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On the Generalized Version of the Extended Standard U-Quadratic Distribution

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Abstract. This paper introduces the Generalized extended Standard U-quadratic (GeSU) distribution, an extension of the extended Standard U-quadratic (eSU) distribution, developed by incorporating an additional parameter. The GeSU distribution effectively models asymmetric data exhibiting bathtub and inverted bathtub shapes over the unit interval [0,1]. Several special cases of the proposed distribution are also derived. Key statistical properties, including moments, the moment-generating function, mean, variance, median, mode, skewness, and kurtosis, are established, along with a random number generation algorithm. The performance of maximum likelihood estimation for the parameters of the GeSU distribution is assessed through simulation studies. Furthermore, the distribution is applied to a real dataset, demonstrating a superior fit compared to the eSU, Unit-Rayleigh, and Unit-Burr XII distributions. These results highlight the flexibility and applicability of the proposed GeSU distribution in modeling unit-interval data.

2020 Mathematics Subject Classifications: 60E05, 62E10, 65C10

Key Words and Phrases: Extended Standard U-quadratic distribution, Kumaraswamy distribution, Generalized distribution, Cubic Transmuted Uniform distribution, bathtub shape distribution

1. Introduction

Probability distributions play a crucial role in describing the behaviour of various phenomena. However, each distribution has inherent strengths and limitations in modeling different types of data. Incorporating additional parameters to any distribution enhances the distribution's flexibility, allowing it to capture more complex data patterns.One important class of data consists of values restricted to the unit interval [0,1], such as rates

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and proportions. This type of data is usually modelled by the beta distribution or Kumaraswamy distribution[1].

Other identified distributions on the interval $[0, 1]$ or $(0, 1)$ are the unit - Lindley distribution [2], unit - Rayleigh distribution [3], new unit - Lindley distribution [4], unit - Weibull distribution [5], among others.

Rahman et al. [6] established a Cubic Transmuted Uniform (CTU) distribution using Cubic Transmuted Family, and they applied this distribution to electronic dataset and compared with the Beta, Kumaraswamy and Skew Uniform distribution. They found out that the CTU distribution provided better fit for the said dataset compared to the said distributions.

Lakibul and Tubo [7] proposed an alternative distribution to beta, Kumaraswamy, and CTU distributions to describe the behavior of the data that has bathtub, constant or inverted bathtub behavior, and this distribution is called the extended Standard U-quadratic distribution.

A random variable X is said to have an extended Standard U-quadratic distribution denoted by "eSU" if the probability density function (pdf) of T is given by

$$f(x) = 1 - \lambda + 3\lambda(2t - 1)^2, t \in [0, 1], \tag{1}$$

where $\lambda \in [-0.5, 1]$. It was observed that the pdf of this distribution can produce three different types of shape, namely the inverted bathtub for $\lambda \in [-0.5, 0)$, constant for $\lambda = 0$ and bathtub for $\lambda \in (0, 1]$. In addition, some properties of this distribution such as the mean, variance, moments, and other properties can be found in the paper of Lakibul and Tubo [8]. Also, the bivariate version of the eSU distribution is written in the paper of Lakibul, Polestico and Supe [9]. However, the eSU distribution is symmetric at $\frac{1}{2}$. Thus, there is a need to generalize this distribution by incorporating an additional parameter to account for the asymmetric behavior of the data.

In this paper, we will expand the extended Standard U-quadratic distribution into a Generalized extended Standard U-quadratic distribution by incorporating an additional parameter to the distribution. We will also derive some properties of the proposed distribution such as the mean, variance, skewness, kurtosis, median, mode, rth moments and moment generating function. The applicability and usefulness of the proposed distribution is investigated by applying it on the tensile strength of polyester fibers dataset.

The rest of the paper is arranged as follows: Section 2 presents the construction of the Generalized extended Standard U-quadratic (GeSU) distribution. Section 3 provides derivations of some properties of the proposed GeSU distribution. Section 4 discusses the maximum likelihood estimation for estimating the parameters of the proposed Generalized distribution. Section 5 deals with the random number generation of the proposed Generalized distribution. Section 6 presents the simulation results for assessing the behavior of the maximum likelihood estimate of the proposed Generalized distribution's parameter. Application of the proposed distribution is presented in Section 7. Section 8 gives some concluding remarks about the paper and recommendations for future studies.

2. The Generalized extended Standard U-quadratic distribution

Developing new models is essential for improving accuracy, efficiency, and robustness by capturing complex data patterns that standard models may overlook. In addition, specialized models are crucial for systems with unique data structures, ensuring better analysis and decision-making.

This section presents the derivation of the new Generalized distribution with support on $[0, 1]$.

The extended Standard U-quadratic (eSU) distribution, while useful for modeling certain forms of data within the unit interval $[0, 1]$ or $(0, 1)$, is limited in its ability to handle asymmetric data or data exhibiting more complex shapes such as the bathtub and inverted bathtub curves. These shapes are commonly observed in a range of practical fields, including economics, engineering, biological studies, and reliability analysis, where data often display significant skewness or asymmetric behavior. In particular, many real-world phenomena, such as survival data, failure rates, and certain types of financial data, do not adhere to the symmetrical distributions assumed by many traditional models. This motivates the development of a more flexible distribution capable of modeling such asymmetries and complex data behaviors. To address this limitation, we propose the Generalized extended Standard U-quadratic (GeSU) distribution, an extension of the eSU distribution that incorporates an additional parameter to provide greater flexibility.

