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Bipolar Soft Minimal Structures

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Abstract. The purpose of this paper is to introduce a new structure of bipolar soft topology called bipolar soft minimal structure. Later, we present most important operators in bipolar soft minimal spaces such as $\widetilde{\widetilde{\mathfrak{m}}}$-interior, $\widetilde{\widetilde{\mathfrak{m}}}$-closure and $\widetilde{\widetilde{\mathfrak{m}}}$-boundary. Moreover, we define $\widetilde{\widetilde{\mathfrak{m}}}$-separated bipolar soft sets and bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected sets. Furthermore, we introduce a new concept of bipolar soft minimal spaces called bipolar soft minimal connected. We prove that the bipolar soft intersection of a pair of bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected spaces over the common universal set is a bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected space. In addition, we show that bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected space is not a bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-hereditary property. Also, we discuss some relations, properties and results of these new concepts of bipolar soft minimal spaces. Finally, some counterexamples are provided.

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1. Introduction

Soft set theory, introduced by Molodtsov [1] in 1999, provides a versatile mathematical framework for handling uncertainty, imprecision, and vagueness in data. Unlike classical set theory, which struggles with uncertainty in decision-making processes, soft set theory offers a more flexible approach by associating parameters with elements, allowing for more nuanced information representation. This adaptability makes soft sets highly applicable in areas such as decision-making, data analysis, engineering, and artificial intelligence, where handling imprecise or incomplete information is critical. Soft set theory's strength lies in its simplicity and generality, as it does not require the strict mathematical constraints of other uncertainty models like fuzzy sets or rough sets. It has been successfully applied to a wide range of disciplines, including medical diagnosis, economics, social sciences, and optimization problems. Researchers have extended soft set theory by redefining operations, introducing new concepts, and developing hybrid approaches that combine soft sets with other mathematical frameworks to solve complex real-world problems. Other researchers,

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like Maji et al. [2], have contributed by refining operations within soft sets, and Çağman and Enginoğlu [3] later redefined these operations to create a uni-int decision-making method. Aktas and Çağman [4] compared soft sets with fuzzy and rough sets. Many researchers have since explored properties and applications of soft set theory (see [5], [6], [7], [8], [9], [10], [11], [12], [13]).

Building on soft set theory, the concept of soft topology emerged as a new field of study to address topological structures within the framework of soft sets. Soft topology, first introduced by Shabir and Naz [14] in 2011, provides a topological structure by defining open and closed sets in terms of soft sets. Çağman [15] further developed this notion. This new form of topology has led to the exploration of various properties and concepts, such as soft open sets, soft continuity, and soft compactness, within a soft set framework. Soft topological spaces extend classical topology and have been applied in fields like decision-making, optimization, and computer science, providing a robust tool for analyzing spaces with uncertain or imprecise boundaries. Over time, researchers have refined and expanded the theory, introducing concepts such as soft minimal spaces and bipolar soft sets, allowing for even greater flexibility in addressing uncertainty in topological and decision-making processes. Numerous studies have since examined soft topological spaces, their properties, and their applications (see [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [14]). Thomas and John [27] expanded on this with the idea of soft minimal spaces ($\mathfrak{sms}s$), examining characteristics like compactness and separation axioms.

Bipolar soft set theory, an extension of soft set theory, was developed to handle situations where uncertainty arises from two opposing perspectives, often referred to as "positive" and "negative" information. Introduced by Shabir and Naz [28] in 2013, bipolar soft sets generalize the classical soft set by associating each parameter with two sets: one representing positive attributes and the other negative attributes. This dual representation makes bipolar soft sets particularly useful in decision-making processes where both favorable and unfavorable factors must be considered simultaneously. Bipolar soft set theory addresses limitations in traditional soft sets by capturing the bipolarity often present in real-world problems, such as preferences, opinions, and criteria that involve both pros and cons. This model has found applications in various fields, including decision analysis, medical diagnosis, and social sciences, where handling conflicting or dual-sided information is critical. Subsequent researchers, including Karaaslan and Karatas [29], explored operations like intersection, union, and complementation within bipolar soft sets, focusing on decision-making applications. Several studies have explored definitions, operations, and applications of bipolar soft sets (see [30], [31], [32]).

Bipolar soft topology builds on the foundation of bipolar soft set theory, introducing a topological framework that accommodates both positive and negative aspects of a given space. This concept allows for the study of topological properties such as continuity, compactness, and separation in a bipolar soft set context. By extending soft topological spaces to incorporate bipolarity, researchers have created a flexible tool for analyzing spaces characterized by dual information. The development of bipolar soft topological spaces has sparked further exploration into concepts like bipolar soft open and closed sets, bipolar soft continuity, and bipolar soft compactness. These advancements allow for a more nuanced understanding of topological structures where both positive and negative relationships between elements must be considered. Bipolar soft topology has potential applications in optimization, decision-making, and other areas where the interaction of opposing factors plays a key role. Further developments have expanded on bipolar soft topological spaces. Moreover, additional research has been conducted on the topological structures of bipolar soft sets, (see [33], [34], [35], [36], [37], [38], [39], [40], [41]). Furthermore, Öztürk [42] investigated closure and interior operations, as well as concepts like basis and subspaces within bipolar soft topological spaces. Musa and Asaad ([43], [44], [45]) presented a novel concept related to bipolar soft sets by extending the hypersoft sets. Additionally, they explored bipolar hypersoft topological spaces, examining various operations and properties associated with them.

The structure of the paper is as follows: Section 2 provides a brief overview of some relevant preliminaries. In Section 3, the new structure of bipolar soft topology, referred to as the bipolar soft minimal structure, is introduced. Additionally, important operators in bipolar soft minimal spaces, such as the $\tilde{\tilde{\mathfrak{m}}}$-interior, $\tilde{\tilde{\mathfrak{m}}}$-closure, and $\tilde{\tilde{\mathfrak{m}}}$-boundary, are explored. Section 4 presents the concepts of $\tilde{\tilde{\mathfrak{m}}}$-separated bipolar soft sets and bipolar soft $\tilde{\tilde{\mathfrak{m}}}$-connected sets, along with their key properties. In Section 5, a new concept of bipolar soft minimal spaces called bipolar soft minimal connected spaces is defined. It is also proven that the intersection of a pair of bipolar soft $\tilde{\tilde{\mathfrak{m}}}$-connected spaces over the common universal set results bipolar soft $\tilde{\tilde{\mathfrak{m}}}$-connected. Furthermore, it is shown that a bipolar soft $\tilde{\tilde{\mathfrak{m}}}$-connected space does not possess the $\tilde{\tilde{\mathfrak{m}}}$-hereditary property. Finally, Section 5 discusses various relations, properties, and examples of these new concepts of bipolar soft minimal spaces and concludes the paper.

2. Preliminaries

Throughout this paper, let Π be an initial universe and $\Sigma(\Pi)$ be the class of all subsets of Π. Let a nonempty set Γ be an entire set of parameters and let $\mu, \nu \subseteq \Gamma$. This section introduces some main notions and definitions on bipolar soft sets.

Definition 1. *[2] Let be a set of parameters. The* ***Not*** *set of the set* $\mu = \{\vartheta_1, \vartheta_2, ..., \vartheta_n\}$ *is denoted by* $\neg\mu = \{\neg\vartheta_1, \neg\vartheta_2, ..., \neg\vartheta_n\}$ *for all i. That is,* $\neg\vartheta_i =$ *Not* ϑ_i.

Definition 2. *[28] A 3-tuple* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is named a bipolar soft (*$\mathfrak{bs}$*) set on* Π*, where* $\ddot{\lambda} : \mu \longrightarrow \Sigma(\Pi)$ *is a mapping and* $\ddot{\zeta} : \neg\mu \longrightarrow \Sigma(\Pi)$ *is a mapping s. t.* $\ddot{\lambda}(\vartheta) \cap \ddot{\zeta}(\neg\vartheta) = \phi$ *for each* $\vartheta \in \mu$ *and* $\neg\vartheta \in \neg\mu$.
That means, a form of a $\mathfrak{bs}$ *set* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is:*

$$(\ddot{\lambda}, \ddot{\zeta}, \mu) = \{(\vartheta, \ddot{\lambda}(\vartheta), \ddot{\zeta}(\neg\vartheta)) : \vartheta \in \mu, \ddot{\lambda}(\vartheta) \cap \ddot{\zeta}(\neg\vartheta) = \phi\}.$$

The class of all bipolar soft sets over Π *along with* Γ *is denoted by* $\mathfrak{bss}(\Pi)$.

Definition 3. *[28] A null* $\mathfrak{bs}$ *set* $(\Phi, \tilde{\tilde{\Pi}}, \mu)$ *is a* $\mathfrak{bs}$ *set* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *if* $\ddot{\lambda}(\vartheta) = \phi$ *for each* $\vartheta \in \mu$ *and* $\ddot{\zeta}(\neg\vartheta) = \Pi$ *for each* $\neg\vartheta \in \neg\mu$.

Definition 4. *[28] A absolute* $\mathfrak{bs}$ *set* $(\ddot{\widetilde{\widetilde{\Pi}}}, \Phi, \mu)$ *is a* $\mathfrak{bs}$ *set* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *if* $\ddot{\lambda}(\vartheta) = \Pi$ *for all* $\vartheta \in \mu$ *and* $\ddot{\zeta}(\neg\vartheta) = \phi$ *for all* $\neg\vartheta \in \neg\mu$.

Definition 5. *[28] Given two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$. *Then* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *is a* $\mathfrak{bs}$ *subset of* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$, *written as* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \,\widetilde{\widetilde{\subseteq}}(\ddot{\lambda}_2, \ddot{\zeta}_2, \mu)$, *if:*

(i) $\mu \subseteq \nu$ *and,*

(ii) $\ddot{\lambda}_1(\vartheta) \subseteq \ddot{\lambda}_2(\vartheta)$ *and* $\ddot{\zeta}_2(\neg\vartheta) \subseteq \ddot{\zeta}_1(\neg\vartheta)$ *for all* $\vartheta \in \mu$ *and* $\neg\vartheta \in \neg\mu$.

Definition 6. *[28] Two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *are said to be equal, written as* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) = (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *if* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *is a* $\mathfrak{bs}$ *subset of* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is a* $\mathfrak{bs}$ *subset of* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

Definition 7. *[28] The complement of a* $\mathfrak{bs}$ *set* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is denoted by* $(\ddot{\lambda}, \ddot{\zeta}, \mu)^c$ *and defined by* $(\ddot{\lambda}, \ddot{\zeta}, \mu)^c$=$(\ddot{\lambda}^c, \ddot{\zeta}^c, \mu)$ *where* $\ddot{\lambda}^c$ *and* $\ddot{\zeta}^c$ *are mappings given by* $\ddot{\lambda}^c(\vartheta) = \ddot{\zeta}(\neg\vartheta)$ *and* $\ddot{\zeta}^c(\neg\vartheta) = \ddot{\lambda}(\vartheta)$ *for all* $\vartheta \in \mu$ *and* $\neg\vartheta \in \neg\mu$.

Definition 8. *[28] The* $\mathfrak{bs}$ *extended intersection between two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is the* $\mathfrak{bs}$ *set* (ξ, η, α) *where* $\alpha = \mu \cup \nu$ *and for each* $\vartheta \in \alpha$,

$$\xi(\vartheta) = \begin{cases} \ddot{\lambda}_1(\vartheta), & \vartheta \in \mu - \nu, \\ \ddot{\lambda}_2(\vartheta), & \vartheta \in \nu - \mu, \\ \ddot{\lambda}_1(\vartheta) \cap \ddot{\lambda}_2(\vartheta), & \vartheta \in \mu \cap \nu. \end{cases}$$

$$\eta(\neg\vartheta) = \begin{cases} \ddot{\zeta}_1(\neg\vartheta), & \neg\vartheta \in \neg\mu - \neg\nu, \\ \ddot{\zeta}_2(\neg\vartheta), & \neg\vartheta \in \neg\nu - \neg\mu, \\ \ddot{\zeta}_1(\neg\vartheta) \cup \ddot{\zeta}_2(\neg\vartheta), & \neg\vartheta \in \neg\mu \cap \neg\nu. \end{cases}$$

It represents by $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \,\widetilde{\widetilde{\sqcap}}\, (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) = (\xi, \eta, \alpha)$.

