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Computational Methods, Existence and Uniqueness for Solving 2-D Fractional Nonlinear Fredholm Integro-Differential Equation

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Abstract. In this paper, we have investigated the two-dimensional fractional nonlinear Fredholm integro-differential equation (**TDFNFI-DE**). These equations are used in many fields, including particle dynamics in physics, biology, and control theory. We have developed an effective combined approach in our work that uses Homotopy analysis (**HAM**) and the Adomian decomposition method (ADM) to solve fractional integro-differential equations in two dimensions. Numerical experiment results show the effectiveness of our recently created technique. We prove the existence and uniqueness of the exact solution. To illustrate the numerical effectiveness of the suggested approach, we provide a number of numerical examples. The suggested approach is accurate and applicable to various nonlinear issues in science, according to theoretical and numerical results.

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Key Words and Phrases: TDFNFI-DE, existence and uniqueness, ADM, HAM

1. Introduction

Integral equations are used in many different fields, such as continuous mechanical engineering, potential concept, geophysics, electricity and magnetism, kinetic theory of gasses, hereditary phenomena in biology and physics, quantum mechanics, radiation, optimisation, which includes optimal system design, theory of communication, economics of mathematics, genetics of populations, medical procedures, computational problems of radiative equilibrium, the particle transport problems of the fields of a and reactor principle, acoustics, steady state heat conduction, and radiative heat transfer troubles. Fredholm integral equation is one of the most important integral equations. Singh and Pippal [1] utilized the Shehu transform combined with the Adomian polynomial to solve effectively nonlinear fractional differential equations. Additionally, Ali et al. [2] presented a numerical solution for fractional integro-differential equations with a linear functional argument,

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employing Chebyshev serious for increased precision. Oyedepo et al. [3] contributed a competitional algorithm specifically designed for fractional Fredholm integro-differential equations, providing a robot and efficacious method for tackling these complex problems. Moreover, Toma and Postavaru [4] proposed a numerical method for solving fractional Fredholm integro-differential equations with a focus on enhancing accuracy. Abdou et al. is performed a comprehensive analysis of the convergence into that Adomian's method when applied to nonlinear equations in [5]. Abbaoui and Cherruault [6] also performed a comprehensive analysis of the convergence into that Adomian's method when applied to nonlinear equations. Behiry et al. [7] developed a new algorithm for the decomposition solution of nonlinear differential equations, further advancing the method. Tair et al. [8] explored two numerical techniques for solving the linear integro-differential Fredholm equation with a weakly singular kernel, while Al-Bugami [9] focused on two-dimensional Fredholm integro-differential equations with singular kernels, providing accurate numerical method for their solution. Abbaszadeh et al. [10] employed Legendre wavelets to solve fractional Fredholm integro-differential equations, while Rawashdeh et al. [11] provided a detailed convergence analysis for the fractional decomposition method as applied to nonlinear fractional Fredholm integro-differential equations. Li and Pang [12]] expanded the application of the ADM to nonlinear systems. In addition, Momani and Noor [13] developed numerical techniques to solve fractional integro-differential equations of fourth order. In the realm of two-dimensional nonlinear Fredholm integro-differential equations, Al-Bugami [14] presented effective numerical solutions. Additionally, Hasan [15] applied the (ADM) to nonlinear systems involving fractional Fredholm integro-differential equations.
We will use the **HAM** and **ADM** for dealing with the **(TDFNFI-DE)** in this study. The TDFNFI-DE form will evaluate.

$$D^{\alpha}u(s,t) = f(s,t) + \lambda \int_a^b \int_c^d K(s,t,\tau,\xi)\gamma(\tau,\xi,D^{\alpha}u(\tau,\xi))d\tau d\xi, \tag{1}$$

where $u(s,t)$ is an unknown function , $f(s,t), K(s,t,\tau,\xi)$ are given functions, and λ is parameter. Also $0 < \alpha \leq 1$ and $(s,t) \in \Omega = [a,b] \times [c,d]$.

2. The Existence and Uniqueness

Theorem 1. Let (**M**,d) be a metric space with **M**$\neq \emptyset$. Suppose $\boldsymbol{M} \neq \emptyset$ is complete and let $I : \mathbf{M} \to \mathbf{M}$ be contraction mapping. Then, I has exactly one fixed point. Furthermore, we can express the equation in (1) in the form of an integral operator as follows:

$$\tilde{T}D^{\alpha}u(s,t) = f(s,t) + TD^{\alpha}u(s,t), \tag{2}$$

where

$$TD^{\alpha}u(s,t) = \lambda \int_a^b \int_c^d K(s,t,\tau,\xi)\gamma(\tau,\xi,D^{\alpha}u(\tau,\xi))d\tau d\xi. \tag{3}$$

Also assume the following conditions:

(i) $K(s,t,\tau,\xi)$ satisfies $|K(s,t,\tau,\xi)| \leq \kappa$, where κ is a constant.

