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Internally-Locating Dominating Sets in Graphs

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Abstract. For a connected graph G, a subset $I \subseteq V(G)$ is a locating-dominating set if it is a dominating set and for every two distinct vertices $x, y \in V(G) \setminus I$, $N(x) \cap I \neq N(y) \cap I$. This paper introduces the concept of an internally-locating dominating set. Specifically, a nonempty set $I \subseteq V(G)$ with $|I| \geq 2$ is an internally-locating set in a nontrivial connected graph G if and only if, for every $u, v \in I$, $N(u) \cap I \neq N(v) \cap I$. Thus, I is an internally-locating dominating set if it is both an internally-locating set and a dominating set. In addition, this paper identifies some properties of this concept, provides characterizations of certain special classes of graphs, including total graphs and shadow graphs with $\Delta(G) = 2$, with their corresponding internally-locating domination number, and cases where $\gamma_{li}(G) = 2$. Moreover, it provides a sufficient condition for $\gamma(G) = \gamma_{li}(G)$, specifically when G is a corona product.

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Key Words and Phrases: Internally-locating set, internally-locating dominating set, internally-locating domination number, total graph, shadow graph, corona of graph

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1. Introduction

Over time, graph theory has evolved into a powerful tool for analyzing networks and solving optimization problems. Two fundamental concepts in this field are dominating sets and locating sets. Domination theory, introduced by Berge and Ore in the 1960s [1], has its applications in various areas, including facility location problems (FLPs) [2]. FLPs focus on optimizing factors such as transportation costs and market share [2]. Locating sets, introduced by Slater [3], are also important, with applications in systems like sonar and long-range navigation (LORAN) stations [4].

In 1998, Slater combined the ideas of domination and location to introduce locating-dominating sets [5], which help identify vertices while also dominating the graph. These sets have applications in fields such as fire location detection and multiprocessor error diagnosis [6]. Slater's work, along with subsequent studies by Canoy and Omega, expanded

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the theory by exploring locating sets in complex graph operations, such as the join and corona of graphs [7]. Canoy and Malacas further characterized locating-dominating sets in graph compositions and corona graphs [8].

Although much progress has been made in understanding dominating and locating sets, there are still unexplored concepts. This led to a question about the distinctness within the set. This study introduces internally-locating dominating sets and explores their properties in special graphs, particularly focusing on unary operations such as total and shadow graphs with $\Delta(G) = 2$, along with their internally-locating domination numbers, and binary operations such as the corona of graphs. Some aspects of the proofs in this study were inspired by [9].

2. Terminology and Notation

This study considers finite, simple, nontrivial, connected, and undirected graphs. Some definitions of the concepts covered in this study is included within. For fundamental graph-theoretic concepts and additional terms, the readers may refer to [10], [11], [4], [12], [13], [14], [7], [15], [16].

Let $G = (V, E)$ be a graph. If the pair u and v is in E, then $e = uv$ is an *edge* of G and the vertices u and v are *adjacent* in G. Two adjacent vertices in G are referred to as *neighbors* of each other. The set of neighbors of a vertex v is called the *open neighborhood* of v denoted by $N_G(v) = N(v)$ [4].

The *degree of a vertex* v in a graph G is the number of vertices that are adjacent to v, denoted by $deg_G(v)$ or simply $deg(v)$. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end-vertex* or *leaf*. The largest degree among the vertices of G is called the *maximum degree* of G, denoted by $\Delta(G)$ [4].

A set $S \subseteq V(G)$ is a *dominating set* of G, if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of dominating set [10].

A subset S of $V(G)$ is a *locating set* in a connected graph G if for any two distinct vertices u and v in $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$ [7].

A set $D \subseteq V$ is a *locating-dominating set* if it is dominating and every two vertices $x, y \in V(G) \setminus S, N(x) \cap D \neq N(y) \cap D$. The *locating-domination number*, denoted by $\gamma_L(G)$ is the minimum cardinality of an locating-dominating set of G [12].

A *gear graph*, denoted by G_n, is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the cycle [16].

Example 1. *Consider the wheel graph* W_4 *with* $V(W_4) = \{u\} \cup \{v_1, v_2, v_3, v_4\}$ *and* $E(W_4) = \{uv_i | 1 \leq i \leq 4\} \cup \{v_1v_2, v_2v_3, v_3, v_4, v_1v_4\}$*. The gear graph* G_4 *in Figure 1, is obtained by adding vertices* a_0*,* a_1*,* a_2*, and* a_3 *between the vertices* v_1 *and* v_2*,* v_2 *and* v_3*,* v_3 *and* v_4*, and* v_4 *and* v_1*, respectively.*

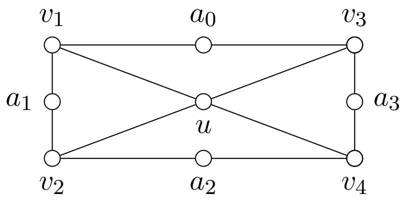

Figure 1: The gear graph G_4

An *banana tree graph* is a graph obtained by connecting one leaf to each of n copies of a star graph $K_{1,k}$ with a single root vertex that is distinct from all the stars, denoted by $B_{n,k+1}$ [17].

Example 2. *Consider the star graph $K_{1,3}$ with $V(K_{1,3}) = \{x_1, y_1, y_2, y_3\}$. Then in Figure 2 is a banana tree graph $B_{3,4}$ obtained by connecting one leaf of three copies of star $K_{1,3}$ with a single root vertex u.*

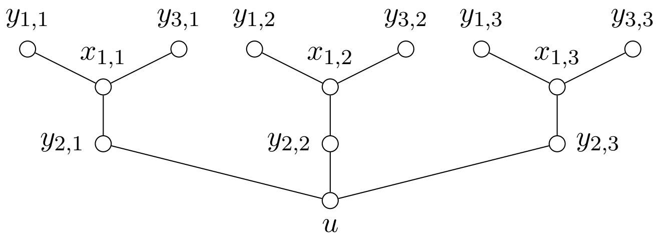

Figure 2: The banana tree graph $B_{3,4}$

A *lollipop graph* is the graph obtained by joining a complete graph K_m to path graph P_n with a bridge, denoted by $L_{m,n}$ [17].

Example 3. *Consider the complete graph K_3 with $V(K_3) = \{v_1, v_2, v_3\}$ and a path graph P_4 with $V(P_4) = \{x_1, x_2, x_3, x_4\}$. Then in Figure 3 is a lollipop graph $L_{3,4}$ obtained from joining the vertex v_3 and x_1.*

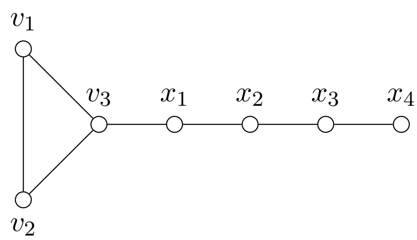

Figure 3: The lollipop graph $L_{3,4}$

The *total graph* $T(G)$ of a graph G, is a graph G such that the vertices set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G. The total graph has a vertex set $V(T(G)) = V(G) \cup E(G)$ [18].

