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Fuzzy Hom-Groups: A New Perspective on Algebraic Generalization

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Abstract. Fuzzy algebraic structures extend classical algebra to model uncertainty, while Hom-groups introduce a twisting map α that modifies associativity and identity conditions. This paper unifies these concepts by introducing *Fuzzy Hom-Groups*, a generalization of fuzzy groups within the Hom-group framework.
We define fuzzy Hom-subgroups and fuzzy Hom-normal subgroups, establishing their fundamental properties. A key result shows that each fuzzy Hom-subgroup induces an upper-level set forming a classical Hom-subgroup, bridging fuzzy group theory and Hom-algebra. We further analyze the structural relationships between fuzzy Hom-subgroups and Hom-subgroups.
Illustrative examples highlight how the twisting map influences fuzzy Hom-structures. This study extends fuzzy algebra and Hom-group theory, with potential applications in decision-making, fuzzy control, and uncertainty modeling.

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1. Introduction

Fuzzy set theory and algebraic structures have been widely studied because they help model uncertainty in a precise mathematical way. Introduced by Lotfi Zadeh in 1965, fuzzy sets provide a mathematical framework for representing and handling imprecise and uncertain information [1]. Since Rosenfeld introduced fuzzy groups [2], researchers have extended this idea to fuzzy rings, fuzzy Lie algebras, and fuzzy modules, which have applications in decision-making, control systems, and artificial intelligence [3–6]. Studies on equivalence relations in fuzzy subgroups [7] and the classification of fuzzy normal subgroups in finite groups [8] have further developed the theory, offering insights into their structure and properties. These contributions provide a useful mathematical framework for both theoretical and practical problems.

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At the same time, algebraic structures have been generalized using *Hom-algebra theory*, which introduces a twisting map α that modifies traditional algebraic properties. This approach has led to new generalizations of groups, Lie algebras, and related structures [9–13]. Hom-algebraic structures have been particularly useful in the study of deformations, quantum symmetries, and non-associative algebraic systems. The development of Hom-group theory by Chen et al. [14] and further extensions such as ordered Hom-groups [15] have expanded algebraic research by adding more flexibility to classical group theory through the twisting map α.

Despite the progress in fuzzy algebra and Hom-group theory, the study of fuzzy Hom-groups and their substructures remains largely unexplored. Fuzzy Hom-groups extend both fuzzy groups [2] and Hom-groups [14] by incorporating a twisting map α, which adds more structural flexibility while maintaining key fuzzy properties. This work systematically develops the theory of fuzzy Hom-subgroups and fuzzy Hom-normal subgroups, linking fuzzy group theory with Hom-algebraic structures. Previous research on fuzzy cosets, fuzzy normal subgroups, and their homomorphic images [8, 16] suggests that combining fuzzy set theory with Hom-groups could lead to significant new results. Additionally, intuitionistic fuzzy structures provide a foundation for extended logic systems [17], which may inspire further generalizations of fuzzy Hom-algebraic systems.

By combining fuzzy set theory with Hom-algebraic structures, we define fuzzy Hom-subgroups and fuzzy Hom-normal subgroups, establishing their fundamental properties and examining their behavior under the structural constraints imposed by the twisting map α. This study builds on recent developments in fuzzy algebraic systems, such as fuzzy (n-)Lie algebras [18, 19], fuzzy Hom-Lie ideals [20], and complex fuzzy Gamma rings [21, 22]. It also aligns with structural studies of Hom-algebras and their fuzzy extensions [9, 10, 15].

The relationships between fuzzy Hom-subgroups, classical fuzzy groups, and upper-level sets of Hom-subgroups are explored, extending the scope of generalized algebraic systems. This work lays the groundwork for further studies on fuzzy Hom-structures, including their links to intuitionistic fuzzy logic and algebraic models in uncertain environments.

This paper is organized as follows. Section 2 provides a brief review of Hom-groups, including their essential properties and examples. Section 3 introduces Hom-subgroups and Hom-normal subgroups, establishing their foundational properties. Section 4 focuses on fuzzy Hom-subgroups, while Section 5 develops the concept of fuzzy Hom-normal subgroups. In Section 6, we explore the relationships between fuzzy Hom-subgroups and classical Hom-subgroups. Section 7 investigates the connections between strong fuzzy Hom-subgroups and Hom-subgroups. Section 8 examines the relationships between fuzzy Hom-normal subgroups and classical Hom-normal subgroups, while Section 9 extends this discussion to strong fuzzy Hom-normal subgroups. Finally, Section 10 concludes the paper and highlights directions for future research, including extensions to intuitionistic fuzzy, bipolar fuzzy, and generalized fuzzy Hom-structures, as well as applications in uncertain environments.