(ii) $f(s,t)$ is continuous in $C[a,b] \times C[c,d]$, with its norm defined as:

$$||f(s,t)|| = \left[\int_a^b \int_c^d |f(s,t)|^2 dsdt\right]^{\frac{1}{2}} = v,$$

where v is a constant.

(iii) The continuous function $\gamma(s,t,D^\alpha u(s,t))$ satisfies the following conditions for some constant $g > g_1 > m, g > m$:

i. $\left[\int_a^b \int_c^d |\gamma(s,t,D^\alpha u(s,t))|^2 dsdt\right]^{\frac{1}{2}} \leq g_1||D^\alpha u(s,t)||$.

ii. $|\gamma(s,t,D^\alpha u_1(s,t)) - \gamma(s,t,D^\alpha u_2(s,t))| \leq \mathbf{M}(s,t) \cdot |D^\alpha u_1(s,t) - D^\alpha u_2(s,t)|$, where $\|\mathbf{M}(s,t)\| = M$ and M is a constant.

(iv) The unknown function $D^\alpha u(s,t)$ behaves similarly to the given function $f(s,t)$ in $C[a,b] \times C[c,d]$, with its norm defined as:

$$\|D^\alpha u(s,t)\| = \left[\int_a^b \int_c^d |D^\alpha u(s,t)|^2 dsdt\right]^{\frac{1}{2}}.$$

Theorem 2. If the first three conditions are satisfied, then eq (1) has a unique solution in $C[a,b] \times C[c,d]$.

Lemma 1. The space $C[a,b] \times C[c,d]$ is mapped into itself by the operator $\tilde{T}$ defined by (2) under the conditions (1)–(3-i).

Proof. Using equation (2) and (3), we have:

$$\|\tilde{T}D^\alpha u(s,t)\| \leq \|f(s,t)\| + |\lambda| \left\| \int_a^b \int_c^d |K(s,t,\tau,\xi)||\gamma(\tau,\xi,D^\alpha u(\tau,\xi))| d\tau d\xi \right\|, \tag{4}$$

by applying condition (2), it follows that:

$$\|\tilde{T}D^\alpha u(s,t)\| \leq v + |\lambda| \cdot \left[|K(s,t,\tau,\xi)|\right] \left[\int_a^b \int_c^d |\gamma(s,t,D^\alpha u(s,t))|^2 dsdt\right]^{\frac{1}{2}}, \tag{5}$$

using conditions (1) and (3-i), we can estimate:

$$\|\tilde{T}D^\alpha u(s,t)\| \leq v + \delta\|D^\alpha u(s,t)\|,$$

when $\delta = |\lambda|\kappa g$. Hence, this inequality (2) confirms that the operator $\tilde{T}$ is bounded, with:

$$\|\tilde{T}D^\alpha u(s,t)\| \leq \delta\|D^\alpha u(s,t)\|.$$

Lemma 2. The operator $\tilde{T}$ is a contraction on the Banach space $C[a,b] \times C[c,d]$, if conditions (1) and (3-ii) are satisfied.

Proof. Suppose we have two unknown function $D^\alpha u_1(s,t)$ and $D^\alpha u_2(s,t)$ in $C[a,b] \times C[c,d]$, we can apply equations (2) and (3) to obtain:

$$\|(\tilde{T}D^\alpha u_1 - \tilde{T}D^\alpha u_2)(s,t)\| \leq |\lambda| \bigg\| \int_a^b \int_c^d |K(s,t,\tau,\xi)| \cdot$$
$$|\gamma(\tau,\xi,D^\alpha u_1(\tau,\xi)) - \gamma(\tau,\xi,D^\alpha u_2(\tau,\xi))| d\tau d\xi \bigg\|,$$

by using condition (3-ii), we get:

$$\|(\tilde{T}D^\alpha u_1 - \tilde{T}D^\alpha u_2)(s,t)\| \leq |\lambda| \cdot \big[|K(s,t,\tau,\xi)|\big] \cdot$$
$$\Big[\int_a^b \int_c^d \mathbf{M}^2(s,t) |D^\alpha u_1(\tau,\xi) - D^\alpha u_1(\tau,\xi)|^2 d\tau d\xi\Big]^{\frac{1}{2}}.$$

Thus, we conclude:

$$\|(\tilde{T}D^\alpha u_1 - \tilde{T}D^\alpha u_2)(s,t)\| \leq \delta \|D^\alpha u_1(s,t) - D^\alpha u_2(s,t)\|.$$

3. ADM

In this section, we will address the **TDFNFI-DE** using ADM.