Example 4. *Consider the path graph P_4, $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and $E(P_4) = \{e_1, e_2, e_3\}$ such that $e_1 = v_1v_2$, $e_2 = v_2v_3$, and $e_3 = v_3v_4$. Then in Figure 4 is the total graph of path graph P_4, denoted by $T(P_4)$ has $V(T(P_4)) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$ with each of the vertices in the vertex set of $T(P_4)$ is joined by an edge if it is adjacent or incident in P_4.*

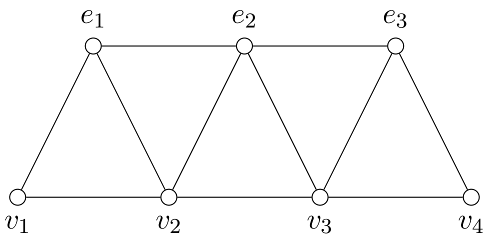

Figure 4: The total graph of path graph P_4

The *shadow graph* $S(G)$ of a graph G is obtained from G by adding, for each vertex v of G, a new vertex v', called the shadow vertex of v, and joining v' to the neighbors of v in G. The set of all shadow vertices is denoted by $V'(G)$ [4].

Example 5. *Consider the path graph P_4 in Figure 4 with $V(P_4) = \{v_1, v_2, v_3, v_4\}$. Then the shadow graph of path graph P_4 is shown in Figure 5 with $V'(P_4) = \{v'_1, v'_2, v'_3, v'_4\}$.*

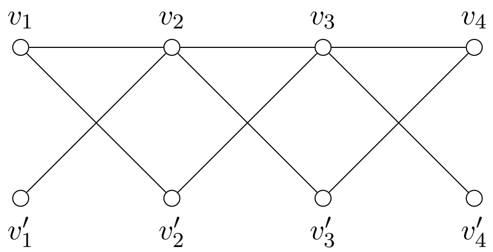

Figure 5: The shadow graph of path graph P_4

3. Known Result

Corollary 1. *[19] Let G be a connected graph of order m, and let H be any graph of order n. Then, $\gamma(G \circ H) = m$, where $\gamma(G \circ H)$ represents the domination number of the corona product $G \circ H$.*

4. Results

This paper uses the following terms to denote specific concepts: $deg_I(v)$ represents the degree of vertex v within the set I, $N_I(v)$ signifies the neighborhood of vertex v within the set I, $\gamma - set$ refers to the minimum dominating set, ILS signifies an internally-locating set, $ILDS$ denotes an internally-locating dominating set, and the $\gamma_{li} - set$ represents the minimum internally-locating dominating set.

Definition 1. *A non-empty set $S \subseteq V(G)$ with $|S| \geq 2$ is an internally-locating set in a connected graph G if and only if for every $u, v \in S$, the $N(u) \cap S \neq N(v) \cap S$.*

Example 6. *Consider the graph G in Figure 6, and a set $I = \{v_2, v_3, v_5\} \subset V(G)$. Note that $N(v_2) \cap I = \{v_3, v_5\}$, $N(v_3) \cap I = \{v_2, v_5\}$, and $N(v_5) \cap I = \{v_2, v_3\}$. By Definition 1, I is an internally-locating set in G.*

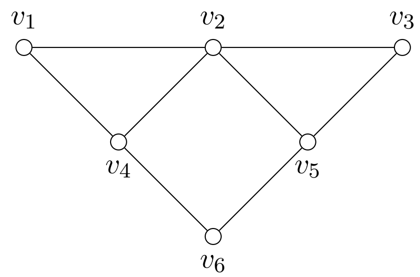

Figure 6: A graph G

Definition 2. *A set $I \subseteq V(G)$ such that $|I| \geq 2$ is an internally-locating dominating set if and only if I is a dominating set and an internally-locating set in G. Moreover, the minimum cardinality of internally-locating dominating set, denoted by $\gamma_{li}(G)$ is called an internally-locating domination number of G.*

Example 7. *Consider the graph G in Figure 6, and a set $I = \{v_2, v_5\} \subset V(G)$. Note that $N(v_2) \cap I = \{v_5\} \neq \{v_2\} = N(v_5) \cap I$. By Definition 1, I is an internally-locating set in G. Note that I is a dominating set in G and also a $\gamma - set$. By Definition 2, I is an internally-locating dominating set and also a $\gamma_{li} - set$ in G. Therefore, $\gamma_{li}(G) = 2$.*

Theorem 1. *Let G be a graph and $I \subseteq V(G)$ such that $|I| \geq 2$ and I is a dominating set. Then I is not an internally-locating dominating set in G if one of the following holds:*

(i) H is a component of $G[I]$ such that $H = P_3$ or C_4;

(ii) I induced more than one isolated vertex or an empty graph of order $n \geq 2$;

(iii) the component of $G[I] = K_{m,n}$, where $m \geq 1$ and $n \geq 3$; or

(iv) the component of $G[I] = D_4$.

Proof. Let G be a graph and $I \subseteq V(G)$ such that $|I| \geq 2$. Assume that $(i)-(iv)$ hold.

If $H = P_3$, consider $V(H) = V(P_3) = \{v_1, v_2, v_3\}$ and $E(H) = E(P_3) = \{v_1v_2, v_2v_3\}$. Note that $H \subseteq G[I]$, thus $V(H) \subseteq I$. Observe that $N(v_1) \cap I = \{v_2\} = N(v_3) \cap I$. Thus, I is neither an ILS nor $ILDS$.

Similarly, if $H = C_4$, let $V(H) = V(C_4) = \{c_1, c_2, c_3, c_4\}$ and $E(H) = E(C_4) = \{c_1c_2, c_2c_3, c_3c_4, c_4c_1\}$. Note that $H \subseteq G[I]$, thus $V(H) \subseteq I$. Observe that $N(c_1) \cap I = \{c_2, c_4\} = N(c_3) \cap I$. Thus, I is neither an ILS nor $ILDS$.

Now, if I induces an isolated vertex v, then $N(v) \cap I = \emptyset$. By (ii), there exists $u \in I$, $u \neq v$, such that $N(u) \cap I = \emptyset$. Thus, $N(v) \cap I = \emptyset = N(u) \cap I$. Hence, I is neither an ILS nor $ILDS$. By the definition of an empty graph, this generally holds if I induces an empty graph of order $n \geq 2$.

