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# Total Safe Domination on Some Known Families of Graphs

Wendel Glenn C. Jumalon[1,*], Isagani S. Cabahug, Jr.[1]

[1] *Department of Mathematics, College of Arts and Sciences, Central Mindanao University, Musuan, Maramag, Bukidnon, 8714 Philippines*

**Abstract.** A *total dominating set* in a graph $G$ is a nonempty set $S \subseteq V(G)$ such that every vertex $v \in V(G)$, including those in $S$, is adjacent to at least one vertex in $S$. A *safe dominating set* in $G$ is a nonempty set $S \subseteq V(G)$ that is a dominating set, and for every component $A$ of the induced subgraph $G[S]$ and every component $B$ of the induced subgraph $G[V(G) \setminus S]$, with $A$ adjacent to $B$, it holds that $|V(A)| \geq |V(B)|$. This study introduces the concept of *total safe domination* in graphs which combines total domination and safe domination. Total safe domination ensures total accessibility and structural resilience. This paper provides characterization of total safe dominating sets for some well-known graph families, including: path, cycle, complete, complete bipartite, friendship, sunlet and helm graphs. It also presents the *total safe domination number* for each of these graph families.

**2020 Mathematics Subject Classifications**: 05C69

**Key Words and Phrases**: Total domination, safe domination, safe set, total safe domination

## 1. Introduction

The study of domination has grown to be one of the most rapidly expanding areas in graph theory. Domination in graphs has wide-ranging applications in network design, resource allocation, and security management. Traditional domination focuses on identifying subsets of vertices that can control or influence the entire graph. However, as systems grow more complex, there is a growing need for more refined approaches. This has led to the emergence of more variations of domination, such as total domination and safe domination, which offer enhanced perspectives and more effective solutions to real-world problems.

The formal study of dominating sets in graph theory began in the 1960s. In 1977, Cockayne published a comprehensive survey of the results on dominating sets known at

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*Email addresses:* `wendeljumalon@gmail.com` (W. G. Jumalon),
`isaganicabahugjr@cmu.edu.ph` (I. Cabahug, Jr.)

 

that time. This survey introduced the notation $\gamma(G)$ for the domination number of a graph $G$, which has since become widely adopted [1].

In 1980, Cockayne, Dawes and Hedetniemi published the first paper on total domination in graphs. They obtained results concerning the total domination number of a graph and the total domatic number of a graph (the largest order of a partition of a graph into total dominating sets) [2].

In recent decades, a lot of variations in total domination have been studied. Some of the researches related to this study are given as follows:

In 2007, Lam and Wei published their paper entitled "On the Total Domination Number of Graphs", where they developed theorems for the bounds of the total domination number of connected graphs of order at least 3, and for connected graphs with degree at least 2 [3].

In 2011, Go and Canoy determined the domination, total domination, and secure total domination numbers in the corona and join of graphs [4]. The study was continued by Eballe and Miranda in 2021, where they presented the domination defect for the join and corona of graphs [5].

In 2021, Jose Sigarreta published his study entitled "Total Domination on Some Graph Operators". In his paper, he introduced bounds for the exact value of the total domination number of some graph operators using some parameters in the original graph [6].

The study published by Klostermeyer in 2008 with the title "Secure Domination and Secure Total Domination in Graphs" has a related title but has a totally different concept. The paper states that a secure (total) dominating set of a graph $G = (V, E)$ is as a (total) dominating set $X \subseteq V$ with the property that for each $u \in V - X$, there exists $x \in X$ adjacent to $u$ such that $(X - \{x\}) \cup \{u\}$ is a (total) dominating set [7].

The study entitled "Defensive Alliances in Graphs" by Gaikwad and Maity which was published in 2022 also seems similar to this study but the concept is also different. As defined, a set $S$ of vertices of a graph is a defensive alliance if, for each element of $S$, the majority of its neighbours are in $S$ [8].