2. Hom-Groups

This section provides an overview of Hom-groups and Hom-subgroups, recalling key definitions and properties while incorporating refinements and generalizations. Some definitions are adapted from [14], but with modifications for consistency. Additionally, certain results originally stated as theorems are reformulated as definitions to better reflect their foundational role. New examples are introduced, including a generalization of a previously studied case, to illustrate structural flexibility within the Hom-group framework. These refinements and additions serve as the basis for developing fuzzy Hom-groups in subsequent sections.

In algebraic structures, Hom-groups are generalizations of classical groups, incorporating a twisting map that modifies the usual associativity and identity properties. The following formal definition outlines the essential components and properties of a Hom-group:

Definition 1 ([14]). *A* ***Hom-group*** *is a triple* $(G, \cdot, \alpha)$*, where:*

- G *is a set,*
- $\cdot : G \times G \to G$ *is a binary operation,*
- $\alpha : G \to G$ *is a bijective map,*

satisfying the following identities for all $g, h, k \in G$*:*

(HG1) ***Hom-associativity****:* $\alpha(g) \cdot (h \cdot k) = (g \cdot h) \cdot \alpha(k)$*,*

(HG2) ***Hom-unitarity****: There exists an element* $e \in G$ *such that for all* $g \in G$*:*

$$g \cdot e = \alpha(g) \quad \text{and} \quad e \cdot g = \alpha(g),$$

and in addition, the condition $\alpha(e) = e$ *holds.*

(HG3) ***Hom-inverses****: For every* $g \in G$*, there exists an inverse element* $g^{-1} \in G$ *such that:*

$$g \cdot g^{-1} = e \quad \text{and} \quad g^{-1} \cdot g = e,$$

where e *is the Hom-identity element.*

(HG4) ***Multiplicativity of*** α*:* $\alpha(g \cdot h) = \alpha(g) \cdot \alpha(h)$ *for all* $g, h \in G$*.*

Remark 1. *In earlier literature, such as in [14], the condition* $\alpha(e) = e$ *was not always required in the definition of Hom-groups. This led to cases where the behavior of the identity element under the twisting map needed to be verified separately. However, in this paper, we adopt a more structured definition by explicitly requiring that* $\alpha(e) = e$*. This condition simplifies many proofs and provides a consistent framework for studying Hom-subgroups and other related structures, ensuring that the identity element behaves uniformly within the Hom-group.*

Example 1 ([14]). *Any classical group $(G, \cdot)$ can be viewed as a Hom-group by defining the twisting map α as the identity map, i.e.,*

$$\alpha(g) = I_G(g) = g \text{ for all } g \in G.$$

Example 2. *Let $G = \mathbb{R}$, and define the binary operation $\oplus_k$ and the twisting map α_k for a fixed constant $k \neq 0$ as follows:*

$$a \oplus_k b = \frac{a+b}{k} \quad \textit{and} \quad \alpha_k(a) = \frac{a}{k}, \quad \textit{for all } a, b \in \mathbb{R}.$$

We verify that $(G = \mathbb{R}, \oplus_k, \alpha_k)$ forms a Hom-group by checking the required properties:

(HG1) ***Hom-associativity:*** *For all $a, b, c \in \mathbb{R}$, we verify:*

$$\alpha_k(a) \oplus_k (b \oplus_k c) = (a \oplus_k b) \oplus_k \alpha_k(c).$$

- The left-hand side (LHS) is:

$$\alpha_k(a) \oplus_k (b \oplus_k c) = \frac{\frac{a}{k} + \frac{b+c}{k}}{k} = \frac{a+b+c}{k^2}.$$

- The right-hand side (RHS) is:

$$(a \oplus_k b) \oplus_k \alpha_k(c) = \frac{\frac{a+b}{k} + \frac{c}{k}}{k} = \frac{a+b+c}{k^2}.$$

Since the LHS equals the RHS, the Hom-associativity condition is satisfied.

(HG2) ***Hom-unitarity:*** *We need to find an element $e \in G$ such that for all $a \in G$, we have:*

$$a \oplus_k e = \alpha_k(a) \quad \textit{and} \quad e \oplus_k a = \alpha_k(a),$$

and $\alpha_k(e) = e$. Let $e = 0$. We check:

$$a \oplus_k 0 = \frac{a+0}{k} = \frac{a}{k} = \alpha_k(a),$$

and

$$0 \oplus_k a = \frac{0+a}{k} = \frac{a}{k} = \alpha_k(a).$$

Also, $\alpha_k(0) = \frac{0}{k} = 0$. Thus, $e = 0$ is the Hom-identity element, and $\alpha_k(e) = e$.