Let's consider (1), in witch $f(s,t)$ is a function that is bounded, and for all (s,t) belongs to the set $\Omega = [a,b] \times [c,d]$. The kernel $|K(s,t,\tau,\xi)| \leq \kappa$. The nonlinear term $\gamma(\tau,\xi,D^\alpha u(\tau,\xi))$ is Lipchitz continuous and satisfies the following condition:

$$|\gamma(D^\alpha u) - \gamma(D^\alpha \tilde{u})| \leq L|D^\alpha u - D^\alpha \tilde{u}|.$$

We define the space $C[a,b] \times C[c,d]$, continuous functions on the rectangle $[a,b] \times [c,d]$, with the associated distance function $d^*(D^\alpha \tilde{u}, D^\alpha u)$ defined by:

$$d^*(D^\alpha \tilde{u}, D^\alpha u) = \max_{(s,t)\in\Omega} |D^\alpha \tilde{u}(s,t) - D^\alpha u(s,t)|. \tag{6}$$

The unknown function $D^\alpha u(s,t)$ is assumed to have the series form:

$$D^\alpha u(s,t) = \sum_{n=0}^{\infty} D^\alpha u_n(s,t). \tag{7}$$

The nonlinear term $\gamma(\tau,\xi,D^\alpha u(\tau,\xi))$ in (1) is similarly decomposed into an infinite series:

$$\gamma(\tau,\xi,D^\alpha u(s,t)) = \sum_{n=0}^{\infty} A_n u_n(s,t), \tag{8}$$

where the traditional form of the Adomian polynomials A_n is [16][17] given by:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}\gamma(\sum_{i=0}^{\infty}\lambda^i D^\alpha u_i)]_{\lambda=0}. \tag{9}$$

Another formulation of the Adomian polynomials is expressed as:

$$A_n = \gamma(S_n) - \sum_{i=0}^{n-1} A_i, \tag{10}$$

where S_n is defined by:

$$S_n = \sum_{i=0}^{n} D^\alpha u_i(s,t). \tag{11}$$

By applying the Adomian decomposition method to equation (1), we obtain the solution in series form:

$$D^\alpha u(s,t) = \sum_{n=0}^{\infty} D^\alpha u_n(s,t), \tag{12}$$

where

$$\begin{aligned} &D^\alpha u_0(s,t) = f(s,t), \\ &D^\alpha u_i(s,t) = \lambda \int_a^b \int_c^d K(s,t,\tau,\xi) A_{i-1} d\tau d\xi, \qquad i \geq 1. \end{aligned} \tag{13}$$

4. HAM

In this section, we aim to solve the nonlinear **(FNT-DFIDE)** using the Homotopy Analysis Method (HAM). This approach is novel because, to the best of our knowledge, previous studies have primarily applied HAM to solve two-dimensional fractional integro-differential equations. Consider (1), where $K(s,t,\tau,\xi)$ and $f(s,t)$ are known functions, and $\gamma(\tau,\xi,D^\alpha u(\tau,\xi))$ is a known nonlinear function of $D^\alpha u$. To describe the method, we define the nonlinear operator N:

$$N[D^\alpha u] = D^\alpha u(x,t) - f(x,t) - \int_a^b \int_c^d K(x,t,\tau,\xi)\gamma(\tau,\xi,D^\alpha u(\tau,\xi)) d\tau d\xi = 0. \tag{14}$$

Let $D^\alpha u_0(s,t)$ denote an initial guess for the exact solution $u(s,t)$, and choose a non-zero auxiliary parameter $h \neq 0$, an auxiliary function $H(s,t)$, and linear operator L, such that $L[e(s,t)] = 0$ whenever $e(s,t) = 0$. For $p \in [0,1]$, we construct the following homotopy:

$$\begin{aligned} &(1-p)[\phi(s,t;p) - D^\alpha u_0(x,t)] - phH(s,t)N[\phi(s,t;p)] \\ &= H[\phi(s,t;p); D^\alpha u_0(s,t), H(s,t), h, p]. \end{aligned} \tag{15}$$

it is important to emphasize that we have significant flexibility in choosing the initial guess $D^{\alpha}u_0(s,t)$, the auxiliary linear operator L, the non-zero parameter h, and the auxiliary function $H(s,t)$. Additionally, let $\tilde{H}$ be a secondary auxiliary function that ensures the homotopy satisfies the condition:

$$\tilde{H}[\phi(s,t;p);D^{\alpha}u_0(s,t),H(s,t),h,p]=0. \tag{16}$$

Thus, the homotopy equation can be written as:

$$(1-p)[\phi(s,t;p)-D^{\alpha}u_0(x,t)]=phH(s,t)N[\phi(s,t;p)], \tag{17}$$

when $p=0$, from equation (17), we get:

$$\phi(s,t;p)=D^{\alpha}u_0(s,t). \tag{18}$$

Furthermore, when $p=1$ and $h\neq 0$, $H(s,t)\neq 0$, equation (17) becomes:

$$\phi(s,t;1)=D^{\alpha}u(s,t). \tag{19}$$

If we observe the value of p from 0 to 1m the embedding parameter gradually increases for $\phi(s,t;p)$ in equations (1) and (19), such that it begins from the initial approximation and reaches the exact solution. We refer to this change in homotopy deformation, and we can be expressed in the following form:

$$\phi(s,t;p)=D^{\alpha}u_0(x,t)+\sum_{r=1}^{\infty}D^{\alpha}u_r(s,t)p^r. \tag{20}$$

Here, $u_r(s,t)$ is given by:

$$D^{\alpha}u_r(s,t)=\frac{1}{r!}\frac{\partial\phi(s,t;p)}{\partial p^r}\bigg|_{p=0} \tag{21}$$

Based on these assumptions, we can write the exact solution as:

$$D^{\alpha}u(s,t)=\phi(s,t;1)=D^{\alpha}u_0(s,t)+\sum_{r=1}^{\infty}D^{\alpha}u_r(s,t)p \tag{22}$$

Thus, the sequence of approximations for $D^{\alpha}u(s,t)$ can be expressed as:

$$D^{\alpha}\bar{u}_n(s,t)=\{D^{\alpha}u_0(s,t),D^{\alpha}u_1(s,t),....,D^{\alpha}u_n(s,t)\} \tag{23}$$

The zero-order deformation equation (21), in accordance with equation (17), can be used to construct the governing equation of $D^{\alpha}u_m(s,t)$. The called mth-order deformation equation gets by differentiating the zero-order deformation equation (17) r times with regard to p, dividing the result by $r!$, and setting $r=0$.

Besed on equation (21), the governing equation for $D^{\alpha}u_m(s,t)$ can be derived from the zero-order deformation equation (17). By differentiating equation (17) r times with respect to r, then dividing by $r!$ and setting $p = 0$, we obtain the mth-order deformation equation as follows:

$$\begin{aligned} L[D^{\alpha}u_r(s,t) - \sigma_r D^{\alpha}u_{r-1}(s,t)] = hH(s,t)\mathcal{J}(D^{\alpha}\bar{u}_{r-1}(s,t)) \\ D^{\alpha}u_r(0,0) = 0 \end{aligned} \tag{24}$$

where

$$\mathcal{J}(D^{\alpha}\bar{u}_{r-1}(s,t)) = \frac{1}{(r-1)!}\frac{\partial^{r-1}N[\phi(s,t;p)]}{\partial p^{r-1}}\bigg|_{p=0} \tag{25}$$

and the condition σ_r is defined as:

$$\sigma_r = \begin{cases} 0 & (r \le 0) \\ 1 & (r > 0) \end{cases} \tag{26}$$

5. Numerical Applications

Application 1. Consider the following Caputo fractional linear Fredholm integro-differential equation

$$D^{0.5}u(s,t) = 2.256758334\ s\sqrt{t} - 0.6790610904 + \int_0^1\int_0^1 \sqrt{\tau\xi}\ D^{0.5}u(\tau,\xi)d\tau d\xi \tag{27}$$

The exact solution $u(s,t) = 2.256758334\ s\sqrt{t}$.

Table 1: Numerical results and absolute error values using (ADM) and (HAM), $N = 10$, at fractional order $\alpha = 0.5$ in Application 1.

s	t	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0	0	0	4×10^{-8}	4×10^{-8}
0.1	0.1	0.07136496464	0.07136496464	0	0.07136500464	4.000×10^{-8}
0.2	0.2	0.2018506018	0.2018506017	1×10^{-10}	0.2018506417	3.99×10^{-8}
0.3	0.3	0.3708232338	0.3708232338	0	0.3708232738	4.00×10^{-8}
0.4	0.4	0.5709197172	0.5709197171	1×10^{-10}	0.5709197571	3.99×10^{-8}
0.5	0.5	0.797884561	0.7978845608	2×10^{-10}	0.7978846008	3.98×10^{-8}
0.6	0.6	1.048846493	1.048846493	0	1.048846533	4.0×10^{-8}
0.7	0.7	1.321697642	1.321697641	1×10^{-9}	1.321697681	3.9×10^{-8}
0.8	0.8	1.614804814	1.614804814	0	1.614804854	4.0×10^{-8}
0.9	0.9	1.926854045	1.926854045	0	1.926854085	4.0×10^{-8}
1.0	1.0	2.256758334	2.256758334	0	2.256758374	4.0×10^{-8}