To start with, let us consider the PDF of the eSU distribution given in Equation (1), which can be rewritten as

$$\begin{aligned} f(t) =& 1 - \lambda + 12\lambda \left(t - \frac{1}{2}\right)^2 \\ =& (1 - \lambda) f_1(t) + \lambda f_2(t), \end{aligned} \tag{2}$$

where $f_1(t) = 1$ and $f_2(t) = 12 \left(t - \frac{1}{2}\right)^2$, $t \in [0, 1]$ and $\lambda \in [-0.5, 1]$. To generalize the eSU distribution, we incorporate an additional parameter a into its PDF so that the eSU distribution becomes a special case of it. Specifically, find a particular form for $f_2(t)$ in Equation (2). Consider a function of the form

$$f^*(t) = C (t - a)^2, \tag{3}$$

where $t \in [0, 1]$, $a \in [0, 1]$, C is a constant. Now, recall that, for any function f, f is said to be a probability distribution function if it satisfies the following conditions:

(*i.*) $f(t) \geq 0$; and

(*ii.*) $\int f(t)dt = 1$.

Let us first solve the constant C for $f^*(t)$ using (*ii.*), that is,

$$\int_0^1 f^*(t)dt = 1$$

$$\int_0^1 C\,(t-a)^2\,dt = 1$$

$$C = \frac{3}{(1-a)^3 + a^3}.$$

Now, $f^*(t)$ becomes

$$f^*(t) = \frac{3}{(1-a)^3 + a^3}(t-a)^2. \tag{4}$$

Observe that $f^*(t)$ is a non-negative function ($f^*(t) \geq 0$) since for $a \in [0,1]$, $t \in [0,1]$, the expression $(t-a)^2 \geq 0$ and $\frac{3}{(1-a)^3+(a)^3} > 0$. Thus, $f^*(t)$ is a PDF. Substituting $f^*(t)$ into Equation (2) for $f_2(t)$, we get

$$f(t) = 1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(t-a)^2. \tag{5}$$

Next, to determine the values for λ, note that from the eSU distribution the λ should be between -0.5 and 1 inclusive, that is, $\lambda \in [-0.5, 1]$. Now, consider the non-negativity of $f(t)$ that is,

$$1 - \lambda + \frac{3\lambda(t-a)^2}{(1-a)^3 + a^3} \geq 0.$$

For $t = 0$ we have

$$\lambda\left[\frac{3a^2 - (1-a)^3 - a^3}{(1-a)^3 + a^3}\right] \geq -1.$$

For $3a^2 - (1-a)^3 - a^3 \neq 0$, and $\dfrac{3a^2}{(1-a)^3 + a^3} - 1 > 0$, we have

$$\lambda \geq -\frac{(1-a)^3 + a^3}{3a^2 - (1-a)^3 - a^3},$$

and for $\dfrac{3a^2}{(1-a)^3 + a^3} - 1 < 0$, we have

$$\lambda \leq \frac{(1-a)^3 + a^3}{3a^2 - (1-a)^3 - a^3}.$$

The relations above imply that

$$\lambda \in \left[-\frac{(1-a)^3 + a^3}{3a^2 - (1-a)^3 - a^3}, \frac{(1-a)^3 + a^3}{3a^2 - (1-a)^3 - a^3}\right].$$

For $t = 1$ we have

$$\lambda\left[\frac{3(1-a)^2 - (1-a)^3 - a^3}{(1-a)^3 + a^3}\right] \geq -1.$$

For $3(1-a)^2-(1-a)^3-a^3 \neq 0$ and $\dfrac{3(1-a)^2}{(1-a)^3+a^3}-1>0$ we have

$$\lambda \geq -\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3},$$

and for $\dfrac{3(1-a)^2}{(1-a)^3+a^3}-1<0$ we have

$$\lambda \leq \frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}.$$

These imply that

$$\lambda \in \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, \frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}\right].$$

Note that we have

$$\begin{cases}(1-a)^2 > a^2 & if\ a<\frac{1}{2}\\ (1-a)^2 = a^2 & if\ a=\frac{1}{2}\\ (1-a)^2 < a^2 & if\ a>\frac{1}{2}.\end{cases}$$

By taking the union of the possible values of λ for $t=0$ and $t=1$, we have

$$\lambda \in \begin{cases}\left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, \frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}\right], & \text{if } a\in[0,\ 0.5]\\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, \frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}\right], & \text{if } a\in[0.5,\ 1].\end{cases}$$

Now, combining or taking the intersection of the possible values of $\lambda \in [-0.5, 1]$ and

$$\lambda \in \begin{cases}\left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, \frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}\right], & \text{if } a\in[0,\ 0.5]\\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, \frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}\right], & \text{if } a\in[0.5,\ 1],\end{cases}$$

we have

$$\lambda \in \begin{cases}\left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \text{if } a\in[0,\ 0.5]\\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \text{if } a\in[0.5,\ 1]\end{cases}$$

since $-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3} \geq -\frac{1}{2}$ for $a\in[0,0.5]$, and $-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3} \geq -\frac{1}{2}$ for $a\in[0.5,1]$.

Definition 1. *A random variable T is said to have a Generalized extended Standard U-quadratic (GeSU) distribution if its pdf is given by*

$$f(t) = 1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(t-a)^2, \; t \in [0,1], \tag{6}$$

where $a \in [0,1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \textit{if } a \in [0,\ 0.5] \\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \textit{if } a \in [0.5,\ 1]. \end{cases}$$

Theorem 1. *Let T be a random variable that follows a GeSU distribution, then the cumulative distribution function of T is given by*

$$F(t) = (1-\lambda)t + \frac{\lambda\left[(t-a)^3 + a^3\right]}{(1-a)^3 + a^3}, \tag{7}$$

where $t \in [0,1]$, $a \in [0,1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \textit{if } a \in [0,\ 0.5] \\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \textit{if } a \in [0.5,\ 1]. \end{cases}$$

Proof. Let T be a random variable that follows a GeSU distribution with parameters $a \in [0,1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0,\ 0.5] \\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0.5,\ 1]. \end{cases}$$

