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The Relationship of Borel Distribution and Horadam Polynomials Leads to Analytical Bi-Univalent Functions

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Abstract. The Borel distribution is a practical and applicable model for a wide range of real-world applications. Using the Borel distribution as a foundation, we create a novel subclass of analytic bi-univalent functions in this study. We employ these functions, which involve the ultraspherical polynomials, to create our new subclass. For functions that fall within the constructed class, we investigate alternative estimations of the Maclaurin coefficients and solve the Fekete-Szego functional problem.

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1. Preliminaries

In many branches of mathematics and physics, particularly in the study of differential equations and approximation theory, orthogonal polynomials are a class of mathematical functions that appear. There are a collection of polynomials that are orthogonal to a particular weight function across a specified range. This indicates that unless the polynomials are equal, the outcome of multiplying the polynomials by one another and integrating across the interval is zero.

Orthogonal polynomials come in a variety of families, each with a unique weight function and interval of orthogonality. The Legendre polynomials, Chebyshev polynomials,

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Hermite polynomials, and Jacobi polynomials are a few of the most well-known families. Each of these families has unique characteristics and uses, see [1–6].

Numerous areas of physics and mathematics, such as numerical analysis, probability theory, and quantum mechanics, all heavily rely on orthogonal polynomials. For instance, these polynomials can be applied to the numerical computation of integrals, the solution of differential equations, and the investigation of the behavior of random variables. Let $\mathcal{A}$ denote the class of functions f that takes the form:

$$f(\varphi) = \varphi + k_2\varphi^2 + k_3\varphi^3 + \cdots, \quad (\varphi \in \mathfrak{B}), \tag{1}$$

that are analytic in the disk $\mathfrak{B} = \{\varphi \in \mathbb{C} : |\varphi| < 1\}$. Also, we represent by $\mathcal{S}$ the subclass of $\mathcal{A}$ comprising functions of the Eq. (1) which are also univalent in $\mathfrak{B}$.

Geometric function theory can benefit greatly from the powerful tools that differential subordination of analytical functions provides. Miller and Mocanu [7] introduced the first differential subordination problem, additionally, see [8]. The majority of the developments in the field are compiled in Miller and Mocanu's book [9].

Every mathematical function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(\varphi)) = \varphi \qquad (\varphi \in \mathfrak{B})$$

and

$$w = f(f^{-1}(w)) \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$g(w) = f^{-1}(w) = w - k_2 w^2 + (-k_3 + 2k_2^2)w^3 - (k_4 + 5k_2^3 - 5k_3k_2)w^4 + \cdots. \tag{2}$$

A function is said to have the property of being bi-univalent in $\mathfrak{B}$ if both $f(\varphi)$ and $f^{-1}(\varphi)$ have the property of being univalent in $\mathfrak{B}$.

Let us refer to the group of bi-univalent functions in $\mathfrak{B}$ as Σ, which is defined by the Eq. (1). Some examples from the class Σ are as follows:

$$\frac{\varphi}{1-\varphi}, \qquad \log\frac{1}{1-\varphi}.$$

However, Σ does not contain the well-known Koebe function. Other examples of functions that are typical in $\mathfrak{B}$ include the following:

$$\frac{2\varphi - \varphi^2}{2} \text{ and } \frac{\varphi}{1-\varphi^2}.$$

Furthermore, it is not a part of Σ. In class Σ and its subclasses, look for intriguing functions ([10]-[11], [12]-[13]).

Also, in [14–20], estimates were made but not sharp for the first two coefficients $|k_2|$ and $|k_3|$ in the Taylor-Maclaurin series expansion (1). These developments were motivated by the groundbreaking work of Srivastava et al [21].

In 2009, Horzum and Kocer published their related to Horadam polynomials $h_m(d)$, [22]. The recurrence relation, which can be seen in the following sentence, gives us these polynomials to work with, that

$$h_m(d) = \vartheta d h_{m-1}(d) + l h_{m-2}(d), \quad (m \in \mathbb{N} \setminus \{1, 2\}), \tag{3}$$

with

$$h_1(d) = a,\ h_2(d) = td \text{ and } h_3(d) = \vartheta t d^2 + \vartheta l, \tag{4}$$

assuming that a, t, ϑ, and l are real constants.