Definition 9. *[28] The* $\mathfrak{bs}$ *intersection between two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is the* $\mathfrak{bs}$ *set* (ξ, η, α) *where* $\alpha = \mu \cap \nu \neq \phi$ *and for all* $\vartheta \in \alpha$,

$$\xi(\vartheta) = \ddot{\lambda}_1(\vartheta) \cap \ddot{\lambda}_2(e) \text{ and } \eta(\neg\vartheta) = \ddot{\zeta}_1(\neg\vartheta) \cup \ddot{\zeta}_2(\neg\vartheta).$$

It represents by $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \,\widetilde{\widetilde{\cap}}\, (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) = (\xi, \eta, \alpha)$.

Definition 10. *[28] The* $\mathfrak{bs}$ *union between two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is the* $\mathfrak{bs}$ *set* (ξ, η, κ) *where* $\kappa = \mu \cup \nu$ *and for all* $\vartheta \in \kappa$,

$$\xi(\vartheta) = \begin{cases} \ddot{\lambda}_1(\vartheta), & \vartheta \in \mu - \nu, \\ \ddot{\lambda}_2(\vartheta), & \vartheta \in \nu - \mu, \\ \ddot{\lambda}_1(\vartheta) \cup \ddot{\lambda}_2(\vartheta), & \vartheta \in \mu \cap \nu \neq \phi. \end{cases}$$

$$\eta(\neg\vartheta) = \begin{cases} \ddot{\zeta}_1(\neg\vartheta), & \neg\vartheta \in \neg\mu - \neg\nu, \\ \ddot{\zeta}_2(\neg\vartheta), & \neg\vartheta \in \neg\nu - \neg\mu, \\ \ddot{\zeta}_1(\neg\vartheta) \cap \ddot{\zeta}_2(\neg\vartheta), & \neg\vartheta \in \neg\mu \cap \neg\nu \neq \phi. \end{cases}$$

It represents by $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) = (\xi, \eta, \alpha)$.

Definition 11. *[28] The* $\mathfrak{bs}$ *extended union between two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is the* $\mathfrak{bs}$ *set* (ξ, η, α) *where* $\alpha = \mu \cap \nu \neq \phi$ *and for all* $\vartheta \in \alpha$,

$$\xi(\vartheta) = \ddot{\lambda}_1(\vartheta) \cup \ddot{\lambda}_2(\vartheta) \text{ and } \eta(\neg\vartheta) = \ddot{\zeta}_1(\neg\vartheta) \cap \ddot{\zeta}_2(\neg\vartheta).$$

It represents by $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) = (\xi, \eta, \alpha)$.

Proposition 1. *[28] If* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$,$(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) \mathrel{\tilde{\tilde{\in}}} \mathfrak{bss}(\Pi)$*, then:*

(i) $((\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu))^c = (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)^c \mathrel{\tilde{\tilde{\cap}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)^c$.

(ii) $((\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu))^c = (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)^c \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)^c$.

(iii) $((\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)^c)^c = (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

(iv) $(\Phi, \tilde{\tilde{\Pi}}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)^c \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)^c \mathrel{\tilde{\tilde{\subseteq}}} (\tilde{\tilde{\Pi}}, \Phi, \mu)$.

Definition 12. *[34] The* $\mathfrak{bs}$ *difference between two* $\mathfrak{bs}$ *sets* $(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$ *and* $(\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)$ *is the* $\mathfrak{bs}$ *set* $(\ddot{\lambda}, \ddot{\zeta}, \kappa)$ *where* $\kappa = \mu \cup \nu$ *is defined as*

$$(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\setminus}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu) = (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\lambda}_2, \ddot{\zeta}_2, \nu)^c.$$

Definition 13. *[32] Let* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *be a* $\mathfrak{bs}$ *set over* Π. *The* $\mathfrak{bs}$ *set* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *is named a* $\mathfrak{bs}$ *point* ($\mathfrak{bsp}$) *if there is* $\alpha, \beta \in \Pi$, $\vartheta \in \mu$ *and* $\neg\vartheta \in \neg\mu$ *s. t.*

$$\ddot{\zeta}(\delta) = \begin{cases} \{\alpha\}, & \delta = \vartheta, \\ \phi, & \delta \in \mu \setminus \{\vartheta\}. \end{cases}$$

$$\ddot{\lambda}(\delta') = \begin{cases} \Pi \setminus \{\alpha, \beta\}, & \delta' = \neg\vartheta, \\ \Pi, & \delta' \in \neg\mu \setminus \{\neg\vartheta\}. \end{cases}$$

We denoted the $\mathfrak{bs}$ *point* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *by* α_β^ϑ, *and* $\mathfrak{bsp}(\Pi)_{(\mu,\neg\mu)}$ *is denoted by the class of all* $\mathfrak{bsp}$*s over* Π *by .*

Proposition 2. *[41] Let* $(\ddot{\lambda}, \ddot{\zeta}, \mu) \mathrel{\tilde{\tilde{\in}}} \mathfrak{bss}(\Pi)$. *Then*

(i) $(\ddot{\lambda}, \ddot{\zeta}, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\lambda}, \ddot{\zeta}, \mu)^c = (\xi, \Phi, \mu)$, *where* $\xi(\vartheta) = \ddot{\lambda}(\vartheta) \cup \ddot{\lambda}^c(\vartheta) \subseteq \Pi$ *for each* $\vartheta \in \mu$ *and* $\Phi(\neg\vartheta) = \ddot{\zeta}(\neg\vartheta) \cap \ddot{\zeta}^c(\neg\vartheta) = \phi$ *for each* $\neg\vartheta \in \neg\mu$.

(ii) $(\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\cap}} (\ddot{\lambda}, \ddot{\zeta}, \mu)^c = (\Phi, \eta, \mu)$, *where* $\Phi(\vartheta) = \ddot{\lambda}(\vartheta) \cap \ddot{\lambda}^c(\vartheta) = \phi$ *for each* $\vartheta \in \mu$ *and* $\eta(\neg\vartheta) = \ddot{\zeta}(\neg\vartheta) \cup \ddot{\zeta}^c(\neg\vartheta) \subseteq \Pi$ *for each* $\neg\vartheta \in \neg\mu$.
Further $(\ddot{\lambda}, \ddot{\zeta}, \mu)$, $(\ddot{\lambda}, \ddot{\zeta}, \mu)^c$ *will always satisfy* $\ddot{\lambda}(\vartheta) \cup \ddot{\lambda}^c(\vartheta) = \ddot{\zeta}(\neg\vartheta) \cup \ddot{\zeta}^c(\neg\vartheta)$ *for all* $\vartheta \in \mu$.

(iii) $(\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\cup}} (\tilde{\tilde{\Pi}}, \Phi, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\cap}} (\tilde{\tilde{\Pi}}, \Phi, \mu) = (\ddot{\lambda}, \ddot{\zeta}, \mu)$.

Definition 14. *A class* $\tilde{\mathfrak{m}}$ *of soft sets of* Π *is said to be soft minimal topology* $\tilde{\mathfrak{m}}$ *on* Π *if* $(\Phi, \mu) \tilde{\in} \tilde{\mathfrak{m}}$ *and* $(\tilde{\tilde{\Pi}}, \mu) \tilde{\in} \tilde{\mathfrak{m}}$. *The triple* $(\Pi, \tilde{\mathfrak{m}}, \mu)$ *is said to be a soft minimal space* ($\mathfrak{sms}$) *over* Π. *The members of* $\tilde{\mathfrak{m}}$ *are called soft* $\tilde{\mathfrak{m}}$*-open sets. The complement of soft* $\tilde{\mathfrak{m}}$*-open sets are soft* $\tilde{\mathfrak{m}}$*-closed.*

3. Bipolar Soft Minimal Structures

This section presents a new structure of bipolar soft topology called bipolar soft minimal structure. It presents some notions and properties of bipolar soft minimal structure.

Definition 15. *A class* $\tilde{\tilde{\mathfrak{m}}}$ *of* $\mathfrak{bs}$ *sets over* Π *with respect to* Γ *is said to be a bipolar soft minimal structure if* $(\Phi, \tilde{\tilde{\Pi}}, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$ *and* $(\tilde{\tilde{\Pi}}, \Phi, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$.

The quadruple $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is called a bipolar soft minimal space* ($\mathfrak{bsms}$). *Every member of* $\tilde{\tilde{\mathfrak{m}}}$ *is called a* $\tilde{\tilde{\mathfrak{m}}}$*-open set in* Π. *The* $\mathfrak{bs}$ *complement of a* $\tilde{\tilde{\mathfrak{m}}}$*-open set is said to be* $\tilde{\tilde{\mathfrak{m}}}$*-closed.*

Theorem 1. *If* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is a* $\mathfrak{bsms}$, *then* $\tilde{\mathfrak{m}} = \{(\ddot{\lambda}, \mu) : (\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}\}$ *is* $\mathfrak{sms}$.

Proof. Assume that $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is $\mathfrak{bsms}$. Then $(\Phi, \tilde{\tilde{\Pi}}, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$ and $(\tilde{\tilde{\Pi}}, \Phi, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$. This implies that $(\Phi, \mu) \tilde{\in} \tilde{\mathfrak{m}}$ and $(\tilde{\tilde{\Pi}}, \mu) \tilde{\in} \tilde{\mathfrak{m}}$. Hence $\tilde{\mathfrak{m}}$ defines a $\mathfrak{sms}$.

The converse of Theorem 1 is not correct as in the next example.

Example 1. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ *and* $\mu = \{\vartheta\}$.
Suppose that $\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \ddot{\zeta}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu)\}$, *where* $(\Phi, \ddot{\zeta}, \mu)$ *is a* $\mathfrak{bs}$ *set defined as follows* $(\Phi, \ddot{\zeta}, \mu) = \{(\vartheta, \phi, \{\epsilon_1, \epsilon_2\})$.
Then $\tilde{\mathfrak{m}} = \{(\Phi, \mu), (\tilde{\tilde{\Pi}}, \mu)\}$ *is* $\mathfrak{sms}$ *defined on* Π. *Meanwhile,* $\tilde{\tilde{\mathfrak{m}}}$ *is not* $\mathfrak{bsms}$.

Theorem 2. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$, *then* $\neg\tilde{\mathfrak{m}} = \{(\ddot{\zeta}, \neg\mu) : (\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}\}$ *is* $\mathfrak{sms}$.

Proof. Similar to Theorem 1.

The converse of Theorem 2 is not correct as in the next example.

Example 2. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ *and* $\mu = \{\vartheta\}$.
Suppose that $\tilde{\tilde{\mathfrak{m}}} = \{(\ddot{\zeta}, \Phi, \mu), (\Phi, \tilde{\tilde{\Pi}}, \mu)\}$, *where* $(\ddot{\zeta}, \Phi, \mu)$ *is a* $\mathfrak{bs}$ *set defined as follows* $(\ddot{\zeta}, \Phi, \mu) = \{(\vartheta, \{\epsilon_1\}, \phi)\}$.
Then $\neg\tilde{\mathfrak{m}} = \{(\Phi, \mu), (\tilde{\tilde{\Pi}}, \mu)\}$ *is* $\mathfrak{sms}$ *defined on* Π. *Meanwhile,* $\tilde{\tilde{\mathfrak{m}}}$ *is not* $\mathfrak{bsms}$.