By (iii), let $H = K_{m,n}$, $m \geq 1$ and $n \geq 3$. By the definition of $K_{m,n}$, $V(H) = V(K_{m,n}) = \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$ and $E(K_{m,n}) = \{x_iy_j | 1 \leq i \leq m, 1 \leq j \leq n\}$. Note that $H \subseteq G[I]$, thus $V(H) \subseteq I$. WLOG, consider x_1 and $x_2 \in V(K_{m,n})$. This implies $N(x_1) \cap I = \{y_1, y_2, \ldots, y_n\}$, and $N(x_2) \cap I = \{y_1, y_2, \ldots, y_n\}$. Thus, $N(x_1) \cap I = N(x_2) \cap I$. Hence, I is neither an ILS nor $ILDS$. This is also true for $H = K_{1,n}$, $n \geq 3$.

Finally, by (iv), if $H = D_4$, let $V(H) = V(D_4) = \{d_1, d_2, d_3, d_4\}$ and $E(H) = E(D_4) = \{d_1d_2, d_2d_3, d_3d_4, d_4d_1, d_1d_3\}$. Note that $H \subseteq G[I]$, thus $V(H) \subseteq I$. Observe that $N(d_2) \cap I = \{d_1, d_3\} = N(d_4) \cap I$. Thus, I is neither an ILS nor $ILDS$.

□

Theorem 2. *Let G be a graph. Then $\gamma_{li}(G) = 2$ if one of the following hold:*

(i) $\gamma(G) = 1$

(ii) $\gamma(G) = 2$, *where $\{u, v\}$ is the $\gamma - set$ and u, v are adjacent.*

Proof. Let G be a graph.

Case 1: If $\gamma(G) = 1$, this means that there exists a single vertex $v \in V(G)$ such that $N(v) = V(G) \setminus \{v\}$. In this case, $\{v\}$ is the γ-set. This implies that for any $u \in V(G)$, $u \neq v$, $\{u, v\}$ is a dominating set. Since $N(v) = V(G) \setminus \{v\}$, and u is dominated by v, this implies

$$N(u) \cap \{u, v\} = \{v\} \quad \text{and} \quad N(v) \cap \{u, v\} = \{u\}.$$

Thus, $N(u) \cap \{u, v\} \neq N(v) \cap \{u, v\}$, which confirms that $\{u, v\}$ is an *internally-locating set*. By definition of $\gamma_{li} - set$, this follows that $\gamma_{li}(G) = 2$ in this case.

Case 2: If $\gamma(G) = 2$, where $\{u, v\}$ is the $\gamma - set$ and u, v are adjacent. Clearly, $\{u, v\}$ is $\gamma_{li} - set$. Hence, $\gamma_{li}(G) = 2$ in this case.

□

Corollary 2. *The $\gamma_{li}(G) = 2$ if*

(i) $G = P_n = C_n$, $2 \leq n \leq 3$;

(ii) $G = K_n$, $n \geq 2$

(iii) $G = K_{1,n}$, $n \geq 2$;

(iv) $G = F_n$, $n \geq 2$; *and*

(v) $G = W_n$, $n \geq 4$.

Proof. This follows from the definition of dominating set and Theorem 2 (i).

□

Theorem 3. *Let G be a graph with $n \geq 4$. Then $\gamma(G) = \gamma_{li}(G)$ if $G = P_n \circ H$, for any graph variation of H.*

Proof. Let G be a graph with $n \geq 4$ and $G = P_n \circ H$, for any graph variation of H. By Corollary 1, $\gamma(G) = n$. By definition of path graph P_n and since $G = P_n \circ H$, $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ is a $\gamma - set$.

Now, observe that

$$\begin{aligned} N(v_1) \cap V(P_n) &= \{v_2\}, \\ N(v_i) \cap V(P_n) &= \{v_{i-1}, v_{i+1}\},\ 2 \leq i \leq n-1 \\ N(v_n) \cap V(P_n) &= \{v_{n-1}\}. \end{aligned}$$

Thus, $V(P_n)$ is an ILS and $\gamma_{li} - set$. Hence $\gamma_{li}(G) = n$. Therefore, $\gamma(G) = \gamma_{li}(G)$.

□

Theorem 4. *Let G be a graph with $\Delta(G) = 2$ and $I \subseteq V(G)$ such that $|I| \geq 2$. Then I is the minimum internally-locating dominating set in G if and only if the following holds:*

(i) *if $deg_I(v) = 1$, then $deg_{V(G)\setminus I}(v) = 1$;*

(ii) *if $deg_I(v) = 0$, then v is unique in I;*

(iii) *if $deg_I(v) = 0$ and $deg(v) = 1$ or 2, then $deg_{V(G)\setminus I}(v) = deg(v)$; and*

(iv) *for all $u \in V(G) \setminus I$, $u \in N(v)$ for some $v \in I$.*

Proof. Let G be a graph with $\Delta(G) = 2$ and $I \subseteq V(G)$ such that $|I| \geq 2$.

Assume I is the $\gamma_{li} - set$ in G. Now, if $\deg_I(v) = 1$, v has one neighbor in I. Since I is dominating and $\Delta(G) = 2$, v must have one neighbor in $V(G) \setminus I$, i.e., $\deg_{V(G)\setminus I}(v) = 1$. Thus, (i) holds.

For (ii), since I is an *ILDS*, we have $N(v) \cap I = \emptyset$ if $deg_I(v) = 0$. Suppose that v is not unique in I, this implies that there exists $u \in I$ where $u \neq v$ such that $\deg_I(u) = 0$. Thus, $N(u) \cap I = \emptyset = N(v) \cap I$. A contradiction, since I is an *ILS*. Thus, v with $deg_I(v) = 0$ is unique in I.

If $\deg_I(v) = 0$ and $\deg(v) = 1$ or 2, $\deg_{V(G)\setminus I}(v) = \deg(v)$, since all neighbors of v must lie in $V(G) \setminus I$. Hence, (iii) holds. Lastly, by definition of dominating set, then for all $u \in V(G) \setminus I$, u is adjacent to at least one $v \in I$, as I is dominating. Thus, (iv) holds.

Conversely, assume I satisfies (i)-(iv). By (iv), clearly, I is a dominating set. By (i), it follows that for every $v \in I$, there exist a unique $u \in I$ such that $N(v) \cap I = u$. By (ii) and (iii), there exist no u, $v \in I$ such that $N(u) \cap I = \emptyset = N(v) \cap I$. Thus, I is an *ILS*. Now, by removing any vertex in I, then it contradicts (iv). Hence, I is the $\gamma_{li} - set$.

□

Corollary 3. *Let G be a path graph P_n or a cycle graph C_n with $n \geq 4$. Then,*

$$\gamma_{li}(G) = \gamma_{li}(P_n) = \gamma_{li}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0, 2 \text{ (mod 4)} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ (mod 4)} \\ \frac{n-1}{2} & \text{if } n \equiv 3 \text{ (mod 4)} \end{cases}$$

Proof. Let G be a path graph P_n or a cycle graph C_n with $n \geq 4$. For convenience, let $V(P_n) = V(C_n) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$, $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

If $G = P_n$, consider the following cases:

Case 1: $n \equiv 0$ (mod 4)
Let $I \subseteq V(P_n)$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$. Observe that $|I| = \frac{n}{2}$. By Theorem 4, I is a $\gamma_{li} - set$. Therefore, $\gamma_{li}(P_n) = |I| = \frac{n}{2}$.