In 2024, Chatterjee, Jent, Osborn and Zhang published their paper "Proper Total Domination in Graphs". The paper states that a total dominating set $S$ in a graph $G$ is called a proper total dominating set if $\sigma_s(u) \neq \sigma_s(v)$ for every two adjacent vertices $u$ and $v$ of $G$ [9].

From available resources and online publications, the authors found none that is of exact same concept as this study. In particular, there is no published study about total domination which incorporates the concept of safe set or safe domination.

The idea of a safe set in graphs was introduced by Fujita, MacGillivray and Sakuma in 2016 [10]. Their work was motivated by applications related to facility location problems, where the goal is to find a "safe" subset of nodes for placing facilities in a network. The paper states that a safe set is a set in which every component of the set has order at least as large as the order of any adjacent component of the complement set.

The concept of safe domination in graphs was introduced by Griño, Maceren, and Cabahug in 2023 [11]. In their paper, they defined a safe dominating set as a set that is both dominating and safe.

This study introduces the concept of total safe dominating set in a graph. This guarantees total coverage and structural resilience. The concept may have unique potential applications in areas such as network coverage, facility location and defense infrastructure.

This study is limited to undirected graphs that are nontrivial, connected and simple.

## 2. Terminology and Notation

A *graph* $G = (V(G), E(G))$ is a finite nonempty set $V(G)$ of objects called *vertices* together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. A graph $H$ is a *subgraph* of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H$ is a subgraph of a graph $G$ where $H \ncong G$, then $H$ is a *proper subgraph* of $G$. If $S$ is a nonempty subset of $V(G)$, then the *induced subgraph* $G[S]$ of $S$ in $G$, is the graph whose vertex set is $S$ and whose edge set consists of all of the edges in $V(E)$ that have both endpoints in $S$ [12].

A *component* of a graph $G$ is defined as a maximal subgraph in which every pair of vertices is connected by a path. The connected graph and the trivial graph both have one component. A subgraph induced by a subset of $V(G)$ may have more than one component. A component $A$ of the induced subgraph $G[S]$ is said to be *adjacent* to a component $B$ of the induced subgraph $G[V(G) \setminus S]$, if there is at least one edge joining a vertex $u \in V(A)$ to a vertex $v \in V(B)$ [12].

For an integer $n \geq 1$, the *path* $P_n$ is a graph of order $n$ and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ [13].

For $n \geq 3$, the *cycle* $C_n$ is a graph of order $n$ and size $n$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1$ [13].

For $n \geq 2$, the *complete graph* $K_n$ is a graph of order $n$ and size $\frac{n(n-1)}{2}$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are represented as $v_iv_j$ for all pairs of vertices where $1 \leq i < j \leq n$ [12].

A graph $G$ is a *complete bipartite graph* if $V(G)$ can be partitioned into two sets $U$ and $W$ (called partite sets) such that $uw$ is an edge of $G$ if and only if $u \in U$ and $w \in W$. If $|U| = m$ and $|W| = n$, then the complete bipartite graph is denoted by $K_{m,n}$ [12].

The *friendship graph* $F_n$, also called *The Dutch windmill graph* ${D_3}^{(n)}$, is the graph with $2n+1$ vertices obtained by taking $n$ copies of the cycle graph $C_3$ with a common vertex, called the central vertex [14].

The *sunlet graph* $S_n$, also called as *n-sunlet graph*, is the graph with $2n$ vertices obtained by attaching a pendant edge at each vertex of a cycle $C_n$ [15].

The *helm graph* $H_n$ is the graph with $2n+1$ vertices obtained by adjoining a pendant edge at each node of the cycle, whose vertices are adjacent to a common vertex, called the hub [15].

For the main concepts included in this study, consider the following definitions:

Let $G$ be a simple connected nontrivial graph. A nonempty set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The minimum cardinality of a dominating set in $G$ is called *domination number* of $G$, denoted by $\gamma(G)$ [16].

A nonempty set $S \subseteq V(G)$ of vertices is a *safe set* if for every component $A$ of $G[S]$ and every component $B$ of $G[V(G) \setminus S]$ adjacent to $A$, it holds that $|V(A)| \geq |V(B)|$. The minimum cardinality of a safe set in $G$ is called *safe number* of $G$, denoted by $s(G)$ [17].