Remark 1. *Special examples of the Horadam polynomials.*
i) If $a = t = \vartheta = l = 1$, the Fibonacci polynomials sequence is obtained

$$F_m(d) = dF_{m-1}(d) + F_{m-2}(d); \quad F_1(d) = 1, F_2(d) = d.$$

ii) If $a = 2, \quad t = \vartheta = l = 1$, the Lucas polynomials sequence is obtained

$$L_{m-1}(d) = dL_{m-2}(d) + L_{m-3}(d); \quad L_0(d) = 2, \quad L_1(d) = d.$$

iii) $a = 1, t = \vartheta = 2, l = -1$, the Chebyshev polynomials of second kind sequence is obtained

$$U_{m-1}(d) = 2dU_{m-2}(d) - U_{m-3}(d); \quad U_0(d) = 1, U_1(d) = 2d.$$

iv) If $a = t = 1, \vartheta = 2, l = -1$, the Chebyshev polynomials of first kind sequence is obtained

$$T_{m-1}(d) = 2dT_{m-2}(d) - T_{m-3}(d); \quad T_0(d) = 1, T_1(d) = d.$$

v) If $a = l = 1, \quad t = \vartheta = 2$, the Pell polynomials sequence is obtained

$$P_m(d) = 2dP_{m-1}(d) + P_{m-2}(d); \quad P_1(d) = 1, P_2(d) = 2d.$$

vi) If $a = t = \vartheta = 2, l = 1$, the Pell-Lucas polynomials sequence is obtained

$$Q_{m-1}(d) = 2dQ_{m-2}(d) + Q_{m-3}(d); \quad Q_0(d) = 2, Q_1(d) = 2d.$$

In general, the Horadam polynomials have a lot of interesting and useful mathematical properties, and they play a big role in a wide range of math, engineering, and physics applications. Many studies, have looked into the properties of these polynomials, both theoretical and practical.

The following expression serves as an example of how to generate the Horadam polynomials $h_m(d)$:

$$\psi(d,\varphi)=\sum_{m=1}^{\infty}h_m(d)\varphi^{n-1}=\frac{a+(t-a\vartheta)d\varphi}{1-\vartheta d\varphi-l\varphi^2}. \tag{5}$$

In recent years, a great number of studies have investigated significant aspects of geometric function theory. These studies have focused on topics such as coefficient estimates, inclusion relations, and propirtses of the classes. These investigations have made use of a wide variety of probability distributions, such as the Poisson, Pascal, and many others (see, [23–25]).

If it is conceivable for d to take on the values $1, 2, 3, ...,$ and so on with the stated probability, then it is said that a discrete random variable, which is indicated by X, should have a Borel distribution.

$$\frac{e^{-v}}{1!},\frac{2ve^{-2v}}{2!},\frac{9v^2e^{-3v}}{3!},..., \tag{6}$$

accordingly, in which cases they are referred to as the parameters. Hence

$$P(d=\partial)=\frac{(v\partial)^{\partial-1}e^{-v\partial}}{\partial!},\qquad \partial=1,2,3,....$$

Finally, we give a power series with Borel distribution coefficients.

$$\mathbb{F}(v,\varphi)=\varphi+\sum_{m=2}^{\infty}\frac{(v\,(m-1))^{m-2}\,e^{-v(m-1)}}{(m-1)!}\varphi^m,\qquad \varphi\in\mathfrak{B}. \tag{7}$$

Take into account the convolution-defined linear operator $\mathbb{P}_\gamma:\mathcal{A}\to\mathcal{A}$.

$$\mathbb{P}_\gamma f(\varphi)=\mathbb{F}(v,\varphi)*f(\varphi)=\varphi+\sum_{m=2}^{\infty}\frac{(v\,(m-1))^{m-2}\,e^{-v(m-1)}}{(m-1)!}k_m\varphi^m,\qquad \varphi\in\mathfrak{B}. \tag{8}$$

Too many scholars to count have studied the relationship between bi-univalent functions and orthogonal polynomials recently, but some of the ones worth mentioning are [26–35]. To the best of our knowledge, we have not been able to locate any previous work in the literature that deals with bi-univalent functions for subordinate Horadam polynomials that use the Borel distribution. We derive bounds for the $|k_2|$ and $|k_3|$ Taylor-Maclaurin coefficients and describe a new subclass of Σ involving the Borel distribution connected to Horadam polynomials. In addition, we address the Fekete-Szegö functional difficulties for this new category of functions.

