EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5914
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Investigating Length and Mean-Fuzzy Subalgebras in Sheffer Stroke Hilbert Algebras

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Abstract. The aim of this paper is to introduce the notions of the length and the mean of an interval-valued fuzzy structure in Sheffer stroke Hilbert algebras. The notions of length-fuzzy subalgebras and mean-fuzzy subalgebras of Sheffer stroke Hilbert algebras are introduced, and related properties are investigated. Characterizations of length-fuzzy subalgebras and mean-fuzzy subalgebras are discussed. Relations between length-fuzzy subalgebras (resp., mean-fuzzy subalgebras) and subalgebras are established. Moreover, we discuss the relationships among length-fuzzy subalgebras (resp., mean-fuzzy subalgebras) and upper and lower-level subsets of the length (resp., mean) of an interval-valued fuzzy structure in Sheffer stroke Hilbert algebras.

2020 Mathematics Subject Classifications: 20N05, 94D05, 03E72

Key Words and Phrases: Sheffer stroke Hilbert algebra, subalgebra, length-fuzzy subalgebra, mean-fuzzy subalgebra

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1. Introduction

The Sheffer operation, also known as the Sheffer stroke or NAND operator, was first introduced by Henry Maurice Sheffer [1]. This operation holds significance because it can be used independently, without any other logical operators, to construct a logical system. This means that any axiom of a logical system can be restated using only the Sheffer operation. Because of this property, it becomes easier to control certain properties of the newly constructed logical system. Additionally, it's worth noting that the axioms of Boolean algebra, which are the algebraic counterpart of classical propositional calculus,

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5914

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can be expressed solely using the Sheffer operation. This highlights the fundamental nature and versatility of the Sheffer operation in logical and algebraic systems. In 2002, McCune et al. [2] applied the Sheffer stroke operation for Boolean algebras, and it is shown that there is no shorter axiom in terms of the Sheffer stroke. Algebraic structures play a prominent role in mathematics, with wide-ranging applications in various disciplines, including theoretical physics, computer science, control engineering, information sciences, coding theory, and topological spaces, among others. This provides sufficient motivation for researchers to simplify axioms for various algebraic structures, e.g., see [3–5]. In 1950, Henkin [6] introduced the notion of "implicative model" as a model of positive implicative propositional calculus. In 1960, Monteiro [7] gave the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, Diego [8] intensively studied and developed some properties of Hilbert algebras. In 2021, Oner et al. [9] investigated the relation between Sheffer stroke and Hilbert algebras. Also, see [10]. In 1965, Zadeh [11] proposed a new theory named fuzzy set theory. Then several researchers studied various extensions and generalizations of this theory, e.g., intuitionistic fuzzy sets [12], L-fuzzy sets [13], type-2 fuzzy sets [14], interval-valued fuzzy sets [15], multi fuzzy sets [16], bipolar-valued fuzzy sets [17], m-polar fuzzy sets [18], and neutrosophic sets [19, 20]. Recently, many researchers have studied and applied concepts of fuzzy sets, including fuzzy (weak) filters and deductive systems, to Sheffer stroke Hilbert algebras [21–24].

This paper aims to introduce and explore the concepts of length and mean within interval-valued fuzzy structures in Sheffer stroke Hilbert algebras. Specifically, we define and analyze the notions of length-fuzzy subalgebras and mean-fuzzy subalgebras, investigating their key properties and characterizations. The study establishes relationships between these fuzzy subalgebras and traditional subalgebras, providing a deeper understanding of their structural interaction. Additionally, we examine the connections between length-fuzzy (resp., mean-fuzzy) subalgebras and their corresponding upper and lower-level subsets within interval-valued fuzzy structures. These findings offer a comprehensive framework for studying gradations of membership and their implications in Sheffer stroke Hilbert algebras, laying the groundwork for further theoretical development and practical applications in fuzzy logic and algebraic systems.

2. Preliminaries

Sheffer stroke Hilbert algebras constitute a pivotal framework within the realms of logic and lattice theory, distinguished by the incorporation of the Sheffer stroke (NAND) operation—a cornerstone of Boolean algebra. This integration extends the classical Hilbert algebra structure, enabling a more versatile exploration of logical systems and their properties. By bridging algebraic theory and practical applications, Sheffer stroke Hilbert algebras provide a robust toolset for analyzing and modeling complex systems characterized by uncertainty, fuzziness, and imprecision. These algebras are particularly relevant in advancing fuzzy logic, decision-making algorithms, and computational frameworks, offering insights that transcend traditional logical paradigms. Moreover, their study contributes to the broader understanding of algebraic hierarchies, enriching both foundational research and real-world problem-solving methodologies. This unique combination of theoretical depth and practical utility underscores their significance in contemporary mathematical and computational research.

Recall the definitions and results that are taken from [1, 9, 15, 25] for the ready reference of the reader.

Definition 1. *[1] Let $\langle A, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in A$,*

$(S1)$ $(x|(y|y))|(x|(y|y)) = y|x$,
$(S2)$ $(x|x)|((x|(y|y))|(x|(y|y))) = x$,
$(S3)$ $x|((y|z)|(y|z)) = (((x|(y|y))|(x|(y|y))))|((x|(y|y))|(x|(y|y)))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2. *[25] An algebra $\langle A, \to, 0\rangle$ of type $(2, 0)$ is called a Hilbert algebra if it satisfies the following axioms: for all $x, y, z \in A$,*

$(H1)$ $x \to (y \to x) = 0$,
$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 0$,
$(H3)$ $x \to y = 0$ *and* $y \to x = 0 \Rightarrow x = y$.