(HG3) ***Hom-inverses:*** *For each $a \in G$, we need to find $a^* \in G$ such that:*

$$a \oplus_k a^* = e \quad \textit{and} \quad a^* \oplus_k a = e,$$

where $e = 0$. Solving for a^*, we have:

$$\frac{a+a^*}{k} = 0 \implies a + a^* = 0 \implies a^* = -a.$$

Thus, the inverse of $a \in G$ under $\oplus_k$ is denoted by a^*, where $a^* = -a$.

(HG4) ***Multiplicativity of*** α_k***:*** *We verify that:*

$$\alpha_k(a \oplus_k b) = \alpha_k(a) \oplus_k \alpha_k(b).$$

- LHS:

$$\alpha_k(a \oplus_k b) = \alpha_k\left(\frac{a+b}{k}\right) = \frac{\frac{a+b}{k}}{k} = \frac{a+b}{k^2}.$$

- RHS:

$$\alpha_k(a) \oplus_k \alpha_k(b) = \frac{\frac{a}{k} + \frac{b}{k}}{k} = \frac{a+b}{k^2}.$$

Since LHS = RHS, the multiplicativity condition is satisfied.

Therefore, the set $G = \mathbb{R}$ *with the binary operation* $\oplus_k$ *and the twisting map* α_k *forms a Hom-group for any* $k \neq 0$.

Remark 2. *In [14], the case* $k = 2$ *was specifically considered in the construction of a Hom-group. However, in Example 2, we generalize this concept by defining the operation and twisting map for an arbitrary nonzero parameter* k*. This generalization provides a broader class of Hom-groups, demonstrating how different choices of* k *influence the underlying structure.*

Definition 2. *Let* $(G, \cdot, \alpha)$ *be a Hom-group. A subset* $H \subseteq G$ *is called a* ***Hom-subgroup*** *if* $(H, \cdot, \alpha|_H)$ *is a Hom-group.*

Theorem 1. *Let* $(G, \cdot, \alpha)$ *be a Hom-group, and let* $H \subseteq G$ *be a non-empty subset. Then* H *is a Hom-subgroup of* G *if and only if the following conditions hold:*

(HS1) H *is closed under the Hom-group operation, i.e., for all* $h_1, h_2 \in H$,

$$h_1 \cdot h_2 \in H,$$

(HS2) H *is closed under the twisting map* α*, i.e., for all* $h \in H$,

$$\alpha(h) \in H,$$

(HS3) H *is closed under inverses, i.e., for all* $h \in H$,

$$h^{-1} \in H,$$

where h^{-1} *denotes the inverse of* h *under the Hom-group operation* $\cdot$.

Proof. Suppose $H \subseteq G$ satisfies the conditions (HS1), (HS2), and (HS3). We will prove that H forms a Hom-subgroup by showing that it satisfies both **Hom-associativity** (HG1) and **Hom-unitarity** (HG2) as inherited from G.

Step 1: Hom-associativity (HG1): Since H is closed under the operation $\cdot$ and G satisfies Hom-associativity, for all $h_1, h_2, h_3 \in H$, we have:

$$\alpha(h_1) \cdot (h_2 \cdot h_3) = (h_1 \cdot h_2) \cdot \alpha(h_3).$$

Therefore, H inherits Hom-associativity from G, and (HG1) is satisfied within H.

Step 2: Hom-unitarity (HG2): We need to check whether the Hom-identity element $e \in G$ belongs to H. Since H is non-empty, take any $h \in H$. By (HS3), we know that the inverse $h^{-1} \in H$.

Now, by (HS1), the operation $h \cdot h^{-1} = e$ must also belong to H because H is closed under the operation $\cdot$. Therefore, the Hom-identity element $e \in H$.

Since $e \in H$ and $\alpha(e) = e$ (inherited from G), H satisfies the Hom-unitarity condition (HG2).

Example 3. *Consider the Hom-group $(\mathbb{Z}, \oplus, \alpha)$, where the operation $\oplus$ and twisting map α are defined as follows:*

$$a \oplus b = -(a+b) \text{ for all } a, b \in \mathbb{Z}, \ \alpha(a) = -a \text{ for all } a \in \mathbb{Z}.$$

This structure satisfies the conditions for a Hom-group, as described in [14].

Now, consider the subset $H = 2\mathbb{Z}$, the set of all even integers. We claim that H is a Hom-subgroup of $(\mathbb{Z}, \oplus, \alpha)$. This is verified as follows:

(HS1) Closure under the operation $\oplus$: For any $h_1, h_2 \in 2\mathbb{Z}$, we have:

$$h_1 \oplus h_2 = -(h_1 + h_2).$$

Since $h_1 + h_2 \in 2\mathbb{Z}$ (the sum of two even integers is even), we know that $-(h_1+h_2) \in 2\mathbb{Z}$. Therefore, H is closed under the operation $\oplus$.

(HS2) Closure under the twisting map α: For any $h \in 2\mathbb{Z}$, we have:

$$\alpha(h) = -h.$$

Since the negative of an even integer is still an even integer, $\alpha(h) \in 2\mathbb{Z}$. Thus, H is closed under the twisting map.