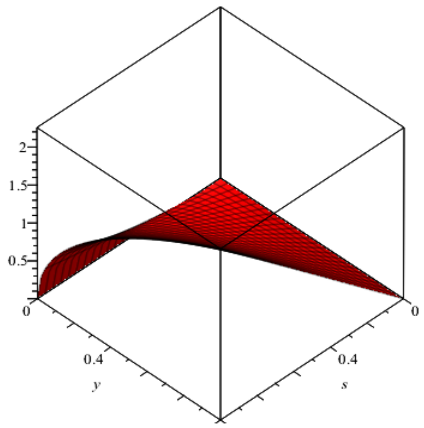

Figure 1: Exact Solution of (ADM) for $\alpha = 0.5$ in Application 1

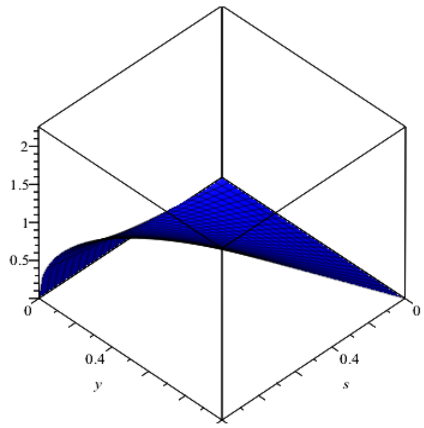

Figure 2: Approximate Solution of (ADM) for $\alpha = 0.5$ in Application 1

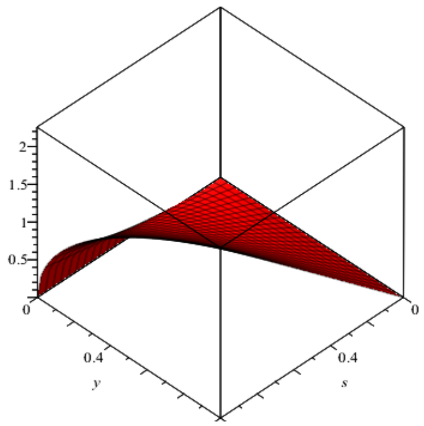

Figure 3: Exact Solution of (HAM) for $\alpha = 0.5$ in Application 1

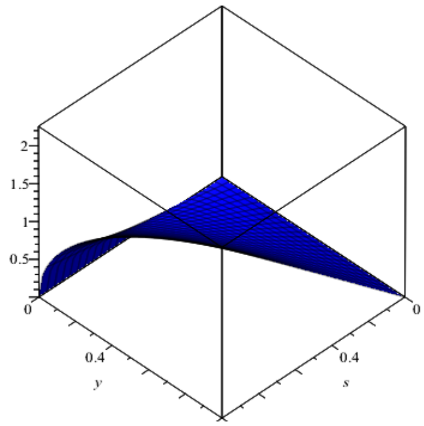

Figure 4: Approximate Solution of (HAM) for $\alpha = 0.5$ in Application 1

Application 2. Consider the following Caputo fractional linear Fredholm integro-differential equation

$$D^{0.8}u(s,t) = 2.178248842\ st^{1/5} - 0.8324154416 + \int_0^1 \int_0^1 \sqrt{\tau\xi}\ D^{0.8}u(\tau,\xi)d\tau d\xi \qquad (28)$$

The exact solution is $u(s,t) = 2.178248842\ st^{1/5}$.

Table 2: Numerical results and absolute error values using (ADM) and (HAM), $N = 10$, at fractional order $\alpha = 0.8$ in Application 2.

s	t	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0	0	0	3×10^{-8}	3×10^{-8}
0.1	0.1	0.1374382105	0.1374382105	0	0.1374382405	3.00×10^{-8}
0.2	0.2	0.3157500926	0.3157500925	1×10^{-10}	0.3157501225	2.99×10^{-8}
0.3	0.3	0.5136330933	0.5136330933	0	0.5136331233	3.00×10^{-8}
0.4	0.4	0.725403224	0.7254032241	1×10^{-10}	0.7254032541	3.01×10^{-8}
0.5	0.5	0.948137878	0.9481378781	1×10^{-10}	0.9481379081	3.01×10^{-8}
0.6	0.6	1.180018979	1.180018979	0	1.180019009	3.0×10^{-8}
0.7	0.7	1.419793357	1.419793357	0	1.419793387	3.0×10^{-8}
0.8	0.8	1.666538980	1.666538980	0	1.666539010	3.0×10^{-8}
0.9	0.9	1.919545908	1.919545908	0	1.919545938	3.0×10^{-8}
1.0	1.0	2.178248842	2.178248842	0	2.178248872	3.0×10^{-8}

