b-irresolute.

Theorem 5. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ be an $\mathcal{F}$-mapping, $r \in I_\circ$, and $s \in I_1$. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute.

(ii) $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-closed, for every $\mathcal{N}$ is (r,s)-$\mathcal{F}$-b-closed.

(iii) $\mathbb{P}(bC_{\Im^*}(\mathcal{M}, r, s)) \le bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(iv) $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \le \mathbb{P}^{-1}(bC_{\digamma^*}(\mathcal{N}, r, s))$.

(v) $\mathbb{P}^{-1}(bI_{\digamma^*}(\mathcal{N}, r, s)) \le bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s)$.

Proof. (i) $\Leftrightarrow$ (ii) The proof follows by $\mathbb{P}^{-1}(\mathcal{N}^c) = (\mathbb{P}^{-1}(\mathcal{N}))^c$ and Definition 11.

(ii) $\Rightarrow$ (iii) Let $\mathcal{M} \in I^G$. By (ii), we have $\mathbb{P}^{-1}(bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s))$ is (r,s)-$\mathcal{F}$-b-closed. Thus,

$$bC_{\Im^*}(\mathcal{M}, r, s) \le bC_{\Im^*}(\mathbb{P}^{-1}(\mathbb{P}(\mathcal{M})), r, s) \le bC_{\Im^*}(\mathbb{P}^{-1}(bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)), r, s) = \mathbb{P}^{-1}(bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)).$$

Therefore, $\mathbb{P}(bC_{\Im^*}(\mathcal{M}, r, s)) \le bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(iii) $\Rightarrow$ (iv) Let $\mathcal{N} \in I^Z$. By (iii), $\mathbb{P}(bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s)) \le bC_{\digamma^*}(\mathbb{P}(\mathbb{P}^{-1}(\mathcal{N})), r, s) \le bC_{\digamma^*}(\mathcal{N}, r, s)$. Thus, $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \le \mathbb{P}^{-1}(\mathbb{P}(bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s))) \le \mathbb{P}^{-1}(bC_{\digamma^*}(\mathcal{N}, r, s))$.

(iv) $\Leftrightarrow$ (v) The proof follows by $\mathbb{P}^{-1}(\mathcal{N}^c) = (\mathbb{P}^{-1}(\mathcal{N}))^c$ and Proposition 3.

(v) $\Rightarrow$ (i) Let $\mathcal{N} \in I^Z$ be an (r,s)-$\mathcal{F}$-b-open set. By (v),

$$\mathbb{P}^{-1}(\mathcal{N}) = \mathbb{P}^{-1}(bI_{\digamma^*}(\mathcal{N}, r, s)) \le bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \le \mathbb{P}^{-1}(\mathcal{N}).$$

Thus, $bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) = \mathbb{P}^{-1}(\mathcal{N})$. Therefore, $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-open, so $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute.

Proposition 5. Let $(G, \Im, \Im^*)$, (Q, η, η^*) and $(Z, \digamma, \digamma^*)$ be $\mathcal{DFTS}s$, and $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Q, \eta, \eta^*)$, $\mathbb{Y} : (Q, \eta, \eta^*) \longrightarrow (Z, \digamma, \digamma^*)$ be two $\mathcal{F}$-mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is $\mathcal{DF}$-b-irresolute (resp. $\mathcal{DF}$-b-continuous) if $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute and $\mathbb{Y}$ is $\mathcal{DF}$-b-irresolute (resp. $\mathcal{DF}$-b-continuous).

Proof. The proof follows from Definitions 10 and 11.

Definition 12. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is called $\mathcal{DF}$-almost b-continuous if $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N}, r, s), r, s)), r, s)$, for every $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$.

Lemma 3. Every $\mathcal{DF}$-b-continuous mapping is $\mathcal{DF}$-almost b-continuous.

Proof. The proof follows by Definitions 10 and 12.

Remark 7. The converse of Lemma 3 fails as Example 8 will show.

Example 8. Let $G = \{g_1, g_2, g_3\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.4}, \frac{g_2}{0.2}, \frac{g_3}{0.4}\}$, $\mathcal{N} = \{\frac{g_1}{0.5}, \frac{g_2}{0.5}, \frac{g_3}{0.4}\}$, $\mathcal{U} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}, \frac{g_3}{0.6}\}$. Define $\Im, \Im^*, \digamma, \digamma^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\digamma(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 0, & \text{otherwise}. \end{cases} \qquad \digamma^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, \digamma, \digamma^*)$ is $\mathcal{DF}$-almost b-continuous, but it is not $\mathcal{DF}$-b-continuous.

Theorem 6. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is $\mathcal{DF}$-almost b-continuous iff for any $g_\theta \in P_\theta(G)$ and any $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ containing $\mathbb{P}(g_\theta)$, there is $\mathcal{M} \in I^G$ that is (r, s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq I_{\digamma^*}(C_{\digamma^*}(\mathcal{N}, r, s), r, s)$.

Proof. ($\Rightarrow$) Let $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ containing $\mathbb{P}(g_\theta)$, and then $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)),r,s)$. Since $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$, then

$$g_\theta \in bI_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)),r,s) = \mathcal{M} \quad \text{(say)}.$$

Therefore, $\mathcal{M} \in I^G$ is (r,s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)$.