2. Delimitations of the class $\varrho^t_\Sigma(d,\vartheta,l,\gamma)$

This section starts off by providing a definition for the new subclass $\varrho^t_\Sigma(d,\vartheta,l,\gamma)$, which is related to the Borel distribution series.

Definition 1. *In the event that the subordinations listed below are satisfied, a function denoted by (1) is considered to be a member of the class* $\varrho_{\Sigma}^{t}(d, \vartheta, l, \gamma)$*:*

$$(1-\gamma)\frac{\varphi \mathbb{P}_{\gamma} f'(\varphi)}{\mathbb{P}_{\gamma} f(\varphi)} + \gamma\left(1 + \frac{\varphi \mathbb{P}_{\gamma} f''(\varphi)}{\mathbb{P}_{\gamma} f'(\varphi)}\right) \prec \psi(d, \varphi) + 1 - a \tag{9}$$

and

$$(1-\gamma)\frac{w \mathbb{P}_{\gamma} f'(w)}{\mathbb{P}_{\gamma} f(w)} + \gamma\left(1 + \frac{w \mathbb{P}_{\gamma} f''(w)}{\mathbb{P}_{\gamma} f'(w)} \prec\right) \psi(d, w) + 1 - a, \tag{10}$$

where $d \in \mathbb{R}$*, and (2) describes the* $function\ g = f^{-1}$.

Example 1. $\varrho_{\Sigma}^{t}(d, \vartheta, l, 0) = \varrho_{\Sigma}^{t}(d, \vartheta, l)$*, is the class of functions* f *that is given by (1) and satisfies the following condition: This holds true with regard to* $\gamma = 0$*.*

$$\frac{\varphi \mathbb{P}_{\gamma} f'(\varphi)}{\mathbb{P}_{\gamma} f(\varphi)} \prec \psi(d, \varphi) + 1 - a \tag{11}$$

and

$$\frac{w \mathbb{P}_{\gamma} f'(w)}{\mathbb{P}_{\gamma} f(w)} \prec \psi(d, w) + 1 - a, \tag{12}$$

where $d \in \mathbb{R}$*, and (2) describes the* $function\ g = f^{-1}$.

Example 2. $\varrho_{\Sigma}^{t}(d, \vartheta, l, 1) = \varrho_{\Sigma}^{t}(d, \vartheta, l)$*, is the class of functions* f *that is given by (1) and satisfies the following condition: This holds true with regard to* $\gamma = 1$*..*

$$1 + \frac{\varphi \mathbb{P}_{\gamma} f''(\varphi)}{\mathbb{P}_{\gamma} f'(\varphi)} \prec \psi(d, \varphi) + 1 - a \tag{13}$$

and

$$1 + \frac{w \mathbb{P}_{\gamma} f''(w)}{\mathbb{P}_{\gamma} f'(w)} \prec \psi(d, w) + 1 - a, \tag{14}$$

where $d \in \mathbb{R}$*, and (2) describes the* $function\ g = f^{-1}$.

We will start by giving the estimated coefficients for class $\varrho_{\Sigma}^{t}(d, \vartheta, l, \gamma)$ from Definition 1.