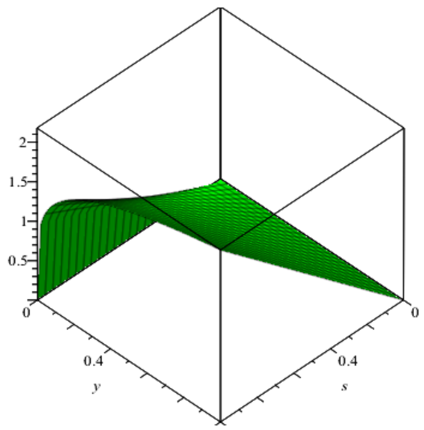

Figure 5: Exact Solution of (ADM) for $\alpha = 0.8$ in Application 2

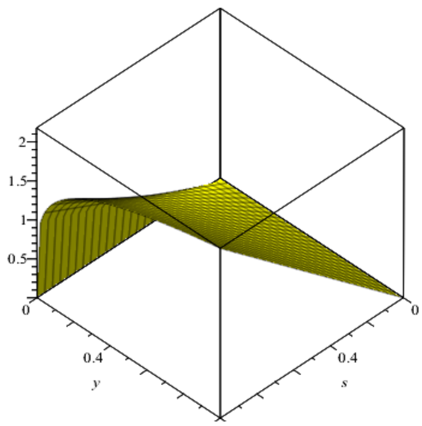

Figure 6: Approximate Solution of (ADM) for $\alpha = 0.8$ in Application 2

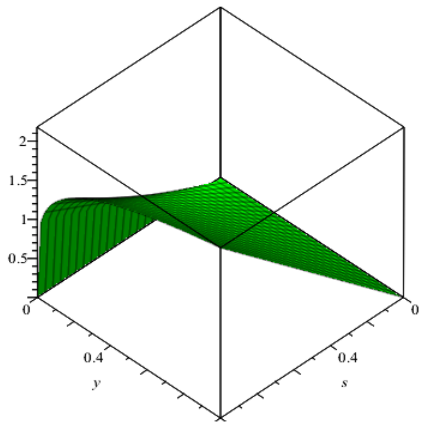

Figure 7: Exact Solution of (HAM) for $\alpha = 0.8$ in Application 2

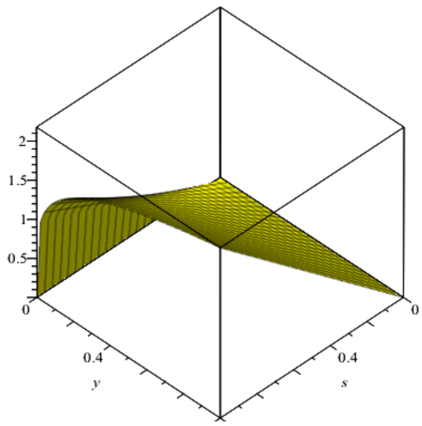

Figure 8: Approximate Solution of (HAM) for $\alpha = 0.8$ in Application 2

Application 3. Consider the following Caputo fractional nonlinear Fredholm integro-differential equation

$$D^{0.5}u(s,t) = 2.256758334\ s\sqrt{t} - 1.235153476 + \int_0^1 \int_0^1 \sqrt{\tau\xi}\ \left(D^{0.5}u(\tau,\xi)\right)^2\ d\tau d\xi \quad (29)$$

The exact solution is $u(s,t) = 2.256758334\ s\sqrt{t}$.

s	t	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0	0	0	-2×10^{-8}	2×10^{-8}
0.1	0.1	0.07136496464	0.07136496464	0	0.07136494464	2.000×10^{-8}
0.2	0.2	0.2018506018	0.2018506017	1×10^{-10}	0.2018505817	2.01×10^{-8}
0.3	0.3	0.3708232338	0.3708232338	0	0.3708232138	2.00×10^{-8}
0.4	0.4	0.5709197172	0.5709197171	1×10^{-10}	0.5709196971	2.01×10^{-8}
0.5	0.5	0.797884561	0.7978845608	2×10^{-10}	0.7978845408	2.02×10^{-8}
0.6	0.6	1.048846493	1.048846493	0	1.048846473	2.0×10^{-8}
0.7	0.7	1.321697642	1.321697641	1×10^{-9}	1.321697621	2.1×10^{-8}
0.8	0.8	1.614804814	1.614804814	0	1.614804794	2.0×10^{-8}
0.9	0.9	1.926854045	1.926854045	0	1.926854025	2.0×10^{-8}
1.0	1.0	2.256758334	2.256758334	0	2.256758314	2.0×10^{-8}

Table 3: Numerical results and absolute error values using (ADM) and (HAM), $N = 10$, at fractional order $\alpha = 0.5$ in Application 3.