($\Leftarrow$) Let $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ such that $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$. According to the assumption there is $\mathcal{M} \in I^G$ that is (r,s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)$. Hence, $g_\theta \in \mathcal{M} \leq \mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s))$ and

$$g_\theta \in bI_{\Im}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)),r,s).$$

Thus, $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)),r,s)$. Therefore, $\mathbb{P}$ is $\mathcal{DF}$-almost b-continuous.

Theorem 7. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent:

(i) $\mathbb{P}$ is $\mathcal{DF}$-almost b-continuous.

(ii) $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-open, for every (r,s)-$\mathcal{F}$-regularly open set $\mathcal{N} \in I^Z$.

(iii) $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-closed, for every (r,s)-$\mathcal{F}$-regularly closed set $\mathcal{N} \in I^Z$.

(iv) $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N},r,s))$, for every (r,s)-$\mathcal{F}$-b-open set $\mathcal{N} \in I^Z$.

(v) $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N},r,s))$, for every (r,s)-$\mathcal{F}$-semi-open set $\mathcal{N} \in I^Z$.

Proof. (i) $\Rightarrow$ (ii) Let $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ be an (r,s)-$\mathcal{F}$-regularly open set with $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$. Hence, by (i), there is $\mathcal{M} \in I^G$ that is (r,s)-$\mathcal{F}$-b-open with $g_\theta \in \mathcal{M}$ and $\mathbb{P}(\mathcal{M}) \leq I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)$. Thus, $\mathcal{M} \leq \mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N},r,s),r,s)) = \mathbb{P}^{-1}(\mathcal{N})$ and $g_\theta \in bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$. Therefore, $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s)$, so $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-open.

(ii) $\Rightarrow$ (iii) If $\mathcal{N} \in I^Z$ is (r,s)-$\mathcal{F}$-regularly closed, then by (ii), $\mathbb{P}^{-1}(\mathcal{N}^c) = (\mathbb{P}^{-1}(\mathcal{N}))^c$ is (r,s)-$\mathcal{F}$-b-open. Thus, $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-closed.

(iii) $\Rightarrow$ (iv) If $\mathcal{N} \in I^Z$ is (r,s)-$\mathcal{F}$-b-open and since $C_{\digamma^*}(\mathcal{N},r,s)$ is (r,s)-$\mathcal{F}$-regularly closed, then by (iii), $\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N},r,s))$ is (r,s)-$\mathcal{F}$-b-closed. Since $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N},r,s))$, hence

$$bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}),r,s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N},r,s)).$$

(iv) $\Rightarrow$ (v) The proof follows from the fact that any (r,s)-$\mathcal{F}$-semi-open set is (r,s)-$\mathcal{F}$-b-open.

(v) $\Rightarrow$ (iii) If $\mathcal{N} \in I^Z$ is (r,s)-$\mathcal{F}$-regularly closed, then $\mathcal{N}$ is (r,s)-$\mathcal{F}$-semi-open. By (v),

$$bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \leq \mathbb{P}^{-1}(C_{F^*}(\mathcal{N}, r, s)) = \mathbb{P}^{-1}(\mathcal{N}).$$

Hence, $\mathbb{P}^{-1}(\mathcal{N})$ is (r,s)-$\mathcal{F}$-b-closed.

(iii) $\Rightarrow$ (i) If $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ with $F(\mathcal{N}) \geq r$ and $F^*(\mathcal{N}) \leq s$ such that $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$, and then $g_\theta \in \mathbb{P}^{-1}(I_{F^*}(C_{F^*}(\mathcal{N}, r, s), r, s))$. Since $[I_{F^*}(C_{F^*}(\mathcal{N}, r, s), r, s)]^c$ is (r,s)-$\mathcal{F}$-regularly closed, then by (iii), we have $\mathbb{P}^{-1}([I_{F^*}(C_{F^*}(\mathcal{N}, r, s), r, s)]^c)$ is (r,s)-$\mathcal{F}$-b-closed. Hence, $\mathbb{P}^{-1}(I_{F^*}(C_{F^*}(\mathcal{N}), r, s))$ is (r,s)-$\mathcal{F}$-b-open and $g_\theta \in bI_{\Im^*}(\mathbb{P}^{-1}(I_{F^*}(C_{F^*}(\mathcal{N}, r, s), r, s)), r, s)$. Thus,

$$\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(I_{F^*}(C_{F^*}(\mathcal{N}, r, s), r, s)), r, s).$$

Therefore, $\mathbb{P}$ is $\mathcal{DF}$-almost b-continuous.

Definition 13. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ is called $\mathcal{DF}$-weakly b-continuous if $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(C_{F^*}(\mathcal{N}, r, s)), r, s)$, for every $\mathcal{N} \in I^Z$ with $F(\mathcal{N}) \geq r$ and $F^*(\mathcal{N}) \leq s$.

Lemma 4. Every $\mathcal{DF}$-b-continuous mapping is $\mathcal{DF}$-weakly b-continuous.

Proof. The proof follows by Definitions 10 and 13.

Remark 8. The converse of Lemma 4 fails as Example 9 will show.