Case 2: $n \equiv 1$ (mod 4)
Let $I \subseteq V(P_n)$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-3}, v_{n-2}, v_n\}$. Observe that $|I| = \frac{n+1}{2}$. By Theorem 4, I is a $\gamma_{li} - set$. Therefore, $\gamma_{li}(P_n) = |I| = \frac{n+1}{2}$.

Case 3: $n \equiv 2$ (mod 4)
Let $I \subseteq V(P_n)$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$. Observe that $|I| = \frac{n}{2}$. By Theorem 4, I is a $\gamma_{li} - set$. Therefore, $\gamma_{li}(P_n) = |I| = \frac{n}{2}$.

Case 4: $n \equiv 3$ (mod 4)
Let $I \subseteq V(P_n)$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-5}, v_{n-4}, v_{n-1}\}$. Observe that $|I| = \frac{n-1}{2}$. By Theorem 4, I is a $\gamma_{li} - set$. Therefore, $\gamma_{li}(P_n) = |I| = \frac{n-1}{2}$.

A similar argument follows for $G = C_n$, as the number of vertices is the same as in the path graph P_n, resulting the same values for $\gamma_{li}(G)$ in all cases.

□

Theorem 5. *Let G be a banana tree graph $B_{n,k+1}$ with $n \geq 2$ and $k \geq 3$, and $I \subseteq V(G)$ with $|I| \geq 2$. Then, I is the minimum internally-locating dominating set of G if and only if the following holds:*

(i) for every $u_i \in V(K_{1_i,k_i})$, then $u_i \in I$;

(ii) there exists exactly one $u_i \in V(K_{1_i,k_i})$ such that $deg_I(u_i) = 0$;

(iii) *if* $deg_I(u_i) \neq 0$, *then* $deg_I(u_j) = 1$, $i \neq j$; *and*

(iv) *there exist* $v \in I$, *such that* $r \in N(v)$.

Where $u_i \in V(K_{1_i,k_i})$ *is the central vertex of the* i^{th} *copy of a star graph* K_{1_i,k_i}, $1 \leq i \leq n$ *and* r *a single root vertex of* G.

Proof. Let $G = B_{n,k+1}$ be a banana tree graph, where $n \geq 2$ and $k \geq 3$, and let $I \subseteq V(G)$ with $|I| \geq 2$. Suppose that $u_i \in V(K_{1_i,k_i})$ be a central vertex of the i^{th} copy of a star graph K_{1_i,k_i}, $1 \leq i \leq n$ and r a single root vertex of G.

Assume I is the $\gamma_{li} - set$ of G. Then, I is a dominating set, so every vertex in G must be adjacent to at least one vertex in I. In K_{1_i,k_i}, the central vertex u_i is connected to all k_i leaves. To dominate the leaves of K_{1_i,k_i}, u_i must belong to I. Thus, (i) holds.

By definition of *ILDS*, for any $u, v \in I$, their neighborhoods within I must satisfy $N(u) \cap I \neq N(v) \cap I$. To minimize $|I|$, exactly one u_i in $V(K_{1_i,k})$ must have $\deg_I(u_i) = 0$. Hence, (ii) holds.

Now, if $\deg_I(u_i) = 0$, since (ii) holds, $\deg_I(u_j) \geq 1$, $i \neq j$. Then, by Theorem 1 (i) and (iii), this follows $\deg_I(u_j) \leq 1$. Thus, $\deg_I(u_j) = 1$, $i \neq j$.

Lastly, by I as an *ILDS*, and since (iii) holds, to dominate r, there must exist $v \in I$ such that $r \in N(v)$.

Conversely, assume that $I \subseteq V(G)$ satisfies conditions $(i) - (iv)$. By (i) and (iv), every vertex in G is dominated by I. By (ii), there exists $u_i \in I$ such that $N_I(u_i) = \emptyset$, and by (iii), for all $u_j \in I$, $i \neq j$, $N_I(u_j)$ is a singleton set. Since $deg_I(u_j) = 1$, $i \neq j$, it follows that u_j is adjacent to one of the leaves of the star graph K_{1_j,k_j}. Note that for all $v_i \in V(K_{1_i,k_i})$, $N(v_i) \neq v_j$, for all $v_j \in V(K_{1_j,k_j})$, $i \neq j$. Consequently, for all u, $v \in I$, $N(u) \cap I \neq N(v) \cap I$. Thus, I is an *ILDS*. Now, by removing any vertex $v \in I$, it violates either domination or *ILS* property. Hence, I is a $\gamma_{li} - set$.

□

Corollary 4. *Let* G *be a banana tree graph* $B_{n,k+1}$ *with* $n \geq 2$ *and* $k \geq 3$. *Then,* $\gamma_{li}(G) = 2n - 1$.

Proof. By Theorem 5 (i)-(iii), $|I| = 2n - 1$. Hence, $\gamma_{li}(G) = 2n - 1$.

□

Theorem 6. *Let* G *be a gear graph* G_n *with* $n \geq 3$, *and* $I \subseteq V(G)$ *such that* $|I| \geq 2$. *Then* I *is a minimum internally-locating dominating set in* G *if and only if the following holds:*

(i) $u \notin I$, *where* $u \in V(G)$ *be a central vertex in* G;

(ii) I *is a minimum internally-locating dominating set of* C_{2n}.

Proof. Let $G = G_n$ be a gear graph with $n \geq 3$, vertex set

$$V(G_n) = \{u\} \cup \{v_i \mid 1 \leq i \leq n\} \cup \{v_{i,i+1} \mid 1 \leq i \leq n\}$$

and edge set

$$E(G_n) = E(C_{2n}) \cup \{uv_i \mid 1 \leq i \leq n\},$$

where u is the central vertex of G_n and C_{2n} is a cycle graph of order $2n$.

Assume $I \subseteq V(G)$ is a γ_{li}-set of G. Then, the central vertex u is adjacent to all v_i, $1 \leq i \leq n$ but is not adjacent to any $v_{i,i+1}$.

Suppose $u \in I$. Then, u dominates all v_i, but it does not dominate $v_{i,i+1}$, which must still be dominated by other vertices in I. Removing u from I does not violate the domination or $ILDS$ conditions. Thus, $u \notin I$, as its inclusion would violate the minimality of I. Now, by definition of G, $V(G) \setminus \{u\}$ is a C_{2n}. Since (i) holds, and by Theorem 4, I is a γ_{li}-set of C_{2n}.

Conversely, assume that $u \notin I$ and I is a $\gamma_{li} - set$ of C_{2n}. Clearly, by Theorem 4, I is the $\gamma_{li} - set$ of G.