The cumulative distribution function of T is computed as

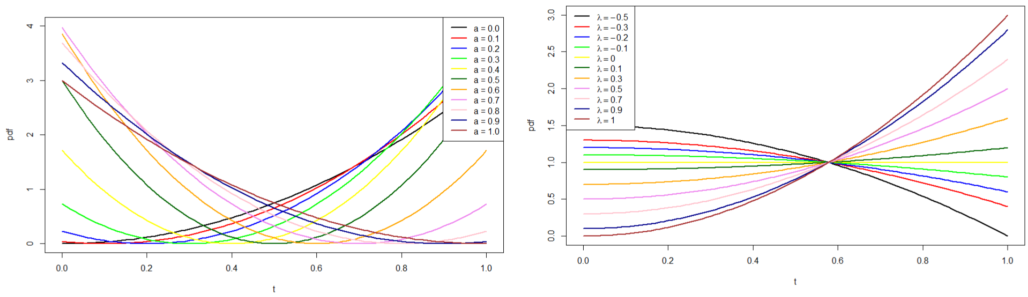
$$\begin{aligned} F(t) &= \int_0^t \left(1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(u-a)^2\right) du \\ &= (1-\lambda)\int_0^t du + \frac{3\lambda}{(1-a)^3 + (a)^3}\int_0^t (u-a)^b du \\ &= (1-\lambda)\, t + \frac{\lambda\left[(t-a)^3 + a^3\right]}{(1-a)^3 + (a)^3}. \end{aligned}$$

2.1. Special Cases of the GeSU Distribution

Here, we construct some special cases of the GeSU distribution based on the values of a and name these cases in accordance with Table 1.

Table 1: Special Cases of the Generalized extended Standard U-quadratic Distribution

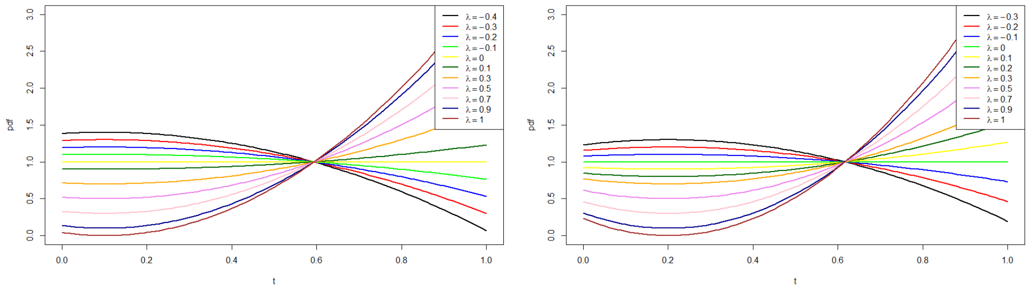
No	Name	a	Support of λ
0	Special eSU ($SPeSU$) Distribution	0.0	$\lambda \in [-0.5, 1]$
1	eSU Type-I Distribution	0.1	$\lambda \in [-\frac{73}{170}, 1]$
2	eSU Type-II Distribution	0.2	$\lambda \in [-\frac{52}{140}, 1]$
3	eSU Type-III Distribution	0.3	$\lambda \in [-\frac{37}{110}, 1]$
4	eSU Type-IV Distribution	0.4	$\lambda \in [-\frac{28}{80}, 1]$
5	eSU Type-V (eSU Distribution)	0.5	$\lambda \in [-0.5, 1]$
6	eSU Type-VI Distribution	0.6	$\lambda \in [-\frac{28}{80}, 1]$
7	eSU Type-VII Distribution	0.7	$\lambda \in [-\frac{37}{110}, 1]$
8	eSU Type-VIII Distribution	0.8	$\lambda \in [-\frac{52}{140}, 1]$
9	eSU Type-IX Distribution	0.9	$\lambda \in [-\frac{73}{170}, 1]$
10	eSU Type-X Distribution	1.0	$\lambda \in [-0.5, 1]$



(a) (b)

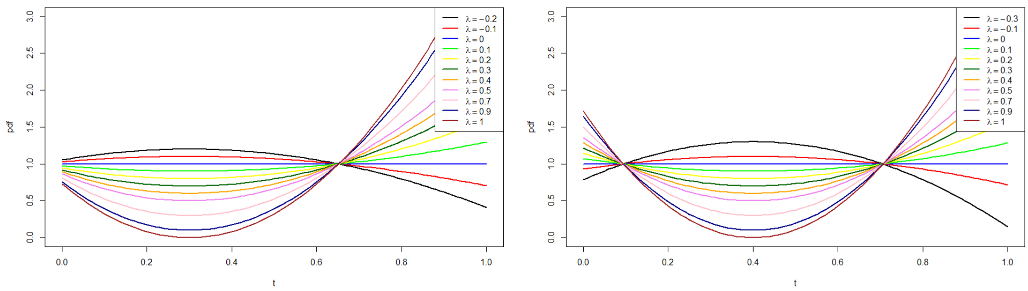
Figure 1: PDF plots of the distributions: (a) GeSU distribution for $\lambda = 1$ and varying values of a; and (b) SPeSU distribution for varying values of λ.

Figure 1a shows the PDFs of GeSU distribution types for a fixed value of λ and varying a. It is important to note that for $a = 0.5$ and $\lambda = 1$, the behavior of the distribution follows a symmetric bathtub shape where the minimum point is equal to 0. As we change the values of a from 0 to 1, we observe the following behaviors: (i) increasing for $a = 0$; (ii) asymmetric to the right for $a \in (0, 0.5)$; (iii) asymmetric to the left for $a \in (0.5, 1)$; and (iv) decreasing for $a = 1$. Figure 1b shows the different shapes of the PDF of the Special eSU distribution for varying values of λ. As we increase the values of λ from -0.5 to 1, it is observed that the behavior of the PDF forms a decreasing behavior for $\lambda \in [-0.5, 0)$, constant for $\lambda = 0$ and increasing for $\lambda \in (0, 1]$.