Example 9. Let $G = \{g_1, g_2, g_3\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.4}, \frac{g_2}{0.2}, \frac{g_3}{0.4}\}$, $\mathcal{N} = \{\frac{g_1}{0.5}, \frac{g_2}{0.5}, \frac{g_3}{0.4}\}$, $\mathcal{U} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}, \frac{g_3}{0.6}\}$. Define $\Im, \Im^*, F, F^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if } \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if } \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise}, \end{cases}$$

$$F(\mathcal{V}) = \begin{cases} 1, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{V} = \mathcal{U}, \\ 0, & \text{otherwise}, \end{cases} \qquad F^*(\mathcal{V}) = \begin{cases} 0, & \text{if } \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{V} = \mathcal{U}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, F, F^*)$ is $\mathcal{DF}$-weakly b-continuous, but it is not $\mathcal{DF}$-b-continuous.

Theorem 8. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is $\mathcal{DF}$-weakly b-continuous iff for any $g_\theta \in P_\theta(G)$ and any $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ containing $\mathbb{P}(g_\theta)$, there is $\mathcal{M} \in I^G$ that is (r, s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq C_{\digamma^*}(\mathcal{N}, r, s)$.

Proof. ($\Rightarrow$) Let $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ containing $\mathbb{P}(g_\theta)$, and then $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s)$. Since $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$, then $g_\theta \in bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s) = \mathcal{M}$ (say). Hence, $\mathcal{M} \in I^G$ is (r, s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq C_{\digamma^*}(\mathcal{N}, r, s)$.

($\Leftarrow$) Let $g_\theta \in P_\theta(G)$ and $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$ such that $g_\theta \in \mathbb{P}^{-1}(\mathcal{N})$. According to the assumption there is $\mathcal{M} \in I^G$ that is (r, s)-$\mathcal{F}$-b-open containing g_θ with $\mathbb{P}(\mathcal{M}) \leq C_{\digamma^*}(\mathcal{N}, r, s)$. Hence, $g_\theta \in \mathcal{M} \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s))$ and $g_\theta \in bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s)$. Thus, $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s)$. Therefore, $\mathbb{P}$ is $\mathcal{DF}$-weakly b-continuous.

Theorem 9. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent:

(i) $\mathbb{P}$ is $\mathcal{DF}$-weakly b-continuous.

(ii) $\mathbb{P}^{-1}(\mathcal{N}) \geq bC_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(\mathcal{N}, r, s)), r, s)$, if $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}^c) \geq r$ and $\digamma^*(\mathcal{N}^c) \leq s$.

(iii) $bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s) \geq \mathbb{P}^{-1}(I_{\digamma^*}(\mathcal{N}, r, s))$.

(iv) $bC_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(\mathcal{N}, r, s)), r, s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s))$.

Proof. (i) $\Leftrightarrow$ (ii) The proof follows by Proposition 3 and Definition 13.

(ii) $\Rightarrow$ (iii) Let $\mathcal{N} \in I^Z$. Hence by (ii),

$$bC_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(C_{\digamma^*}(\mathcal{N}^c, r, s), r, s)), r, s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}^c, r, s)).$$

Thus, $\mathbb{P}^{-1}(I_{\digamma^*}(\mathcal{N}, r, s)) \leq bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s)$.

(iii) $\Leftrightarrow$ (iv) The proof follows from Proposition 3.

(iv) $\Rightarrow$ (i) Let $\mathcal{N} \in I^Z$ with $\digamma(\mathcal{N}) \geq r$ and $\digamma^*(\mathcal{N}) \leq s$. Hence by (iv), $bC_{\Im^*}(\mathbb{P}^{-1}(I_{\digamma^*}(\mathcal{N}^c, r, s)), r, s) \leq \mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}^c, r, s)) = \mathbb{P}^{-1}(\mathcal{N}^c)$. Thus, $\mathbb{P}^{-1}(\mathcal{N}) \leq bI_{\Im^*}(\mathbb{P}^{-1}(C_{\digamma^*}(\mathcal{N}, r, s)), r, s)$, so $\mathbb{P}$ is $\mathcal{DF}$-weakly b-continuous.

Lemma 5. Every $\mathcal{DF}$-almost b-continuous mapping is $\mathcal{DF}$-weakly b-continuous.

Proof. The proof follows by Definitions 12 and 13.

Remark 9. The converse of Lemma 5 fails as Example 10 will show.

Example 10. Let $G = \{g_1, g_2, g_3\}$ and define $\mathcal{M}, \mathcal{N}, \mathcal{U} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.6}, \frac{g_2}{0.2}, \frac{g_3}{0.4}\}$, $\mathcal{N} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}, \frac{g_3}{0.5}\}$, $\mathcal{U} = \{\frac{g_1}{0.3}, \frac{g_2}{0.2}, \frac{g_3}{0.4}\}$. Define $\Im, \Im^*, \digamma, \digamma^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{U}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\digamma(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \digamma^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, \digamma, \digamma^*)$ is $\mathcal{DF}$-weakly b-continuous, but it is not $\mathcal{DF}$-almost b-continuous.

Remark 10. From the previous discussions and definitions, we have the following diagram.

$\mathcal{DF}$-b-continuity $\longrightarrow$ **$\mathcal{DF}$-almost b-continuity** $\longrightarrow$ **$\mathcal{DF}$-weakly b-continuity**

Proposition 6. Let $(G, \Im, \Im^*)$, (Q, η, η^*) and $(Z, \digamma, \digamma^*)$ be $\mathcal{DFTS}s$, and $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Q, \eta, \eta^*)$, $\mathbb{Y} : (Q, \eta, \eta^*) \longrightarrow (Z, \digamma, \digamma^*)$ be two $\mathcal{F}$-mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is $\mathcal{DF}$-almost b-continuous if $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute (resp. $\mathcal{DF}$-b-continuous) and $\mathbb{Y}$ is $\mathcal{DF}$-almost b-continuous (resp. $\mathcal{DF}$-continuous).