□

Corollary 5. *Let G be a gear graph G_n with $n \geq 3$. Then, $\gamma_{li}(G) = n$.*

Proof. Note that $|V(G_n)| - 1 = |V(C_{2n})| = 2n$, and also by Theorem 6, $|I| = 2n$, where I is the $\gamma_{li} - set$ of G_n. This implies, $2n \equiv 0, 2 \pmod 4$. By Corollary 3, if $n \equiv 0, 2 \pmod 4$, then $\gamma_{li}(C_{2n}) = \frac{n}{2}$. Let $l \in \mathbb{Z}$ such that $l = 2n$. With this, $\gamma_{li}(G_n) = \frac{l}{2} = \frac{2n}{2} = n$. Hence, $\gamma_{li}(G) = n$.

□

Theorem 7. *Let G be a lollipop graph $L_{m,n}$ with $m \geq 3$ and $n \geq 4$, and $I \subseteq V(G)$ such that $|I| \geq 2$. Then I is a minimum internally-locating dominating set in G if and only if the following holds:*

(i) $u \in I$ and $deg_I(u) = 0$ or $deg_I(u) = 1$;

(ii) for every $v \in I$, if $deg_I(v) = 0$ or 1, then $deg_{V(G)\setminus I}(v) = 1$;

(iii) if $deg_I(v) = 0$, then v is unique in I; and

(iv) for all $x \in V(G) \setminus I$, $x \in N(y)$ for some $y \in I$.

Where, $u \in V(K_m)$ and incident to a bridge and $v \in V(P_n)$.

Proof. Let $G = L_{m,n}$ be a lollipop graph with $m \geq 3$ and $n \geq 4$, and $I \subseteq V(G)$ such that $|I| \geq 2$.

Assume I is a $\gamma_{li} - set$ in G. Let $u \in V(K_m)$ be incident to the bridge. Since I is a $\gamma_{li} - set$, $u \in I$ to dominate a vertex in P_n. If $\deg_I(u) > 1$, then multiple vertices in I are unnecessarily dominated, contradicting the minimality of I. Hence, $\deg_I(u) = 0$ or $\deg_I(u) = 1$.

Note that $V(L_{m,n}) \setminus V(K_m)$ induces a path graph P_n. Thus, by Theorem 4, (ii), (iii) and (iv) holds.

Conversely, assume conditions $(i)-(iv)$ hold. Then, by Theorem 4, I is a $\gamma_{li}-set$ in G.

□

Corollary 6. *Let G be a lollipop graph $L_{m,n}$ with $m \geq 3$ and $n \geq 4$. Then,*

$$\gamma_{li}(G) = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0, 2 (\text{mod } 4) \\ \frac{n+1}{2} & \text{if } n \equiv 1 (\text{mod } 4) \\ \frac{n+3}{2} & \text{if } n \equiv 3 (\text{mod } 4) \end{cases}$$

Proof. Let G be a lollipop graph $L_{m,n}$ with $m \geq 3$ and $n \geq 4$. For convenience, $V(L_{m,n}) = V(K_m) \cup V(P_n)$, where $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ such that $u_1v_1 \in E(L_{m,n})$. Now, consider the following cases:

Case 1: $n \equiv 0 \pmod 4$
Let $I \subseteq V(L_{m,n})$ with $I = \{u_1, v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$. Observe that $|I| = \frac{n+2}{2}$. By Theorem 7, I is a $\gamma_{li}-set$. Therefore, $\gamma_{li}(L_{m,n}) = |I| = \frac{n+2}{2}$.

Case 2: $n \equiv 1 \pmod 4$
Let $I \subseteq V(L_{m,n})$ with $I = \{u_1, v_3, v_4, v_7, v_8, \ldots, v_{n-2}, v_{n-1}\}$. Observe that $|I| = \frac{n+1}{2}$. By Theorem 7, I is a $\gamma_{li}-set$. Therefore, $\gamma_{li}(L_{m,n}) = |I| = \frac{n+1}{2}$.

Case 3: $n \equiv 2 \pmod 4$
Let $I \subseteq V(L_{m,n})$ with $I = \{u_1, v_1, v_4, v_5, \ldots, v_{n-2}, v_{n-1}\}$. Observe that $|I| = \frac{n+2}{2}$. By Theorem 7, I is a $\gamma_{li}-set$. Therefore, $\gamma_{li}(L_{m,n}) = |I| = \frac{n+2}{2}$.

Case 4: $n \equiv 3 \pmod 4$
Let $I \subseteq V(L_{m,n})$ with $I = \{u_1, v_1, v_4, v_5, \ldots, v_{n-3}, v_{n-2}, v_n\}$. Observe that $|I| = \frac{n+3}{3}$. By Theorem 7, I is a $\gamma_{li}-set$. Therefore, $\gamma_{li}(L_{m,n}) = |I| = \frac{n+3}{2}$.

□

Theorem 8. *Let G be a graph with $\Delta(G) = 2$, and $I \subseteq V(T(G))$ such that $|I| \geq 2$. Suppose that $I \subseteq V(G)$. Then I is an internally-locating dominating set in $T(G)$ if and only if I is an internally-locating dominating set in $V(G)$ such that the component of $G[V(G) \setminus I] \neq P_2$.*

Proof. Let G be a graph with $\Delta(G) = 2$, and $I \subseteq V(T(G))$ such that $|I| \geq 2$. Suppose $I \subseteq V(G)$.

Assume I is an *ILDS* in $T(G)$. Then, for all $u, v \in I$, $N_{T(G)}(u) \cap I \neq N_{T(G)}(u) \cap I$. Thus, $N_{V(G)}(u) \cap I \neq N_{V(G)}(v) \cap I$, making I an *ILS* in $V(G)$.

Suppose that $G[V(G) \setminus I] = P_2$, the edge corresponding to this path would not be dominated by I in $T(G)$. This violates the domination condition in $T(G)$, so $G[V(G) \setminus I] \neq P_2$. Consequently, since $I \subseteq V(G)$, this implies I is a dominating set in $V(G)$. Hence, I is an *ILDS* in $V(G)$.

Conversely, assume that I is an *ILDS* in $V(G)$ and $G[V(G) \setminus I] \neq P_2$. Now, for all $u, v \in I$, $N_{V(G)}(u) \cap I \neq N_{V(G)}(v) \cap I$. This implies $N_{T(G)}(u) \cap I \neq N_{T(G)}(v) \cap I$, making I an *ILS* in $T(G)$. Note that by the definition of $T(G)$, $V(T(G)) = V(G) \cup E(G)$. By assumption that I is an *ILDS* in $V(G)$, and since $G[V(G) \setminus I] \neq P_2$, this implies that for all $e_i \in E(G)$, $e_i \in N(u)$, for some $u \in I$. Hence, I is an *ILS* in $T(G)$.

□

Remark 1. *Let G be a path graph P_n or cycle graph C_n with $n = 2$ or 3. Then,*

$$\gamma_{li}(T(G)) = \gamma_{li}(T(P_n)) = \gamma_{li}(T(C_n)) = 2.$$

The result is straighforward and can be directly verified from the total graph of a path graph or cycle graph of order 2 or 3.