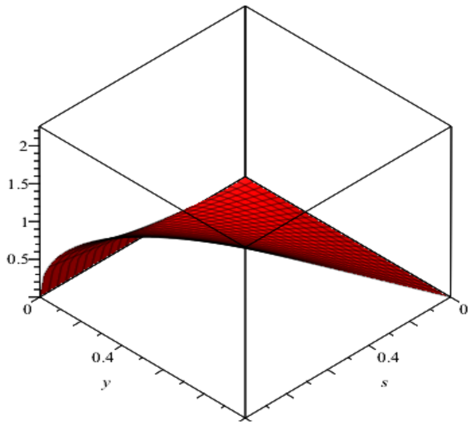

Figure 9: Exact Solution of (ADM) for $\alpha = 0.5$ in Application 3

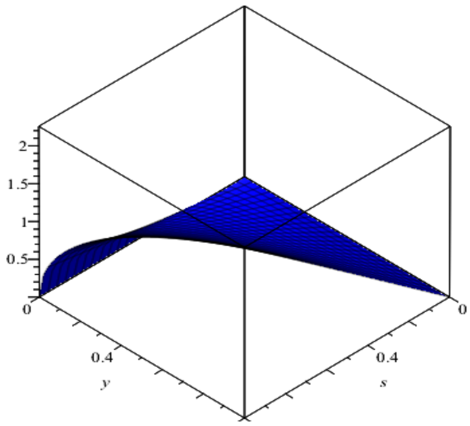

Figure 10: Approximate Solution of (ADM) for $\alpha = 0.5$ in Application 3

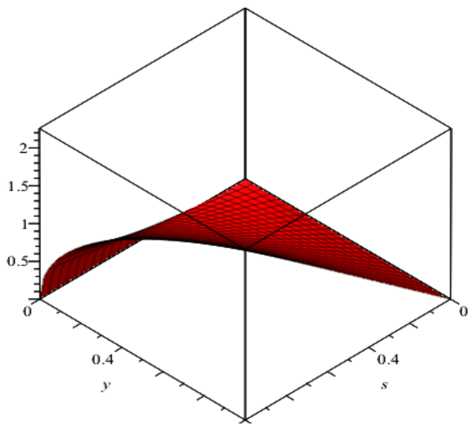

Figure 11: Exact Solution of (HAM) for $\alpha = 0.5$ in Application 3

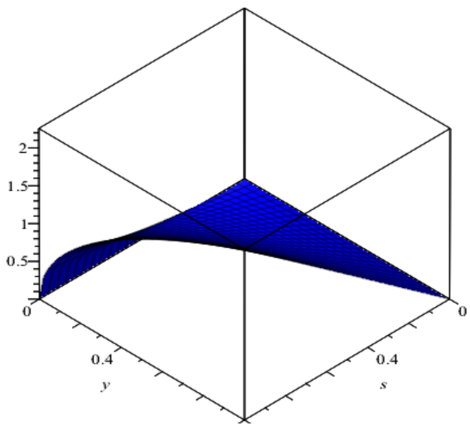

Figure 12: Approximate Solution of (HAM) for $\alpha = 0.5$ in Application 3

Application 4. Consider the following Caputo fractional nonlinear Fredholm integro-differential equation

$$D^{0.8}u(s,t) = 2.178248842\ st^{1/5} - 1.631364025 + \int_0^1 \int_0^1 \sqrt{\tau\xi}\ (D^{0.8}u(\tau,\xi))^2\ \ d\tau d\xi \quad (30)$$

The exact solution is $u(s,t) = 2.178248842\ s\ t^{1/5}$.

Table 4: Numerical results and absolute error values using (ADM) and (HAM), $N = 10$, at fractional order $\alpha = 0.8$ in Application 4.

s	t	u_{Exact}	**ADM**		**HAM**	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0	0	0	-5.5×10^{-8}	5.5×10^{-8}
0.1	0.1	0.1374382105	0.1374382105	0	0.1374381555	5.50×10^{-8}
0.2	0.2	0.3157500926	0.3157500925	1×10^{-10}	0.3157500375	5.51×10^{-8}
0.3	0.3	0.5136330933	0.5136330933	0	0.5136330383	5.50×10^{-8}
0.4	0.4	0.725403224	0.7254032241	1×10^{-10}	0.7254031691	5.49×10^{-8}
0.5	0.5	0.948137878	0.9481378781	1×10^{-10}	0.9481378231	5.49×10^{-8}
0.6	0.6	1.180018979	1.180018979	0	1.180018924	5.5×10^{-8}
0.7	0.7	1.419793357	1.419793357	0	1.419793302	5.5×10^{-8}
0.8	0.8	1.66653898	1.66653898	0	1.666538925	5.5×10^{-8}
0.9	0.9	1.919545908	1.919545908	0	1.919545853	5.5×10^{-8}
1.0	1.0	2.178248842	2.178248842	0	2.178248787	5.5×10^{-8}