Proof. The proof follows by the previous definitions.

5. Some applications

Here, we present and study some new $\mathcal{DF}$-mappings between $\mathcal{DFTS}s$ $(G, \Im, \Im^*)$ and $(Z, \digamma, \digamma^*)$ based on Šostak's sense [3]. Next, we introduce and discuss new types of $\mathcal{DF}$-separation axioms via (r, s)-$\mathcal{F}$-b-closed sets, called (r, s)-$\mathcal{F}$-b-regular and (r, s)-$\mathcal{F}$-b-normal spaces.

Definition 14. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is called $\mathcal{DF}$-b-open if $\mathbb{P}(\mathcal{M})$ is an (r,s)-$\mathcal{F}$-b-open set, for each $\mathcal{M} \in I^G$ with $\Im(\mathcal{M}) \geq r$ and $\Im^*(\mathcal{M}) \leq s$.

Definition 15. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ is called $\mathcal{DF}$-b-irresolute open if $\mathbb{P}(\mathcal{M})$ is an (r,s)-$\mathcal{F}$-b-open set, for each (r,s)-$\mathcal{F}$-b-open set $\mathcal{M} \in I^G$.

Lemma 6. Each $\mathcal{DF}$-b-irresolute open mapping is $\mathcal{DF}$-b-open.

Proof. The proof follows from Definitions 14 and 15.

Remark 11. The converse of Lemma 6 fails as Example 11 will show.

Example 11. Let $G = \{g_1, g_2\}$ and define $\mathcal{M}, \mathcal{N} \in I^G$ as follows: $\mathcal{M} = \{\frac{g_1}{0.5}, \frac{g_2}{0.5}\}$, $\mathcal{N} = \{\frac{g_1}{0.5}, \frac{g_2}{0.4}\}$. Define $\Im, \Im^*, \digamma, \digamma^* : I^G \longrightarrow I$ as follows:

$$\Im(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 0, & \text{otherwise}, \end{cases} \qquad \Im^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{V} = \mathcal{M}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\digamma(\mathcal{V}) = \begin{cases} 1, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \digamma^*(\mathcal{V}) = \begin{cases} 0, & \text{if} \quad \mathcal{V} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{V} = \mathcal{N}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (G, \digamma, \digamma^*)$ is $\mathcal{DF}$-b-open, but it is not $\mathcal{DF}$-b-irresolute open.

Theorem 10. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-open.

(ii) $\mathbb{P}(I_{\Im^*}(\mathcal{M}, r, s)) \leq bI_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(iii) $I_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \leq \mathbb{P}^{-1}(bI_{\digamma^*}(\mathcal{N}, r, s))$.

(iv) For every $\mathcal{N}$ and every $\mathcal{M}$ with $\Im(\mathcal{M}^c) \geq r$, $\Im^*(\mathcal{M}^c) \leq s$ and $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{M}$, there is $\mathcal{U} \in I^Z$ is (r,s)-$\mathcal{F}$-b-closed with $\mathcal{N} \leq \mathcal{U}$ and $\mathbb{P}^{-1}(\mathcal{U}) \leq \mathcal{M}$.

Proof. (i) $\Rightarrow$ (ii) Since $\mathbb{P}(I_{\Im^*}(\mathcal{M}, r, s)) \leq \mathbb{P}(\mathcal{M})$, hence by (i), $\mathbb{P}(I_{\Im^*}(\mathcal{M}, r, s))$ is (r, s)-$\mathcal{F}$-b-open. Thus,

$$\mathbb{P}(I_{\Im^*}(\mathcal{M}, r, s)) \leq bI_{F^*}(\mathbb{P}(\mathcal{M}), r, s).$$

(ii) $\Rightarrow$ (iii) Set $\mathcal{M} = \mathbb{P}^{-1}(\mathcal{N})$, hence by (ii), $\mathbb{P}(I_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s)) \leq bI_{F^*}(\mathbb{P}(\mathbb{P}^{-1}(\mathcal{N})), r, s) \leq bI_{F^*}(\mathcal{N}, r, s)$. Thus, $I_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \leq \mathbb{P}^{-1}(bI_{F^*}(\mathcal{N}, r, s))$.

(iii) $\Rightarrow$ (iv) Let $\mathcal{N} \in I^Z$ and $\mathcal{M} \in I^G$ with $\Im(\mathcal{M}^c) \geq r$ and $\Im^*(\mathcal{M}^c) \leq s$ such that $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{M}$. Since $\mathcal{M}^c \leq \mathbb{P}^{-1}(\mathcal{N}^c)$, $\mathcal{M}^c = I_{\Im^*}(\mathcal{M}^c, r, s) \leq I_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}^c), r, s)$. Hence by (iii), $\mathcal{M}^c \leq I_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}^c), r, s) \leq \mathbb{P}^{-1}(bI_{F^*}(\mathcal{N}^c, r, s))$. Then, we have

$$\mathcal{M} \geq (\mathbb{P}^{-1}(bI_{F^*}(\mathcal{N}^c, r, s)))^c = \mathbb{P}^{-1}(bC_{F^*}(\mathcal{N}, r, s)).$$

Thus, $bC_{F^*}(\mathcal{N}, r, s) \in I^Z$ is (r, s)-$\mathcal{F}$-b-closed with $\mathcal{N} \leq bC_{F^*}(\mathcal{N}, r, s)$ and $\mathbb{P}^{-1}(bC_{F^*}(\mathcal{N}, r, s)) \leq \mathcal{M}$.