(HS3) Closure under inverses: For any $h \in 2\mathbb{Z}$, the inverse under the operation $\oplus$ is $h^* = -h$. Since $-h \in 2\mathbb{Z}$ (the set of even integers is closed under negation), H is closed under inverses.

Therefore, $H = 2\mathbb{Z}$ is a nontrivial Hom-subgroup of $(\mathbb{Z}, \oplus, \alpha)$.

Definition 3. *[14] Let $(G, \cdot, \alpha)$ be a Hom-group. A subset $N \subseteq G$ is called a* ***Hom-normal subgroup*** *if:*

(HN1) N is a Hom-subgroup of G,

(HN2) N is closed under conjugation, i.e., for all $g \in G$ and $n \in N$,

$$g \cdot n \cdot g^{-1} \in N.$$

Example 4. *Consider the Hom-group $(G = \mathbb{Z}, \oplus, \alpha)$, where:*

- *The operation is defined as* $a \oplus b = -(a+b)$ *for all* $a, b \in \mathbb{Z}$,
- *The twisting map is* $\alpha(a) = -a$ *for all* $a \in \mathbb{Z}$.

Let $N = 2\mathbb{Z}$*, the set of all even integers. We previously showed that* N *is a Hom-subgroup of* $(\mathbb{Z}, \oplus, \alpha)$ *(see Example 3). Now, we will verify that* N *is also a Hom-normal subgroup.*

Hom-normal subgroup property (HN2): *To check that* N *is closed under conjugation, take any* $g \in \mathbb{Z}$ *and* $n \in N$*. We compute the conjugation:*

$$g \oplus n \oplus g^* = -(g+n) \oplus (-g).$$

Using the operation $\oplus$*, this simplifies to:*

$$-(g+n) \oplus (-g) = -(-(g+n)+g) = -n.$$

Since $n \in 2\mathbb{Z}$*, we know that* $-n \in 2\mathbb{Z}$*, so the conjugation result* $g \oplus n \oplus g^* \in 2\mathbb{Z}$*.*

Therefore, $N = 2\mathbb{Z}$ *is a Hom-normal subgroup of* G*.*

3. Fuzzy Hom-Subgroups and Fuzzy Hom-Normal Subgroups

In this section, we introduce the concept of fuzzy Hom-subgroups (and Hom-normal subgroups), extending the classical notion of fuzzy subgroups (and fuzzy normal subgroups) to the Hom-group framework. This construction builds on the foundations of fuzzy set theory and Hom-groups. While fuzzy subsets have been widely studied in algebraic structures, their integration with Hom-group structures has not been explicitly explored.

Before proceeding, we recall the definition of a fuzzy set. A fuzzy set μ on a set G is a function $\mu : G \to [0, 1]$ that assigns to each element $g \in G$ a membership degree $\mu(g)$, representing the extent to which g belongs to the set. Using this foundation, we develop the notion of fuzzy Hom-subgroups and establish their key properties.

Definition 4. *Let* $(G, \cdot, \alpha)$ *be a Hom-group and* $\mu : G \to [0, 1]$ *be a fuzzy set on* G*. The fuzzy set* μ *is called a* ***Fuzzy Hom-subgroup*** *if for all* $g, h \in G$*:*

(i) $\mu(g \cdot h) \geq \min\{\mu(g), \mu(h)\}$,

(ii) $\mu(\alpha(g)) \geq \mu(g)$,

(iii) $\mu(g^{-1}) \geq \mu(g)$.

Example 5. *Consider the real numbers* $G = \mathbb{R}$ *equipped with the binary operation* $\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by:*

$$a \oplus b = \frac{a+b}{2},$$

and the twisting map $\alpha : \mathbb{R} \to \mathbb{R}$ *defined by:*

$$\alpha(a) = \frac{a}{2}.$$

We have shown in Example 2 that the pair $(G = \mathbb{R}, \oplus, \alpha)$ *forms a Hom-group, satisfying the properties of Hom-associativity, Hom-unitarity, and Hom-inverses.*

Now, let us define a fuzzy set $\mu : \mathbb{R} \to [0,1]$ *over this Hom-group, where* $\mu(a)$ *represents the degree of membership of the element* a *in the fuzzy subset. We define the fuzzy membership function* μ *as:*

$$\mu(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{1+|a|} & \text{if } a \neq 0. \end{cases}$$

This function assigns full membership to the identity element 0*, and the membership decreases as* $|a|$ *increases for other elements.*