Definition 3. *[9] A Sheffer stroke Hilbert algebra (abbreviated SHA) is a structure $\langle A, |, 0\rangle$ of type $(2, 0)$, in which A is a nonempty set, $|$ is a Sheffer stroke operation on A, and 0 is the fixed element in A such that the following identities are satisfied for all $x, y, z \in A$,*

(1) $(x|(P|P))|(Q|(R|R))|(Q|(R|R)) = x|(x|x)$,
where $P := y|(z|z)$, $Q := x|(y|y)$ *and* $R := x|(z|z)$,

(2) $x|(y|y) = y|(x|x) = x|(x|x) \Rightarrow x = y$.

Proposition 1. [9] *Let $\langle A, |, 0\rangle$ be a Sheffer stroke Hilbert algebra. Then the binary relation $x \leq y$ if and only if $x|(y|y) = 0$ is a partial order on A.*

Definition 4. *[9] A nonempty subset G of a Sheffer stroke Hilbert algebra $\langle A, |, 0\rangle$ is called a subalgebra of A if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.*

Definition 5. *[15] An interval-valued intuitionistic fuzzy set X over a nonempty set A is an object having the form $X = \{\langle x, \mu_X(x), \gamma_X(x)\rangle : x \in A\}$, where $\mu_X(x) : A \to D[0, 1]$ and $\gamma_X(x) : A \to D[0, 1]$ and $D[0, 1]$ is the set of all intervals of $[0, 1]$. The intervals $\mu_X(x)$ and $\gamma_X(x)$ denote the intervals of the degree of belongingness and non-belongingness of the element x to X, where*

$$\mu_X(x) = [\mu^l_X(x), \mu^u_X(x)]$$

and

$$\gamma_X(x) = [\gamma^l_X(x), \gamma^u_X(x)]$$

for all $x \in A$ with the condition $0 \leq \mu^l_X(x) + \gamma^u_X(x) \leq 1$.

For the sake of simplicity, we shall use the symbol $X = (\mu_X, \gamma_X)$ for the interval-valued intuitionistic fuzzy set $X = \{\langle x, \mu_X(x), \gamma_X(x)\rangle : x \in A\}$.

Note that $\overline{\mu_X}(x) = [1 - \mu^u_X(x), 1 - \mu^l_X(x)]$ and $\overline{\gamma_X}(x) = [1 - \gamma^u_X(x), 1 - \gamma^l_X(x)]$, where $[\overline{\mu_X}(x), \overline{\gamma_X}(x)]$ represents the complement of x in X.

3. Length of an interval-valued fuzzy structure in Sheffer stroke Hilbert algebras

In this section, we present the concept of the length of an interval-valued fuzzy structure within the framework of Sheffer stroke Hilbert algebras. We introduce the notion of length-fuzzy subalgebras, which are specific to these algebras, and explore their fundamental properties and interrelationships. This analysis aims to deepen the understanding of how interval-valued fuzziness interacts with the algebraic operations in Sheffer stroke Hilbert algebras, providing new insights into their structural characteristics.

Throughout this discussion, we assume $A = \langle A, |, 0\rangle$ to be a Sheffer stroke Hilbert algebra, serving as the foundational structure for the concepts and results developed herein.

Definition 6. *Given an interval-valued fuzzy structure $(A, \widetilde{f})$ over a nonempty set A, we define two fuzzy structures $(A, \widetilde{f}_{\inf})$ and $(A, \widetilde{f}_{\sup})$ in A as follows:*

$$\widetilde{f}_{\inf} : A \to [0, 1]; x \mapsto \inf\{\widetilde{f}(x)\},$$

and

$$\widetilde{f}_{\sup} : A \to [0, 1]; x \mapsto \sup\{\widetilde{f}(x)\}.$$

Example 1. *[9] Let $A = \{0, u, v, 1\}$ be a set with the binary operation $|$ given in the following table:*

$\vert$	1	u	v	0
1	0	v	u	1
u	v	v	1	1
v	u	1	u	1
0	1	1	1	1

Then $(A, |)$ is a Sheffer stroke Hilbert algebra. Define an interval-valued fuzzy structure $(A, \widetilde{f})$ over A by the table below:

A	1	u	v	0
$\widetilde{f}$	$\{0.3, 0.7\}$	$[0.2, 0.4]$	$[0.3, 0.7]$	$[0.1, 0.4]$

Then

A	1	u	v	0
$\widetilde{f}_{\inf}$	0.3	0.2	0.3	0.1
$\widetilde{f}_{\sup}$	0.7	0.4	0.7	0.4

Definition 7. *[26] Given an interval-valued fuzzy structure $(A, \widetilde{f})$ over A, we define a fuzzy structure $(A, \widetilde{f}_{\ell})$ in A as follows:*

$$\widetilde{f}_{l} : A \to [0, 1]; x \mapsto \widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x),$$

which is called the length of $\widetilde{f}$.