Theorem 1. *Recognize that class* $\varrho_{\Sigma}^{t}(d, \vartheta, l)$ *is a member of the function* $f \in \Sigma$ *defined by reference (1) . Then*

$$|k_2| \leq \frac{|td| \sqrt{t\,|d|}}{\sqrt{\left|2\,(1+2\gamma)\, v e^{-2v}\,(td)^2 - (1+\gamma)^2\, e^{-2v}\,(\vartheta t d^2 + a l)\right|}},$$

and

$$|k_3| \leq \frac{t^2 d^2}{(1+\gamma)^2\, e^{-2v}} + \frac{t|d|}{2\,(1+2\gamma)\, v e^{-2v}}.$$

Proof. Let $f \in \varrho_{\Sigma}^{t}(d, \vartheta, l, \gamma)$. From Definition 1, we can write

$$(1-\gamma)\frac{\varphi \mathbb{P}_{\gamma} f'(\varphi)}{\mathbb{P}_{\lambda} f(\varphi)} + \gamma\left(1 + \frac{\varphi \mathbb{P}_{\lambda} f''(\varphi)}{\mathbb{P}_{\lambda} f'(\varphi)}\right) = \psi(d, \varkappa(\varphi)) + 1 - a \tag{15}$$

and

$$(1-\gamma)\frac{w \mathbb{P}_{\gamma} f'(w)}{\mathbb{P}_{\gamma} f(w)} + \gamma\left(1 + \frac{w \mathbb{P}_{\gamma} f''(w)}{\mathbb{P}_{\gamma} f'(w)}\right) = \psi(d, \tau(w)) + 1 - a, \tag{16}$$

the point at which the analytical functions $\varkappa$ and τ assume the form

$$\varkappa(\varphi) = b_1\varphi + b_2\varphi^2 + b_3\varphi^3 + \cdots, \quad (\varphi \in \mathfrak{B})$$

and

$$\tau(w) = i_1 w + i_2 w^2 + i_3 w^3 + \cdots, \quad (w \in \mathfrak{B}),$$

such that $\varkappa(0) = \tau(0) = 0$ and $|\varkappa(\varphi)| < 1$, $|\tau(w)| < 1$ for all $\varphi, w \in \mathfrak{B}$.
From the equalities (15) and (16), it is what we get

$$(1-\gamma)\frac{\varphi \mathbb{P}_{\gamma} f'(\varphi)}{\mathbb{P}_{\gamma} f(\varphi)} + \gamma\left(1 + \frac{\varphi \mathbb{P}_{\gamma} f''(\varphi)}{\mathbb{P}_{\gamma} f'(\varphi)}\right) = 1 + h_2(d) b_1 \varphi + \left[h_2(d) b_2 + h_3(d) b_1^2\right]\varphi^2 + \cdots \tag{17}$$

and

$$(1-\gamma)\frac{w \mathbb{P}_{\gamma} f'(w)}{\mathbb{P}_{\gamma} f(w)} + \gamma\left(1 + \frac{w \mathbb{P}_{\gamma} f''(w)}{\mathbb{P}_{\gamma} f'(w)}\right) = 1 + h_2(d) i_1 w + \left[h_2(d) i_2 + h_3(d) i_1^2\right] w^2 + \cdots. \tag{18}$$

It is common knowledge that if

$$|\varkappa(\varphi)| = \left|b_1\varphi + b_2\varphi^2 + b_3\varphi^3 + \cdots\right| < 1, \quad (\varphi \in \mathfrak{B})$$

and

$$|\tau(w)| = \left|i_1 w + i_2 w^2 + i_3 w^3 + \cdots\right| < 1, \quad (w \in \mathfrak{B}),$$

then

$$|b_j| \le 1 \text{ and } |i_j| \le 1 \text{ for all } j \in \mathbb{N}. \tag{19}$$

When we compare the relevant coefficients in (17) and (18), we get the following:

$$(1+\gamma)\, e^{-v} k_2 = h_2(d) b_1, \tag{20}$$

$$2\,(1+2\gamma)\, v e^{-2v} k_3 = h_2(d) b_2 + h_3(d) b_1^2, \tag{21}$$

$$-\,(1+\gamma)\, e^{-v} k_2 = h_2(d) i_1, \tag{22}$$

and

$$2\,(1+2\gamma)\, v e^{-2v}\left(2k_2^2 - k_3\right) = h_2(d) i_2 + h_3(d) i_1^2. \tag{23}$$

According to(20) and (22),

$$b_1 = -i_1 \tag{24}$$

and

$$2(1+\gamma)^2 e^{-2v} k_2^2 = [h_2(d)]^2 \left(b_1^2 + i_1^2\right). \tag{25}$$