Definition 16. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\in}}\, \mathfrak{bss}(\Pi)$. *The* $\tilde{\tilde{\mathfrak{m}}}$*-interior of* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is the* $\mathfrak{bs}$ *union of all* $\tilde{\tilde{\mathfrak{m}}}$*-open subsets of* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *and it is denoted by* $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu)$.

Properties of $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu)$ are as follows.

Theorem 3. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu)$,$(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \,\tilde{\tilde{\in}}\, \mathfrak{bss}(\Pi)$. *Then*

(i) $\tilde{\tilde{m}}Int\,(\Phi, \tilde{\tilde{\Pi}}, \mu) = (\Phi, \tilde{\tilde{\Pi}}, \mu)$ *and* $\tilde{\tilde{m}}Int\,(\tilde{\tilde{\Pi}}, \Phi, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$.

(ii) $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\lambda}, \ddot{\zeta}, \mu)$.

(iii) If $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is* $\tilde{\tilde{\mathfrak{m}}}$*-open, then* $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) = (\ddot{\lambda}, \ddot{\zeta}, \mu)$.

(iv) $\tilde{\tilde{m}}Int(\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu)) = \tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu)$.

(v) If $(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$, *then* $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{m}}Int(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

(vi) $\tilde{\tilde{m}}Int((\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)) \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\cap}}\, \tilde{\tilde{m}}Int(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

(vii) $\tilde{\tilde{m}}Int((\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)) \,\tilde{\tilde{\supseteq}}\, \tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\cup}}\, \tilde{\tilde{m}}Int(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

Proof. (i), (iv), (v), (vi) and (vii) Obvious.
(ii) Since $\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) = \tilde{\tilde{\bigcup}}\{(\ddot{\lambda}_j, \ddot{\zeta}_j, \mu) : (\ddot{\lambda}_j, \ddot{\zeta}_j, \mu) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{m}}},\ (\ddot{\lambda}_j, \ddot{\zeta}_j, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\lambda}, \ddot{\zeta}, \mu), j \in \mathcal{J}\}$. Then $\ddot{\lambda}_j(\vartheta) \subseteq \ddot{\lambda}(\vartheta)$ and $\ddot{\zeta}(\neg\vartheta) \subseteq \ddot{\zeta}_j(\neg\vartheta)$ for all $j \in \mathcal{J}$. So, $\bigcup_{j\in\mathcal{J}} \ddot{\lambda}_j(\vartheta) \subseteq \ddot{\lambda}(\vartheta)$ and $\ddot{\zeta}(\neg\vartheta) \subseteq \bigcap_{j\in\mathcal{J}} \ddot{\zeta}_j(\neg\vartheta)$. Therefore, $\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\lambda}, \ddot{\zeta}, \mu)$.
(iii) Let $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ be a $\tilde{\tilde{\mathfrak{m}}}$-open set. Then $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ is the $\mathfrak{bs}$ union of all $\tilde{\tilde{\mathfrak{m}}}$-open sets of $(\ddot{\lambda}, \ddot{\zeta}, \mu)$. From (ii), we have $\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\lambda}, \ddot{\zeta}, \mu)$. Therefore, $\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) = (\ddot{\lambda}, \ddot{\zeta}, \mu)$.

The converse of point (iii) in Theorem 3 is not true in general and the equality of parts (vi) and (vii) in Theorem 3 do not hold as shown in the example below.

Example 3. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta\}$ *and*
$\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu), (\ddot{\lambda}_2, \ddot{\zeta}_2, \mu), (\ddot{\lambda}_3, \ddot{\zeta}_3, \mu)\}$, *where*

$$(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) = \{(\vartheta, \{\epsilon_1\}, \{\epsilon_2\})\},$$
$$(\ddot{\lambda}_2, \ddot{\zeta}_2, \mu) = \{(\vartheta, \{\epsilon_1\}, \{\epsilon_3\})\},$$
$$(\ddot{\lambda}_3, \ddot{\zeta}_3, \mu) = \{(\vartheta, \{\epsilon_2\}, \{\epsilon_1\})\}.$$

For the converse of point (iii), let $(\ddot{\lambda}, \ddot{\zeta}, \mu) = \{(\vartheta, \{\epsilon_1\}, \phi)\}$, *then*
$\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu) = \tilde{\tilde{m}}Int\{(\vartheta, \{\epsilon_1\}, \phi)\} = \{(\vartheta, \{\epsilon_1\}, \phi)\}$.
But $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is not* $\tilde{\tilde{\mathfrak{m}}}$*-open. For the equality of parts (vi) and (vii), if we take* $(\xi_1, \eta_1, \mu) = \{(\vartheta, \{\epsilon_1, \epsilon_2\}, \{\epsilon_3\})\}$ *and* $(\xi_2, \eta_2, \mu) = \{(\vartheta, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\})\}$.
Then, $\tilde{\tilde{m}}Int(\xi_1, \eta_1, \mu) = (\ddot{\lambda}_2, \ddot{\zeta}_2, \mu)$ *and* $\tilde{\tilde{m}}Int(\xi_2, \eta_2, \mu) = (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)$.

Thus, $\widetilde{\widetilde{m}}Int(\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cap}}\ \widetilde{\widetilde{m}}Int(\xi_2,\eta_2,\mu)=\{(\vartheta,\{\epsilon_1\},\{\epsilon_2,\epsilon_3\})\}$.
Also, $\widetilde{\widetilde{m}}Int((\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cap}}\ (\xi_2,\eta_2,\mu))=\widetilde{\widetilde{m}}Int\{(\vartheta,\{\epsilon_1\},\{\epsilon_2,\epsilon_3\})\}=(\Phi,\widetilde{\widetilde{\Pi}},\mu)$.
Therefore,

$$\widetilde{\widetilde{m}}Int(\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cap}}\ \widetilde{\widetilde{m}}Int(\xi_2,\eta_2,\mu)\neq\widetilde{\widetilde{m}}Int((\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cap}}\ (\xi_2,\eta_2,\mu)).$$

Now, $\widetilde{\widetilde{m}}Int(\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cup}}\ \widetilde{\widetilde{m}}Int(\xi_2,\eta_2,\mu)=\{(\vartheta,\{\epsilon_1\},\phi)\}$.
Also, $\widetilde{\widetilde{m}}Int((\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cup}}\ (\xi_2,\eta_2,\mu))=\widetilde{\widetilde{m}}Int(\widetilde{\widetilde{\Pi}},\Phi,\mu)=(\widetilde{\widetilde{\Pi}},\Phi,\mu)$.
So, $\widetilde{\widetilde{m}}Int\left((\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cup}}\ (\xi_2,\eta_2,\mu)\right)\neq\widetilde{\widetilde{m}}Int(\xi_1,\eta_1,\mu)\ \widetilde{\widetilde{\cup}}\ \widetilde{\widetilde{m}}Int(\xi_2,\eta_2,\mu)$.

Definition 17. *Let* $(\Pi,\widetilde{\widetilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\in}}\ \mathfrak{bss}(\Pi)$. *Then the* $\widetilde{\widetilde{\mathfrak{m}}}$*-closure of* $(\ddot{\lambda},\ddot{\zeta},\mu)$ *is the* $\mathfrak{bs}$ *intersection of all* $\widetilde{\widetilde{\mathfrak{m}}}$*-closed sets containing* $(\ddot{\lambda},\ddot{\zeta},\mu)$ *and it is denoted by* $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.

Properties of $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$ are as follows.

Theorem 4. *Let* $(\Pi,\widetilde{\widetilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu),(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)\ \widetilde{\widetilde{\in}}\ \mathfrak{bss}(\Pi)$. *Then*

(i) $\widetilde{\widetilde{m}}Cl(\Phi,\widetilde{\widetilde{\Pi}},\mu)=(\Phi,\widetilde{\widetilde{\Pi}},\mu)$ *and* $\widetilde{\widetilde{m}}Cl(\widetilde{\widetilde{\Pi}},\Phi,\mu)=(\widetilde{\widetilde{\Pi}},\Phi,\mu)$.

(ii) $(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.

(iii) If $(\ddot{\lambda},\ddot{\zeta},\mu)$ *is* $\widetilde{\widetilde{\mathfrak{m}}}$*-closed, then* $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)=(\ddot{\lambda},\ddot{\zeta},\mu)$.

(iv) $\widetilde{\widetilde{m}}Cl(\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu))=\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.

(v) If $(\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\subseteq}}(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)$, *then* $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)$.

(vi) $\widetilde{\widetilde{m}}Cl((\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\cap}}\ (\ddot{\lambda}_1,\ddot{\zeta}_1,\mu))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\cap}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)$.

(vii) $\widetilde{\widetilde{m}}Cl((\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\cup}}\ (\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)\ \widetilde{\widetilde{\supseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\cup}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)$.

Proof. (i), (iv), (v), (vi) and (vii) Obvious.
(ii) Since $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)=\widetilde{\widetilde{\bigcap}}\{(\ddot{\lambda}_i,\ddot{\zeta}_i,\mu):(\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)^c\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{m}}},(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\subseteq}}\ (\ddot{\lambda}_i,\ddot{\zeta}_i,\mu),i\in\mathcal{I}\}$. Then $\ddot{\lambda}(\vartheta)\subseteq\ddot{\lambda}_i(\vartheta)$ and $\ddot{\zeta}_i(\neg\vartheta)\subseteq\ddot{\zeta}(\neg\vartheta)$ for all $i\in\mathcal{I}$. So, $\ddot{\lambda}(\vartheta)\subseteq\bigcap_{i\in\mathcal{I}}\ddot{\zeta}_i(\vartheta)$ and $\bigcup_{i\in\mathcal{I}}\ddot{\zeta}_i(\neg\vartheta)\subseteq\ddot{\zeta}(\neg\vartheta)$. Thus, $(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.
(iii) Let $(\ddot{\lambda},\ddot{\zeta},\mu)$ be a $\widetilde{\widetilde{\mathfrak{m}}}$-closed set. Then $(\ddot{\lambda},\ddot{\zeta},\mu)$ is the $\mathfrak{bs}$ intersection of all $\widetilde{\widetilde{\mathfrak{m}}}$-closed sets containing $(\ddot{\lambda},\ddot{\zeta},\mu)$. From (ii), we have $(\ddot{\lambda},\ddot{\zeta},\mu)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$. Therefore, $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)=(\ddot{\lambda},\ddot{\zeta},\mu)$.

The converse of point (iii) in Theorem 4 is not true in general and the equality of parts (vi) and (vii) in Theorem 4 do not hold as shown in the example below.