Example 2. *Consider Example 1, we have*

A	1	u	v	0
$\widetilde{f}_l$	0.4	0.2	0.4	0.3

Definition 8. *A fuzzy structure (A, f) in A is called*

(1) *a fuzzy subalgebra of A with type 1 (briefly, 1-fuzzy subalgebra of A) if*

$$(\forall x, y \in A)(f((x|(y|y))|(x|(y|y))) \geq \min\{f(x), f(y)\}),$$

(2) *a fuzzy subalgebra of A with type 2 (briefly, 2-fuzzy subalgebra of A) if*

$$(\forall x, y \in A)(f((x|(y|y))|(x|(y|y))) \leq \min\{f(x), f(y)\}),$$

(3) *a fuzzy subalgebra of A with type 3 (briefly, 3-fuzzy subalgebra of A) if*

$$(\forall x, y \in A)(f((x|(y|y))|(x|(y|y))) \geq \max\{f(x), f(y)\}),$$

(4) *a fuzzy subalgebra of A with type 4 (briefly, 4-fuzzy subalgebra of A) if*

$$(\forall x, y \in A)(f((x|(y|y))|(x|(y|y))) \leq \max\{f(x), f(y)\}).$$

Example 3. *Consider Example 1, we have 3 cases as follows:*

Case 1: *Let $x = u$ and $y = v$. Then $f((x|(y|y))|(x|(y|y))) = f((u|(v|v))|(u|(v|v))) = f((u|u)|(u|u)) = f(v|v) = f(u)$. Since $f(u) = [0.2, 0.4]$ and $f(v) = [0.3, 0.7]$, it follows that $\min\{f(x), f(y) \ = \min\{[0.2, 0.4], [0.3, 0.7]\} = [0.2, 0.4]$. The condition $f((x|(y|y))|(x|(y|y))) \geq \min\{f(x), f(y)\}$ is satisfied, as the result $f(u) = [0.2, 0.4]$ is less than or equal to $\min\{f(x), f(y)\}$. Therefore, $(A, \widetilde{f})$ over A forms a 1-fuzzy subalgebra of A.*

Case 2: *Let $x = 1$ and $y = u$. Then $f((x|(y|y))|(x|(y|y))) = f((1|(u|u))|(1|(u|u))) = f((1|v)|(1|v)) = f(u|u) = f(u)$. Given that $f(1) = [0.3, 0.7]$ and $f(u) = [0.2, 0.4]$, we can compute the minimum of the two fuzzy sets: $\min\{f(x), f(y) \ = \min\{[0.3, 0.7], [0.2, 0.4]\} = [0.2, 0.4]$. The condition $f((x|(y|y))|(x|(y|y))) \leq \min\{f(x), f(y)\})$ is satisfied, as the result $f(u) = [0.2, 0.4]$ is indeed less than or equal to $\min\{f(x), f(y)\}$. Therefore, $(A, \widetilde{f})$ over A forms a 2-fuzzy subalgebra of A.*

Case 3: *Similarly, by choosing $x = 1$ and $y = u$, we have $(A, \widetilde{f})$ over A forms a 3-fuzzy subalgebra of A, and by selecting $x = 0$ and $y = v$, we have $(A, \widetilde{f})$ over A forms a 4-fuzzy subalgebra of A.*

Definition 9. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is called a length 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) subalgebra of A if a fuzzy structure $(A, \widetilde{f}_\ell)$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) subalgebra of A.*

Proposition 2. *If $(A, \widetilde{f})$ is a length k-fuzzy subalgebra of A for $k \in \{1, 3\}$, then*

$$(\forall x \in A)(\widetilde{f}_\ell(0) \geq \widetilde{f}_\ell(x)). \tag{1}$$

Proof. Let $(A, \widetilde{f})$ be a length 1-fuzzy subalgebra of A and $x \in A$. Then

$$\widetilde{f}_\ell(0) = \widetilde{f}_\ell(1|1) = \widetilde{f}_\ell((x|(x|x))|(x|(x|x))) \geq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(x)\} = \widetilde{f}_\ell(x).$$

Let $(A, \widetilde{f})$ be a length 3-fuzzy subalgebra of A. Then

$$\widetilde{f}_\ell(0) = \widetilde{f}_\ell(1|1) = \widetilde{f}_\ell((x|(x|x))|(x|(x|x))) \geq \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(x)\} = \widetilde{f}_\ell(x).$$

Proposition 3. *If $(A, \widetilde{f})$ is a length k-fuzzy subalgebra of A for $k \in \{2, 4\}$, then*

$$(\forall x \in A)(\widetilde{f}_\ell(0) \leq \widetilde{f}_\ell(x)). \tag{2}$$

Proof. Let $(A, \widetilde{f})$ be a length 2-fuzzy subalgebra of A and $x \in A$. Then

$$\widetilde{f}_\ell(0) = \widetilde{f}_\ell(1|1) = \widetilde{f}_\ell((x|(x|x))|(x|(x|x))) \leq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(x)\} = \widetilde{f}_\ell(x).$$

Let $(A, \widetilde{f})$ be a length 4-fuzzy subalgebra of A. Then

$$\widetilde{f}_\ell(0) = \widetilde{f}_\ell(1|1) = \widetilde{f}_\ell((x|(x|x))|(x|(x|x))) \leq \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(x)\} = \widetilde{f}_\ell(x).$$

Theorem 1. *Every length 3-fuzzy subalgebra of A is a length 1-fuzzy subalgebra.*

Proof. Let $(A, \widetilde{f})$ be a length 3-fuzzy subalgebra of A and $x, y \in A$. Then

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} \geq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.