(a) (b)

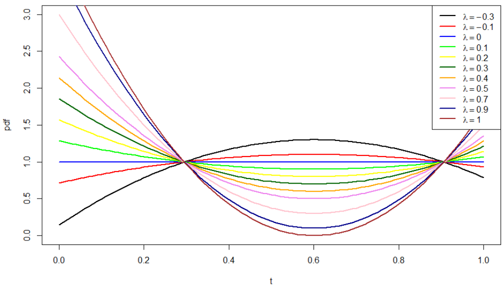
Figure 2: PDF plots of the distributions for varying values of λ: (a) eSU Type-I distribution; and (b) eSU Type-II distribution.



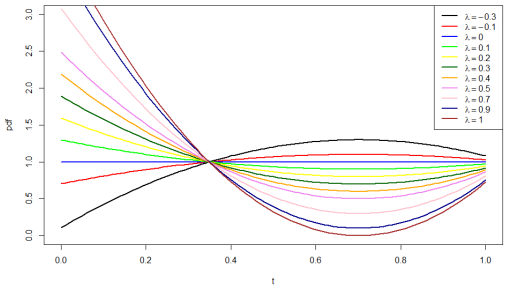
(a) (b)

Figure 3: PDF plots of the distributions for varying values of λ: (a) eSU Type-III distribution; and (b) eSU Type-IV distribution.

Figures 2a-3b show the behavior of the PDF of the eSU Type-I, eSU Type-II, eSU Type-III and eSU Type-IV distributions, respectively. It can be seen from the said figures that for varying values of λ, the behavior of each of the said distributions follows some decreasing function for $\lambda \in (-0.5, 0)$, constant for $\lambda = 0$, and asymmetric to the right bathtub for $\lambda \in (0, 1]$. In addition, it is also observed that the parameter a is the minimum of the distribution for bathtub shapes and it is the maximum for the inverted bathtub shapes.

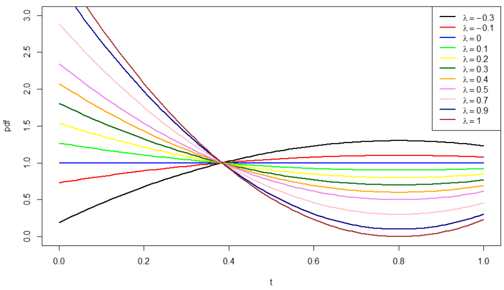


(a)

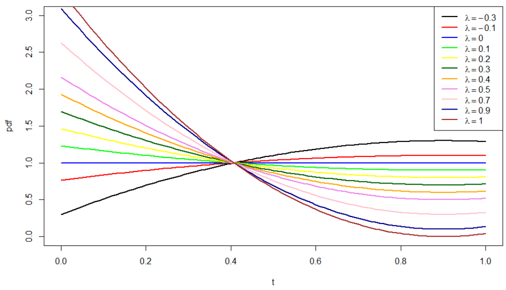


(b)

Figure 4: PDF plots of the distributions for varying values of λ: (a) eSU Type-VI distribution; and (b) eSU Type-VII distribution.



(a)



(b)

Figure 5: PDF plots of the distributions for varying values of λ: (a) eSU Type - VIII distribution; and (b) eSU Type-IX distribution.

Figures 4a-5b show almost same observations with the previous figures. The difference is asymmetric to the left bathtub shapes are produced in this case.

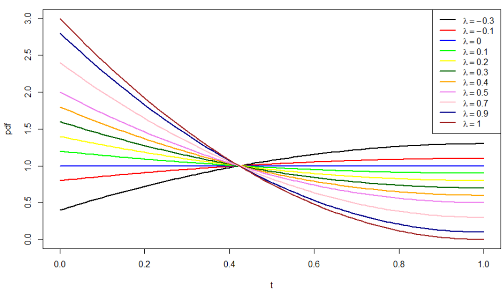


Figure 6: PDF plots of the eSU Type-X distribution for varying values of λ.

Figure 6 presents plots of the PDF of the eSU Type-X distribution for varying values of λ. It is observed from the said figures that the behavior of the distributions is increasing for $\lambda \in [-0.1, 0)$, constant for $\lambda = 0$, and decreasing for $\lambda \in (0, 1]$.

3. Some properties of the Generalized extended Standard U-quadratic distribution

In this section, we present some properties of the proposed generalized distribution such as the limiting behavior of the pdf, mode, moments, mean, variance, skewness, kurtosis, and the moment generating function.

Lemma 1. *Let T be a random variable that follows a GeSU distribution, then*

$$\lim_{t\to 0} f(t) = 1 - \lambda + \frac{3\lambda a^2}{(1-a)^3 + a^3},$$

and

$$\lim_{t\to 1} f(t) = 1 - \lambda + \frac{3\lambda (1-a)^2}{(1-a)^3 + a^3},$$

where $t \in [0,1]$, $a \in [0,1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0,\ 0.5] \\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0.5,\ 1]. \end{cases}$$

Proof. The limit of $f(t)$ as $t \to 0$ and $t \to 1$ are, respectively, computed as

$$\begin{aligned} \lim_{t\to 0} f(t) =& \lim_{t\to 0}\left[1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(t-a)^2\right] \\ =& 1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3} a^2, \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to 1} f(t) =& \lim_{t\to 1}\left[1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(t-a)^2\right] \\ =& 1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3} (1-a)^2, \end{aligned}$$

where $t \in [0,1]$, $a \in [0,1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0, 0.5] \\ \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0.5, 1]. \end{cases}$$

Remark 1. *If* $a = 0.5$*, then*

$$\lim_{t\to 0} f(t) = \lim_{t\to 1} f(t) = 1 + 2\lambda.$$

Lemma 2. *Let* T *be a random variable that follows a GeSU distribution, then the mode of* T *is* a *if*