Example 4. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta\}$ *and*
$\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu), (\ddot{\lambda}_2, \ddot{\zeta}_2, \mu), (\ddot{\lambda}_3, \ddot{\zeta}_3, \mu)\}$, *where*

$$\begin{aligned}(\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) &= \{(\vartheta, \{\epsilon_1, \epsilon_2\}, \{\epsilon_3\})\},\\ (\ddot{\lambda}_2, \ddot{\zeta}_2, \mu) &= \{(\vartheta, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1\})\},\\ (\ddot{\lambda}_3, \ddot{\zeta}_3, \mu) &= \{(\vartheta, \{\epsilon_3\}, \{\epsilon_2\})\}.\end{aligned}$$

Then

$$\tilde{\tilde{\mathfrak{m}}}^c = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\lambda}_4, \ddot{\zeta}_4, \mu), (\ddot{\lambda}_5, \ddot{\zeta}_5, \mu), (\ddot{\lambda}_6, \ddot{\zeta}_6, \mu)\},$$

where

$$\begin{aligned}(\ddot{\lambda}_4, \ddot{\zeta}_4, \mu) &= (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu)^c = \{(\vartheta, \{\epsilon_3\}, \{\epsilon_1, \epsilon_2\})\},\\ (\ddot{\lambda}_5, \ddot{\zeta}_5, \mu) &= (\ddot{\lambda}_2, \ddot{\zeta}_2, \mu)^c = \{(\vartheta, \{\epsilon_1\}, \{\epsilon_2, \epsilon_3\})\},\\ (\ddot{\lambda}_6, \ddot{\zeta}_6, \mu) &= (\ddot{\lambda}_3, \ddot{\zeta}_3, \mu)^c = \{(\vartheta, \{\epsilon_2\}, \{\epsilon_3\})\}.\end{aligned}$$

For the converse of point (iii), let $(\ddot{\lambda}, \ddot{\zeta}, \mu) = \{(\vartheta, \phi, \{\epsilon_2, \epsilon_3\})\}$, *then*
$\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu) = \tilde{\tilde{m}}Cl\{(\vartheta, \phi, \{\epsilon_2, \epsilon_3\})\} = \{(\vartheta, \phi, \{\epsilon_2, \epsilon_3\})\}$.
But $(\ddot{\lambda}, \ddot{\zeta}, \mu)$ *is not* $\tilde{\tilde{\mathfrak{m}}}$*-closed.*
For the equality of parts (vi) and (vii), suppose that $(\xi_1, \eta_1, \mu) = \{(\vartheta, \{\epsilon_3\}, \{\epsilon_2\})\}$, $(\xi_2, \eta_2, \mu) = \{(\vartheta, \{\epsilon_2\}, \{\epsilon_1, \epsilon_3\})\}$ *and* $(\xi_3, \eta_3, \mu) = (\ddot{\lambda}_4, \ddot{\zeta}_4, \mu)$.
Then, $\tilde{\tilde{m}}Cl(\xi_1, \eta_1, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$, $\tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) = (\ddot{\lambda}_6, \ddot{\zeta}_6, \mu)$ *and* $\tilde{\tilde{m}}Cl(\xi_3, \eta_3, \mu) = (\ddot{\lambda}_4, \ddot{\zeta}_4, \mu)$.
Thus, $\tilde{\tilde{m}}Cl(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) = (\ddot{\lambda}_6, \ddot{\zeta}_6, \mu)$.
Also, $\tilde{\tilde{m}}Cl((\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} (\xi_2, \eta_2, \mu)) = \tilde{\tilde{m}}Cl(\Phi, \tilde{\tilde{\Pi}}, \mu) = (\Phi, \tilde{\tilde{\Pi}}, \mu)$.
Therefore,

$$\tilde{\tilde{m}}Cl(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) \neq \tilde{\tilde{m}}Cl((\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} (\xi_2, \eta_2, \mu)).$$

Now, $\tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) \tilde{\tilde{\cup}} \tilde{\tilde{m}}Cl(\xi_3, \eta_3, \mu) = \{(\vartheta, \{\epsilon_2, \epsilon_3\}, \phi)\}$.
Also, $\tilde{\tilde{m}}Cl((\xi_2, \eta_2, \mu) \tilde{\tilde{\cup}} (\xi_3, \eta_3, \mu)) = \tilde{\tilde{m}}Cl\{(\vartheta, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1\})\} = (\tilde{\tilde{\Pi}}, \Phi, \mu)$.
So, $\tilde{\tilde{m}}Cl((\xi_2, \eta_2, \mu) \tilde{\tilde{\cup}} (\xi_3, \eta_3, \mu)) \neq \tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) \tilde{\tilde{\cup}} \tilde{\tilde{m}}Cl(\xi_3, \eta_3, \mu)$.

Proposition 3. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$. *Then*
$\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\subseteq}} (\ddot{\lambda}, \ddot{\zeta}, \mu) \tilde{\tilde{\subseteq}} \tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)$.

Proof. Follows directly from Theorem 3 (ii) and Theorem 4 (ii).

Theorem 5. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu), (\ddot{\lambda}_1, \ddot{\zeta}_1, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$. *Then*

(i) $\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c = (\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu))^c$.

(ii) $\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)^c = (\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu))^c$.

(iii) $\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu) = (\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)^c)^c$.

(iv) $\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu) = (\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)^c)^c$.

(v) $\widetilde{\widetilde{m}}Int\,((\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\setminus}}\;(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu))\;\widetilde{\widetilde{\subseteq}}\;\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\setminus}}\;\widetilde{\widetilde{m}}Int\,(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)$.

Proof. We prove the parts (i) and (v) because the proof of the remaining points are similar.

(i) Since $(\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu))^c = (\widetilde{\widetilde{\bigcup}}\{(\ddot{\lambda}_i,\ddot{\zeta}_i,\mu) : (\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)\;\widetilde{\widetilde{\in}}\;\widetilde{\widetilde{\mathfrak{m}}},\ (\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)\;\widetilde{\widetilde{\subseteq}}\;(\ddot{\lambda},\ddot{\zeta},\mu),\ i\in\mathcal{I}\})^c$ $= \widetilde{\widetilde{\bigcap}}\;\{(\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)^c\colon\ (\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)\;\widetilde{\widetilde{\in}}\;\widetilde{\widetilde{\mathfrak{m}}},\ (\ddot{\lambda},\ddot{\zeta},\mu)^c\;\widetilde{\widetilde{\subseteq}}\;(\ddot{\lambda}_i,\ddot{\zeta}_i,\mu)^c,\ i\in\mathcal{I}\} = \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)^c$.

(v) Since

$$\begin{aligned}
\widetilde{\widetilde{m}}Int\,((\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\setminus}}(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)) &= \widetilde{\widetilde{m}}Int((\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\cap}}\,(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)^c)\\
&\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\cap}}\,\widetilde{\widetilde{m}}Int\,(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu)^c\,(by\ Theorem\,3(vi))\\
&= \widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\,\widetilde{\widetilde{\cap}}(\widetilde{\widetilde{m}}Cl(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu))^c\\
&\widetilde{\widetilde{\subseteq}}\;\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\,\widetilde{\widetilde{\cap}}(\widetilde{\widetilde{m}}Int\,(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu))^c\\
&= \widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\widetilde{\widetilde{\setminus}}\widetilde{\widetilde{m}}Int\,(\ddot{\lambda}_1,\ddot{\zeta}_1,\mu).
\end{aligned}$$

Definition 18. *Let* $(\Pi,\widetilde{\widetilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\in}}\;\mathfrak{bss}(\Pi)$. *Then the* $\widetilde{\widetilde{\mathfrak{m}}}$*-boundary of* $(\ddot{\lambda},\ddot{\zeta},\mu)$*, denoted by* $b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)$*, is defined as*

$$b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu) = \widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\cap}}\;\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)^c.$$

Proposition 4. *It is clear that* $b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu) = b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)^c$.

Proof. Follows directly from Definition 18.

Theorem 6. *Let* $(\Pi,\widetilde{\widetilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\in}}\;\mathfrak{bss}(\Pi)$. *Then*

(i) $b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\subseteq}}\;\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.

(ii) $(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\cup}}\;b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\subseteq}}\;\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$.

(iii) $\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\subseteq}}\;(\ddot{\lambda},\ddot{\zeta},\mu)\;\widetilde{\widetilde{\setminus}}\;b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)$.

(iv) $b_{\widetilde{\widetilde{\mathfrak{m}}}}(\widetilde{\widetilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu))\;\widetilde{\widetilde{\subseteq}}\;b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)$.

(v) $b_{\widetilde{\widetilde{\mathfrak{m}}}}(\widetilde{\widetilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu))\;\widetilde{\widetilde{\subseteq}}\;b_{\widetilde{\widetilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)$.

Proof.

(i) Since $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu) = \tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu) \;\tilde{\tilde{\cap}}\; \tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c$. Then $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu) \;\tilde{\tilde{\subseteq}}\; \tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)$.

(ii)

$$\begin{aligned}
(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\, b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu) &= (\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\,(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c) \\
&= ((\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\,\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu))\tilde{\tilde{\cap}}((\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\,\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c) \\
&= \tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}((\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\,\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c) \\
&\tilde{\tilde{\subseteq}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu).
\end{aligned}$$

(iii)

$$\begin{aligned}
(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\setminus}} b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu) &= (\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}(b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu))^c \\
&= (\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\,\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c)^c \\
&= (\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}(\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu)^c\tilde{\tilde{\cup}}\,\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu)) \\
&= ((\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu)^c)\tilde{\tilde{\cup}}\,((\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu)) \\
&= (\Phi, \ddot{\zeta}, \mu)\tilde{\tilde{\cup}}\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) \\
&\tilde{\tilde{\supseteq}}\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu).
\end{aligned}$$

(iv)

$$\begin{aligned}
b_{\tilde{\tilde{\mathfrak{m}}}}(\tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu)) &= \tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu))\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu))^c \\
&= \tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Int(\ddot{\lambda}, \ddot{\zeta}, \mu))\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c) \\
&\tilde{\tilde{\subseteq}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c \\
&= b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu).
\end{aligned}$$

(v)

$$\begin{aligned}
b_{\tilde{\tilde{\mathfrak{m}}}}(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)) &= \tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu))\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu))^c \\
&\tilde{\tilde{\subseteq}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\ddot{\lambda}, \ddot{\zeta}, \mu)^c \\
&= b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu).
\end{aligned}$$

Theorem 7. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda}, \ddot{\zeta}, \mu) \;\tilde{\tilde{\in}}\; \mathfrak{bss}(\Pi)$.
Then $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda}, \ddot{\zeta}, \mu) \;\tilde{\tilde{\cap}}\; \tilde{\tilde{m}}Int\,(\ddot{\lambda}, \ddot{\zeta}, \mu) = (\Phi, \ddot{\zeta}, \mu)$.

Proof. We start by

$$
\begin{aligned}
& b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\\
&= (\tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\setminus}}\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu))\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\\
&= \tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)^{c}\tilde{\tilde{\cap}}\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\\
&= \tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}(\Phi,\ddot{\zeta},\mu)\\
&= (\Phi,\ddot{\zeta},\mu).
\end{aligned}
$$

Theorem 8. *Let* $(\Pi,\tilde{\tilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\in}}\ \mathfrak{bss}(\Pi)$. *The following points are true:*

(i) If $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{m}}}$*, then* $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\cap}}\ b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)=(\Phi,\ddot{\zeta},\mu)$.

(ii) If $(\ddot{\lambda},\ddot{\zeta},\mu)$ *is* $\tilde{\tilde{\mathfrak{m}}}$*-closed, then* $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\subseteq}}\ (\ddot{\lambda},\ddot{\zeta},\mu)$.

Proof.

(i) Suppose $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{m}}}$. Then by Theorem 3 (iii), we have $(\ddot{\lambda},\ddot{\zeta},\mu)=\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)$. Since $\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\subseteq}}\ (b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu))^{c}$. That means $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\subseteq}}\ (b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu))^{c}$. Therefore, $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\cap}}\ b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)=(\Phi,\ddot{\zeta},\mu)$.

(ii) By Theorem 6 (i), we have $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\subseteq}}\ \tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)$. Since $(\ddot{\lambda},\ddot{\zeta},\mu)$ is an $\tilde{\tilde{\mathfrak{m}}}$-closed set, then $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\subseteq}}(\ddot{\lambda},\ddot{\zeta},\mu)$.