Theorem 2. *Every length 2-fuzzy subalgebra of A is a length 4-fuzzy subalgebra.*

Proof. Let $(A, \widetilde{f})$ be a length 2-fuzzy subalgebra of A and $x, y \in A$. Then

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} \leq \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.

Theorem 3. *Length 2-fuzzy subalgebra and length 3-fuzzy subalgebra of A coincide.*

Proof. It is straightforward by Theorems 1 and 2.

Theorem 4. *Given a subalgebra S of A and $B_1, B_2 \in D[0, 1]$, let $(A, \widetilde{f})$ be an interval-valued fuzzy structure over A given by:*

$$\widetilde{f} : A \to D[0, 1]; x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise.} \end{cases}$$

(1) *If* $B_1 \subset B_2$, *then* $(A, \widetilde{f})$ *is a length* 1*-fuzzy subalgebra of* A.

(2) *If* $B_2 \subset B_1$, *then* $(A, \widetilde{f})$ *is a length* 4*-fuzzy subalgebra of* A.

Proof. If $x \in S$, then $\widetilde{f}(x) = B_2$. Hence,

$$\widetilde{f}_\ell(x) = \widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x) = \sup \widetilde{f}(x) - \inf \widetilde{f}(x) = \sup B_2 - \inf B_2.$$

If $x \notin S$, then $\widetilde{f}(x) = B_1$. Hence,

$$\widetilde{f}_\ell(x) = \widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x) = \sup \widetilde{f}(x) - \inf \widetilde{f}(x) = \sup B_1 - \inf B_1.$$

(1) Assume that $B_1 \subset B_2$. Then $\sup B_2 - \inf B_2 \geq \sup B_1 - \inf B_1$.

Case 1: Let $x, y \in S$. Then $\widetilde{f}_\ell(x) = \sup B_2 - \inf B_2$ and $\widetilde{f}_\ell(y) = \sup B_2 - \inf B_2$. Thus, $\min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_2 - \inf B_2$. Since S is a subalgebra of A, we have $(x|(y|y))|(x|(y|y)) \in S$ and so $\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) = \sup B_2 - \inf B_2$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) = \sup B_2 - \inf B_2 = (\geq) \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_\ell(x) = \sup B_1 - \inf B_1$ and $\widetilde{f}_\ell(y) = \sup B_1 - \inf B_1$, and so $\min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_\ell(x) = \sup B_1 - \inf B_1$ and $\widetilde{f}_\ell(y) = \sup B_2 - \inf B_2$, and so $\min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_\ell(x) = \sup B_2 - \inf B_2$ and $\widetilde{f}_\ell(y) = \sup B_1 - \inf B_1$, and so $\min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq \sup B_1 - \inf B_1 = \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $\widetilde{f}_\ell$ is a 1-fuzzy subalgebra of A and so $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.

(2) Assume that $B_2 \subset B_1$. Then $\sup B_2 - \inf B_2 \leq \sup B_1 - \inf B_1$.

Case 1: Let $x, y \in S$. Then $\widetilde{f}_\ell(x) = \sup B_2 - \inf B_2$ and $\widetilde{f}_\ell(y) = \sup B_2 - \inf B_2$. Thus, $\max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_2 - \inf B_2$. Since S is a subalgebra of A, we have $(x|(y|y))|(x|(y|y)) \in S$ and so $\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) = \sup B_2 - \inf B_2$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) = \sup B_2 - \inf B_2 = (\leq) \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_\ell(x) = \sup B_1 - \inf B_1$ and $\widetilde{f}_\ell(y) = \sup B_1 - \inf B_1$, so $\max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_\ell(x) = \sup B_1 - \inf B_1$ and $\widetilde{f}_\ell(y) = \sup B_2 - \inf B_2$, so $\max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_\ell(x) = \sup B_2 - \inf B_2$ and $\widetilde{f}_\ell(y) = \sup B_1 - \inf B_1$, so $\max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} = \sup B_1 - \inf B_1$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \sup B_1 - \inf B_1 = \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $\widetilde{f}_\ell$ is a 4-fuzzy subalgebra of A and so $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.

Definition 10. *Let (A, f) be a fuzzy structure in A. For any $t \in [0,1]$, the sets*

$$U(f;t) = \{x \in A : f(x) \geq t\},$$

$$L(f;t) = \{x \in A : f(x) \leq t\},$$

are called an upper t-level subset and a lower t-level subset of f, respectively.

Example 4. *Consider Example 2, and let $t = 0.3$. Then $U(\widetilde{f}_l; 0.3) = \{1, v, 0\}$ and $L(\widetilde{f}_l; 0.3) = \{u, 0\}$. If $t = 0.5$, then $U(\widetilde{f}_l; 0.5) = \emptyset$ and $L(\widetilde{f}_l; 0.5) = A$. If $t = 0.1$, then $U(\widetilde{f}_l; 0.1) = A$ and $L(\widetilde{f}_l; 0.1) = \emptyset$.*

Theorem 5. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 1-fuzzy subalgebra of A if and only if the set $U(\widetilde{f}_\ell; t)$ is a subalgebra of A for all $t \in [0,1]$ with $U(\widetilde{f}_\ell; t) \neq \emptyset$.*

Proof. Assume that $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A. Let $t \in [0,1]$ be such that $U(\widetilde{f}_\ell; t) \neq \emptyset$ and let $x, y \in U(\widetilde{f}_\ell; t)$. Then $\widetilde{f}_\ell(x) \geq t$ and $\widetilde{f}_\ell(y) \geq t$. Since $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A, we have

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} \geq t.$$

Thus, $(x|(y|y))|(x|(y|y)) \in U(\widetilde{f}_\ell; t)$. Hence, $U(\widetilde{f}_\ell; t)$ is a subalgebra of A.