(iv) $\Rightarrow$ (i) Let $\mathcal{V} \in I^G$ with $\Im(\mathcal{V}) \geq r$ and $\Im^*(\mathcal{V}) \leq s$. Set $\mathcal{N} = (\mathbb{P}(\mathcal{V}))^c$ and $\mathcal{M} = \mathcal{V}^c$, then $\mathbb{P}^{-1}(\mathcal{N}) = \mathbb{P}^{-1}((\mathbb{P}(\mathcal{V}))^c) \leq \mathcal{M}$. Hence by (iv), there is $\mathcal{U} \in I^Z$ is (r, s)-$\mathcal{F}$-b-closed with $\mathcal{N} \leq \mathcal{U}$ and $\mathbb{P}^{-1}(\mathcal{U}) \leq \mathcal{M} = \mathcal{V}^c$. Thus, $\mathbb{P}(\mathcal{V}) \leq \mathbb{P}(\mathbb{P}^{-1}(\mathcal{U}^c)) \leq \mathcal{U}^c$. On the other hand, since $\mathcal{N} \leq \mathcal{U}$, $\mathbb{P}(\mathcal{V}) = \mathcal{N}^c \geq \mathcal{U}^c$. Hence, $\mathbb{P}(\mathcal{V}) = \mathcal{U}^c$, so $\mathbb{P}(\mathcal{V})$ is an (r, s)-$\mathcal{F}$-b-open set. Therefore, $\mathbb{P}$ is $\mathcal{DF}$-b-open.

Theorem 11. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute open.

(ii) $\mathbb{P}(bI_{\Im^*}(\mathcal{M}, r, s)) \leq bI_{F^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(iii) $bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \leq \mathbb{P}^{-1}(bI_{F^*}(\mathcal{N}, r, s))$.

(iv) For every $\mathcal{N}$ and every $\mathcal{M}$ is an (r, s)-$\mathcal{F}$-b-closed set with $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{M}$, there is $\mathcal{U} \in I^Z$ is (r, s)-$\mathcal{F}$-b-closed with $\mathcal{N} \leq \mathcal{U}$ and $\mathbb{P}^{-1}(\mathcal{U}) \leq \mathcal{M}$.

Proof. The proof is similar to that of Theorem 10.

Definition 16. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ is called $\mathcal{DF}$-b-closed if $\mathbb{P}(\mathcal{M})$ is an (r, s)-$\mathcal{F}$-b-closed set, for each $\mathcal{M} \in I^G$ with $\Im(\mathcal{M}^c) \geq r$ and $\Im^*(\mathcal{M}^c) \leq s$.

Definition 17. An $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ is called $\mathcal{DF}$-b-irresolute closed if $\mathbb{P}(\mathcal{M})$ is an (r, s)-$\mathcal{F}$-b-closed set, for each (r, s)-$\mathcal{F}$-b-closed set $\mathcal{M} \in I^G$.

Lemma 7. Each $\mathcal{DF}$-b-irresolute closed mapping is $\mathcal{DF}$-b-closed.

Proof. The proof follows from Definitions 16 and 17.

Theorem 12. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-closed.

(ii) $bC_{F^*}(\mathbb{P}(\mathcal{M}), r, s) \leq \mathbb{P}(C_{\Im^*}(\mathcal{M}, r, s))$.

(iii) $\mathbb{P}^{-1}(bC_{F^*}(\mathcal{N}, r, s)) \leq C_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s)$.

(iv) For every $\mathcal{N}$ and every $\mathcal{M}$ with $\Im(\mathcal{M}) \geq r$, $\Im^*(\mathcal{M}) \leq s$ and $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{M}$, there is $\mathcal{U} \in I^Z$ is (r, s)-$\mathcal{F}$-b-open with $\mathcal{N} \leq \mathcal{U}$ and $\mathbb{P}^{-1}(\mathcal{U}) \leq \mathcal{M}$.

Proof. The proof is similar to that of Theorem 10.

Theorem 13. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be an $\mathcal{F}$-mapping. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute closed.

(ii) $bC_{F^*}(\mathbb{P}(\mathcal{M}), r, s) \leq \mathbb{P}(bC_{\Im^*}(\mathcal{M}, r, s))$.

(iii) $\mathbb{P}^{-1}(bC_{F^*}(\mathcal{N}, r, s)) \leq bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s)$.

(iv) For every $\mathcal{N}$ and every $\mathcal{M}$ is an (r, s)-$\mathcal{F}$-b-open set with $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{M}$, there is $\mathcal{U} \in I^Z$ is (r, s)-$\mathcal{F}$-b-open with $\mathcal{N} \leq \mathcal{U}$ and $\mathbb{P}^{-1}(\mathcal{U}) \leq \mathcal{M}$.

Proof. The proof is similar to that of Theorem 10.