Verification of the Fuzzy Hom-subgroup Properties

(i) ***Closure under the operation:***

We need to verify that:

$$\mu(g \oplus h) \geq \min\{\mu(g), \mu(h)\}.$$

Using the operation $g \oplus h = \frac{g+h}{2}$*, the membership function evaluates to:*

$$\begin{aligned} \mu\left(g \oplus h\right) &= \mu\left(\frac{g+h}{2}\right) \\ &= \frac{2}{2+|g+h|}. \end{aligned}$$

Since $|g+h| \leq 2\max(|g|,|h|)$*, we have:*

$$\begin{aligned} \frac{2}{2+|g+h|} &\geq \frac{2}{2+2\max(|g|,|h|)} \\ &= \frac{1}{1+\max(|g|,|h|)} \\ &= \min\left(\frac{1}{1+|g|}, \frac{1}{1+|h|}\right). \end{aligned}$$

Thus, we have shown that:

$$\mu(g \oplus h) \geq \min\{\mu(g), \mu(h)\},$$

which confirms that the closure property is satisfied.

(ii) ***Compatibility with the twisting map:***

We need to verify that:

$$\mu(\alpha(g)) \geq \mu(g).$$

Since $\alpha(g) = \frac{g}{2}$*, Simple computations show that:*

$$\mu(\alpha(g)) = \frac{4}{4+|g|}.$$

Now, comparing this with:

$$\mu(g) = \frac{1}{1+|g|},$$

we need to verify the inequality:

$$\frac{4}{4+|g|} \geq \frac{1}{1+|g|}.$$

To prove this inequality, we cross-multiply:

$$4 \cdot (1+|g|) \geq 1 \cdot (4+|g|).$$

Expanding both sides:

$$4 + 4|g| \geq 4 + |g|.$$

Subtracting 4 from both sides:

$$4|g| \geq |g|.$$

This inequality holds for all $g \in \mathbb{R}$ *because* $4|g|$ *is always greater than or equal to* $|g|$*. Thus, the inequality:*

$$\frac{4}{4+|g|} \geq \frac{1}{1+|g|}$$

is satisfied for all $g \in \mathbb{R}$*.*

Hence, we conclude that:

$$\mu(\alpha(g)) \geq \mu(g).$$

(iii) ***Closure under inverses:*** *We need to confirm that:*

$$\mu(g^*) \geq \mu(g).$$

Since $g^{-1} = -g$ *and the membership function depends only on the absolute value of the element, we have:*

$$\mu(g^*) = \mu(-g) = \mu(g).$$

Therefore, the closure under inverses is satisfied.

The concept of fuzzy normal subgroups has been extensively studied in classical group theory, with foundational work by Rosenfeld [2] and later refinements by Ajmal and Prajapati [16]. These studies explore how fuzzy subsets of groups maintain their structure under conjugation, forming the basis for fuzzy normal subgroups.

In this paper, we extend this notion to Hom-groups, which introduce an additional twisting map α that affects the algebraic structure. As far as we know, the definition of **Fuzzy Hom-Normal Subgroups** presented here is novel and offers a new framework for studying fuzzy subsets in Hom-group theory. Below, we formalize this concept.

Definition 5. *Let $(G, \cdot, \alpha)$ be a Hom-group, and let $\mu : G \to [0, 1]$ be a fuzzy set on G. The fuzzy set μ is called a* ***Fuzzy Hom-Normal Subgroup*** *if it is a Fuzzy Hom-subgroup (Definition 4) and satisfies the additional condition:*

(FHN3) For all $g, h \in G$,

$$\mu(g \cdot h \cdot g^{-1}) \geq \mu(h),$$

ensuring invariance under conjugation.

Example 6. *Consider the Hom-group $(G = \mathbb{R}, \oplus, \alpha)$, where:*

$$a \oplus b = \frac{a+b}{2}, \quad \alpha(a) = \frac{a}{2}.$$

Define the fuzzy membership function $\mu : \mathbb{R} \to [0, 1]$ as:

$$\mu(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{1+|a|} & \text{if } a \neq 0. \end{cases}$$

In Example 5, we verified that μ satisfies the conditions required for a fuzzy Hom-subgroup of G, specifically:

- *Closure under the operation $\oplus$,*
- *Compatibility with the twisting map α,*
- *Closure under inverses.*

To confirm that μ is a fuzzy Hom-normal subgroup, we now verify the additional condition:

- *Invariance under Conjugation (FHN3): For any $g, h \in G$, we need to show that:*

$$\mu(g \oplus h \oplus (-g)) \geq \mu(h).$$

Since $g \oplus h \oplus (-g) = \frac{h}{2}$, we calculate:

$$\mu\left(\frac{h}{2}\right) = \frac{2}{2+|h|}.$$

Given $\mu(h) = \frac{1}{1+|h|}$, we need to verify that:

$$\frac{2}{2+|h|} \geq \frac{1}{1+|h|}.$$

To confirm this inequality, we cross-multiply:

$$2(1+|h|) \geq 1(2+|h|).$$

Expanding both sides:

$$2 + 2|h| \geq 2 + |h|,$$

and simplifying, we obtain:

$$2|h| \geq |h|,$$

which holds for all $h \in \mathbb{R}$. Thus, the invariance under conjugation condition (FHN3) is satisfied.