Conversely, assume that for all $t \in [0,1]$, the set $U(\widetilde{f}_\ell; t)$ is a subalgebra of A if $U(\widetilde{f}_\ell; t) \neq \emptyset$. Let $x, y \in A$. Then $\widetilde{f}_\ell(x), \widetilde{f}_\ell(y) \in [0,1]$. If we take $t = \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}$, then $\widetilde{f}_\ell(x) \geq t$ and $\widetilde{f}_\ell(y) \geq t$. Hence, $x, y \in U(\widetilde{f}_\ell; t) \neq \emptyset$. By assumption, we have $U(\widetilde{f}_\ell; t)$ is a subalgebra of A, and so $(x|(y|y))|(x|(y|y)) \in U(\widetilde{f}_\ell; t)$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \geq t = \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $(A, \widetilde{f}_\ell)$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.

Corollary 1. *If $(A, \widetilde{f})$ is a length 3-fuzzy subalgebra of A, then the set $U(\widetilde{f}_\ell; t)$ is a subalgebra of A for all $t \in [0,1]$ with $U(\widetilde{f}_\ell; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 1 and 5.

Theorem 6. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a length 4-fuzzy subalgebra of A if and only if the set $L(\widetilde{f}_\ell; t)$ is a subalgebra of A for all $t \in [0,1]$ with $L(\widetilde{f}_\ell; t) \neq \emptyset$.*

Proof. Assume that $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A. Let $t \in [0,1]$ be such that $L(\widetilde{f}_\ell; t) \neq \emptyset$ and let $x, y \in L(\widetilde{f}_\ell; t)$. Then $\widetilde{f}_\ell(x) \leq t$ and $\widetilde{f}_\ell(y) \leq t$. Since $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A, we have

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} \leq t.$$

Thus, $(x|(y|y))|(x|(y|y)) \in L(\widetilde{f}_\ell; t)$. Hence, $L(\widetilde{f}_\ell; t)$ is a subalgebra of A.

Conversely, assume that for all $t \in [0,1]$, the set $L(\widetilde{f}_\ell; t)$ is a subalgebra of A if $L(\widetilde{f}_\ell; t) \neq \emptyset$. Let $x, y \in A$. Then $\widetilde{f}_\ell(x), \widetilde{f}_\ell(y) \in [0,1]$. If we take $t = \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}$. Thus, $\widetilde{f}_\ell(x) \leq t$ and $\widetilde{f}_\ell(y) \leq t$, and so $x, y \in L(\widetilde{f}_\ell; t) \neq \emptyset$. By assumption, $L(\widetilde{f}_\ell; t)$ is a subalgebra of A, and so $(x|(y|y))|(x|(y|y)) \in L(\widetilde{f}_\ell; t)$. Thus,

$$\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq t = \max\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\}.$$

Hence, $(A, \widetilde{f}_\ell)$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.

Corollary 2. *If $(A, \widetilde{f})$ is a length 2-fuzzy subalgebra of A, then the set $L(\widetilde{f}_\ell; t)$ is a subalgebra of A for all $t \in [0,1]$ with $L(\widetilde{f}_\ell; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 2 and 6.

Theorem 7. *If $(A, \widetilde{f})$ is a length 2-fuzzy subalgebra of A, then $U_\ell(\widetilde{f}; t)^c$ is a subalgebra of A for all $t \in [0,1]$ with $U_\ell(\widetilde{f}; t)^c \neq \emptyset$.*

Proof. Assume that $(A, \widetilde{f})$ is a length 2-fuzzy subalgebra of A and let $x, y \in A$ be such that $x \in U_\ell(\widetilde{f}; t)^c$ and $y \in U_\ell(\widetilde{f}; t)^c$. This shows that $\widetilde{f}_\ell(x) < t$ and $\widetilde{f}_\ell(y) < t$. This implies that $\widetilde{f}_\ell((x|(y|y))|(x|(y|y))) \leq \min\{\widetilde{f}_\ell(x), \widetilde{f}_\ell(y)\} < t$, that is, $(x|(y|y))|(x|(y|y)) \in U_\ell(\widetilde{f}; t)^c$. Therefore, $U_\ell(\widetilde{f}; t)^c$ is a subalgebra of A for all $t \in [0,1]$ with $U_\ell(\widetilde{f}; t)^c \neq \emptyset$.

Theorem 8. *If $(A, \widetilde{f})$ is a length 3-fuzzy subalgebra of A, then $L_\ell(\widetilde{f}; t)^c$ is a subalgebra of A for all $t \in [0,1]$ with $L_\ell(\widetilde{f}; t)^c \neq \emptyset$.*

Proof. It is similar to the proof of Theorem 7.