When we combine (21) and (23), we obtain

$$4(1+2\gamma) v e^{-2v} k_2^2 = h_2(d)(b_2 + i_2) + h_3(d)\left(b_1^2 + i_1^2\right). \tag{26}$$

We can find out what it is by changing the value of $\left(b_1^2 + i_1^2\right)$ in (25) onto the right side of (26) .

$$\begin{aligned} &2\left(2(1+2\gamma)v - (1+\gamma)^2 \frac{h_3(d)}{[h_2(d)]^2}\right) e^{-2v} k_2^2 \\ &= h_2(d)(b_2 + i_2). \end{aligned} \tag{27}$$

Moreover computations using (4), and (27), we find that

$$|k_2| \leq \frac{td\sqrt{td}}{\sqrt{\left|2(1+2\gamma) v e^{-2v} (td)^2 - (1+\gamma)^2 e^{-2v} \left(\vartheta t d^2 + al\right)\right|}}.$$

In addition to this, the result that we get when we take away (23) from (21) is.

$$4(1+2\gamma) v e^{-2v} \left(k_3 - k_2^2\right) = h_2(d)(b_2 - i_2) + h_3(d)\left(b_1^2 - i_1^2\right). \tag{28}$$

So, if you take into account (24) and (25), the equation on (28) can be rewritten as

$$\begin{aligned} k_3 = &\frac{[h_2(d)]^2}{2(1+\gamma)^2 e^{-2v}} \left(b_1^2 + i_1^2\right) \\ &+ \frac{h_2(d)}{4(1+2\gamma) v e^{-2v}} (b_2 - i_2). \end{aligned}$$

So, using(4), we come to the conclusion that

$$|k_3| \leq \frac{t^2 d^2}{(1+\gamma)^2 e^{-2v}} + \frac{td}{2(1+2\gamma) v e^{-2v}}.$$

.

An exact limit on the functional space $\left|k_3 - \eta k_2^2\right|$ was obtained by Fekete and Szego in 1933 [36] . This limit was specific to a univalent function f and η that belongs to the interval $[0, 1]$.

Using the values of k_2^2 and k_3, we prove the functional $\left|k_3 - \eta k_2^2\right|$ for class functions $\varrho_{\Sigma}^{t}(d, \vartheta, l, \gamma)$.

Theorem 2. *Recognize that class $\varrho_{\Sigma}^{t}(d,\vartheta,l,\gamma)$ is a member of the function $f\in\Sigma$ defined by reference (1) . Then*

$$\left|k_3-\eta k_2^2\right|\leq\begin{cases}\frac{|td|}{2(1+2\gamma)ve^{-2v}}, & |\eta-1|\leq\theta\\ \frac{(td)^3|1-\eta|}{e^{-2v}\left|\left[2(1+2\gamma)v[td]^2-(1+\lambda)^2(\vartheta td^2+al)\right]\right|}, & |\eta-1|\geq\theta,\end{cases}$$

where

$$\theta=\left|1-\frac{2\left(1+\gamma\right)^2e^{-2v}\left(\vartheta td^2+al\right)}{4\left(1+2\gamma\right)t^2d^2ve^{-2v}}\right|.$$

Proof. From (27) and (28)

$$\begin{aligned}&k_3-\eta k_2^2\\&=(1-\eta)\frac{\left[h_2(d)\right]^3\left(b_2+i_2\right)}{2e^{-2v}\left[2v\left(1+2\gamma\right)\left[h_2(d)\right]^2-\left(1+\gamma\right)^2h_3(d)\right]}\\&+\frac{h_2(d)}{4\left(1+2\gamma\right)ve^{-2v}}\left(b_2-i_2\right)\\&=h_2(d)\left[\mho(\eta)+\frac{1}{4\left(1+2\gamma\right)ve^{-2v}}\right]b_2\\&+h_2(d)\left[\mho(\eta)-\frac{1}{4\left(1+2\gamma\right)ve^{-2v}}\right]i_2,\end{aligned}$$

where

$$\mho(\eta)=\frac{\left[h_2(d)\right]^2\left(1-\eta\right)}{2e^{-2v}\left[2v\left(1+2\gamma\right)\left[h_2(d)\right]^2-\left(1+\gamma\right)^2h_3(d)\right]},$$