Proposition 7. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be a bijective $\mathcal{F}$-mapping, then $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute open iff $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute closed.

Proof. The proof follows from:

$$\mathbb{P}^{-1}(bC_{F^*}(\mathcal{N}, r, s)) \leq bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) \iff \mathbb{P}^{-1}(bI_{F^*}(\mathcal{N}^c, r, s)) \leq bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}^c), r, s).$$

Definition 18. A bijective $\mathcal{F}$-mapping $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ is called $\mathcal{DF}$-b-irresolute homeomorphism if $\mathbb{P}$ and $\mathbb{P}^{-1}$ are $\mathcal{DF}$-b-irresolute.

The proof of the following corollary is easy and so is omitted.

Corollary 4. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, \digamma, \digamma^*)$ be a bijective $\mathcal{F}$-mapping. Then the following statements are equivalent for every $\mathcal{M} \in I^G$ and $\mathcal{N} \in I^Z$:

(i) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute homeomorphism.

(ii) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute closed and $\mathcal{DF}$-b-irresolute.

(iii) $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute open and $\mathcal{DF}$-b-irresolute.

(iv) $\mathbb{P}(bI_{\Im^*}(\mathcal{M}, r, s)) = bI_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(v) $\mathbb{P}(bC_{\Im^*}(\mathcal{M}, r, s)) = bC_{\digamma^*}(\mathbb{P}(\mathcal{M}), r, s)$.

(vi) $bI_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) = \mathbb{P}^{-1}(b_{\digamma^*}(\mathcal{N}, r, s))$.

(vii) $bC_{\Im^*}(\mathbb{P}^{-1}(\mathcal{N}), r, s) = \mathbb{P}^{-1}(bC_{\digamma^*}(\mathcal{N}, r, s))$.

Definition 19. Let $g_\theta \in P_\theta(G)$, $\mathcal{M} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{DFTS}$ $(G, \Im, \Im^*)$ is called an (r, s)-$\mathcal{F}$-b-regular space if $g_\theta \, \overline{q} \, \mathcal{M}$ for each (r, s)-$\mathcal{F}$-b-closed set $\mathcal{M}$, there is $\mathcal{U}_i \in I^G$ with $\Im(\mathcal{U}_i) \geq r$ and $\Im^*(\mathcal{U}_i) \leq s$ for $i = 1, 2$, such that $g_\theta \in \mathcal{U}_1$, $\mathcal{M} \leq \mathcal{U}_2$, and $\mathcal{U}_1 \, \overline{q} \, \mathcal{U}_2$.

Definition 20. Let $\mathcal{M}, \mathcal{N} \in I^G$, $r \in I_\circ$, and $s \in I_1$. An $\mathcal{DFTS}$ $(G, \Im, \Im^*)$ is called an (r, s)-$\mathcal{F}$-b-normal space if $\mathcal{M} \, \overline{q} \, \mathcal{N}$ for each (r, s)-$\mathcal{F}$-b-closed sets $\mathcal{M}$ and $\mathcal{N}$, there is $\mathcal{U}_i \in I^G$ with $\Im(\mathcal{U}_i) \geq r$ and $\Im^*(\mathcal{U}_i) \leq s$ for $i = 1, 2$, such that $\mathcal{M} \leq \mathcal{U}_1$, $\mathcal{N} \leq \mathcal{U}_2$, and $\mathcal{U}_1 \, \overline{q} \, \mathcal{U}_2$.

Theorem 14. Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $g_\theta \in P_\theta(G)$, and $\mathcal{M} \in I^G$. Then the following statements are equivalent:

(i) $(G, \Im, \Im^*)$ is an (r, s)-$\mathcal{F}$-b-regular space.

(ii) If $g_\theta \in \mathcal{M}$ for every (r, s)-$\mathcal{F}$-b-open set $\mathcal{M}$, there is $\mathcal{N} \in I^G$ with $\Im(\mathcal{N}) \geq r$, $\Im^*(\mathcal{N}) \leq s$, and $g_\theta \in \mathcal{N} \leq C_{\Im^*}(\mathcal{N}, r, s) \leq \mathcal{M}$.

(iii) If $g_\theta \, \overline{q} \, \mathcal{M}$ for each (r, s)-$\mathcal{F}$-b-closed set $\mathcal{M}$, there is $\mathcal{O}_i \in I^G$ with $\Im(\mathcal{O}_i) \geq r$ and $\Im^*(\mathcal{O}_i) \leq s$ for $i = 1, 2$, such that $g_\theta \in \mathcal{O}_1$, $\mathcal{M} \leq \mathcal{O}_2$, and $C_{\Im^*}(\mathcal{O}_1, r, s) \, \overline{q} \, C_{\Im^*}(\mathcal{O}_2, r, s)$.

Proof. (i) ⇒ (ii) Let $g_\theta \in \mathcal{M}$ for every (r,s)-$\mathcal{F}$-b-open set $\mathcal{M}$, then $g_\theta \, \overline{q} \, \mathcal{M}^c$. Since $(G, \Im, \Im^*)$ is (r,s)-$\mathcal{F}$-b-regular, then there is $\mathcal{N}, \mathcal{O} \in I^G$ with $\Im(\mathcal{N}) \geq r$, $\Im^*(\mathcal{N}) \leq s$, $\Im(\mathcal{O}) \geq r$, and $\Im^*(\mathcal{O}) \leq s$, such that $g_\theta \in \mathcal{N}$, $\mathcal{M}^c \leq \mathcal{O}$, and $\mathcal{N} \, \overline{q} \, \mathcal{O}$. Thus, $g_\theta \in \mathcal{N} \leq \mathcal{O}^c \leq \mathcal{M}$, so $g_\theta \in \mathcal{N} \leq C_{\Im^*}(\mathcal{N}, r, s) \leq \mathcal{M}$.