A nonempty subset $S$ of $V(G)$ is a *safe dominating set* if and only if it is both a dominating set and a safe set. The minimum cardinality of a safe dominating set in $G$ is called *safe domination number* of $G$, denoted by $\gamma_s(G)$ [11].

A *total dominating set* $S$ in a graph $G$ is a nonempty subset of $V(G)$ such that every vertex in $V(G)$, including those in $S$, is adjacent to at least one vertex in $S$. The minimum cardinality of a total dominating set in $G$ is called *total domination number* of $G$, denoted by $\gamma_t(G)$ [2].

## 3. Results

**Definition 1.** *A nonempty subset $S$ of $V(G)$ is a* ***total safe dominating set*** *in $G$ if it is both a total dominating set and a safe dominating set in $G$. The minimum cardinality of a total safe dominating set in $G$ is called* ***total safe domination number*** *of $G$, denoted by $\gamma_{ts}(G)$.*

**Example 1.** Consider Figure 1 that shows a graph $G$ together with different dominating sets as in $G_1$, $G_2$, and $G_3$. In $G_1$, $S_1 = \{v_3, v_4\}$ shows a total safe dominating set in $G$ since it is a total dominating set and the order of the component of $G[S_1]$ is at least as large as the order of any adjacent component of $G[V(G) \setminus S_1]$. In $G_2$, $S_2 = \{v_3, v_5\}$ is not a total safe dominating set. It is a total dominating set but not a safe dominating set since $G[V(G) \setminus S_2]$ has a component with order 3 which is greater than the order of the component of $G[S_2]$. Finally, in $G_3$, $S_3 = \{v_1, v_2, v_4, v_5\}$ is not a total safe dominating set. It is a safe dominating set but not a total dominating set since $v_2$ is isolated.

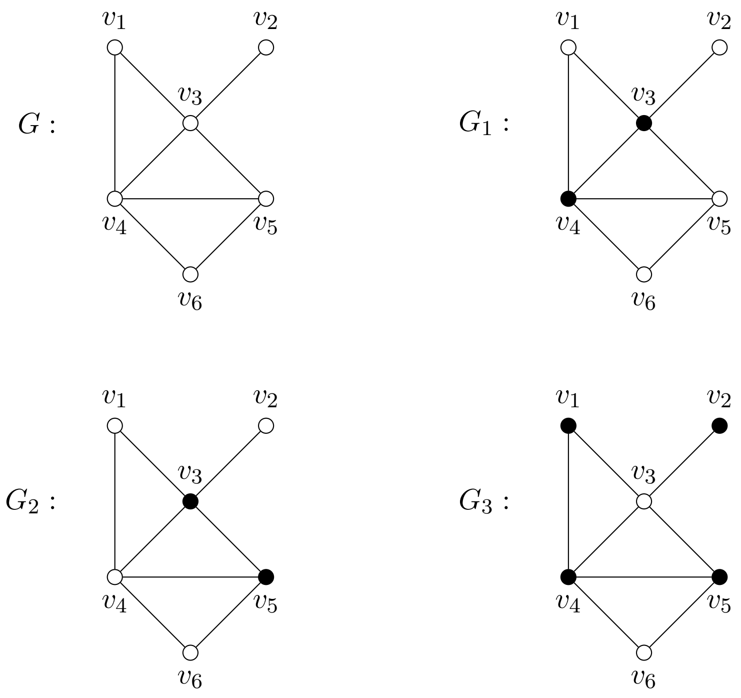


Figure 1: A graph $G$ with examples of dominating sets, where $G_1$ shows a total safe dominating set.

### 3.1. Some Realization Results on Total Safe Domination

By assumption, $G$ is a nontrivial connected graph. Hence, the following remark follows.

**Remark 1.** *Let $G$ be a nontrivial connected graph. Then $V(G)$ is a total safe dominating set.*

By definition, the following remarks also follow.