Theorem 9. *Let* $(\Pi,\tilde{\tilde{\mathfrak{m}}},\mu,\neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\lambda},\ddot{\zeta},\mu)\ \tilde{\tilde{\in}}\ \mathfrak{bss}(\Pi)$ *If* $(\ddot{\lambda},\ddot{\zeta},\mu)$ *is both* $\tilde{\tilde{\mathfrak{m}}}$*-open and* $\tilde{\tilde{\mathfrak{m}}}$*-closed. Then* $b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu)=(\Phi,\ddot{\zeta},\mu)$.

Proof. Suppose that $(\ddot{\lambda},\ddot{\zeta},\mu)$ is $\tilde{\tilde{\mathfrak{m}}}$-open and $\tilde{\tilde{\mathfrak{m}}}$-closed. Then,

$$
\begin{aligned}
b_{\tilde{\tilde{\mathfrak{m}}}}(\ddot{\lambda},\ddot{\zeta},\mu) &= \tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}\tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)^{c}\\
&= \tilde{\tilde{m}}Cl(\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}(\tilde{\tilde{m}}Int\,(\ddot{\lambda},\ddot{\zeta},\mu))^{c}\\
&= (\ddot{\lambda},\ddot{\zeta},\mu)\tilde{\tilde{\cap}}(\ddot{\lambda},\ddot{\zeta},\mu)^{c}\\
&= (\Phi,\ddot{\zeta},\mu).
\end{aligned}
$$

4. $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-Connected Sets

This section presents $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$ using $\mathfrak{bsms}$ and it gives some of their properties. Also, it presents $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected sets in terms of $\mathfrak{bsms}$ and it obtains some properties and relations.

Definition 19. *Let* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *be two* $\mathfrak{bss}s$ *in* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *which are not null. Then* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *are named* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *(* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$*) if* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \vartheta) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \vartheta) = (\Phi, \ddot{\lambda}, \vartheta)$ *and* $\tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \vartheta) \tilde{\tilde{\cap}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \vartheta) = (\Phi, \ddot{\lambda}, \vartheta)$.

Proposition 5. *Any two* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *are disjoint* $\mathfrak{bss}s$*.*

Proof. Obvious.

Note that disjoint $\mathfrak{bss}s$ may not be $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$; it means that the converse of Proposition 5 does not true as shown by the next example.

Example 5. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, $\mu = \{\vartheta_1, \vartheta_2\}$ *and* $\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)\}$ *be a* $\mathfrak{bsms}$ *over* Π *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$,$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$,$(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$, *defined as follows*

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \{(\vartheta_1, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1\}), (\vartheta_2, \{\epsilon_3, \epsilon_4\}, \{\epsilon_2\})\},$$
$$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \{(\vartheta_1, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2, \epsilon_4\}), (\vartheta_2, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\})\},$$
$$(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) = \{(\vartheta_1, \{\epsilon_1, \epsilon_2, \epsilon_3\}, \phi), (\vartheta_2, \{\epsilon_1, \epsilon_3, \epsilon_4\}, \{\epsilon_2\})\}.$$

Now, assume that (ξ_1, η_1, μ) *and* (ξ_2, η_2, μ) *are disjoint* $\mathfrak{bss}s$ *over* Π *defined by*

$$(\xi_1, \eta_1, \mu) = \{(\vartheta_1, \{\epsilon_1, \epsilon_2, \epsilon_4\}, \{\epsilon_3\}), (\vartheta_2, \{\epsilon_1, \epsilon_2, \epsilon_4\}, \{\epsilon_3\})\} \text{ and}$$
$$(\xi_2, \eta_2, \mu) = \{(\vartheta_1, \{\epsilon_3\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_3\}, \{\epsilon_2\})\}.$$

Then $\tilde{\tilde{m}}Cl(\xi_1, \eta_1, \mu) = \tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$ *and* $(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl(\xi_2, \eta_2, \mu) = (\xi_1, \eta_1, \mu)$, $\tilde{\tilde{m}}Cl(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} (\xi_2, \eta_2, \mu) = (\xi_2, \eta_2, \mu)$*. But* $(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} (\xi_2, \eta_2, \mu) = (\Phi, \eta, \mu)$*. Thus,* $\mathfrak{bss}s$ (ξ_1, η_1, μ), (ξ_2, η_2, μ) *are disjoint* $\mathfrak{bss}s$ *but not* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$*.*

Proposition 6. *If* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *are two* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *over* Π *with* $(\xi_1, \eta_1, \mu) \tilde{\tilde{\subseteq}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\xi_2, \eta_2, \mu) \tilde{\tilde{\subseteq}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$*. Then,* (ξ_1, η_1, μ) *and* (ξ_2, η_2, μ) *also are* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *over* Π*.*

Proof. Given $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$ $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$. Then

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \tilde{\tilde{\cap}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu).$$

Since $(\xi_1, \eta_1, \mu) \tilde{\tilde{\subseteq}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\xi_2, \eta_2, \mu) \tilde{\tilde{\subseteq}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, then $\tilde{\tilde{m}}Cl\,(\xi_1, \eta_1, \mu) \tilde{\tilde{\subseteq}} \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $\tilde{\tilde{m}}Cl\,(\xi_2, \eta_2, \mu) \tilde{\tilde{\subseteq}} \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$. Therefore,

$$(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} \tilde{\tilde{m}}Cl\,(\xi_2, \eta_2, \mu) = \tilde{\tilde{m}}Cl\,(\xi_1, \eta_1, \mu) \tilde{\tilde{\cap}} (\xi_2, \eta_2, \mu) = (\Phi, \eta, \mu).$$

Hence, (ξ_1, η_1, μ) and (ξ_2, η_2, μ) are $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$ over Π.

Theorem 10. *Two* $\tilde{\tilde{\mathfrak{m}}}$*-closed subsets* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *of* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *over* Π *are* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *if and only if they are disjoint* $\mathfrak{bss}s$.

Proof. The first condition is obvious. Conversely, assume that $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are disjoint $\tilde{\tilde{\mathfrak{m}}}$-closed. So, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu)$ and $\tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $\tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ and hence

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu)$$

showing that $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$ over Π.

Remark 1. *Two disjoint* $\tilde{\tilde{\mathfrak{m}}}$*-open sets* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *need not be* $\tilde{\tilde{\mathfrak{m}}}$*-separated.*

Example 6. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, $\mu = \{\vartheta_1, \vartheta_2\}$ *and* $\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)\}$ *be a* $\mathfrak{bsms}$ *over* Π *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, $(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)$ $\tilde{\tilde{\in}}$ $\mathfrak{bss}(\Pi)$, *defined as follows*

$$\begin{aligned}
(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) &= \{(\vartheta_1, \{\epsilon_2\}, \{\epsilon_1, \epsilon_4\}), (\vartheta_2, \{\epsilon_2\}, \{\epsilon_1, \epsilon_4\})\},\\
(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) &= \{(\vartheta_1, \{\epsilon_3\}, \{\epsilon_1, \epsilon_4\}), (\vartheta_2, \{\epsilon_3\}, \{\epsilon_1, \epsilon_4\})\},\\
(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) &= \{(\vartheta_1, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1, \epsilon_4\}), (\vartheta_2, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1, \epsilon_4\})\}.
\end{aligned}$$

Obviously, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *are disjoint* $\tilde{\tilde{\mathfrak{m}}}$*-open but not* $\tilde{\tilde{\mathfrak{m}}}$*-separated as* $\tilde{\tilde{m}}Cl(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \tilde{\tilde{m}}Cl(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$, *which implies that* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, \tilde{\tilde{m}}Cl(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $\tilde{\tilde{m}}Cl(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$. *Hence the conclusion.*

Definition 20. *A* $\mathfrak{bs}$ *subset* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *of* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *over* Π *is called* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected over* Π *if there are no a* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$ *of* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$. *Otherwise, a* $\mathfrak{bs}$ *set* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *is called* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected over* Π.

Remark 2. *The set* $(\Phi, \tilde{\tilde{\Pi}}, \mu)$ *is always* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected. Also, every* $\mathfrak{bss}$ *in which* α_β^ϑ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected as it cannot written as a* $\mathfrak{bs}$ *union of a pair of nonnull* $\tilde{\tilde{\mathfrak{m}}}$*-separated sets.*

Definition 21. *Let* α_β^ϑ, $\alpha_{\beta'}^{\prime\mu'}$ $\tilde{\tilde{\in}}$ $\mathfrak{bsp}(\Pi)_{(\mu,\neg\mu)}$ *of a* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$. *Then,* α_β^ϑ *and* $\alpha_{\beta'}^{\prime\mu'}$ *are called* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected points if they are contained in* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected set over* Π.

Proposition 7. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *over* Π *and* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *be a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected set s.t.* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *are* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$. *Then* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *or* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$.

Proof. From $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$, then $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, \tilde{\tilde{m}}Cl(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu)$ and $\tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu)$. Since $(\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, then $(\ddot{\zeta}, \ddot{\lambda}, \mu) = (\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\cap}}\, \big((\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big) = \big((\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)\big) \,\tilde{\tilde{\cup}}$

$((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu))$. We states that at least one of the $\mathfrak{bss}s$ $\big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)\big)$ and $\big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big)$ is null $\mathfrak{bs}$. Now, suppose that if possible non of these $\mathfrak{bss}s$ is null, hence,

$$(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \neq (\Phi, \ddot{\lambda}, \mu) \text{ and } (\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \neq (\Phi, \ddot{\lambda}, \mu).$$

Thus,

$$\begin{aligned}
((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}}\ & (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)) \mathrel{\tilde{\tilde{\cap}}} \tilde{\tilde{m}}Cl\big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big) \\
&\mathrel{\tilde{\tilde{\subseteq}}} \big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)\big) \mathrel{\tilde{\tilde{\cap}}} \big(\tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big) \\
&= \big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu)\big) \mathrel{\tilde{\tilde{\cap}}} \big((\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big) \\
&= (\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\Phi, \ddot{\lambda}, \mu) \\
&= (\Phi, \ddot{\lambda}, \mu).
\end{aligned}$$

Similarly,

$$\tilde{\tilde{m}}Cl\big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)\big) \mathrel{\tilde{\tilde{\cap}}} \big((\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\big) = (\Phi, \ddot{\lambda}, \mu).$$

Therefore, $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$. Thus, $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ can be expressed as $\mathfrak{bs}$ union of a pair of $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$. So, $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected. Which is contradiction. Hence, at least one of the $\mathfrak{bss}s$ $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ is null $\mathfrak{bss}$. Now, if $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = (\Phi, \ddot{\lambda}, \mu)$, then $(\ddot{\zeta}, \ddot{\lambda}, \mu) = (\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ which implies that $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$. If $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu)$, then $(\ddot{\zeta}, \ddot{\lambda}, \mu) = (\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ which implies that $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$. Therefore, either $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ or $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$.

Proposition 8. *Let* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *be* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected and* $(\xi, \eta, \mu) \mathrel{\tilde{\tilde{\in}}} \mathfrak{bss}(\Pi)$ *s. t.* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\xi, \eta, \mu) \mathrel{\tilde{\tilde{\subseteq}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu)$. *Then* (ξ, η, μ) *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected. In particular,* $\tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *is also* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Suppose that (ξ, η, μ) is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected. Then, there exist nonnull $\mathfrak{bss}s$ $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ in which

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \tilde{\tilde{m}}Cl\,(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu) \text{ and}$$
$$(\xi, \eta, \mu) = (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu).$$

From $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\xi, \eta, \mu) = (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cup}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, it follows from Proposition 7 that $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ or $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$. Let $(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ thus, $c_{\tilde{\tilde{\mathfrak{m}}}}\tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\subseteq}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ then,

$$\tilde{\tilde{m}}Cl(\ddot{\zeta}, \ddot{\lambda}, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \mathrel{\tilde{\tilde{\subseteq}}} \tilde{\tilde{m}}Cl(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \mathrel{\tilde{\tilde{\cap}}} (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = (\Phi, \ddot{\lambda}, \mu),$$

but $(\Phi,\ddot{\lambda},\mu)\;\tilde{\tilde{\subseteq}}\;\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$, therefore,

$$\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)=(\Phi,\ddot{\lambda},\mu).$$

So, $(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)\;\tilde{\tilde{\cup}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)=(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)$ then, $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)\;\tilde{\tilde{\subseteq}}\;(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)$ implies that $\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)=(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$. Hence, $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)=(\Phi,\ddot{\lambda},\mu)$. This is a contradiction because $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$ is nonnull $\mathfrak{bss}$. Therefore, (ξ,η,μ) is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected. Also, from $(\ddot{\zeta},\ddot{\lambda},\mu)\;\tilde{\tilde{\subseteq}}\;(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)$, implies that $\tilde{\tilde{m}}Cl(\ddot{\zeta},\ddot{\lambda},\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected.