Theorem 9. *Let G be a path graph P_n with $n \geq 4$ Then,*

$$\gamma_{li}(T(G)) = \begin{cases} \frac{4n}{7} & \text{if } n \equiv 0 \text{ (mod 7)} \\ \left\lceil \frac{4n}{7} \right\rceil & \text{if } n \equiv 1, 5 \text{ (mod 7)} \\ \left\lfloor \frac{4n}{7} \right\rfloor & \text{if } n \equiv 2, 3, 4, 6 \text{ (mod 7)} \end{cases}$$

Proof. Let G be a path graph of order $n \geq 4$ and $T(G)$ be a total graph of G. For convenience, let $V(G) = V(P_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, $E(G) = E(P_n) = \{e_1, e_2, e_3 \ldots, e_{n-2}, e_{n-1}\}$ where e_1 is the edge incident with v_1 and v_2, e_2 is the edge incident to v_2 and v_3, and so on and so forth. By definition of total graph, $V(T(G)) = V(G) \cup E(G)$.

Suppose I is an internally-locating dominating set of minimum cardinality, i.e., I is a $\gamma_{li} - set$. Now, consider the following cases:

Case 1: $n \equiv 0 \, (mod\, 7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-5}, v_{n-4}\} \cup \{e_5, e_6, e_{12}, e_{13}, \ldots, e_{n-9}, e_{n-8}, e_{n-2}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \frac{4n}{7}$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \frac{4n}{7}$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\frac{4n}{7}$ vertices in $T(G)$. Hence, $|I| \geq \frac{4n}{7}$. Therefore, $|I| = \frac{4n}{7}$.

Case 2: $n \equiv 1 \, (mod\, 7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_n\} \cup \{e_5, e_6, e_{11}, e_{12}, \ldots, e_{n-10}, e_{n-9}, e_{n-3}, e_{n-2}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 3: $n \equiv 2\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_n\} \cup \{e_5, e_6, e_{12}, e_{13}, \ldots, e_{n-11}, e_{n-10}, e_{n-4}, e_{n-3}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

Case 4: $n \equiv 3\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-1}\} \cup \{e_5, e_6, e_{12}, e_{13}, \ldots, e_{n-12}, e_{n-11}, e_{n-5}, e_{n-4}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

Case 5: $n \equiv 4\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-2}, v_{n-1}\} \cup \{e_5, e_6, \ldots, e_{n-6}, e_{n-5}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

Case 6: $n \equiv 5\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_8, v_9, \ldots, v_{n-4}, v_{n-3}\} \cup \{e_5, e_6, \ldots, e_{n-7}, e_{n-6}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 7: $n \equiv 6\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-4}, v_{n-3}\} \cup \{e_5, e_6, \ldots, e_{n-8}, e_{n-7}, e_{n-2}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

□

Example 8. *Consider Figure 7. Clearly $I = \{v_2, v_3, e_5\}$ is a dominating set. Observe that, $N(v_2) \cap I = \{v_3\}$, $N(v_3) \cap I = \{v_2\}$, and $N(e_5) \cap I = \emptyset$. Hence, I is an internally-locating dominating set of minimum cardinality, so $\gamma_{li}(T(P_6)) = 3$. By Theorem 9, for $n = 6 \equiv 6\,(mod\,7)$, $\gamma_{li}(T(P_6)) = \lfloor \frac{4(n)}{7} \rfloor = \lfloor \frac{4(6)}{7} \rfloor = \lfloor \frac{24}{7} \rfloor = 3$.*

$T(P_6)$:

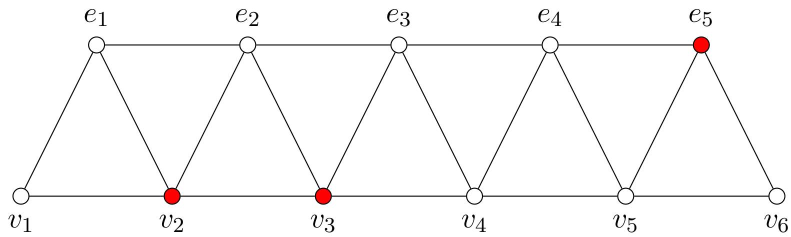

Figure 7: The minimum internally-locating dominating set of $T(P_6)$

Theorem 10. *Let G be a cycle graph C_n with $n \geq 4$ Then,*

$$\gamma_{li}(T(G)) = \begin{cases} \frac{4n}{7} & \text{if } n \equiv 0 \ (mod\ 7) \\ \left\lceil \frac{4n}{7} \right\rceil & \text{if } n \equiv 1,3,4,5 \ (mod\ 7) \\ \left\lfloor \frac{4n}{7} \right\rfloor & \text{if } n \equiv 2,6 \ (mod\ 7) \end{cases}$$

Proof. Let G be a cycle graph of order $n \geq 4$ and $T(G)$ be a total graph of G. For convenience, let $V(G) = V(C_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, $E(G) = E(C_n) = \{e_1, e_2, e_3 \ldots, e_{n-1}, e_n\}$ where e_1 is the edge incident with v_1 and v_2, e_2 is the edge incident to v_2 and v_3, and so on, up to e_n which is incident with v_1 and v_n. By definition of total graph, $V(T(G)) = V(G) \cup E(G)$.

Suppose I is an internally-locating dominating set of minimum cardinality, i.e., I is a $\gamma_{li} - set$. Now, consider the following cases:

Case 1: $n \equiv 0\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-5}, v_{n-4}\} \cup \{e_5, e_6, e_{12}, e_{13}, \ldots, e_{n-9}, e_{n-8}, e_{n-2}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \frac{4n}{7}$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \frac{4n}{7}$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\frac{4n}{7}$ vertices in $T(G)$. Hence, $|I| \geq \frac{4n}{7}$. Therefore, $|I| = \frac{4n}{7}$.

Case 2: $n \equiv 1\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_n\} \cup \{e_5, e_6, e_{11}, e_{12}, \ldots, e_{n-10}, e_{n-9}, e_{n-3}, e_{n-2}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 3: $n \equiv 2\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_n\} \cup \{e_5, e_6, e_{12}, e_{13}, \ldots, e_{n-11}, e_{n-10}, e_{n-4}, e_{n-3}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

Case 4: $n \equiv 3\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-1}, v_n\} \cup \{e_5, e_6, \ldots, e_{n-12}, e_{n-11}, e_{n-5}, e_{n-4}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 5: $n \equiv 4\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_1, v_2, v_8, v_9, \ldots, v_{n-3}, v_{n-2}\} \cup \{e_4, e_5, \ldots, e_{n-7}, e_{n-6}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 6: $n \equiv 5\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-3}, v_{n-2}\} \cup \{e_5, e_6, \ldots, e_{n-7}, e_{n-6}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lceil \frac{4n}{7} \rceil$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lceil \frac{4n}{7} \rceil$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lceil \frac{4n}{7} \rceil$ vertices in $T(G)$. Hence, $|I| \geq \lceil \frac{4n}{7} \rceil$. Therefore, $|I| = \lceil \frac{4n}{7} \rceil$.