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 0\right) & \textit{if } a \in [0, 0.5] \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 0\right) & \textit{if } a \in [0.5, 1]. \end{cases}$$

Proof. Let T be a random variable that follows a GeSU distribution. To compute for the mode of a GeSU distribution, let us first take the first derivative of $f(t)$, that is,

$$\frac{df(t)}{dt} = \frac{6\lambda}{(1-a)^3 + a^3}(t - a).$$

Equating $\frac{df(t)}{dt} = 0$, it follows that

$$\begin{aligned} \frac{6\lambda(t-a)}{(1-a)^3 + a^3} &= 0 \\ \implies t &= a \end{aligned}$$

The second derivative of $f(t)$ is given by

$$f''(t) = \left[\frac{6}{(1-a)^3 + a^3}\right]\lambda.$$

Note that, for $a \in [0, 1]$ and $t \in [0, 1]$, we have

$$\frac{6}{(1-a)^3 + a^3} > 0.$$

So, $f''(t) < 0$ if

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0,\ 0.5] \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0.5,\ 1]. \end{cases}$$

Thus, the mode of T is a if

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0,\ 0.5] \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0.5,\ 1]. \end{cases}$$

Theorem 2. *Let T be a random variable that follows a GeSU distribution, then the rth moment of T is given by*

$$\mathbb{E}[T^r] = \frac{1-\lambda}{r+1} + \frac{3\lambda}{(1-a)^3 + a^3} \sum_{k=0}^{2} \frac{\binom{2}{k}(-a)^k}{r-k+3}, \tag{8}$$

where $a \in [0, 1]$,

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0,\ 0.5] \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 0\right), & \text{if } a \in [0.5,\ 1], \end{cases}$$

and $r = 1, 2, 3, ...$

Proof. Let T be a random variable that follows a GeSU distribution. The rth moment of T is defined as

$$\begin{aligned} \mathbb{E}[T^r] &= \int_0^1 t^r \left[1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3}(t-a)^2\right] dt \\ &= (1-\lambda) \int_0^1 t^r dt + \frac{3\lambda}{(1-a)^3 + a^3} \int_0^1 (t-a)^2 t^r dt. \end{aligned}$$

But since the binomial expansion of $(t-a)^2$ is given by

$$(t-a)^2 = \sum_{k=0}^{2} \binom{2}{k} t^{2-k}(-a)^k,$$

we have

$$\begin{aligned} \mathbb{E}[T^r] &= (1-\lambda) \int_0^1 t^r dt + \frac{3\lambda}{(1-a)^3 + a^3} \int_0^1 \sum_{k=0}^{2} \binom{2}{k} t^{2-k}(-a)^k t^r dt \\ &= (1-\lambda)\frac{1^{r+1}}{r+1} + \frac{3\lambda}{(1-a)^3 + a^3} \sum_{k=0}^{2} \binom{2}{k}(-a)^k \frac{1^{r-k+3}}{r-k+3} \\ &= \frac{1-\lambda}{r+1} + \frac{3\lambda}{(1-a)^3 + a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{r-k+3}. \end{aligned}$$

Corollary 1. *Let T be a random variable that follows a GeSU distribution, then the first to the fourth raw moment of T are given by:*

(*i.*) $\mathbb{E}[T] = \frac{1-\lambda}{2} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{4-k}$;

(*ii.*) $\mathbb{E}[T^2] = \frac{1-\lambda}{3} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{5-k}$;

(*iii.*) $\mathbb{E}[T^3] = \frac{1-\lambda}{4} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{6-k}$; *and*

(*iv.*) $\mathbb{E}[T^4] = \frac{1-\lambda}{5} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{7-k}$.

The proof is straight forward from the rth moments of a GeSU distribution by taking $r = 1$, $r = 2$, $r = 3$ and $r = 4$, respectively. These moments are needed for the mean, variance, skewness and kurtosis of T as given in the following remark.

Remark 2. *Let T be a random variable that follows a GeSU distribution, then*

(*i.*) *The mean of T is given by*

$$\mathbb{E}[T] = \frac{1-\lambda}{2} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{4-k};$$

(*ii.*) *The variance of T is given by*

$$\begin{aligned} Var(T) =& \mathbb{E}[T^2] - (\mathbb{E}[T])^2 \\ =& \frac{1-\lambda}{3} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{5-k} \\ & - \left[\frac{1-\lambda}{2} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{4-k} \right]^2 ; \end{aligned}$$

(*iii.*) *The skewness of T, denoted by β_2, is given by*

$$\beta_2 = \frac{\mathbb{E}[T^3] - 3\mathbb{E}[T^2]\mathbb{E}[T] + 2\left(\mathbb{E}[T]\right)^3}{(Var(T))^{\frac{3}{2}}},$$

where $\mathbb{E}[T^3] = \frac{1-\lambda}{4} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{6-k}$; *and*

(*iii.*) *The kurtosis of T, denoted by κ, is given by*

$$\kappa = \frac{\mathbb{E}[T^4] - 4\mathbb{E}[T]\mathbb{E}[T^3] + 6(\mathbb{E}[T])^2\mathbb{E}[T^2] - 3(\mathbb{E}[T])^4}{(Var(T))^2},$$

where $\mathbb{E}[T^4] = \frac{1-\lambda}{5} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \binom{2}{k} \frac{(-a)^k}{7-k}$.