Therefore, μ defines a fuzzy Hom-normal subgroup for the Hom-group $(\mathbb{R}, \oplus, \alpha)$.

4. Relationships Between Fuzzy Hom-Substructures and Hom-Substructures

Let G be a Hom-group equipped with a fuzzy set μ. We define the set $U(\mu, t)$ as the upper level set of μ for $t \in [0, 1]$, denoted by:

$$U(\mu, t) = \{g \in G \mid \mu(g) \geq t\}.$$

In the setting of Hom-groups, we seek to extend the result of Theorem 1 from [20], which explored the relationship between fuzzy subsets and structural subgroups. By establishing the connection between fuzzy Hom-subgroups of G and classical Hom-subgroups of G, we can better understand how fuzzy membership functions relate to subgroup properties within the framework of Hom-groups, supporting further analysis on fuzzy structures in Hom-algebraic systems.

Theorem 2. *Let μ be a fuzzy subset on a Hom-group $(G, \cdot, \alpha)$. The following statements are equivalent:*

(i) μ is a fuzzy Hom-subgroup of G,

(ii) For every $t \in Im(\mu)$, the set $U(\mu, t) = \{g \in G \mid \mu(g) \geq t\}$ is a Hom-subgroup of G.

Proof. To show the equivalence, we proceed as follows:

(i) ⇒ (ii): Suppose μ is a fuzzy Hom-subgroup of G. For each $t \in \mathrm{Im}(\mu)$, let $h_1, h_2 \in U(\mu, t)$ and $h \in U(\mu, t)$. By definition of a fuzzy Hom-subgroup:

- **Closure under $\cdot$:** Since $\mu(h_1 \cdot h_2) \geq \min\{\mu(h_1), \mu(h_2)\} \geq t$, we conclude $h_1 \cdot h_2 \in U(\mu, t)$.
- **Closure under α:** Similarly, $\alpha(h) \in U(\mu, t)$ as $\mu(\alpha(h)) \geq \mu(h) \geq t$.
- **Existence of Inverses:** For $h \in U(\mu, t)$, $h^{-1} \in U(\mu, t)$ since $\mu(h^{-1}) \geq t$.

Thus, $U(\mu, t)$ is a Hom-subgroup of G.

(ii) ⇒ (i): Conversely, assume each $U(\mu, t)$ is a Hom-subgroup of G for every $t \in \mathrm{Im}(\mu)$. We want to show that μ satisfies the properties required for a fuzzy Hom-subgroup.

Take any $g, h \in G$ and let $t_1 = \min\{\mu(g), \mu(h)\}$. This implies that $g, h \in U(\mu, t_1)$ since $\mu(g) \geq t_1$ and $\mu(h) \geq t_1$. Since $U(\mu, t_1)$ is a Hom-subgroup of G, it has the following properties:

- **Closure under the operation:** Since $g, h \in U(\mu, t_1)$ and $U(\mu, t_1)$ is closed under the operation $\cdot$, we have $g \cdot h \in U(\mu, t_1)$. Therefore, $\mu(g \cdot h) \geq t_1 = \min\{\mu(g), \mu(h)\}$, satisfying the closure condition for fuzzy Hom-subgroups.

- **Closure under the twisting map α:** Since $g \in U(\mu, t_1)$ and $U(\mu, t_1)$ is closed under α, we also have $\alpha(g) \in U(\mu, t_1)$, which implies $\mu(\alpha(g)) \geq \mu(g)$, satisfying the compatibility with α for fuzzy Hom-subgroups.

- **Closure under inverses:** For any $g \in U(\mu, t_1)$, since $U(\mu, t_1)$ is closed under inverses, we have $g^{-1} \in U(\mu, t_1)$, which ensures that $\mu(g^{-1}) \geq \mu(g)$, fulfilling the inverse condition for fuzzy Hom-subgroups.

Thus, by the properties of each upper level set $U(\mu, t)$ as Hom-subgroups, μ satisfies all the conditions to be a fuzzy Hom-subgroup.

In analogy with the results of Theorem 3 in [20], which addressed strong upper level sets in the context of fuzzy subsets, we investigate similar conditions within the framework of Hom-groups. By establishing a connection between strong fuzzy Hom-subgroups of G and strong Hom-subgroups of G, we aim to understand the interplay between fuzzy membership functions and subgroup properties in Hom-algebraic systems.