**Remark 2.** *Every total safe dominating set of a graph $G$ is a safe dominating set. Thus, $\gamma_s(G) \leq \gamma_{ts}(G)$.*

**Remark 3.** *Every total safe dominating set of a graph $G$ is a total dominating set. Hence, $\gamma_t(G) \leq \gamma_{ts}(G)$.*

**Theorem 1.** *Let $G$ be a nontrivial connected graph and $S \subsetneq V(G)$ be a total dominating set in $G$. Then $\gamma_{ts}(G) = 2$ if and only if $\gamma_t(G) = 2$ and for every component $B$ of $G[V(G) \setminus S]$, $|V(B)| \leq 2$.*

*Proof.* Assume that $G$ is a nontrivial connected graph and $S \subsetneq V(G)$ is a total dominating set in $G$. Suppose that $\gamma_{ts}(G) = 2$. Clearly, $\gamma_t(G) = 2$. Also, for every component $B$ of $G[V(G) \setminus S]$, $|V(G[S])| \geq |V(B)|$. Thus, $|V(B)| \leq 2$.

For the converse, suppose that $\gamma_t(G) = 2$ and for every component $B$ of $G[V(G) \setminus S]$, $|V(B)| \leq 2$. Then, $\gamma_{ts}(G) \geq 2$. Since $\gamma_t(G) = 2$, there exists a total dominating set S with $|S| = 2$. Since $|V(B)| \leq 2$ for every component $B$ of $G[V(G) \setminus S]$, and since $|S| = 2$, it follows that $S$ is a safe dominating set in $G$. So, $S$ is a total safe dominating set in $G$. This implies that $\gamma_{ts}(G) \leq |S| = 2$. Hence, $\gamma_{ts}(G) = 2$. □

In the following results, we show by construction the existence of a graph where the parameters $\gamma_{ts}(G)$ and $\gamma_t(G)$ are equal, strictly unequal and whose difference can be made arbitrarily large.

**Theorem 2.** *Let $a$ and $b$ be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph $G$ such that $\gamma_t(G) = a$ and $\gamma_{ts}(G) = b$.*

*Proof.* Let $a, b \in \mathbb{Z}^+$ and consider the following cases:
Case 1: $a = b$

Consider graph $G$ in Figure 2. Take $a \geq m$. Clearly, $P = \{x_i : i = 1, 2, ..., a\}$ is both a minimum total dominating set and a minimum total safe dominating set. Thus, $2 \leq \gamma_t(G) = |P| = a = b = \gamma_{ts}(G)$.

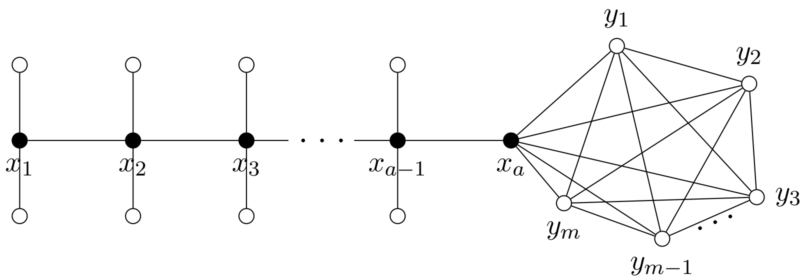


Figure 2: A graph $G$ with $\gamma_t(G) = a = b = \gamma_{ts}(G)$, where $a \geq m$.

Case 2: $a < b$

Consider graph $G$ in Figure 3. Take $a < m$. Let $b = |P \cup Q|$, where $P = \{x_i : i = 1, 2, ..., a\}$ and $Q = \{y_j : j = 1, 2, ..., y_{\lceil \frac{m-a}{2} \rceil}\}$. Then, $\gamma_t(G) = |P| = a$ and $\gamma_{ts}(G) = b = |P \cup Q| = a + \lceil \frac{m-a}{2} \rceil = \lceil \frac{a+m}{2} \rceil$. Hence, $\gamma_t(G) = a < b = \gamma_{ts}(G)$.