Case 7: $n \equiv 6\,(mod\,7)$. Let $S \subseteq V(T(G))$ with

$$S = \{v_2, v_3, v_9, v_{10}, \ldots, v_{n-4}, v_{n-3}\} \cup \{e_5, e_6, \ldots, e_{n-8}, e_{n-7}, e_{n-1}\}.$$

Observe that S is a dominating set and for all u, $v \in S$, $N(u) \cap S \neq N(v) \cap S$. Hence, S is an $ILDS$. To end this, note that $|S| = \lfloor \frac{4n}{7} \rfloor$. Since I is a $\gamma_{li} - set$, $|S| \geq |I|$. So, $|I| \leq |S| = \lfloor \frac{4n}{7} \rfloor$. On the other hand, since I is a $\gamma_{li} - set$ of $T(G)$, then I must have atleast $\lfloor \frac{4n}{7} \rfloor$ vertices in $T(G)$. Hence, $|I| \geq \lfloor \frac{4n}{7} \rfloor$. Therefore, $|I| = \lfloor \frac{4n}{7} \rfloor$.

□

Example 9. *Consider Figure 8. Clearly* $I = \{v_1, v_2, e_3\}$ *is a dominating set. Observe that,* $N(v_1) \cap I = \{v_2\}$, $N(v_2) \cap I = \{v_1\}$, *and* $N(e_3) \cap I = \emptyset$. *Hence,* I *is an internally-locating dominating set of minimum cardinality, so* $\gamma_{li}(T(C_4)) = 3$. *By Theorem 10, for* $n = 4 \equiv 4\,(mod\,7)$,

$$\gamma_{li}(T(C_4)) = \lceil \frac{4(n)}{7} \rceil = \lceil \frac{4(4)}{7} \rceil = \lceil \frac{16}{7} \rceil = 3.$$

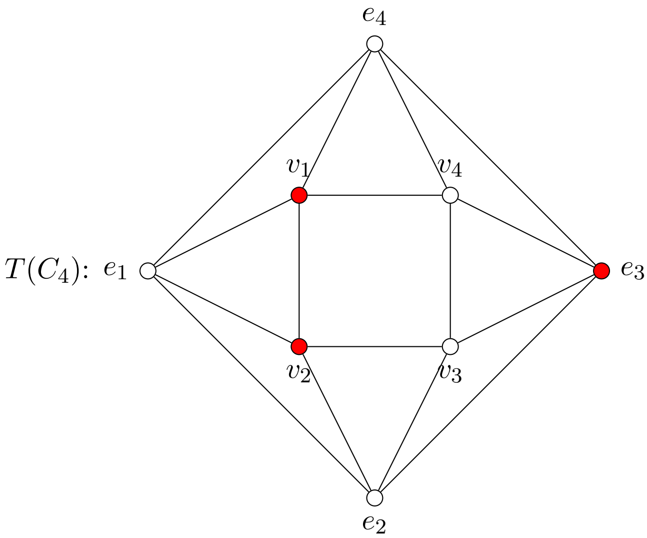

Figure 8: The minimum internally-locating dominating set in $T(C_4)$

Theorem 11. *Let* G *be a graph with* $\Delta(G) = 2$, *and* $I \subseteq V(T(G))$ *such that* $|I| \geq 2$. *Suppose that* $I \subseteq V(G)$. *Then* I *is an internally-locating dominating set in* $S(G)$ *if and only if the following holds:*

(i) I *is an internally-locating dominating set in* $V(G)$*; and*

(ii) for all $u \in I$, $deg_I(u) \neq 0$.

Proof. Let G be a graph with $\Delta(G) = 2$, and $I \subseteq V(S(G))$ such that $|I| \geq 2$. Suppose that $I \subseteq V(G)$.

Assume $I \subseteq V(G)$ is an $ILDS$ in $S(G)$. Then, by definition of $S(G)$, I must dominate all vertices in $V(S(G)) = V(G) \cup V'(G)$. In particular, I dominates $V(G)$, as $I \subseteq V(G)$. Also, the property of I as $ILDS$ for $S(G)$ holds for $V(G)$ since $I \subseteq V(G)$. Thus, (i) holds.

Now, if $\deg_I(u) = 0$ for some $u \in I$, then u would not contribute to the domination of $S(G)$. Specifically, u would fail to dominate $u' \in V'(G)$, the shadow vertex corresponding to u, violating the domination property of I in $S(G)$. Hence, $\deg_I(u) \neq 0$ for all $u \in I$.

Conversely, assume $I \subseteq V(G)$ satisfies (i) and (ii). By Condition (i), I dominates $V(G)$. This follows, each shadow vertex $v' \in V'(G)$ is adjacent to the neighbors of its corresponding vertex $v \in V(G)$. Since I dominates $V(G)$, it also dominates $V'(G)$. Hence, I dominates all of $V(S(G)) = V(G) \cup V'(G)$.

Now, by (i), for any distinct $u, v \in I$, their neighborhoods in $V(G)$ satisfy $N(u) \cap I \neq N(v) \cap I$. This uniqueness extends to $S(G)$ because the neighborhoods of vertices in $V'(G)$ are determined by their neighbors in $V(G)$. Thus, $N(u) \cap I \neq N(v) \cap I$ for any $u, v \in I$ in $S(G)$.

Finally, by (ii), $\deg_I(u) \neq 0$ for all $u \in I$, ensuring that every vertex in I actively contributes to the domination of $S(G)$. Therefore, I satisfies the conditions to be an $ILDS$ in $S(G)$.

□

Lemma 1. *Let G be a path graph P_n or cycle graph C_n of order $n \geq 2$ such that $n \equiv 0$ (mod 4). Then $I \subseteq V(S(G))$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$ is a minimum internally-locating dominating set in G.*

Proof. Let $I \subseteq V(S(G))$ with $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$. Note that I is an internally-locating dominating set in $V(G)$ by Theorem 4. So, Theorem 11 (i) is satisfied. Additionally, for all $u \in I$, $deg_I(u) \neq 0$. This implies, Theorem 11 (i) is satisfied.

Now, removing a vertex v in I, contradicts Theorem 11 (ii). Hence, I is a $\gamma_{li} - set$.

□

Lemma 2. *Let G be a path graph P_n or cycle graph C_n of order $n \geq 2$ such that $n \equiv 1$ (mod 4). Then $I \subseteq V(S(G))$ with $I = \{v_1, v_2, v_5, v_6, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$ is a minimum internally-locating dominating set in G.*

Proof. Let $I \subseteq V(S(G))$ with $I = \{v_1, v_2, v_5, v_6, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$. Note that I is an internally-locating dominating set in $V(G)$ by Theorem 4. So, Theorem 11 (i) is satisfied. Additionally, for all $u \in I$, $deg_I(u) \neq 0$. This implies, Theorem 11 (i) is satisfied.