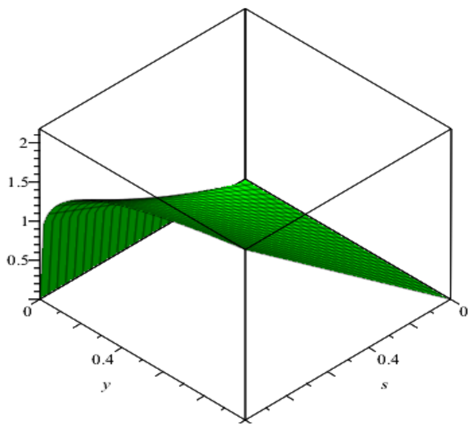

Figure 13: Exact Solution of (ADM) for $\alpha = 0.8$ in Application 4

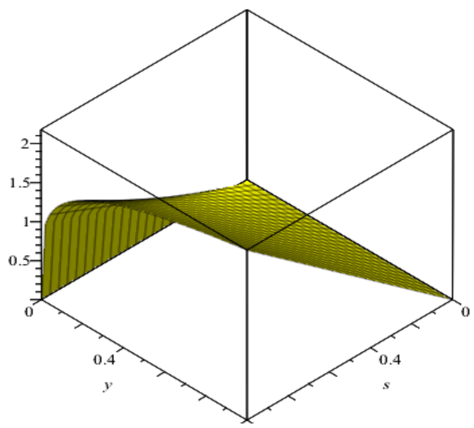

Figure 14: Approximate Solution of (ADM) for $\alpha = 0.8$ in Application 4

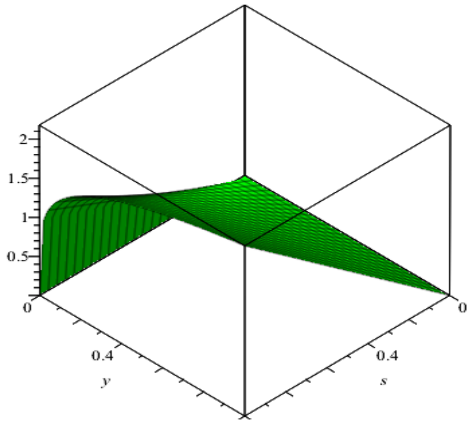

Figure 15: Exact Solution of (HAM) for $\alpha = 0.8$ in Application 4

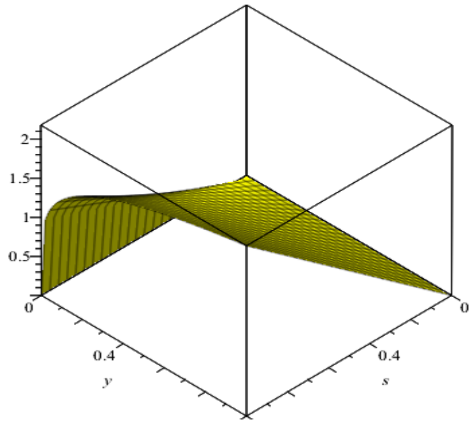

Figure 16: Approximate Solution of (HAM) for $\alpha = 0.8$ in Application 4

6. Conclusion

From the previous study, we concluded the following: as s, t are increasing in [0, 1]×[0, 1] the error for (ADM) is increasing and for (HAM) is remain constant at 108. In nonlinear case the using 0.8 is a smaller than by 0.5. The error using 0.8 is it smaller than in (ADM) and (HAM),(ADM) using 0.8 is more accurate than using 0.5. At N = 10 the error is decreasing in linear and nonlinear case.When using the same fractional order (ADM) and (HAM), exactly the same error, but using fractional order 0.8, (ADM) convergences faster than (HAM). The error decreases when the fractional order of (ADM) increases and the error for (HAM) remains the same. When the fractional order of (ADM) is 0.8, the error increases, whereas when the fractional order is 0.8, the error decreases. The Maple program was used to write the codes. Future Work. Other methods, such as the Homomtopy perturbation method, the Galerkin method, the collocation method and Chypechev polynomial, will be used to solve the **TDFNFI-DE**, and **compare** with ADM and HAM.

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