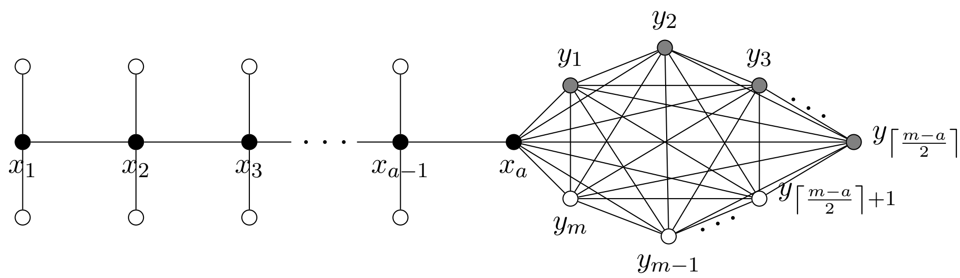


Figure 3: A graph $G$ with $\gamma_t(G) = a < b = \gamma_{ts}(G)$, where $b = \lceil \frac{a+m}{2} \rceil$.

This completes the proof. □

From Theorem 2, the following Corollary immediately follows.

**Corollary 1.** *For each positive integer $n$, there exists a connected graph $G$ such that $\gamma_{ts}(G) - \gamma_t(G) = n$, that is, the difference between $\gamma_{ts}(G)$ and $\gamma_t(G)$ can be made arbitrarily large.*

## 3.2. Results on Total Safe Domination on Some Known Graph Families

The following results give the characterization of total safe dominating sets on some well-known graph families.

**Theorem 3.** *Let $G$ be a nontrivial connected graph of order $n$ such that $\Delta(G) = 2$. Then a nonempty set $S \subsetneq V(G)$ is a total safe dominating set in $G$ if and only if every component of $G[S]$ is a $P_k$, $2 \leq k \leq n-1$, and every component of $G[V(G) \setminus S]$ is a trivial graph or a $P_2$ with no end vertex in $G$.*

*Proof.* Assume that $S$ is a total safe dominating set in $G$. Suppose that there exists a component of $G[S]$ that is a $P_1$, or there exists a component of $G[V(G) \setminus S]$ that is not a trivial graph and a $P_2$ with an end vertex in $G$. The first part implies that $G[S]$ contains an isolated vertex and the latter implies that $S$ is not a dominating set. Both implications contradict the assumption of $S$. Thus, every component of $G[S]$ is a $P_k$, $2 \leq k \leq n-1$, and every component of $G[V(G) \setminus S]$ is a trivial graph or a $P_2$ with no end vertex in $G$.

For the converse, suppose that $S \subsetneq V(G)$ such that every component of $G[S]$ is a $P_k$, $2 \leq k \leq n-1$, and every component of $G[V(G) \setminus S]$ is a trivial graph or a $P_2$ with no end vertex in $G$. Clearly, $G[S]$ has no isolated vertex and $S$ is a dominating set. If $A$ and $B$ are components of $G[S]$ and $G[V(G) \setminus S]$, respectively, then $|V(A)| \geq |V(B)|$. Therefore, $S$ is a total safe dominating set in $G$. □

**Corollary 2.** *For $n \geq 3$,*

$$\gamma_{ts}(P_n) = \gamma_{ts}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 (\text{mod } 4) \\ \frac{n+1}{2}, & \text{if } n \equiv 1 \text{ or } 3 (\text{mod } 4) \\ \frac{n+2}{2}, & \text{if } n \equiv 2 (\text{mod } 4). \end{cases}$$

*Proof.* Let $P_n = [v_1, v_2, ..., v_n]$ and $S \subsetneq V(P_n)$. Consider the following cases:
Case 1: $n \equiv 0(\text{mod } 4)$

Choose $S = \{v_2, v_3, v_6, v_7, ..., v_{n-2}, v_{n-1}\}$. Then, $|S| = \frac{n}{2}$. By Theorem 3, $S$ is a total safe dominating set in $P_n$. Thus, $\gamma_{ts}(P_n) \leq |S| = \frac{n}{2}$. By vertex selection and labeling, we cannot find a total safe dominating set in $P_n$ with cardinality less than the cardinality of $S$. Therefore, $\gamma_{ts}(P_n) = |S| = \frac{n}{2}$.