(ii) ⇒ (iii) Let $g_\theta \, \overline{q} \, \mathcal{M}$ for each (r,s)-$\mathcal{F}$-b-closed set $\mathcal{M}$, then $g_\theta \in \mathcal{M}^c$. By (ii), there is $\mathcal{O} \in I^G$ with $\Im(\mathcal{O}) \geq r$, $\Im^*(\mathcal{O}) \leq s$ and $g_\theta \in \mathcal{O} \leq C_{\Im^*}(\mathcal{O}, r, s) \leq \mathcal{M}^c$. Since $\Im(\mathcal{O}) \geq r$ and $\Im^*(\mathcal{O}) \leq s$, then $\mathcal{O}$ is an (r,s)-$\mathcal{F}$-b-open set and $g_\theta \in \mathcal{O}$. Again, by (ii), there is $\mathcal{V} \in I^G$ with $\Im(\mathcal{V}) \geq r$, $\Im^*(\mathcal{V}) \leq s$, and $g_\theta \in \mathcal{V} \leq C_{\Im^*}(\mathcal{V}, r, s) \leq \mathcal{O} \leq C_{\Im^*}(\mathcal{O}, r, s) \leq \mathcal{M}^c$. Hence, $\mathcal{M} \leq (C_{\Im^*}(\mathcal{O}, r, s))^c = I_{\Im^*}(\mathcal{O}^c, r, s) \leq \mathcal{O}^c$. Set $\mathcal{U} = I_{\Im^*}(\mathcal{O}^c, r, s)$, thus $\Im(\mathcal{U}) \geq r$ and $\Im^*(\mathcal{U}) \leq s$. Then, $C_{\Im^*}(\mathcal{U}, r, s) \leq \mathcal{O}^c \leq (C_{\Im^*}(\mathcal{V}, r, s))^c$. Therefore, $C_{\Im^*}(\mathcal{U}, r, s) \, \overline{q} \, C_{\Im^*}(\mathcal{V}, r, s)$.

(iii) ⇒ (i) This is easily proved by Definition 19.

Theorem 15. Let $(G, \Im, \Im^*)$ be an $\mathcal{DFTS}$, $\mathcal{M}, \mathcal{N} \in I^G$. Then the following statements are equivalent:

(i) $(G, \Im, \Im^*)$ is an (r,s)-$\mathcal{F}$-b-normal space.

(ii) If $\mathcal{N} \leq \mathcal{M}$ for every (r,s)-$\mathcal{F}$-b-closed set $\mathcal{N}$ and (r,s)-$\mathcal{F}$-b-open set $\mathcal{M}$, there is $\mathcal{O} \in I^G$ with $\Im(\mathcal{O}) \geq r$, $\Im^*(\mathcal{O}) \leq s$, and $\mathcal{N} \leq \mathcal{O} \leq C_{\Im^*}(\mathcal{O}, r, s) \leq \mathcal{M}$.

(iii) If $\mathcal{M} \, \overline{q} \, \mathcal{N}$ for each (r,s)-$\mathcal{F}$-b-closed sets $\mathcal{M}$ and $\mathcal{N}$, there is $\mathcal{O}_i \in I^G$ with $\Im(\mathcal{O}_i) \geq r$, $\Im^*(\mathcal{O}_i) \leq s$ for $i = 1, 2$, such that $\mathcal{M} \leq \mathcal{O}_1$, $\mathcal{N} \leq \mathcal{O}_2$, and $C_{\Im^*}(\mathcal{O}_1, r, s) \, \overline{q} \, C_{\Im^*}(\mathcal{O}_2, r, s)$.

Proof. The proof is similar to that of Theorem 14.

Theorem 16. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be a bijective $\mathcal{DF}$-b-irresolute and $\mathcal{DF}$-open mapping. If $(G, \Im, \Im^*)$ is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space), then (Z, F, F^*) is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space).

Proof. If $z_\theta \, \overline{q} \, \mathcal{N}$ for every (r,s)-$\mathcal{F}$-b-closed set $\mathcal{N} \in I^Z$ and $\mathbb{P}$ is $\mathcal{DF}$-b-irresolute, then $\mathbb{P}^{-1}(\mathcal{N})$ is an (r,s)-$\mathcal{F}$-b-closed set. Set $z_\theta = \mathbb{P}(g_\theta)$, and then $g_\theta \, \overline{q} \, \mathbb{P}^{-1}(\mathcal{N})$. Since $(G, \Im, \Im^*)$ is (r,s)-$\mathcal{F}$-b-regular, there is $\mathcal{O}_1, \mathcal{O}_2 \in I^G$ with $\Im(\mathcal{O}_1) \geq r$, $\Im^*(\mathcal{O}_1) \leq s$, $\Im(\mathcal{O}_2) \geq r$, and $\Im^*(\mathcal{O}_2) \leq s$ such that $g_\theta \in \mathcal{O}_1$, $\mathbb{P}^{-1}(\mathcal{N}) \leq \mathcal{O}_2$, and $\mathcal{O}_1 \, \overline{q} \, \mathcal{O}_2$. Since $\mathbb{P}$ is a bijective $\mathcal{DF}$-open mapping, hence $z_\theta \in \mathbb{P}(\mathcal{O}_1)$, $\mathcal{N} = \mathbb{P}(\mathbb{P}^{-1}(\mathcal{N})) \leq \mathbb{P}(\mathcal{O}_2)$, and $\mathbb{P}(\mathcal{O}_1) \, \overline{q} \, \mathbb{P}(\mathcal{O}_2)$. Therefore, (Z, F, F^*) is an (r,s)-$\mathcal{F}$-b-regular space. The other case also follows similar lines.