Proposition 9. *Let* $\{(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu):\delta\in\Delta\}$ *be the collection of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected sets s. t.* $\tilde{\tilde{\bigcap}}_{\delta\in\delta}(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\neq(\Phi,\ddot{\lambda},\mu)$*. Then* $\tilde{\tilde{\bigcup}}_{\delta\in\Delta}(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Assume $(\xi,\eta,\mu)=\tilde{\tilde{\bigcup}}_{\delta\in\Delta}(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ is not $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected. Thus, there exist two nonnull disjoint $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-open sets (ξ_1,η_1,μ) and (ξ_2,η_2,μ) s. t. $(\xi,\eta,\mu)=(\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cup}}\;(\xi_2,\eta_2,\mu)$. For each $\delta\in\Delta$, $(\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ and $(\xi_2,\eta_2,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ are disjoint $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-open sets in $(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ in which

$$\begin{aligned}&\big((\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\big)\;\tilde{\tilde{\cup}}\;\big((\xi_2,\eta_2,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\big)\\&=\big((\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cup}}\;(\xi_2,\eta_2,\mu)\big)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)=(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu).\end{aligned}$$

Now, from $(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected set, one of the $\mathfrak{bss}s$ $(\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ and $(\xi_2,\eta_2,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ is a null $\mathfrak{bss}s$, say, $(\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)=(\Phi,\eta,\mu)$. Then, $(\xi_2,\eta_2,\mu)\;\tilde{\tilde{\cap}}\;(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)=(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)$ which implies that $(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\;\tilde{\tilde{\subseteq}}\;(\xi_2,\eta_2,\mu)$ for all $\delta\in\Delta$ and hence $\tilde{\tilde{\bigcup}}_{\delta\in\Delta}(\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\;\tilde{\tilde{\subseteq}}\;(\xi_2,\eta_2,\mu)$, that is, $(\xi_1,\eta_1,\mu)\;\tilde{\tilde{\cup}}\;(\xi_2,\eta_2,\mu)\;\tilde{\tilde{\subseteq}}\;(\xi_2,\eta_2,\mu)$. This given, $(\xi_1,\eta_1,\mu)=(\Phi,\eta,\mu)$. This is a contradiction because (ξ_1,η_1,μ) is nonnull $\mathfrak{bss}$. Hence, (ξ,η,μ) is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected.

Proposition 10. *For any two* $\mathfrak{bsp}s$ $\alpha^{\vartheta}_{\beta}$, $\alpha'^{\mu'}_{\beta'}\;\tilde{\tilde{\in}}\;(\ddot{\zeta},\ddot{\lambda},\mu)\;\tilde{\tilde{\in}}\;\mathfrak{bss}(\Pi)$ *in a* $\mathfrak{bsms}$ $(\Pi,\tilde{\tilde{\mathfrak{m}}},\mu,\neg\mu)$ *are contained in some* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected set* $(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;(\ddot{\zeta},\ddot{\lambda},\mu)$*. Then* $(\ddot{\zeta},\ddot{\lambda},\mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Let $(\ddot{\zeta},\ddot{\lambda},\mu)$ be a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected set. Thus, there exist a $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$ $(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$ and $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$ of $(\ddot{\zeta},\ddot{\lambda},\mu)$. Then, there are two $\mathfrak{bsp}s$ $\alpha^{\vartheta}_{\beta}$, $\alpha'^{\mu'}_{\beta'}$ in which $\alpha^{\vartheta}_{\beta}\;\tilde{\tilde{\in}}\;(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$ and $\alpha'^{\mu'}_{\beta'}\;\tilde{\tilde{\in}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$. By using the assumption,, there is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected set (ξ,η,μ) containing $\alpha^{\vartheta}_{\beta}$, $\alpha'^{\mu'}_{\beta'}$ s. t.

$$(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;(\ddot{\zeta},\ddot{\lambda},\mu)=(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)\;\tilde{\tilde{\cup}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu).$$

Thus, by Proposition 7, we have $(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$ or $(\xi,\eta,\mu)\;\tilde{\tilde{\subseteq}}\;(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$. This implies that

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \neq (\Phi, \ddot{\lambda}, \mu).$$

This is contradiction since $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\tilde{\tilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$. So, $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected.

Proposition 11. *Let* $\{(\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu) : \delta \in \Delta\}$ *be the class of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected sets s. t. one of the members of this class intersects every other member. Then,* $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Let $(\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu)$ be a fixed member of the given class s. t. $(\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cap}}\, (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)$ $\neq (\Phi, \ddot{\lambda}\mu)$ for every $\delta \in \Delta$. Then, $(\xi_\delta, \eta_\delta, \mu) = (\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected for each $\delta \in \Delta$, hence by Proposition 10. Now,

$$\begin{aligned}\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu) &= \tilde{\tilde{\bigcup}}_{\delta\in\Delta} \left((\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)\right) \\ &= (\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cup}}\, \left(\tilde{\tilde{\bigcup}}_{\delta\in\Delta}(\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)\right).\end{aligned}$$

Since $(\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu)$ is one of the family $\{(\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu) : \delta \in \Delta\}$ and

$$\begin{aligned}\tilde{\tilde{\bigcap}}_{\delta\in\Delta} (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu) &= \tilde{\tilde{\bigcap}}_{\delta\in\Delta} \left((\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)\right) \\ &= (\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \,\tilde{\tilde{\cap}}\, \left(\tilde{\tilde{\bigcup}}_{\delta\in\Delta}(\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)\right) \neq (\Phi, \tilde{\tilde{\Pi}}, \mu).\end{aligned}$$

From $(\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu)$ intersects every $(\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)$. Therefore, $(\ddot{\zeta}_{\delta_0}, \ddot{\lambda}_{\delta_0}, \mu) \neq (\Phi, \tilde{\tilde{\Pi}}, \mu)$. Hence, by Proposition 9, $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\ddot{\zeta}_\delta, \ddot{\lambda}_\delta, \mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected.

Proposition 12. *If* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected subset of a* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *s. t.* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \,\tilde{\tilde{\subseteq}}\, (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) \,\tilde{\tilde{\cup}}\, (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *are both* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-closed and nonnull disjoint* $\mathfrak{bss}s$*. Then,* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *are* $\tilde{\tilde{\mathfrak{m}}}$*-separated* $\mathfrak{bss}s$*.*

Proof. Follows directly from Proposition 7 and Theorem 10.

Proposition 13. *For each two* α_β^ϑ*,* $\alpha'^{\mu'}_{\beta'} \,\tilde{\tilde{\in}}\, \mathfrak{bsp}(\Pi)_{(\mu,\neg\mu)}$ *of a* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *are* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected, then* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Let α_β^ϑ be a fixed $\mathfrak{bsp}$ in a $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$. Then, for each α_β^ϑ $\mathfrak{bs}$ different than $\alpha'^{\mu'}_{\beta'}$, we have $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected, say, $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ containing α_β^ϑ and $\alpha'^{\mu'}_{\beta'}$. Since $\alpha_\beta^\vartheta \,\tilde{\tilde{\in}}$ $\tilde{\tilde{\bigcap}}_{\alpha_\beta^\vartheta \tilde{\tilde{\in}} (\tilde{\tilde{\Pi}}, \Phi, \mu)} (\ddot{\zeta}, \ddot{\lambda}, \mu)$, it follows from Proposition 9 that $\tilde{\tilde{\bigcup}}_{\alpha_\beta^\vartheta \tilde{\tilde{\in}} (\tilde{\tilde{\Pi}}, \Phi, \mu)} (\ddot{\zeta}, \ddot{\lambda}, \mu) = (\tilde{\tilde{\Pi}}, \Phi, \mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected.

5. $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-Connected Spaces

This section presents the concept of bipolar soft minimal connected ($\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected) space. Also, it discusses some properties and results of this new concept of $\mathfrak{bsms}$.

Definition 22. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$. *A* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ *is defined to be the nonnull disjoint* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-open sets* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *s. t.* $\ddot{\zeta}_1(\vartheta) \cup \ddot{\zeta}_1(\vartheta) = \Pi$ *for each* $\vartheta \in \mu$.

Definition 23. *A* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is called* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected if* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ *has no* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation. That is, there exist no nonnull disjoint* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-open sets* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *s. t.* $\ddot{\zeta}_1(\mu) \cup \ddot{\zeta}_2(\mu) = \Pi$ *for all* $\mu \in \mu$. *Otherwise,* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is said to be* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected. Note that if* $\mid \Pi \mid= 1$*, there are only two* $\mathfrak{bsms}$ *in* Π *(that is,* $(\Phi, \tilde{\tilde{\Pi}}, \mu)$, $(\tilde{\tilde{\Pi}}, \Phi, \mu)$*) are a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected space. Hence, we suppose* $\mid \Pi \mid> 1$.

Example 7. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, $\mu = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ *and* $\neg\mu = \{\neg\vartheta_1, \neg\vartheta_2, \neg\vartheta_3\}$. *Suppose that* $\tilde{\tilde{\mathfrak{m}}} = \{(\tilde{\tilde{\Pi}}, \Phi, \mu), (\Phi, \tilde{\tilde{\Pi}}, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)\}$ *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, $(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$ *defined as follows:*

$$
\begin{aligned}
(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) &= \{(\vartheta_1, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1, \epsilon_4\}), (\vartheta_3, \{\epsilon_1, \epsilon_2\}, \{\epsilon_3\})\},\\
(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) &= \{(\vartheta_1, \{\epsilon_3, \epsilon_4\}, \{\epsilon_1, \epsilon_2\}), (\vartheta_2, \{\epsilon_1, \epsilon_2, \epsilon_3\}, \{\epsilon_4\}), (\vartheta_3, \{\epsilon_1, \epsilon_4\}, \phi\},\\
(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) &= \{(\vartheta_1, \{\epsilon_1, \epsilon_3, \epsilon_4\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1, \epsilon_2, \epsilon_3\}, \{\epsilon_4\}), (\vartheta_3, \{\epsilon_1, \epsilon_2, \epsilon_4\}, \phi)\}.
\end{aligned}
$$

Thus $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected space since there does not exist* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$.

Example 8. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ *and* $\mu = \{\vartheta_1, \vartheta_2\}$. *So, the* $\mathfrak{bsms}$ $\tilde{\tilde{\mathfrak{m}}}$ *over* Π *is given by* $\tilde{\tilde{\mathfrak{m}}} = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)\}$ *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, $(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$ *defined as follows:*

$$
\begin{aligned}
(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) &= \{(\vartheta_1, \{\epsilon_1\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1\}, \{\epsilon_2\})\},\\
(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) &= \{(\vartheta_1, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1\}), (\vartheta_2, \{\epsilon_2, \epsilon_3\}, \{\epsilon_1\})\},\\
(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) &= \{(\vartheta_1, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\})\}.
\end{aligned}
$$

Therefore, $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected since* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *form a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$.