Case 2: $n \equiv 1(\text{mod } 4)$

Choose $S = \{v_2, v_3, v_4, v_7, v_8, ..., v_{n-2}, v_{n-1}\}$. Then, $|S| = \frac{n+1}{2}$. By Theorem 3, $S$ is a total safe dominating set in $P_n$. By the same argument as in Case 1, we have $\gamma_{ts}(P_n) = \frac{n+1}{2}$.

Case 3: $n \equiv 2(\text{mod } 4)$

Choose $S = \{v_2, v_3, v_5, v_6, ..., v_{n-1}, v_n\}$. By similar argument as in the previous cases, we have $\gamma_{ts}(P_n) = \frac{n+2}{2}$.

Case 4: $n \equiv 3(\text{mod } 4)$

Choose $S = \{v_2, v_3, v_6, v_7, ..., v_{n-1}, v_n\}$. So, $|S| = \frac{n+1}{2}$. Again, by Theorem 3, $S$ is a total safe dominating set in $P_n$. Hence, $\gamma_{ts}(P_n) \leq |S| = \frac{n+1}{2}$. By vertex selection and labeling, we cannot find a total safe dominating set in $P_n$ with cardinality less than the cardinality of $S$. Thus, $\gamma_{ts}(P_n) = |S| = \frac{n+1}{2}$.

Similarly, for $C_n = [v_1, v_2, ..., v_n, v_1]$ and $S \subsetneq V(C_n)$, choose the same vertex labeling for $S$ in each case above. By the same arguments as above, we have

$$\gamma_{ts}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0(\text{mod } 4) \\ \frac{n+1}{2}, & \text{if } n \equiv 1 \text{ or } 3(\text{mod } 4) \\ \frac{n+2}{2}, & \text{if } n \equiv 2(\text{mod } 4). \end{cases}$$

□

**Theorem 4.** *Let $K_n$ be a complete graph of order $n \geq 3$. Then a nonempty set $S \subsetneq V(K_n)$ is a total safe dominating set in $K_n$ if and only if $|S| \geq \lceil \frac{n}{2} \rceil$.*

*Proof.* Assume that $S$ is a total safe dominating set in $K_n$. Now, suppose that $|S| < \lceil \frac{n}{2} \rceil$. Then we have $|V(K_n) \setminus S| \geq \lceil \frac{n}{2} \rceil$. Since $K_n[S]$ and $K_n[V(K_n) \setminus S]$ both consist of a single component, this implies that $S$ is not a safe set in $K_n$. This contradicts our assumption that $S$ is a total safe dominating set. Thus, $|S| \geq \lceil \frac{n}{2} \rceil$.

Conversely, suppose that $|S| \geq \lceil \frac{n}{2} \rceil$. Since any vertex in $V(K_n)$ is adjacent to a vertex in $S$, $S$ is a total dominating set. Additionally, since $|S| \geq \lceil \frac{n}{2} \rceil$, we have enough vertices in $S$ to satisfy the safe set condition. Hence, $S$ is a total safe dominating set in $K_n$. □

**Corollary 3.** *For $K_n$, $n \geq 3$,*

$$\gamma_{ts}(K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

*Proof.* By Theorem 4, the lower bound for a total safe dominating set in $K_n$ is $\lceil \frac{n}{2} \rceil$. Hence, $\gamma_{ts}(K_n) = \lceil \frac{n}{2} \rceil$. □

**Theorem 5.** *Let $K_{m,n}$ be a complete bipartite graph with partite sets $U$ and $W$ such that $|U| = m$, $|W| = n$, and $m + n \geq 3$. Then a nonempty set $S \subsetneq V(K_{m,n})$ is a total safe dominating set in $K_{m,n}$ if and only if the following hold:*