Theorem 3. *Let T be a random variable that follows a GeSU distribution, then the moment-generating function of T is given by*

$$\mathbb{E}[e^{sT}] = \sum_{r=0}^{\infty} \frac{s^r}{r!} \left[\frac{1-\lambda}{r+1} + \frac{3\lambda}{(1-a)^3+a^3} \sum_{k=0}^{2} \frac{\binom{2}{k}(-a)^k}{r-k+3} \right].$$

Proof. By definition of moment generating function, we have

$$\mathbb{E}[e^{sT}] = \int_0^1 e^{st} f(t)dt.$$

Now, since

$$e^{st} = \sum_{r=0}^{\infty} \frac{s^r}{r!} t^r,$$

it follows that

$$\mathbb{E}[e^{sT}] = \sum_{r=0}^{\infty} \frac{s^r}{r!} \mathbb{E}[T^r].$$

Using the rth moment of T, we get

$$\mathbb{E}[e^{sT}] = \sum_{r=0}^{\infty} \frac{s^r}{r!} \left[\frac{1-\lambda}{r+1} + \frac{3\lambda}{(1-a)^3 + a^3} \sum_{k=0}^{2} \frac{\binom{2}{k}(-a)^k}{r-k+3} \right].$$

4. Maximum Likelihood Estimation

Let X_1,X_2,...,X_n be a random sample of size n from a Generalized extended Standard U-quadratic (GeSU) Distribution. Then the likelihood function is defined by

$$L = \prod_{i=1}^{n} \left[1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3} (t_i - a)^2 \right],$$

with its log-likelihood function is given by

$$\log L = \sum_{i=1}^{n} \log \left[1 - \lambda + \frac{3\lambda}{(1-a)^3 + a^3} (t_i - a)^2 \right].$$

The partial derivative of $\log L$ with respect to the parameters λ and a are given by

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\frac{3}{(1-a)^3+a^3}(t_i - a)^2 - 1}{1 - \lambda + \frac{3\lambda}{(1-a)^3+a^3}(t_i - a)^2},$$

and

$$\frac{\partial \log L}{\partial a} = \sum_{i=1}^{n} \frac{3\lambda(t_i - a)\left[(1-a)^3 + a^3\right]^{-2} \left\{ 3(t_i - a)\left[(1-a)^2 - a^2\right] - 2\left[(1-a)^3 + a^3\right] \right\}}{1 - \lambda + \frac{3\lambda}{(1-a)^3+a^3}(t_i - a)^2}.$$

Remark 3. *The maximum likelihood estimates of λ and a are obtained by solving the system of two score equations given as follows:*

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\frac{3}{(1-a)^3+a^3}(t_i-a)^2-1}{1-\lambda+\frac{3\lambda}{(1-a)^3+a^3}(t_i-a)^2} = 0 \\ \displaystyle\sum_{i=1}^{n} \frac{3\lambda(t_i-a)\left[(1-a)^3+a^3\right]^{-2}\left\{3(t_i-a)\left[(1-a)^2-a^2\right]-2\left[(1-a)^3+a^3\right]\right\}}{1-\lambda+\frac{3\lambda}{(1-a)^3+a^3}(t_i-a)^2} = 0. \end{cases}$$

5. Random Number Generation and Median of the Distribution

This section presents the algorithm for the generation of random numbers from the Generalized extended Standard U-quadratic (GeSU) distribution.

Considering the inversion method, let q be generated from a uniform distribution on $(0, 1)$. Then,

$$\begin{aligned} F(t) &= q \\ \implies \quad t &= F^{-1}(q), \end{aligned}$$

where $F(t)$ is the CDF of the random variable T. For the GeSU distribution, the CDF is given by

$$F(t) = (1-\lambda)t + \frac{\lambda\left[(t-a)^3+a^3\right]}{(1-a)^3+a^3},$$

where $t \in [0, 1]$, $a \in [0, 1]$ and

$$\lambda \in \begin{cases} \left[-\frac{(1-a)^3+a^3}{3(1-a)^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0,\ 0.5] \\ \left[-\frac{(1-a)^3+a^3}{3a^2-(1-a)^3-a^3}, 1\right], & \text{if } a \in [0.5,\ 1]. \end{cases}$$

Thus, we have

$$\begin{aligned} (1-\lambda)t + \frac{\lambda\left[(t-a)^3+a^3\right]}{(1-a)^3+a^3} &= q \\ \implies \quad (1-\lambda)t + \frac{\lambda\left[(t-a)^3+a^3\right]}{(1-a)^3+a^3} - q &= 0. \end{aligned} \tag{9}$$

Observe that Equation (9) cannot be solved analytically. So, we use numerical method to solve for t. Here, we consider the Newton-Raphson method. A Newton - Raphson method is an iterative numerical technique that is used to find roots of real-valued functions. The algorithm for the Newton - Raphson method is given as follows:

Let $f(t)$ be a real-valued function with first derivative $f'(t)$. Then

Steps	Description
1	Choose an initial guess (t_0), a tolerance level (tol), and a maximum number of iteration $(maxiter)$.
2	Evaluate $f(t_n)$ and $(f'(t_n))$.
3	Check for convergence: if $(\lvert f(t_n)\rvert < tol)$, stop and return (t_n).
4	Update the estimate using the following formula $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}, n = 0, 1, 2, 3, ...$
5	Repeat until the function value is within the desired tolerance or the maximum number of iterations is reached.