Theorem 9. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, we have $\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A, we have $\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) \geq \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}$. Thus,

$$\begin{aligned} \widetilde{f}_{\ell}((x|(y|y))|(x|(y|y))) &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \\ &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}(0) \\ &\geq \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\} - \widetilde{f}_{\inf}(0) \\ &= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(0), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(0)\} \\ &= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\} \\ &= \min\{\widetilde{f}_{\ell}(x), \widetilde{f}_{\ell}(y)\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_{\ell})$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.

Theorem 10. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, we have $\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A, we have $\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) \leq \max\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}$. Thus,

$$\begin{aligned} \widetilde{f}_{\ell}((x|(y|y))|(x|(y|y))) &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \\ &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}(0) \\ &\leq \max\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\} - \widetilde{f}_{\inf}(0) \\ &= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(0), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(0)\} \\ &= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\} \\ &= \max\{\widetilde{f}_{\ell}(x), \widetilde{f}_{\ell}(y)\}. \end{aligned}$$

Hence, $(A, \widetilde{f}_{\ell})$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.

Theorem 11. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A, we have $\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \le \max\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}$. Thus,

$$\begin{aligned}\widetilde{f}_{\ell}((x|(y|y))|(x|(y|y))) &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))\\ &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))\\ &\ge \widetilde{f}_{\sup}(0) - \max\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}\\ &= \min\{\widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(y)\}\\ &= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\}\\ &= \min\{\widetilde{f}_{\ell}(x), \widetilde{f}_{\ell}(y)\}.\end{aligned}$$

Hence, $(A, \widetilde{f}_{\ell})$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 1-fuzzy subalgebra of A.

Theorem 12. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A, we have $\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \ge \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}$. Thus,

$$\begin{aligned}\widetilde{f}_{\ell}((x|(y|y))|(x|(y|y))) &= \widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))\\ &= \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))\\ &\le \widetilde{f}_{\sup}(0) - \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}\\ &= \max\{\widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(0) - \widetilde{f}_{\inf}(y)\}\\ &= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\}\\ &= \max\{\widetilde{f}_{\ell}(x), \widetilde{f}_{\ell}(y)\}.\end{aligned}$$

Hence, $(A, \widetilde{f}_{\ell})$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a length 4-fuzzy subalgebra of A.

4. Mean of an interval-valued fuzzy structure in Sheffer stroke Hilbert algebras

This section introduces the concept of the mean of an interval-valued fuzzy structure within Sheffer stroke Hilbert algebras, along with the corresponding notion of mean-fuzzy subalgebras. The fundamental properties of these subalgebras are examined, shedding light on their intrinsic algebraic behavior. We further explore the connections between mean-fuzzy subalgebras and classical subalgebras, offering a comparative perspective on their structural interplay. Additionally, the relationships between mean-fuzzy subalgebras and various level subsets—namely, upper and lower-level subsets—of the mean of an interval-valued fuzzy structure are analyzed, providing a comprehensive framework for understanding their hierarchical and interval-dependent dynamics in the context of Sheffer stroke Hilbert algebras.

Definition 11. *[26] Given an interval-valued fuzzy structure $(A, \widetilde{f})$ over A, we define a fuzzy structure $(A, \widetilde{f}_m)$ in A as follows:*

$$\widetilde{f}_m : A \to [0,1]; x \mapsto \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2},$$

which is called the mean of $\widetilde{f}$.

Example 5. *Consider Example 1, we have*

A	1	u	v	0
$\widetilde{f}_m$	0.5	0.3	0.5	0.25

Definition 12. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is called a mean 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) subalgebra of A if a fuzzy structure $(A, \widetilde{f}_m)$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) subalgebra of A.*

Proposition 4. *If $(A, \widetilde{f})$ is a mean k-fuzzy subalgebra of A for $k \in \{1, 3\}$, then*

$$(\forall x \in A)(\widetilde{f}_m(0) \geq \widetilde{f}_m(x)). \tag{3}$$

Proof. Let $(A, \widetilde{f})$ be a mean k-fuzzy subalgebra of A for $k \in \{1, 3\}$. Then

$$\begin{aligned} \widetilde{f}_m(0) &= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(0)}{2} \\ &\geq \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2} \\ &= \widetilde{f}_m(x). \end{aligned}$$

Proposition 5. *If $(A, \widetilde{f})$ is a mean k-fuzzy subalgebra of A for $k \in \{2, 4\}$, then*

$$(\forall x \in A)(\widetilde{f}_m(0) \leq \widetilde{f}_m(x)). \tag{4}$$

Proof. Let $(A, \widetilde{f})$ be a mean k-fuzzy subalgebra of A for $k \in \{2, 4\}$. Then

$$\begin{aligned} \widetilde{f}_m(0) &= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}(0)}{2} \\ &\leq \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2} \\ &= \widetilde{f}_m(x). \end{aligned}$$

Theorem 13. *Every mean 3-fuzzy subalgebra of A is a mean 1-fuzzy subalgebra.*

Proof. Let $(A, \widetilde{f})$ be a mean 3-fuzzy subalgebra of A. Then

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y)))}{2} + \frac{\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&\geq \max\left\{\frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2}\right\} + \max\left\{\frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&\geq \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2}\right\} + \min\left\{\frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(y) + \widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.

Theorem 14. *Every mean 2-fuzzy subalgebra of A is a mean 4-fuzzy subalgebra.*

Proof. Let $(A, \widetilde{f})$ be a mean 2-fuzzy subalgebra of A. Then

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y)))}{2} + \frac{\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&\leq \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2}\right\} + \min\left\{\frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&\leq \max\left\{\frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2}\right\} + \max\left\{\frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(y) + \widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.