Theorem 11. *A* $\mathfrak{bsms}$ $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *over* Π *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected space if and only if there are two* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-closed sets* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *s. t.* $\ddot{\lambda}_1(\neg\vartheta) \neq \phi$, $\ddot{\lambda}_2(\neg\vartheta) \neq \phi$ *for some* $\neg\vartheta \in \neg\mu$*, and* $\ddot{\lambda}_1(\neg\vartheta) \cup \ddot{\lambda}_2(\neg\vartheta) = \Pi$ *for each* $\neg\vartheta \in \neg\mu$ *and* $\ddot{\lambda}_1(\neg\vartheta) \cap \ddot{\lambda}_2(\neg\vartheta) = \phi$ *for each* $\neg\vartheta \in \neg\mu$.

Proof. Suppose that $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected. Then, there exist $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$, say, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$. Then,

$$\begin{aligned}
&\ddot{\zeta}_1(\vartheta) \cup \ddot{\zeta}_2(\vartheta) = \Pi \textit{ for all } \vartheta \in \mu,\\
&\ddot{\zeta}_1(\vartheta) \cap \ddot{\zeta}_2(\vartheta) = \phi \textit{ for all } \vartheta \in \mu \textit{ and}\\
&\ddot{\zeta}_1(\vartheta) \neq \phi, \ddot{\zeta}_2(\vartheta) \neq \phi \textit{ for some } \vartheta \in \mu.
\end{aligned}$$

Since $\ddot{\zeta}_1(\vartheta) = \ddot{\lambda}_1^c(\neg\vartheta)$ and $\ddot{\zeta}_2(\vartheta) = \ddot{\lambda}_2^c(\neg\vartheta)$. Now, we get

$$\begin{aligned}
&\ddot{\lambda}_1^c(\neg\vartheta) \cup \ddot{\lambda}_2^c(\neg\vartheta) = \Pi \textit{ for all } \vartheta \in \mu,\\
&\ddot{\lambda}_1^c(\neg\vartheta) \cap \ddot{\lambda}_2^c(\neg\vartheta) = \phi \textit{ for all } \vartheta \in \mu \textit{ and}\\
&\ddot{\lambda}_1^c(\vartheta) \neq \phi, \ddot{\zeta}_2(\vartheta) \neq \phi \textit{ for some } \vartheta \in \mu.
\end{aligned}$$

From, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$, then $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)^c$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)^c$ are $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-closed sets. Conversely, assume that there are $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-closed sets $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ s. t.

$$\begin{aligned}
&\ddot{\lambda}_1(\neg\vartheta) \cup \ddot{\lambda}_2(\neg\vartheta) = \Pi \textit{ for all } \neg\vartheta \in \neg\mu,\\
&\ddot{\lambda}_1(\neg\vartheta) \cap \ddot{\lambda}_2(\neg\vartheta) = \phi \textit{ for all } \neg\vartheta \in \neg\mu \textit{ and}\\
&\ddot{\lambda}_1(\neg\vartheta) \neq \phi, \ddot{\lambda}_2(\neg\vartheta) \neq \phi \textit{ for some } \neg\vartheta \in \neg\mu.
\end{aligned}$$

Then $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)^c$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)^c$ are $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-open sets s. t.

$$\begin{aligned}
&\ddot{\zeta}_1^c(\vartheta) = \ddot{\lambda}_1(\neg\vartheta) \neq \phi \textit{ and } \ddot{\zeta}_2^c(\vartheta) = \ddot{\lambda}_2(\neg\vartheta) \neq \phi \textit{ for some } \vartheta \in \mu,\\
&\ddot{\zeta}_1^c(\vartheta) \cup \ddot{\zeta}_2^c(\vartheta) = \ddot{\lambda}_1(\neg\vartheta) \cup \ddot{\lambda}_2(\neg\vartheta) = \Pi \textit{ for all } \vartheta \in \mu \textit{ and}\\
&\ddot{\zeta}_1^c(\vartheta) \cap \ddot{\zeta}_2^c(\vartheta) = \ddot{\lambda}_1(\neg\vartheta) \cap \ddot{\lambda}_2(\neg\vartheta) = \phi \textit{ for all } \vartheta \in \mu.
\end{aligned}$$

Thus, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)^c$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)^c$ form $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$. Thus, $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected space.

Theorem 12. *The* $\mathfrak{bs}$ *intersection of a pair of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected spaces over a common universal set is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Proof. Let $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ and $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ be two $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_i$-connected spaces over Π, $i = 1, 2$ and $\tilde{\tilde{\mathfrak{m}}} = \tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cap}} \tilde{\tilde{\mathfrak{m}}}_2$. We need to show that the space $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected. If we say that $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is not $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected. Then there exist two $\mathfrak{bss}s$ $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$,$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ $\tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$, which forms a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ in $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$. From $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ $\tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}$, then $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}_1$ and $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{m}}}_2$. This lead to $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ form a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ in $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ and also $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ form a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ in $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ which is the contradiction to given hypothesis. Therefore, $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-connected space over Π.

Remark 3. *The* $\mathfrak{bs}$ *union of a pair of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected spaces over the common universal set may not be* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected.*

Example 9. *Let* $\Pi = \{\epsilon_1, \epsilon_2\}$, $\mu = \{\vartheta_1, \vartheta_2\}$, $\tilde{\tilde{\mathfrak{m}}}_1 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)\}$ *and* $\tilde{\tilde{\mathfrak{m}}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\}$, *where*

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \{(\vartheta_1, \phi, \Pi), (\vartheta_2, \Pi, \phi)\} \text{ and}$$
$$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \{(\vartheta_1, \Pi, \phi), (\vartheta_2, \phi, \Pi)\}.$$

Clearly $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ *and* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ *are* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected spaces over* Π *where* $\tilde{\tilde{\mathfrak{m}}} = \tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cup}} \tilde{\tilde{\mathfrak{m}}}_2$. *But we note that* $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cup}} \tilde{\tilde{\mathfrak{m}}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\}$ *is not a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-connected space over* Π *since* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *and* $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ *form a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ *in* $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cup}} \tilde{\tilde{\mathfrak{m}}}_2$.

Proposition 14. *The* $\mathfrak{bs}$ *union of a pair of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected spaces over the common universal set is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected.*

Proof. Obvious.

Remark 4. *The* $\mathfrak{bs}$ *intersection of a pair of* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected spaces over the common universal set need not be a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected space.*

Example 10. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta_1, \vartheta_2\}$, $\tilde{\tilde{\mathfrak{m}}}_1 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\}$ *and* $\tilde{\tilde{\mathfrak{m}}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu), (\ddot{\zeta}_4, \ddot{\lambda}_4, \mu)\}$, *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, $(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)$, $(\ddot{\zeta}_4, \ddot{\lambda}_4, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$ *defined as follows*

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \{(\vartheta_1, \{\epsilon_1\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1, \epsilon_2\}, \{\epsilon_3\})\},$$
$$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \{(\vartheta_1, \{\epsilon_2, \epsilon_3\}, \phi), (\vartheta_2, \{\epsilon_3\}, \{\epsilon_1\})\},$$
$$(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) = \{(\vartheta_1, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\})\} \text{ and}$$
$$(\ddot{\zeta}_4, \ddot{\lambda}_4, \mu) = \{(\vartheta_1, \{\epsilon_2\}, \{\epsilon_1\}), (\vartheta_2, \{\epsilon_2\}, \{\epsilon_1\})\}.$$

Clearly $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ *and* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ *are* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected spaces over* Π *where* $\tilde{\tilde{\mathfrak{m}}} = \tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cap}} \tilde{\tilde{\mathfrak{m}}}_2$. *But we note that* $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cap}} \tilde{\tilde{\mathfrak{m}}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu)\}$ *is not a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected space over* Π *since there is no two* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ *in* $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\cap}} \tilde{\tilde{\mathfrak{m}}}_2$.

Proposition 15. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *over* Π. *If there exist a nonnull, non-absolute* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-clopen set* $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ *over* Π *with* $\ddot{\zeta}(\vartheta) \cup \ddot{\zeta}^c(\vartheta) = \Pi$ *for each* $\vartheta \in \mu$, *then* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected.*

Proof. Since $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ is a nonnull, non-absolute $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-clopen set, then $(\ddot{\zeta}, \ddot{\lambda}, \mu)^c$ is a nonnull non-absolute $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-clopen set. By Proposition 2 and the assumption, we get

$$\ddot{\zeta}(\vartheta) \cup \ddot{\zeta}^c(\vartheta) = \Pi \text{ for each } \vartheta \in \mu \text{ and } \ddot{\lambda}(\neg\vartheta) \cap \ddot{\lambda}^c(\neg\vartheta) = \phi \text{ for each } \neg\vartheta \in \neg\mu,$$

and

$$\ddot{\zeta}(\vartheta) \cap \ddot{\zeta}^c(\vartheta) = \phi \text{ for each } \vartheta \in \mu \text{ and } \ddot{\lambda}(\neg\vartheta) \cup \ddot{\lambda}^c(\neg\vartheta) = \Pi \text{ for each } \neg\vartheta \in \neg\mu,$$

Therefore, $(\ddot{\zeta}, \ddot{\lambda}, \mu)$ and $(\ddot{\zeta}, \ddot{\lambda}, \mu)^c$ form a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$. Hence, $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$-disconnected space.

Remark 5. *If there exist a nonnull, non-absolute* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-clopen set, then* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *may not be a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected space.*

Example 11. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta_1, \vartheta_2\}$, $\tilde{\tilde{\mathfrak{m}}}_1 = \{(\Phi, \tilde{\tilde{\Pi}}, \mu), (\tilde{\tilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu), (\ddot{\zeta}_3, \ddot{\lambda}_3, \mu)\}$, *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$, $(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$ *defined as follows*

$$\begin{aligned}
&(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \{(\vartheta_1, \{\epsilon_1, \epsilon_2\}, \{\epsilon_3\}), (\vartheta_2, \{\epsilon_1\}, \{\epsilon_3\})\},\\
&(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \{(\vartheta_1, \{\epsilon_3\}, \{\epsilon_1, \epsilon_2\}), (\vartheta_2, \{\epsilon_3\}, \{\epsilon_1\})\} \textit{ and}\\
&(\ddot{\zeta}_3, \ddot{\lambda}_3, \mu) = \{(\vartheta_1, \Pi, \phi), (\vartheta_2, \{\epsilon_1, \epsilon_3\}, \phi)\}.
\end{aligned}$$

Obviously, $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ *is nonnull, non-absolute* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-clopen but* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is not a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-disconnected space since there does not exist* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}$*-separation of* $(\tilde{\tilde{\Pi}}, \Phi, \mu)$.

Proposition 16. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ *and* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ *be two* $\mathfrak{bsms}$s *over* Π. *Then,*

(i) If $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$*-connected s. t.* $\tilde{\tilde{\mathfrak{m}}}_2 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_1$, *then* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$*-connected.*

(ii) If $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$*-disconnected s. t.* $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_2$, *then* $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$*-disconnected.*

Proof.

(i) Assume that $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$-connected s. t. $\tilde{\tilde{\mathfrak{m}}}_2 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_1$. Assume the contrary that $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$ and $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ in $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$. Since $\tilde{\tilde{\mathfrak{m}}}_2 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_1$, then $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu)$, $(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)$ are $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \mu)$ in $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$. This is contradiction. Therefore, $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$-connected.