Now, note that if $\lambda = 0$, then the GeSU distribution simplifies to the uniform distribution defined on $(0, 1)$. So, we can use the uniform distribution as initial guess (t_0). Thus, we have the following modified algorithm to generate random numbers from the GeSU distribution. Set true values for λ and a. Then,

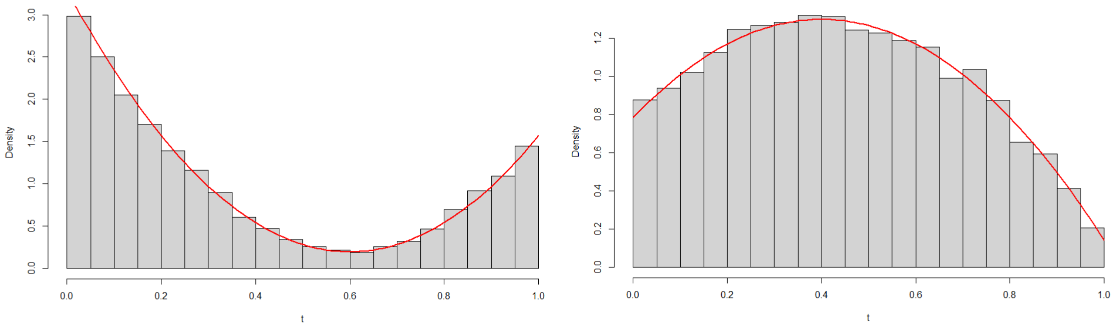
Steps	Description
1	Set $t_0 = q$, $tol = 1 \times 10^{-8}$, and $maxiter = 100$.
2	Evaluate $f(t_n) = (1-\lambda)t_n + \frac{\lambda\left[(t_n-a)^3+a^3\right]}{(1-a)^3+a^3} - q$, and $f'(t_n) = 1 - \lambda + \frac{3\lambda}{(1-a)^3+a^3}(t_n - a)^2$.
3	Check for convergence: if $(\lvert f(t_n)\rvert < tol)$, stop and return (t_n).
4	Update the estimate using the following formula $t_{n+1} = t_n - \frac{(1-\lambda)t_n + \frac{\lambda\left[(t_n-a)^3+a^3\right]}{(1-a)^3+a^3} - q}{1-\lambda+\frac{3\lambda}{(1-a)^3+a^3}(t_n-a)^2}, n = 0, 1, 2, 3, ...$
5	Repeat until the function value is within the desired tolerance or the maximum number of iterations is reached.

In addition, the median (t_{med}) of the GeSU distribution can be computed numerically using the Newton - Raphson method by solving

$$F(t_{med}) = (1-\lambda)t_{med} + \frac{\lambda\left[(t_{med}-a)^3 + a^3\right]}{(1-a)^3 + a^3} = \frac{1}{2}.$$

Now, to assess the above algorithm for generating random numbers from the GeSU distribution, let us consider the following examples.

Example 1. *Random numbers Generated from GeSU distribution of size* $N = 10000$.



(a) (b)

Figure 7: Histogram of the simulated data set and fitted density of the true values of the parameters for GeSU distribution with the following values: (a) $\lambda = 0.8$ and $a = 0.6$; and (b) $\lambda = -0.3$ and $a = 0.4$.

Figures 7a and 7b show that the true plots fitted the simulated histogram for GeSU distribution. This implies that the above algorithm works well for generating random numbers from the GeSU distribution.

6. Simulation Study

This section presents the simulation results to assess the behavior of the maximum likelihood estimate of the parameter of the proposed Generalized distribution. The simulation algorithm is given as follows:

Steps	Description
1	Draw sample of size n, $n = 50, 100, 200, 500, 800, 1000$, from a Generalized extended Standard U-quadratic (GeSU) distribution with parameter λ, a using the algorithm given in previous Subsection.
2	Using the sample x obtained in Step 1 above, compute the maximum likelihood estimate of λ and a.
3	Repeat the preceding Steps 1-2 $N = 1000$ times to get 1000 estimates of λ and a, respectively.
4	Compute the mean, bias, and mean squared error (MSE) of the 1000 estimates obtained in Step 3 to get the desired results.

The mean (AE), Bias, and MSE are, respectively, defined by

$$AE = \sum_{i=1}^{N} \frac{\theta_i}{N}, \ Bias = AE - \theta \text{ and } MSE = \sum_{i=1}^{N} \frac{(\theta_i - \theta)^2}{N},$$

where $\theta_i = \lambda_i, a_i$, and $\theta = \lambda, a$, respectively.

Here, we consider the following set of values of the parameters for the b-GeSU distribution, $\lambda = 0.7$ and $a = 0.4$, and $\lambda = -0.3$ and $a = 0.4$.

Table 2: The Average of Estimates (AE), Bias and MSE based on 1,000 Simulations of the GeSU Distribution for $\lambda = 0.7$ and $a = 0.4$.

$\lambda = 0.7$ and $a = 0.4$				
n	MLE	AE	$Bias$	MSE
50	$\hat{\lambda}$	0.799	0.099	0.017
100	$\hat{\lambda}$	0.771	0.071	0.009
200	$\hat{\lambda}$	0.750	0.050	0.004
500	$\hat{\lambda}$	0.735	0.035	0.002
800	$\hat{\lambda}$	0.733	0.033	0.002
1000	$\hat{\lambda}$	0.732	0.032	0.002
50	$\hat{a}$	0.393	−0.007	0.001
100	$\hat{a}$	0.395	−0.005	0.001
200	$\hat{a}$	0.397	−0.003	0.000
500	$\hat{a}$	0.398	−0.002	0.000
800	$\hat{a}$	0.400	0.000	0.000
1000	$\hat{a}$	0.400	0.000	0.000

Table 3: The Average of Estimates (AE), Bias and MSE based on 1,000 Simulations of the GeSU Distribution for $\lambda = -0.3$ and $a = 0.4$.

$\lambda = -0.3$ and $a = 0.4$				
n	MLE	AE	$Bias$	MSE
50	$\hat{\lambda}$	−0.255	0.045	0.011
100	$\hat{\lambda}$	−0.277	0.023	0.003
200	$\hat{\lambda}$	−0.289	0.011	0.002
500	$\hat{\lambda}$	−0.301	−0.001	0.001
800	$\hat{\lambda}$	−0.305	−0.005	0.000
1000	$\hat{\lambda}$	−0.306	−0.006	0.000
50	$\hat{a}$	0.379	−0.021	0.034
100	$\hat{a}$	0.394	−0.006	0.010
200	$\hat{a}$	0.415	0.015	0.003
500	$\hat{a}$	0.421	0.021	0.001
800	$\hat{a}$	0.422	0.022	0.001
1000	$\hat{a}$	0.421	0.021	0.001

The above tables show the behavior of the ML estimates of the GeSU distribution parameters as the sample size n becomes large. It reveals that the ML estimates of the said distribution parameters is consistent for all parameters since the MSE decays toward zero. A faster rate of convergence of the parameter estimates of a is observed.