Theorem 15. *Mean 2-fuzzy subalgebra and mean 3-fuzzy subalgebra of A coincide.*

Proof. It is straightforward by Theorems 13 and 14.

Theorem 16. *Given a subalgebra S of A and $B_1, B_2 \in D[0,1]$, let $(A, \widetilde{f})$ be an interval-valued fuzzy structure over A given by*

$$\widetilde{f} : A \to D[0,1]; x \mapsto \begin{cases} B_2 & \textit{if } x \in S, \\ B_1 & \textit{otherwise.} \end{cases}$$

(1) *If* $\sup B_2 \geq \sup B_1$ *and* $\inf B_2 \geq \inf B_1$, *then* $(A, \widetilde{f})$ *is a mean* 1*-fuzzy subalgebra of* A.

(2) *If* $\sup B_2 \leq \sup B_1$ *and* $\inf B_2 \leq \inf B_1$, *then* $(A, \widetilde{f})$ *is a mean* 4*-fuzzy subalgebra of* A.

Proof. If $x \in S$, then $\widetilde{f}(x) = B_2$ and so

$$\widetilde{f}_m(x) = \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2} = \frac{\sup \widetilde{f}(x) + \inf \widetilde{f}(x)}{2} = \frac{\sup B_2 + \inf B_2}{2}.$$

If $x \notin S$, then $\widetilde{f}(x) = B_1$ and so

$$\widetilde{f}_m(x) = \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2} = \frac{\sup \widetilde{f}(x) + \inf \widetilde{f}(x)}{2} = \frac{\sup B_1 + \inf B_1}{2}.$$

(1) Assume that $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \geq \frac{\sup B_1 + \inf B_1}{2}.$$

Case 1: Let $x, y \in S$. Then $\widetilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $f_m(y) = \frac{\sup B_2 + \inf B_2}{2}$. Thus, $\min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is a subalgebra of A, we have $(x|(y|y))|(x|(y|y)) \in S$ and so $\widetilde{f}_m((x|(y|y))|(x|(y|y))) = \frac{\sup B_2 + \inf B_2}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) = \frac{\sup B_2 + \inf B_2}{2} = (\geq) \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Hence, $\widetilde{f}_m$ is a 1-fuzzy subalgebra of A and so $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.

(2) Assume that $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \leq \frac{\sup B_1 + \inf B_1}{2}.$$

Case 1: Let $x, y \in S$. Then $\widetilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_2 + \inf B_2}{2}$. Since S is a subalgebra of A, we have $(x|(y|y))|(x|(y|y)) \in S$ and so $\widetilde{f}_m((x|(y|y))|(x|(y|y))) = \frac{\sup B_2 + \inf B_2}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) = \frac{\sup B_2 + \inf B_2}{2} = (\leq) \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_m(x) = \frac{\sup B_1 + \inf B_1}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_2 + \inf B_2}{2}$, so $\max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_m(x) = \frac{\sup B_2 + \inf B_2}{2}$ and $\widetilde{f}_m(y) = \frac{\sup B_1 + \inf B_1}{2}$, so $\max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} = \frac{\sup B_1 + \inf B_1}{2}$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \leq \frac{\sup B_1 + \inf B_1}{2} = \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Hence, $\widetilde{f}_m$ is a 4-fuzzy subalgebra of A and so $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.

Theorem 17. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a mean 1-fuzzy subalgebra of A if and only if the set $U(\widetilde{f}_m; t)$ is a subalgebra of A for all $t \in [0, 1]$ with $U(\widetilde{f}_m; t) \neq \emptyset$.*

Proof. Assume that $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A. Let $t \in [0, 1]$ be such that $U(\widetilde{f}_m; t) \neq \emptyset$ and let $x, y \in U(\widetilde{f}_m; t)$. Then $\widetilde{f}_m(x) \geq t$ and $\widetilde{f}_m(y) \geq t$. Since $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A, we have

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \geq \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} \geq t.$$

Thus, $(x|(y|y))|(x|(y|y)) \in U(\widetilde{f}_m; t)$. Hence, $U(\widetilde{f}_m; t)$ is a subalgebra of A.

Conversely, assume that for all $t \in [0, 1]$, the set $U(\widetilde{f}_m; t)$ is a subalgebra of A if $U(\widetilde{f}_m; t) \neq \emptyset$. Let $x, y \in A$. Then $\widetilde{f}_m(x), \widetilde{f}_m(y) \in [0, 1]$. Choose $t = \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}$. Thus, $\widetilde{f}_m(x) \geq t$ and $\widetilde{f}_m(y) \geq t$. It follows that $x, y \in U(\widetilde{f}_m; t) \neq \emptyset$. By assumption, we have $U(\widetilde{f}_m; t)$ is a subalgebra of A and so $(x|(y|y))|(x|(y|y)) \in U(\widetilde{f}_m; t)$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \geq t = \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Hence, $(A, \widetilde{f}_m)$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.

Corollary 3. *If $(A, \widetilde{f})$ is a mean 3-fuzzy subalgebra of A, then $U(\widetilde{f}_m; t)$ is a subalgebra of A for all $t \in [0, 1]$ with $U(\widetilde{f}_m; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 13 and 17.