Theorem 3. *Let μ be a fuzzy subset on a Hom-group $(G, \cdot, \alpha)$. The following statements are equivalent:*

(i) μ is a strong fuzzy Hom-subgroup of G,

(ii) For every $t \in (0, 1]$, the set $U^*(\mu, t) = \{g \in G \mid \mu(g) > t\}$ is a Hom-subgroup of G.

Proof. We will show the equivalence between (i) and (ii) as stated in the theorem.

(i) $\Rightarrow$ (ii): Suppose μ is a fuzzy Hom-subgroup of G. For each $t \in \mathrm{Im}(\mu)$, consider the set $U(\mu, t) = \{g \in G \mid \mu(g) \geq t\}$.

To prove that $U(\mu, t)$ is a Hom-subgroup, we verify that it satisfies the Hom-group properties:

- **Closure under the operation $\cdot$:** For any $g, h \in U(\mu, t)$, since μ is a fuzzy Hom-subgroup, we have:
$$\mu(g \cdot h) \geq \min\{\mu(g), \mu(h)\} > t.$$
This implies $g \cdot h \in U(\mu, t)$, so $U(\mu, t)$ is closed under $\cdot$.

- **Closure under the twisting map α:** For any $g \in U(\mu, t)$, we have $\mu(\alpha(g)) \geq \mu(g) > t$. Hence, $\alpha(g) \in U(\mu, t)$, confirming closure under α.

- **Closure under inverses:** For any $g \in U(\mu, t)$, since $\mu(g^{-1}) \geq \mu(g) > t$, we have $g^{-1} \in U(\mu, t)$. Thus, $U(\mu, t)$ is closed under inverses.

Therefore, $U(\mu, t)$ is a Hom-subgroup of G for each $t \in \text{Im}(\mu)$.

(ii) ⇒ (i): Now, assume that for each $t \in \text{Im}(\mu)$, the strict upper-level set $U^*(\mu, t) = \{g \in G \mid \mu(g) > t\}$ is a Hom-subgroup of G. We want to show that μ satisfies the conditions required for a fuzzy Hom-subgroup.

- **Case of Zero Membership:** If $\mu(g) = 0$ or $\mu(h) = 0$, then by definition of a fuzzy Hom-subgroup, we have $\mu(g \cdot h) \geq 0 = \min\{\mu(g), \mu(h)\}$, which fulfills the closure condition.

- **Case of Positive Membership:** Assume $\mu(g) \neq 0$ and $\mu(h) \neq 0$. Let $t_1 = \min\{\mu(g), \mu(h)\}$. Then $g, h \in U^*(\mu, t_0)$ for some $t_0 < t_1$, since $\mu(g) > t_0$ and $\mu(h) > t_0$. Since $U^*(\mu, t_0)$ is a Hom-subgroup of G, it is closed under the group operation. Assume, for contradiction, that $\mu(g \cdot h) < t_1$. Let t_0 be the greatest lower bound of values strictly below t_1. Since $g, h \in U^*(\mu, t_0)$, it follows that $g \cdot h \in U^*(\mu, t_0)$ as well (by closure under the operation). This implies $\mu(g \cdot h) > t_0$, which contradicts our assumption that $\mu(g \cdot h) < t_1$ because there would be no value between t_0 and t_1 that $\mu(g \cdot h)$ could take. Therefore, $\mu(g \cdot h) \geq t_1$, satisfying the closure condition.

- **Compatibility with Twisting Map α:** Similarly, to show compatibility with α, assume for contradiction that $\mu(\alpha(g)) < \mu(g)$. Then $g \in U^*(\mu, t_0)$ would imply $\alpha(g) \in U^*(\mu, t_0)$, so $\mu(\alpha(g)) > t_0$, again contradicting the assumption $\mu(\alpha(g)) < \mu(g)$.

Thus, by showing that each strict upper level set $U^*(\mu, t)$ is a Hom-subgroup, we conclude that μ satisfies all conditions to be a fuzzy Hom-subgroup.

We aim to extend the relationship between fuzzy subsets and structural subgroups for Hom-normal subgroups, building on Theorem 2 in [20]. By establishing the connection between fuzzy Hom-normal subgroups of G and classical Hom-normal subgroups of G, we can deepen our understanding of fuzzy membership functions within the Hom-group framework.

Theorem 4. *Let μ be a fuzzy subset on a Hom-group $(G, \cdot, \alpha)$. The following statements are equivalent:*

(i) *μ is a fuzzy Hom-normal subgroup of G,*

(ii) *For every $t \in Im(\mu)$, the set $U(\mu, t) = \{g \in G \mid \mu(g) \geq t\}$ is a Hom-normal subgroup of G.*

Proof. **(i) ⇒ (ii):** Suppose μ is a fuzzy Hom-normal subgroup of G. For each $t \in \text{Im}(\mu)$, we need to show that $U(\mu, t)$ is a Hom-normal subgroup of G.