(i) *$S = S_1 \cup S_2$ such that $\emptyset \neq S_1 \subseteq U$ and $\emptyset \neq S_2 \subseteq W$;*

(ii) *$|S| \geq \lceil \frac{m+n}{2} \rceil$.*

*Proof.* Assume that $S$ is a total safe dominating set in $K_{m,n}$. Suppose that $S \subseteq U$ or $S \subseteq W$. In either case, $K_{m,n}[S]$ is an empty graph and so $S$ is not a total safe dominating set in $K_{m,n}$. This is a contradiction to the assumption of $S$. So, $S$ cannot be contained entirely in either $U$ or $W$. Thus, $S = S_1 \cup S_2$ such that $\emptyset \neq S_1 \subseteq U$ and $\emptyset \neq S_2 \subseteq W$. Now, suppose that $|S| < \lceil \frac{m+n}{2} \rceil$. Then, $|V(K_{m,n}) \setminus S| \geq \lceil \frac{m+n}{2} \rceil$. Since $K_{m,n}[S]$ and $K_{m,n}[V(K_{m,n}) \setminus S]$ both consist of a single component, this means that $S$ is not a safe set in $K_{m,n}$, a contradiction to the assumption of $S$. Therefore, $|S| \geq \lceil \frac{m+n}{2} \rceil$.

For the converse, suppose that $S = S_1 \cup S_2$ such that $\emptyset \neq S_1 \subseteq U$ and $\emptyset \neq S_2 \subseteq W$, and $|S| \geq \lceil \frac{m+n}{2} \rceil$. The first part guarantees total domination of $S$ in $K_{m,n}$, and together with the latter, it follows that $S$ is a safe dominating set in $K_{m,n}$. Thus, $S$ is a total safe dominating set in $K_{m,n}$.

□

**Corollary 4.** *For a complete bipartite graph $K_{m,n}$ such that $m + n \geq 3$,*
$$\gamma_{ts}(K_{m,n}) = \left\lceil \frac{m+n}{2} \right\rceil.$$

*Proof.* By Theorem 5 (ii), the lower bound for a total safe dominating set in $K_{m,n}$ is $\lceil \frac{m+n}{2} \rceil$. Thus, $\gamma_{ts}(K_{m,n}) = \lceil \frac{m+n}{2} \rceil$. □

**Remark 4.** *Let $K_{m,n}$ be a complete bipartite graph with partite sets $U$ and $W$ such that $|U| = m$, $|W| = n$. If $S \subseteq U$ or $S \subseteq W$, then $S$ is not a total safe dominating set in $K_{m,n}$.*

**Theorem 6.** *Let $F_n$ be a friendship graph of order $2n+1$ with central vertex $x$. Then a nonempty set $S \subsetneq V(F_n)$ is a total safe dominating set in $F_n$ if and only if one of the following holds:*

*(i)* $|S| \geq 2$, *if* $x \in S$

*(ii)* $|S| = 2n$, *otherwise*

*Proof.* Let $S$ be a total safe dominating set in $F_n$. Suppose that $x \in S$ and $|S| < 2$. Then $|S| = 1$, that is, $S = \{x\}$. So, $S$ is not a total dominating set in $F_n$, a contradiction to the assumption of $S$. Therefore, $|S| \geq 2$ if $x \in S$.

Now, suppose that $x \notin S$ and $|S| \neq 2n$. Then $|S| < 2n$. Without loss of generality, suppose $|S| = 2n-1$. Let $v \notin S$ with $v \neq x$. Since $S$ is a dominating set in $F_n$, then there exists a vertex $u \in S$ such that $v \in N(u)$. This implies that $F_n[S]$ has an isolated vertex, a contradiction to the assumption of $S$. Thus, $|S| = 2n$ if $x \notin S$.