Theorem 18. *An interval-valued fuzzy structure $(A, \widetilde{f})$ over A is a mean 4-fuzzy subalgebra of A if and only if the set $L(\widetilde{f}_m; t)$ is a subalgebra of A for all $t \in [0, 1]$ with $L(\widetilde{f}_m; t) \neq \emptyset$.*

Proof. Assume that $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A. Let $t \in [0, 1]$ be such that $L(\widetilde{f}_m; t) \neq \emptyset$ and let $x, y \in L(\widetilde{f}_m; t)$. Then $\widetilde{f}_m(x) \leq t$ and $\widetilde{f}_m(y) \leq t$. Since $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A, we have

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \leq \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\} \leq t.$$

Thus, $(x|(y|y))|(x|(y|y)) \in L(\widetilde{f}_m; t)$. Hence, $L(\widetilde{f}_m; t)$ is a subalgebra of A.

Conversely, assume that for all $t \in [0, 1]$, the set $L(\widetilde{f}_m; t)$ is a subalgebra of A if $L(\widetilde{f}_m; t) \neq \emptyset$. Let $x, y \in A$. Then $\widetilde{f}_m(x), \widetilde{f}_m(y) \in [0, 1]$. Choose $t = \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}$. Thus, $\widetilde{f}_m(x) \leq t$ and $\widetilde{f}_m(y) \leq t$, and so $x, y \in L(\widetilde{f}_m; t) \neq \emptyset$. By assumption, we have $L(\widetilde{f}_m; t)$ is a subalgebra of A and so $(x|(y|y))|(x|(y|y)) \in L(\widetilde{f}_m; t)$. Thus,

$$\widetilde{f}_m((x|(y|y))|(x|(y|y))) \leq t = \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.$$

Hence, $(A, \widetilde{f}_m)$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.

Corollary 4. *If $(A, \widetilde{f})$ is a mean 2-fuzzy subalgebra of A, then $L(\widetilde{f}_m; t)$ is a subalgebra of A for all $t \in [0, 1]$ with $L(\widetilde{f}_m; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 15 and 18.

Theorem 19. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, we have $\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of A, we have

$$\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) \geq \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.$$

Thus,

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y)))}{2} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&\geq \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2} + \frac{\widetilde{f}_{\inf}(x)}{2}\right\} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(y) + \widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.

Theorem 20. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\inf})$ is constant and $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\inf})$ is constant, we have $\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of A, we have

$$\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) \leq \max\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.$$

Thus,

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y)))}{2} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&\geq \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2} + \frac{\widetilde{f}_{\inf}(x)}{2}\right\} + \frac{\widetilde{f}_{\inf}(0)}{2} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x)}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2} + \frac{\widetilde{f}_{\inf}(0)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(y) + \widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.

Theorem 21. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of A, we have

$$\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \leq \max\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}.$$

Thus,

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&\leq \frac{\widetilde{f}_{\sup}(0)}{2} + \max\left\{\frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\inf}(x)}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \max\left\{\frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\sup}(y) + \widetilde{f}_{\inf}(yx)}{2}\right\} \\
&= \max\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 4-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 4-fuzzy subalgebra of A.

Theorem 22. *If $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A, then $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.*

Proof. Assume that $(A, \widetilde{f})$ is an interval-valued fuzzy structure over A in which $(A, \widetilde{f}_{\sup})$ is constant and $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A. Let $x, y \in A$. Since $(A, \widetilde{f}_{\sup})$ is constant, we have $\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of A, we have

$$\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y))) \geq \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}.$$

Thus,

$$\begin{aligned}
\widetilde{f}_m((x|(y|y))|(x|(y|y))) &= \frac{\widetilde{f}_{\sup}((x|(y|y))|(x|(y|y))) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&= \frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\inf}((x|(y|y))|(x|(y|y)))}{2} \\
&\geq \frac{\widetilde{f}_{\sup}(0)}{2} + \min\left\{\frac{\widetilde{f}_{\inf}(x)}{2}, \frac{\widetilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(0)}{2} + \frac{\widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(0)}{2}, \frac{\widetilde{f}_{\sup}(y)}{2}\right\} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\sup}(x)}{2}, \frac{\widetilde{f}_{\sup}(0) + \widetilde{f}_{\sup}(y)}{2}\right\} \\
&= \min\{\widetilde{f}_m(x), \widetilde{f}_m(y)\}.
\end{aligned}$$

Hence, $(A, \widetilde{f}_m)$ is a 1-fuzzy subalgebra of A, that is, $(A, \widetilde{f})$ is a mean 1-fuzzy subalgebra of A.

5. Conclusion

This study advances the theoretical framework of Sheffer stroke Hilbert algebras by introducing the concepts of length-fuzzy subalgebras and mean-fuzzy subalgebras within interval-valued fuzzy structures. These new constructs deepen the understanding of fuzzy logic in algebraic systems, particularly by elucidating the interplay between fuzzy and traditional subalgebras. The investigation reveals key properties and relationships, including their alignment with upper and lower-level subsets, offering a refined perspective on the gradations of membership functions. This framework not only enriches algebraic theory but also underscores the practical relevance of fuzzy subalgebras in fields such as logic, computer science, and uncertainty modeling. The findings pave the way for future research to explore these ideas in more complex fuzzy systems or adapt them to other algebraic structures, broadening their applicability and potential impact across diverse domains.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

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