Now, removing a vertex v in I, contradicts Theorem 11 (ii). Hence, I is a $\gamma_{li} - set$.

□

Lemma 3. *Let G be a path graph P_n or cycle graph C_n of order $n \geq 2$ such that $n \equiv 2$ (mod 4). Then $I \subseteq V(S(G))$ with $I = \{v_1, v_2, v_5, v_6, \ldots, v_{n-5}, v_{n-4}, v_{n-1}, v_n\}$ is a minimum internally-locating dominating set in G.*

Proof. Let $I \subseteq V(S(G))$ with $I = \{v_1, v_2, v_5, v_6, \ldots, v_{n-5}, v_{n-4}, v_{n-1}, v_n\}$. Note that I is an internally-locating dominating set in $V(G)$ by Theorem 4. So, Theorem 11 (i) is satisfied. Additionally, for all $u \in I$, $deg_I(u) \neq 0$. This implies, Theorem 11 (i) is satisfied.

Now, removing a vertex v in I, contradicts Theorem 11 (ii). Hence, I is a $\gamma_{li} - set$.

□

Lemma 4. *Let G be a path graph P_n or cycle graph C_n of order $n \geq 2$ such that $n \equiv 3$ (mod 4). Then $I \subseteq V(S(G))$ with $I = \{v_2, v_3, v_6, v_7, \dots, v_{n-2}, v_{n-1}\}$ is a minimum internally-locating dominating set in G.*

Proof. Let $I \subseteq V(S(G))$ with $I = \{v_2, v_3, v_6, v_7, \dots, v_{n-2}, v_{n-1}\}$. Note that I is an internally-locating dominating set in $V(G)$ by Theorem 4. So, Theorem 11 (i) is satisfied. Additionally, for all $u \in I$, $deg_I(u) \neq 0$. This implies, Theorem 11 (i) is satisfied.

Now, removing a vertex v in I, contradicts Theorem 11 (ii). Hence, I is a $\gamma_{li} - set$. □

Corollary 7. *Let G be a path graph P_n or cycle graph C_n with $n \geq 2$ Then,*

$$\gamma_{li}(S(G)) = \gamma_{li}(S(P_n)) = \gamma_{li}(S(C_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \ (mod\ 4) \\ \frac{n+3}{2} & \text{if } n \equiv 1 \ (mod\ 4) \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (mod\ 4) \\ \frac{n+1}{2} & \text{if } n \equiv 3 \ (mod\ 4) \end{cases}$$

Proof. Let G be a path graph P_n or a cycle graph C_n with $n \geq 2$. For convenience, let $V(G) = V(P_n) = V(C_n) = \{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$ and $V'(G) = V'(P_n) = V'(C_n) = \{v'_1, v'_2, v'_3, \dots, v'_{n-1}, v'_n\}$ as the set of all shadow vertices of $V(G)$ where joining v'_i to the neighbors of $v_i \in V(G)$. By definition of shadow graph, $V(S(G)) = V(G) \cup V'(G)$.

Now, observe the following cases:

Case 1: $n \equiv 0$ (mod 4)

Let $I = \{v_2, v_3, v_6, v_7, \dots, v_{n-2}, v_{n-1}\}$. By Lemma 1, I is a $\gamma_{li} - set$ in G. Therefore,

$$\begin{aligned} \gamma_{li}(G) &= |I| \\ &= |\{v_2, v_3, v_6, v_7, \dots, v_{n-2}, v_{n-1}\}| \\ &= \frac{n}{2}. \end{aligned}$$

Case 2: $n \equiv 1$ (mod 4)

Let $I = \{v_2, v_3, v_6, v_7, \dots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$. By Lemma 2, I is a $\gamma_{li} - set$ in G. Therefore,

$$\begin{aligned} \gamma_{li}(G) &= |I| \\ &= |\{v_2, v_3, v_6, v_7, \dots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}| \\ &= \frac{n+3}{2}. \end{aligned}$$

Case 3: $n \equiv 2$ (mod 4)

Let $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-5}, v_{n-4}, v_{n-1}, v_n\}$. By Lemma 3, I is a $\gamma_{li}-set$ in G. Therefore,

$$\begin{aligned}\gamma_{li}(G) &= |I| \\ &= |\{v_2, v_3, v_6, v_7, \ldots, v_{n-5}, v_{n-4}, v_{n-1}, v_n\}| \\ &= \frac{n+2}{2}.\end{aligned}$$

Case 4: $n \equiv 3 \pmod 4$
Let $I = \{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$. By Lemma 4, I is a $\gamma_{li} - set$ in G. Therefore,

$$\begin{aligned}\gamma_{li}(G) &= |I| \\ &= |\{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}| \\ &= \frac{n+1}{2}.\end{aligned}$$

□

Example 10. *Consider Figure 9. Clearly* $I = \{a, b, d, e\}$ *is a dominating set. Observe that,* $N(a) \cap I = \{b\}$, $N(b) \cap I = \{a\}$, $N(d) \cap I = \{e\}$, *and* $N(d) \cap I = \{e\}$. *Hence,* I *is an internally-locating dominating set of minimum cardinality, so* $\gamma_{li}(S(P_5)) = 4$. *By Corollary 7, for* $n = 5 \equiv 1\,(mod\,4)$,

$$\gamma_{li}(S(P_5)) = \tfrac{n+3}{2} = \tfrac{5+3}{2} = \tfrac{8}{2} = 4.$$

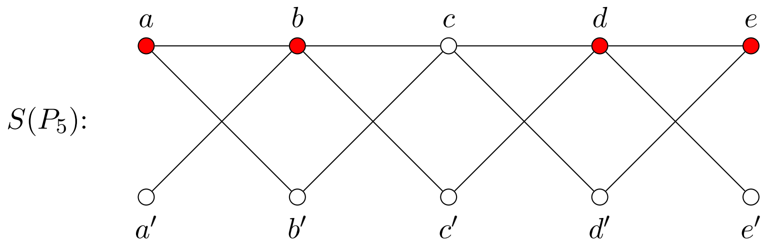

Figure 9: The minimum internally-locating dominating set in $S(P_5)$

5. Conclusion

It was shown in this paper that certain properties of this concept are identified, and characterizations of special classes of graphs are provided, including total graphs and shadow graphs with $\Delta(G) = 2$, along with their corresponding internally-locating domination numbers. The paper also examines cases where $\gamma_{li}(G) = 2$ and $\gamma(G) = \gamma_{li}(G)$. Additionally, the authors intend to explore other properties of internally-locating sets and internally-locating dominating sets, as well as extend the study to other binary operations on graphs, such as the join of graphs, along with their corresponding internally-locating domination numbers.

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