Consequently, based on (4), we deduce that

$$\left|k_3-\eta k_2^2\right|\leq\begin{cases}\frac{2|h_2(d)|}{4(1+2\gamma)ve^{-2v}} & |\mho(\eta)|\leq\frac{1}{4(1+2\gamma)ve^{-2v}},\\ 2\left|h_2(d)\right|\left|\mho(\eta)\right| & |\mho(\eta)|\geq\frac{1}{4(1+2\gamma)ve^{-2v}}.\end{cases}$$

.

3. Corollaries

As a result of the theorems called 1 and 2, the following corollaries are true. These corollaries generally correspond to the examples referred to as 1 and 2.

Corollary 1. *Recognize that class $\varrho_{\Sigma}^{t}(d,\vartheta,l)$ is a member of the function $f\in\Sigma$ defined by reference (1) . Then*

$$|k_2| \leq \frac{td\sqrt{td}}{\sqrt{\left|2ve^{-2v}\left(td\right)^2 - e^{-2v}\left(\vartheta td^2 + al\right)\right|}},$$

$$|k_3| \leq \frac{t^2d^2}{e^{-2v}} + \frac{td}{2ve^{-2v}}.$$

and

$$\left|k_3 - \eta k_2^2\right| \leq \begin{cases} \frac{|td|}{2ve^{-2v}}, & |\eta - 1| \leq \left|1 - \frac{2e^{-2v}\left(\vartheta td^2+al\right)}{4t^2d^2ve^{-2v}}\right| \\ \frac{(td)^3|1-\eta|}{e^{-2v}|[2vt^2d^2-(\vartheta td^2+al)]|}, & |\eta - 1| \geq \left|1 - \frac{2e^{-2v}\left(\vartheta td^2+al\right)}{4t^2d^2ve^{-2v}}\right|. \end{cases}$$

Corollary 2. *Recognize that class $\varrho_{\Sigma}^{t}(d,\vartheta,l)$ is a member of the function $f \in \Sigma$ defined by reference (1) . Then*

$$|k_2| \leq \frac{td\sqrt{td}}{\sqrt{\left|6ve^{-2v}\left(td\right)^2 - 4e^{-2v}\left(\vartheta td^2 + al\right)\right|}},$$

$$|k_3| \leq \frac{t^2d^2}{4e^{-2v}} + \frac{td}{6ve^{-2v}}.$$

and

$$\left|k_3 - \eta k_2^2\right| \leq \begin{cases} \frac{|td|}{6ve^{-2v}}, & |\eta - 1| \leq \left|1 - \frac{8e^{-2v}\left(\vartheta td^2+al\right)}{12t^2d^2ve^{-2v}}\right| \\ \frac{2(td)^3|1-\eta|}{e^{-2v}|[6vt^2d^2-4(\vartheta td^2+al)]|}, & |\eta - 1| \geq \left|1 - \frac{8e^{-2v}\left(\vartheta td^2+al\right)}{12t^2d^2ve^{-2v}}\right|. \end{cases}$$

4. Conclusions

In this important study, we created a new category of normalised analytic and bi-univalent functions that are closely related to the famous Borel distribution series $\varrho_{\Sigma}^{t}(d,\vartheta,l,\gamma)$. With our new way of thinking, we were able to find accurate values for the Taylor-Maclaurin coefficients $|k_2|$ and $|k_3|$ and solve the hard Fekete-Szego functional problems. We were able to figure out the results for subclasses $\varrho_{\Sigma}^{t}(d,\vartheta,l,1)$ and $\varrho_{\Sigma}^{t}(d,\vartheta,l,0)$, which are shown in Examples 1 and 2, by cleverly changing the parameters γ. They are connected in a complicated way to the Borel series of distributions. Based on our ground-breaking use of the Borel distribution series (8), future researchers will be able to use the extraordinary Horadam polynomials associated with this distribution series to solve Fekete-Szego functional problems and estimate Taylor-Maclaurin coefficients for new classes of bi-univalent functions.

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