Since μ is a fuzzy Hom-normal subgroup, we know from the fuzzy Hom-subgroup conditions that $U(\mu, t)$ satisfies closure under the operation $\cdot$, closure under α, and closure under inverses. To establish that $U(\mu, t)$ is also normal, we verify:

- **Invariance under Conjugation:** For any $g, h \in G$, we have $\mu(g \cdot h \cdot g^{-1}) \geq \mu(h)$. Thus, if $h \in U(\mu, t)$ (i.e., $\mu(h) \geq t$), it follows that $g \cdot h \cdot g^{-1} \in U(\mu, t)$, ensuring $U(\mu, t)$ is closed under conjugation.

Therefore, $U(\mu, t)$ is a Hom-normal subgroup of G for each $t \in \text{Im}(\mu)$.

(ii) $\Rightarrow$ **(i):** Conversely, assume that each $U(\mu, t)$ is a Hom-normal subgroup of G for every $t \in \text{Im}(\mu)$. We want to show that μ satisfies the properties of a fuzzy Hom-normal subgroup.

Since each $U(\mu, t)$ is a Hom-subgroup, we know from the previous case that μ satisfies the closure conditions for a fuzzy Hom-subgroup. To verify the additional normality condition:

- **Invariance under Conjugation:** Take any $g, h \in G$ and let $t_1 = \min\{\mu(g), \mu(h)\}$. Then $g, h \in U(\mu, t_1)$, and because $U(\mu, t_1)$ is a Hom-normal subgroup, it follows that $g \cdot h \cdot g^{-1} \in U(\mu, t_1)$, which implies $\mu(g \cdot h \cdot g^{-1}) \geq \mu(h)$.

This shows that μ satisfies the conjugation invariance required for a fuzzy Hom-normal subgroup.

In analogy with the results established for strong fuzzy Hom-subgroups in Theorem 3, we extend our investigation to strong fuzzy Hom-normal subgroups. The proof of the following theorem follows the same reasoning and structure as that for strong fuzzy Hom-subgroups, with the only additional consideration being the normality condition. Therefore, we omit repetitive details and focus only on the modifications necessary for the Hom-normal subgroup case.

Theorem 5. *Let μ be a fuzzy subset on a Hom-group $(G, \cdot, \alpha)$. The following statements are equivalent:*

(i) μ is a strong fuzzy Hom-normal subgroup of G,

(ii) For every $t \in (0, 1]$, the set $U^*(\mu, t) = \{g \in G \mid \mu(g) > t\}$ is a Hom-normal subgroup of G, with equality in the conjugation condition.

5. Conclusions

This paper introduced the concept of *fuzzy Hom-groups*, combining fuzzy set theory with the structural flexibility of Hom-groups. By defining fuzzy Hom-subgroups and fuzzy Hom-normal subgroups, we extended classical fuzzy group theory to the Hom-algebraic setting, where the twisting map α modifies associativity and identity conditions. We established key properties, including structural behaviors and the connection between fuzzy Hom-subgroups and upper-level sets of classical Hom-subgroups.

Our results contribute to the growing research on fuzzy algebraic systems and Hom-algebras, building on Rosenfeld's fuzzy groups [2] and Chen et al.'s Hom-groups [14]. The examples provided illustrate how fuzzy Hom-subgroups behave under different algebraic conditions. In particular, the generalization using a parameter $k \neq 0$ demonstrates the adaptability of fuzzy Hom-group structures beyond previously known cases.

Future research directions include:

- Extending fuzzy Hom-groups to other algebraic structures, such as *fuzzy Hom-Lie algebras* and *fuzzy Hom-rings.*

- Investigating *fuzzy nilpotent* and *fuzzy solvable Hom-groups*, inspired by classical fuzzy group results [4, 6].

- Exploring advanced fuzzy extensions, including:
  - *Complex fuzzy Hom-groups*, with membership values in the complex unit disk.
  - *Intuitionistic fuzzy Hom-groups*, incorporating membership and non-membership degrees.
  - *Bipolar fuzzy Hom-groups*, handling positive and negative membership values.
  - *Pythagorean fuzzy Hom-groups*, allowing squared membership and non-membership values.
  - *Interval-valued fuzzy Hom-groups*, considering ranges of membership values.
  - *Picture fuzzy Hom-groups*, introducing a neutral membership degree alongside standard values.

- Investigating fuzzy Hom-groups in automorphism theory to understand symmetry and invariance under fuzzy Hom-structures.

- Developing computational approaches for fuzzy Hom-groups, with applications in decision-making, fuzzy control systems, and uncertainty modeling.

This work lays a foundation for further studies in the intersection of fuzzy set theory and Hom-algebraic structures. We anticipate that fuzzy Hom-groups will continue to inspire research in both theoretical and applied mathematics.

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