For the converse, suppose that $x \in S$ and $|S| \geq 2$. Clearly, $S$ is a total safe dominating set in $F_n$ since $x$ is adjacent to all other vertices of $F_n$. Now, suppose $x \notin S$ and $|S| = 2n$. Again, clearly, $S$ is a total safe dominating set in $F_n$. □

**Corollary 5.** *For any friendship graph $F_n$ of order $2n+1$,*

$$\gamma_{ts}(F_n) = 2.$$

*Proof.* By Theorem 6 (i), clearly, $\gamma_{ts}(F_n) = |S| = 2$. □

**Theorem 7.** *Let $S_n$ be a sunlet graph of order $2n$ with vertex set $V(C_n) \cup P$, where $P$ is the set of pendant vertices in $S_n$. Then a nonempty set $S \subsetneq V(S_n)$ is a total safe dominating set in $S_n$ if and only if $V(C_n) \subseteq S$.*

*Proof.* Assume that $S$ is a total safe dominating set in $S_n$. Suppose that $V(C_n) \nsubseteq S$. Then there exists a vertex, say $v \in V(C_n)$, that is not in $S$. Let $u \in V(S_n)$ be a pendant vertex such that $u \in N(v)$. Then $u$ must be in $S$ since $v \notin S$ and $S$ is a dominating set in $S_n$. Thus, $u$ is an isolated vertex in $S_n[S]$, and so $S$ is not a total dominating set in $S_n$, a contradiction to the assumption of $S$. Hence, $V(C_n) \subseteq S$.

For the converse, let $V(C_n) \subseteq S$. Clearly, $V(C_n)$ is a total safe dominating set in $S_n$. Therefore, $S$ is a total safe dominating set in $S_n$. □

**Corollary 6.** *For any sunlet graph $S_n$ of order $2n$,*

$$\gamma_{ts}(S_n) = n.$$

*Proof.* By Theorem 7, $V(C_n)$ is the minimum total safe dominating set in $S_n$. Therefore, $\gamma_{ts}(S_n) = |V(C_n)| = n$. □

**Theorem 8.** *Let $H_n$ be a helm graph of order $2n+1$ with central vertex $x$ and vertex set $V(C_n) \cup \{x\} \cup P$, where $P$ is the set of pendant vertices in $H_n$. Then a nonempty set $S \subsetneq V(H_n)$ is a total safe dominating set in $H_n$ if and only if $V(C_n) \subseteq S$.*

*Proof.* Let $S$ be a total safe dominating set in $H_n$. Suppose that $V(C_n) \nsubseteq S$. Then there exists a vertex $v \in V(C_n)$ with $v \notin S$. Let $u \in V(H_n)$ be a pendant vertex such that $u \in N(v)$. Then $u \in S$ since $S$ is a dominating set in $H_n$. So, $u$ is an isolated vertex in $H_n[S]$, and so $S$ is not a total dominating set in $S_n$, a clear contradiction to the assumption of $S$. Hence, $V(C_n) \subseteq S$.

Conversely, let $V(C_n) \subseteq S$. Clearly, $V(C_n)$ is a total safe dominating set in $H_n$. Thus, $S$ is a total safe dominating set in $H_n$. □

**Corollary 7.** *For any helm graph $H_n$ of order $2n+1$,*

$$\gamma_{ts}(H_n) = n.$$

*Proof.* By Theorem 8, $V(C_n)$ is the minimum total safe dominating set in $H_n$. Hence, $\gamma_{ts}(H_n) = |V(C_n)| = n$. □

## Conclusions

The concept of total safe domination has been introduced and explored in this study. Some realizations on how the total safe domination number relate with the total domination number and the safe domination number are presented. Characterizations of total safe dominating sets in several well-known families of graphs are provided and used to determine the exact values of total safe domination number of those graphs.

For those interested in further study on this topic, it would be valuable to explore other variations of total safe dominating sets, such as restrained, forcing, and locating sets. The study of domination in graphs resulting from unary operations is relatively underexplored and less studied. Exploring total safe domination in such graphs is worthwhile. With existing results on safe domination in ladder graphs, a special type of grid graph (denoted as $(P_m \times Pn)$), exploration of total safe domination within these graphs is also interesting. Finally, the authors encourage other researchers to investigate inequalities related to Vizing's conjecture and Nordhaus-Gaddum-type inequalities.

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