(ii) Let $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ be a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$-disconnected s. t. $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_2$. Assume the contrary that $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ is a $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$-connected space. Since $\tilde{\tilde{\mathfrak{m}}}_1 \tilde{\tilde{\subseteq}} \tilde{\tilde{\mathfrak{m}}}_2$, then by (1), we get $(\Pi, \tilde{\tilde{\mathfrak{m}}}_1, \mu, \neg\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_1$-connected. This is contradiction. Therefore, $(\Pi, \tilde{\tilde{\mathfrak{m}}}_2, \mu, \neg\mu)$ is $\mathfrak{bs}$ $\tilde{\tilde{\mathfrak{m}}}_2$-disconnected.

Definition 24. *Let* $(\Pi, \tilde{\tilde{\mathfrak{m}}}, \mu, \neg\mu)$ *be a* $\mathfrak{bsms}$ *and* $(\ddot{\zeta}, \ddot{\lambda}, \mu) \tilde{\tilde{\in}} \mathfrak{bss}(\Pi)$. *Then the collection*

$$\widetilde{\widetilde{\mathfrak{m}}}_{(\ddot{\zeta},\ddot{\lambda},\mu)} = \{(\ddot{\zeta},\ddot{\lambda},\mu)\ \widetilde{\widetilde{\cap}}\ (\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\colon (\ddot{\zeta}_\delta,\ddot{\lambda}_\delta,\mu)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{m}}}, \delta\in\Delta\}.$$

Then $(\epsilon_{(\ddot{\zeta},\ddot{\lambda},\mu)}, \widetilde{\widetilde{\mathfrak{m}}}_{(\ddot{\zeta},\ddot{\lambda},\mu)}, \mu, \neg\mu)$ *is called a* $\mathfrak{bs}$ *minimal subspace of* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *and denoted by* $\mathfrak{bsmss}$.

Proposition 17. *Let* $\big((\ddot{\zeta},\ddot{\lambda},\mu), \widetilde{\widetilde{\mathfrak{m}}}_{(\ddot{\zeta},\ddot{\lambda},\mu)}, \mu, \neg\mu\big)$ *be* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected, then* $(\ddot{\zeta},\ddot{\lambda},\mu)$ *is* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected.*

Proof. Let $\big((\ddot{\zeta},\ddot{\lambda},\mu), \widetilde{\widetilde{\mathfrak{m}}}_{(\ddot{\zeta},\ddot{\lambda},\mu)}, \mu, \neg\mu\big)$ be a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-connected space. Assume $(\ddot{\zeta},\ddot{\lambda},\mu)$ is $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-disconnected, then there exist $\widetilde{\widetilde{\mathfrak{m}}}$-separated $\mathfrak{bss}s$, say, $(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$ and $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$ of $(\ddot{\zeta},\ddot{\lambda},\mu)$, so by Proposition 10 that $(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$ and $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$ are $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-separation of $(\ddot{\zeta},\ddot{\lambda},\mu)$. This is a contradiction. Thus, $(\ddot{\zeta},\ddot{\lambda},\mu)$ is a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-connected space.

Definition 25. *A property* $\mathcal{P}$ *of a* $\mathfrak{bsms}$ $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is said to be a* $\mathfrak{bs}$ *minimal hereditary property* ($\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-hereditary property*) *if every* $\mathfrak{bsmss}$ $(\Omega, \widetilde{\widetilde{\mathfrak{m}}}_\Omega, \mu, \neg\mu)$ *of* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *also has the property* $\mathcal{P}$.

Proposition 18. *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{m}}}_\Omega, \mu, \neg\mu)$ *be a* $\mathfrak{bsmss}$ *of* $\mathfrak{bsms}$ $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *over* Π *and* $({}^\Omega\ddot{\zeta}, {}^\Omega\ddot{\lambda}, \mu)$ *be a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-closed set in* Ω*. Then* $(\ddot{\zeta},\ddot{\lambda},\mu)$ *is a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-closed set in* Π.

Proof. Suppose that $({}^\Omega\ddot{\zeta}, {}^\Omega\ddot{\lambda}, \mu)$ is a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}_\Omega$-closed set in Ω. Thus $({}^\Omega\ddot{\zeta}, {}^\Omega\ddot{\lambda}, \mu)^c = ({}^\Omega\ddot{\lambda}, {}^\Omega\ddot{\zeta}, \mu)$ is a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}_\Omega$-open set in Ω, where $(\ddot{\lambda},\ddot{\zeta},\mu)$ is a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-open set in Π. Thus, $(\ddot{\lambda},\ddot{\zeta},\mu)^c = (\ddot{\zeta},\ddot{\lambda},\mu)$ is a $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$-closed set in Π.

Remark 6. *The* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected space* (*resp.* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-disconnected space*) *is not a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-hereditary property.*

Example 12. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta_1, \vartheta_2\}$ *and* $\widetilde{\widetilde{\mathfrak{m}}} = \{(\Phi, \widetilde{\widetilde{\Pi}}, \mu), (\ddot{\zeta}_1,\ddot{\lambda}_1,\mu), (\ddot{\zeta}_2,\ddot{\lambda}_2,\mu), (\ddot{\zeta}_3,\ddot{\lambda}_3,\mu)\}$ *where* $(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu)$, $(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu)$, $(\ddot{\zeta}_3,\ddot{\lambda}_3,\mu)\ \widetilde{\widetilde{\in}}\ \mathfrak{bss}(\Pi)$*, defined as follows*

$$(\ddot{\zeta}_1,\ddot{\lambda}_1,\mu) = \{(\vartheta_1, \{\epsilon_1\}, \{\epsilon_2,\epsilon_3\}), (\vartheta_2, \{\epsilon_1\}, \{\epsilon_2,\epsilon_3\})\},$$
$$(\ddot{\zeta}_2,\ddot{\lambda}_2,\mu) = \{(\vartheta_1, \{\epsilon_2\}, \{\epsilon_1,\epsilon_3\}), (\vartheta_2, \{\epsilon_2\}, \{\epsilon_1,\epsilon_3\})\} \text{ and}$$
$$(\ddot{\zeta}_3,\ddot{\lambda}_3,\mu) = \{(\vartheta_1, \{\epsilon_1,\epsilon_2\}, \{\epsilon_3\}), (\vartheta_2, \{\epsilon_1,\epsilon_2\}, \{\epsilon_3\})\}.$$

Therefore, $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected space.*
Now let $\Omega = \{\epsilon_1, \epsilon_2\}$*, then* $\widetilde{\widetilde{\mathfrak{m}}}_\Omega = \{(\Phi, \widetilde{\widetilde{\Omega}}, \mu), ({}^\Omega\ddot{\zeta}_1, {}^\Omega\ddot{\lambda}_1, \mu), ({}^\Omega\ddot{\zeta}_2, {}^\Omega\ddot{\lambda}_2, \mu), ({}^\Omega\ddot{\zeta}_3, {}^\Omega\ddot{\lambda}_3, \mu)\}$*, s. t.*

$$({}^\Omega\ddot{\zeta}_1, {}^\Omega\ddot{\lambda}_1, \mu) = \{(\vartheta_1, \{\epsilon_1\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_1\}, \{\epsilon_2,\})\},$$
$$({}^\Omega\ddot{\zeta}_2, {}^\Omega\ddot{\lambda}_2, \mu) = \{(\vartheta_1, \{\epsilon_2\}, \{\epsilon_1\}), (\vartheta_2, \{\epsilon_2\}, \{\epsilon_1\})\} \text{ and}$$
$$({}^\Omega\ddot{\zeta}_3, {}^\Omega\ddot{\lambda}_3, \mu) = \{(\vartheta_1, \Omega, \phi), (\vartheta_2, \Omega, \phi)\} = (\widetilde{\widetilde{\Omega}}, \Phi, \mu).$$

Clearly, $(\Omega, \widetilde{\widetilde{\mathfrak{m}}}_\Omega, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-disconnected subspace of* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$*. While* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected space.*

Example 13. *Let* $\Pi = \{\epsilon_1, \epsilon_2, \epsilon_3\}$, $\mu = \{\vartheta_1, \vartheta_2\}$ *and* $\widetilde{\widetilde{\mathfrak{m}}} = \{(\Phi, \widetilde{\widetilde{\Pi}}, \mu), (\widetilde{\widetilde{\Pi}}, \Phi, \mu), (\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu)\}$ *where* $(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu), (\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) \,\widetilde{\widetilde{\in}}\, \mathfrak{bss}(\Pi)$, *defined as follows*

$$(\ddot{\zeta}_1, \ddot{\lambda}_1, \mu) = \{(\vartheta_1, \{\epsilon_1\}, \{\epsilon_2\}), (\vartheta_2, \{\epsilon_2\}, \{\epsilon_1, \epsilon_3\})\} \text{ and}$$
$$(\ddot{\zeta}_2, \ddot{\lambda}_2, \mu) = \{(\vartheta_1, \{\epsilon_2, \epsilon_3\}, \phi), (\vartheta_2, \{\epsilon_1, \epsilon_3\}, \{\epsilon_2\})\}.$$

Therefore, $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-disconnected space.*
Let $\Omega = \{\epsilon_3\}$, *then* $\widetilde{\widetilde{\mathfrak{m}}}_\Omega = \{(\Phi, \widetilde{\widetilde{\Omega}}, \mu), ({}^\Omega\ddot{\zeta}_1, {}^\Omega\ddot{\lambda}_1, \mu), ({}^\Omega\ddot{\zeta}_2, {}^\Omega\ddot{\lambda}_2, \mu)\}$, *s. t.*

$$({}^\Omega\ddot{\zeta}_1, {}^\Omega\ddot{\lambda}_1, \mu) = \{(\vartheta_1, \phi, \phi), (\vartheta_2, \phi, \Omega)\},$$
$$({}^\Omega\ddot{\zeta}_2, {}^\Omega\ddot{\lambda}_2, \mu) = \{(\vartheta_1, \Omega, \phi), (\vartheta_2, \Omega, \phi)\}.$$

Clearly, $(\Omega, \widetilde{\widetilde{\mathfrak{m}}}_\Omega, \mu, \neg\mu)$ *is* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected subspace of* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$. *While* $(\Pi, \widetilde{\widetilde{\mathfrak{m}}}, \mu, \neg\mu)$ *is a* $\mathfrak{bs}$ $\widetilde{\widetilde{\mathfrak{m}}}$*-connected space.*

6. Conclusions and Future Research

In this paper, we have presented a comprehensive study on bipolar soft minimal structures in the context of bipolar soft topology. We began by providing a brief overview of relevant preliminaries. The new structure, termed the bipolar soft minimal structure, was introduced, and key operators in bipolar soft minimal spaces, such as the $\widetilde{\widetilde{\mathfrak{m}}}$-interior, $\widetilde{\widetilde{\mathfrak{m}}}$-closure, and $\widetilde{\widetilde{\mathfrak{m}}}$-boundary, were explored in detail. Additionally, we introduced the concepts of $\widetilde{\widetilde{\mathfrak{m}}}$-separated bipolar soft sets and bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected sets, along with their essential properties. A new concept, called bipolar soft minimal connected spaces, was also defined, and it was demonstrated that the bipolar soft intersection of a pair of bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected spaces over the common universal set results in a bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected space. However, we showed that the bipolar soft $\widetilde{\widetilde{\mathfrak{m}}}$-connected space does not exhibit the $\widetilde{\widetilde{\mathfrak{m}}}$-hereditary property. Finally, we discussed various relationships, properties, and examples of these new concepts within bipolar soft minimal spaces, providing a foundation for further research in this area.

Future research on bipolar soft minimal spaces can focus on several key aspects, including compactness, continuous mappings, and separation axioms. By exploring these aspects—compactness, continuous mappings, and separation axioms—future research can significantly contribute to the development of bipolar soft minimal spaces and their applications across a range of fields.

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