Theorem 17. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be an injective $\mathcal{DF}$-continuous and $\mathcal{DF}$-b-irresolute closed mapping. If (Z, F, F^*) is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space), then $(G, \Im, \Im^*)$ is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space).

Proof. If $g_\theta\ \overline{q}\ \mathcal{M}$ for each (r,s)-$\mathcal{F}$-b-closed set $\mathcal{M} \in I^G$ and $\mathbb{P}$ is injective $\mathcal{DF}$-b-irresolute closed, hence $\mathbb{P}(\mathcal{M})$ is an (r,s)-$\mathcal{F}$-b-closed set and $\mathbb{P}(g_\theta)\ \overline{q}\ \mathbb{P}(\mathcal{M})$. Since (Z, F, F^*) is (r,s)-$\mathcal{F}$-b-regular, there is $\mathcal{O}_1, \mathcal{O}_2 \in I^Z$ with $F(\mathcal{O}_1) \geq r$, $F^*(\mathcal{O}_1) \leq s$, $F(\mathcal{O}_2) \geq r$, and $F^*(\mathcal{O}_2) \leq s$ such that $\mathbb{P}(g_\theta) \in \mathcal{O}_1$, $\mathbb{P}(\mathcal{M}) \leq \mathcal{O}_2$, and $\mathcal{O}_1\ \overline{q}\ \mathcal{O}_2$. Since $\mathbb{P}$ is an $\mathcal{DF}$-continuous mapping, then $g_\theta \in \mathbb{P}^{-1}(\mathcal{O}_1)$ and $\mathcal{M} \leq \mathbb{P}^{-1}(\mathcal{O}_2)$ with $\Im(\mathbb{P}^{-1}(\mathcal{O}_1)) \geq r$, $\Im^*(\mathbb{P}^{-1}(\mathcal{O}_1)) \leq s$, $\Im(\mathbb{P}^{-1}(\mathcal{O}_2)) \geq r$, $\Im^*(\mathbb{P}^{-1}(\mathcal{O}_2)) \leq s$, and $\mathbb{P}^{-1}(\mathcal{O}_1)\ \overline{q}\ \mathbb{P}^{-1}(\mathcal{O}_2)$. Hence, $(G, \Im, \Im^*)$ is an (r,s)-$\mathcal{F}$-b-regular space. The other case also follows similar lines.

Theorem 18. Let $\mathbb{P} : (G, \Im, \Im^*) \longrightarrow (Z, F, F^*)$ be a surjective $\mathcal{DF}$-b-irresolute, $\mathcal{DF}$-open, and $\mathcal{DF}$-closed mapping. If $(G, \Im, \Im^*)$ is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space), then (Z, F, F^*) is an (r,s)-$\mathcal{F}$-b-regular space (resp. (r,s)-$\mathcal{F}$-b-normal space).

Proof. The proof is similar to that of Theorem 16.

6. Conclusions

In the present paper, a novel class of generalized $\mathcal{F}$-open sets, called (r,s)-$\mathcal{F}$-b-open sets, has been introduced in $\mathcal{DFTS}$ based on Šostak's sense [3]. Furthermore, some characterizations of (r,s)-$\mathcal{F}$-b-open sets along with their mutual relationships have been discussed. In addition, the notions of $\mathcal{DF}$-b-closure operators and $\mathcal{DF}$-b-interior operators have been presented and investigated. Thereafter, the notion of $\mathcal{DF}$-b-continuity between $\mathcal{DFTS}s$ $(G, \Im, \Im^*)$ and (Z, F, F^*) has been defined and discussed. Moreover, the concepts of $\mathcal{DF}$-almost b-continuity and $\mathcal{DF}$-weakly b-continuity, which are weaker forms of $\mathcal{DF}$-b-continuity, have been explored and characterized. After that, some new $\mathcal{DF}$-mappings using (r,s)-$\mathcal{F}$-b-closed sets and (r,s)-$\mathcal{F}$-b-open sets have been defined and studied. Lastly, we introduced new types of $\mathcal{DF}$-separation axioms using (r,s)-$\mathcal{F}$-b-closed sets, and some properties have been specified.

In upcoming works might look into the following topics: (i) defining upper (lower) b-continuous $\mathcal{DF}$-multifunctions and (r,s)-$\mathcal{F}$-b-connected sets; (ii) introducing these novel notions given here in the frame of fuzzy ideals as defined in [41–43]; and (iii) extending these novel notions given here in the frame of fuzzy soft topological (r-minimal) spaces as defined in [44–46].

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