7. Application

In this section, we apply the GeSU distribution into a real dataset and compare with the eSU distribution. Also, we compare the GeSU distribution with the following distributions:

- Unit- Rayleigh (UR) distribution [3]

$$f(x) = -\frac{2\alpha}{x}\log(x)e^{-\alpha(\log(x))^2},$$

where $x \in (0,1)$ and $\alpha > 0$.

- Unit - Burr XII (UBXII) distribution [10]

$$f(x) = \alpha\beta x^{-1}(-\log(x))^{\beta-1}(1+(-\log(x))^{\beta})^{-\alpha-1},$$

where $x \in (0,1)$, $\alpha > 0$ and $\beta > 0$.

Here, we use a dataset from the study of Quesenberry and Hales [11]. This dataset is related to 30 measurements of tensile strength of polyester fibers. The observations are given as follow: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926.

In this application, we use the following diagnostic statistics, namely, the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov - Smirnov (K-S) as the basis for comparing the proposed distribution with the said competing distributions. The model which has the smallest values of the said diagnostic statistics is chosen to be the best model for the data. In addition, we use a package "fitdistrplus" in R software to perform the analysis of the data.

Table 4: Estimates and some diagnostic values of the fitted models for the given dataset.

Distribution	*Estimate*	*logLik*	*AIC*	*BIC*	$K-S$
GeSU	$\hat{\lambda} = 0.5426148$ $\hat{a} = 1.0000000$	**3.250599**	**-2.501198**	**0.3011964**	**0.06615338**
eSU	$\hat{\lambda} = 0.06649717$	0.05187171	1.896257	3.297454	0.23958860
UR	$\hat{\alpha} = 0.3481808$	−0.271881	2.543762	3.944959	0.19666520
UBXII	$\hat{\alpha} = 1.033106$ $\hat{\beta} = 1.846484$	1.038986	1.922029	4.724423	0.09925780

Table 4 lists the ML estimates and the result of the said diagnostics statistics of the fitted models for the said dataset and it suggests that the proposed distribution is selected to be a best model for the said dataset since it has the smallest values of AIC, BIC, and

K-S as compared with the eSU, UR and UBXII distributions. Moreover, the results of the K-S statistic are consistent with values of the AIC and BIC. In addition, the plot of the fitted models for the given dataset is given in the following Figure 8 and same result is observed.

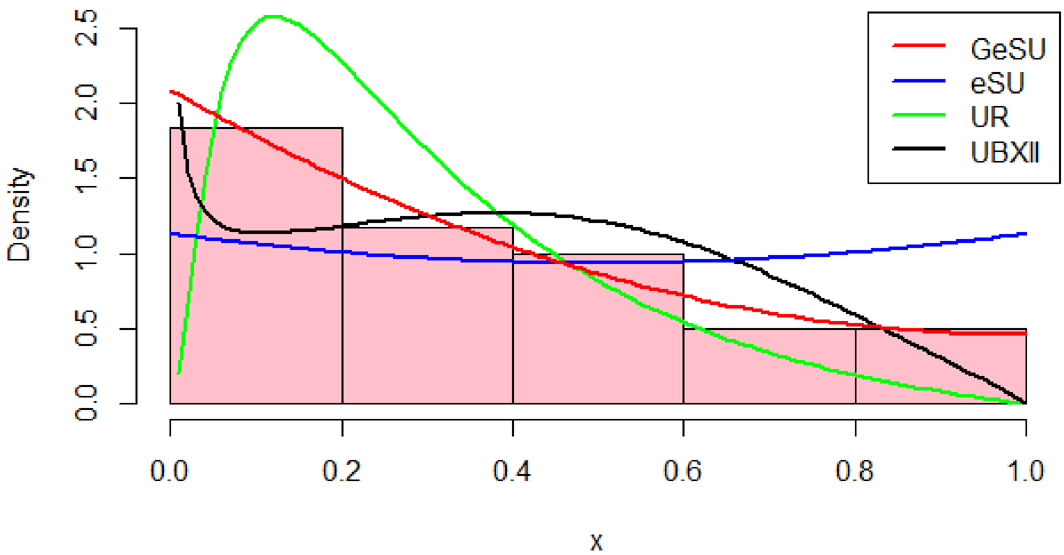


Figure 8: Estimated pdf of the fitted models for the given data.

8. Conclusions and Recommendations

In this paper, the generalized version of the extended Standard U-quadratic distribution has been derived. Some special cases of the generalized distribution such as the Special eSU, eSU Type-I, eSU Type-II, eSU Type-III, eSU Type-IV, eSU Type-V, eSU Type-VI, eSU Type-VII, eSU Type-VIII, eSU Type-IX and eSU Type-X distributions were generated. Moreover, some properties of the proposed distribution such as the moments, mean, variance, mode, median, moment generating function were computed. Further, a Newton-Raphson method was used to generate a random numbers from the proposed distribution. In addition, maximum likelihood method was used to estimate the parameters of the distribution, and simulation study was carried out to evaluate the performance of the maximum likelihood estimates of the distribution. Finally, the applicability of the proposed distribution was tested by applying on a real dataset and it was revealed that the GeSU distribution provides better estimate as compared with the eSU, Unit-Rayleigh and the Unit-Burr XII distributions. For future studies in this field, it is recommended to incorporate additional parameters to this proposed distribution in order to model more complex behavior of the data on the interval $[0, 1]$ or $(0, 1)$.

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